Least–squares Geopotential Approximation by Windowed Fourier and Wavelet Transform

by

WILLI FREEDEN AND VOLKER MICHEL

University of Kaiserslautern
Laboratory of Technomathematics
Geomathematics Group
67653 Kaiserslautern
P.O. Box 3049
Germany

phone: ++49 631 205-2852
fax: ++49 631 29081
email: freeden@mathematik.uni-kl.de
michel@mathematik.uni-kl.de
www: http://www.mathematik.uni-kl.de/~wwwgeo

Abstract. Two possible substitutes of the Fourier transform in geopotential determination are windowed Fourier transform (WFT) and wavelet transform (WT). In this paper we introduce harmonic WFT and WT and show how it can be used to give information about the geopotential simultaneously in the space domain and the frequency (angular momentum) domain. The counterparts of the inverse Fourier transform are derived, which allow us to reconstruct the geopotential from its WFT and WT, respectively. Moreover, we derive a necessary and sufficient condition that an otherwise arbitrary function of space and frequency has to satisfy to be the WFT or WT of a potential. Finally, least–squares approximation and minimum norm (i.e. least–energy) representation, which will play a particular role in geodetic applications of both WFT and WT, are discussed in more detail.

1 Introduction

The spectral representation of a geopotential U such as the earth's external gravitational potential by means of outer harmonics is essential to solving many problems in today's physical geodesy. In future research, however, Fourier expansions

$$U = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \int_A U(y) H_{n,j}(\alpha; y) \ d\omega(y) H_{n,j}(\alpha, \cdot)$$

($d\omega$ denotes the suface element) in tems of outer harmonics

$$\{H_{n,j}(j; \cdot)\}_{\substack{n=0,1,\ldots, \\ j=1,\ldots,2n+1}}$$

will not be the most natural or useful way of representing a harmonic function such as the earths gravitational potential. In order to explain this in more detail we think of the earth's gravitational potential as a signal in which the spectrum evolves over space in significant way. We imagine that at each point on the sphere A around the origin with radius α. The potential refers to a certain combination of frequencies, and that in dependence of the mass distribution inside the earth, these frequencies are constantly changing. This space–evolution of the frequencies is not reflected in the Fourier transform in terms of non–space localizing outer harmonics, at least not directly.

In theory, a member U of the Sobolev space $\mathcal{H}_0$ (of harmonic functions in the outer space A_{ext} of the sphere A with square–integrable restrictions on A) can be reconstructed from its Fourier transform, i.e. the 'amplitude spactrum'

$$\left\{ (U, H_{n,j}(\alpha; \cdot))_{\mathcal{H}_0} \right\}_{\substack{n=0,1,\ldots, \\ j=1,\ldots,2n+1}}$$

with

$$(U, H_{n,j}(\alpha, \cdot))_{\mathcal{H}_0} = \int_A U(x) H_{n,j}(\alpha; x) \ d\omega(x),$$

but the Fourier transform contains information about the frequencies of the potential over all positions instead of showing how the frequencies vary in space.

This paper will present two methods of achieving a space–dependent frequency analysis in geopotential determination which we refer to as the windowed Fourier transform and the wavelet transform. Essential tool is the concept of a harmonic scaling function $\rho \in (0, \infty)$. Roughly speaking, a scaling function is a kernel $\Phi_\rho^{(2)} : \overline{A_{\text{ext}}} \times \overline{A_{\text{ext}}} \to \mathbb{R}$ of the form

$$\Phi_\rho^{(2)}(x, y) = \sum_{n=0}^{\infty} (\varphi_\rho(n))^2 \sum_{j=1}^{2n+1} H_{n,j}(\alpha; x) H_{n,j}(\alpha; y)$$

converging (in $\mathcal{H}_0$–sense) to the 'Dirac–kernel' δ as $\rho \to 0$. As is well–known, the Dirac kernel is given by

$$\delta(x, y) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} H_{n,j}(\alpha; x) H_{n,j}(\alpha; y) \ .$$

In conclusion, $\{\varphi_p(n)\}_{n=0,1,\ldots}$ essentially is a sequence satisfying

$$\lim_{\substack{\rho\to 0\\ \rho>0}} \varphi_\rho(n) = 1$$

for all $n = 0, 1, \ldots$. According to this construction principle, $\{\Phi_\rho^{(2)}\}, \rho \in (0, \infty)$, constitutes an approximate convolution identity, i.e. the convolution integral

$$(\Phi_\rho^{(2))} * U(x) = \int_A \Phi_\rho^{(2)}(x, y)U(y)\ d\omega(y), \qquad x \in \overline{A_{\text{ext}}},$$

formally converges to

$$U(x)(\delta * U)(x) = \int_A \delta(x, y)U(y)\ d\omega(y), \qquad x \in \overline{A_{\text{ext}}},$$

as ρ tends to 0. Therefore, if U is a potential of class $\mathcal{H}_0$, then

$$\Phi_\rho^{(2)}||_{\mathcal{H}_0} * \lim_{\substack{\rho\to 0\\ \rho>0}} ||U - U = 0, \tag{1}$$

where the convolution of the potential U against the kernel $\Phi_\rho^{(2)}$ is defined by

$$\Phi_\rho^{(2)} * U = \int_A \Phi_\rho^{(2)}(\cdot, y)U(y)d\omega(y).$$

The windowed Fourier transform and the wavelet transform are two–parameter representations of a one–parameter (spatial) potential in $\mathcal{H}_0$. This indicates the existence of some redendancy in the both transforms which in turn gives rise to establish the promised least–squares approximation property.

The *windowed Fourier transform (WFT)* can be formulated as a technique known as 'short–space Fourier transform'. This transform works by first dividing a potential (signal) into short consecutive (space) segments and then computing the Fourier coefficients of each segment. The windowed Fourier transform is a space–frequency localization technique in that it determines the frequencies associated with small space portions of the potential. The windowed Fourier segments are constructed by shifting in space and modulating in frequency the 'window kernel' Φ_ρ given by

$$\Phi_\rho(x, y) = \sum_{n=0}^{\infty} \varphi_\rho(n) \sum_{j=1}^{2n+1} H_{n,j}(\alpha; x)H_{n,j}(\alpha; y), \qquad (x, y) \in \overline{A_{\text{ext}}} \times \overline{A_{\text{ext}}}\ .$$

Note that

$$Phi_\rho(x, y) = (Phi_\rho * Phi_\rho)(x, y) = \int_A Phi_\rho(x, z)Phi_\rho(z, y)\ d\omega(z)$$

for all $(x,y) \in \overline{A_{\text{ext}}} \times \overline{A_{\text{ext}}}$. One again, the way of describing the windowed Fourier transform (i.e. the 'short–space Fourier transform') is as follows: Let U be a potential of class $\mathcal{H}_0$. Assume that

$$(FT)(U)(n,j) = \int\limits_A U(y) H_{n,j}(\alpha; y)\ d\omega(y)$$

$((n,j) \in \mathcal{N}, \mathcal{N} = \{(k,l) \mid k = 0,1,\ldots; l = 1,\ldots,2n+1\})$. Φ_ρ with ρ arbitrary, but fixed is a space window ('cutoff kernel'), i.e. Φ_ρ generates an approximate convolution identity $\{\Phi_\rho^{(2)}\}$ in the sense of (??51). Chopping up the potential amounts to multiplying U by the kernel Φ_ρ, i.e. $U(y)\Phi_\rho(x,y)$ with $x \in \overline{A_{\text{ext}}}$, $y \in A$, where the fixed value ρ determines the length of the window (i.a., the 'cutoff cap') on the sphere A. The Fourier coefficients of the product in terms of outer harmonics $\{H_{n,j}(\alpha;\cdot)\}_{\substack{n=0,1,\ldots,\\ j=1,\ldots,2n+1}}$ are then given by

$$\int\limits_A U(y)\Phi_\rho(x,y) H_{n,j}(\alpha;y)\ d\omega(y), \qquad (n,j) \in \mathcal{N}, \quad x \in \overline{A_{\text{ext}}}\ .$$

In other words, we have defined the $\mathcal{H}_0$–inner product of U with a discrete set of 'shifts' and 'modulations' of U. The windowed Fourier transform is the operator $(WFT)_{\Phi_\rho}$, which is defined for potentials $U \in \mathcal{H}_0$ by

$$(WFT)_{\Phi_\rho}(n,j;x) = \left(\Phi_\rho^{(2)}(1)\right)^{-1/2} \int\limits_A U(y)\Phi_\rho(x,y) H_{n,j}(\alpha;y)\ d\omega(y),$$

for $(n,j) \in \mathcal{N}$ and $x \in \overline{A_{\text{ext}}}$ ($\Phi_\rho(1)$ is a normalization constant). If Φ_ρ is concentrated in space at a point $x \in A$, then $(WFT)_{\Phi_\rho}(n,j;x)$ gives information of U at position $x \in A$ and frequency $(n,j) \in \mathcal{N}$. The potential $U \in \mathcal{H}_0$ is completely characterized by the values of $(WFT)_{\Phi_\rho}(n,j;x)$ and can be recovered via the reconstruction formula

$$U = \left(\Phi_\rho^{(2)}(1)\right)^{-1/2} \sum_{n=0}^{N} \sum_{j=1}^{2n+1} \int\limits_A (GT)_{\Phi_\rho}(U)(n,j;x)\Phi_\rho(\cdot,x)\ d\omega(x) H_{n,j}(\alpha;\cdot)$$

in the $\|\cdot\|_{\mathcal{H}_0}$–sense.

Obviously, $(GT)_{\Phi_\rho}$, ρ converts a potential U of one spatial variable into a function of two variables $x \in \overline{A_{\text{ext}}}$ and $(n,j) \in \mathcal{N}$ without changing its total energy, i.e.:

$$\begin{aligned} \|U\|^2_{\mathcal{H}_0} &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \int\limits_A \left| \left((WFT)_{\Phi_\rho}(U)\right)(n,j;y)\right|^2\ d\omega(y) \\ &= \|\,(U, \Phi_\rho(\cdot,\cdot)H_{\cdot,\cdot}(\alpha;\cdot))_{\mathcal{H}_0}\,\|^2_{\mathcal{H}_0(\mathcal{N}\times\overline{A_{\text{ext}}})}\ . \end{aligned}$$

But, as we shall see in this paper, the space $\mathcal{G} = (WFT)_{\Phi_\rho}(\mathcal{H}_0)$ of all windowed Fourier transforms is a *proper subspace* of the space $\mathcal{H}_0(\mathcal{N} \times \overline{A_{\text{ext}}})$ of all (two–parameter) functions $G : (n,j;x) \mapsto G(n,j;x)$, $(n,j) \in \mathcal{N}$, $x \in \overline{A_{\text{ext}}}$, such that $G(n,j;\cdot) \in \mathcal{H}_0$ for all $(n,j) \in \mathcal{N}$ and $\|G\|_{\mathcal{H}_0(\mathcal{N}\times\overline{A_{\text{ext}}})} < \infty$ (this simply means that $\mathcal{G}$ is a subspace of $\mathcal{H}_0(\mathcal{N} \times \overline{A_{\text{ext}}})$ but not equal to the latter). Thus being a member of $\mathcal{H}_0(\mathcal{N} \times \overline{A_{\text{ext}}})$ is a necessary but not sufficient condition for $G \in \mathcal{G}$ (note that the extra condition that is both necessary and sufficient is called consistency condition). The essential meaning of $\mathcal{G} = (WFT)_{\Phi_\rho}\mathcal{H}_0$ in the framework of $\mathcal{H}_0(\mathcal{N} \times \overline{A_{\text{ext}}})$ can be described by the following least–squares property: Suppose we want a potential with certain properties in both space and frequency. In other words, we look for a potential $U \in \mathcal{H}_0$ such that $(WFT)_{\Phi_\rho}(U)(n,j;x)$ is closest to $H(n,j;x)$, $(n,j) \in \mathcal{N}$, $x \in \overline{A_{\text{ext}}}$ in the '$\mathcal{H}_0(\mathcal{N} \times \overline{A_{\text{ext}}})$–metric', where $H \in \mathcal{H}_0(\mathcal{N} \times \overline{A_{\text{ext}}})$ is given. Our investigations will show that the solutions to the *least–squares problem* is provided by the function U_H given by

$$U_H = \left(\Phi_\rho^{(2)}(1)\right)^{-1/2} \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \int_A H(n,j;y)\Phi_\rho(\cdot,y)\; d\omega(y) H_{n,j}(\alpha;\cdot), \qquad (2)$$

which indeed is the unique potential in $\mathcal{H}_0$ that minimizes the '$\mathcal{H}_0(\mathcal{N} \times \overline{A_{\text{ext}}})$–error':

$$\begin{aligned} &\|H - (WFT)_{\Phi_\rho}(U_H)\|_{\mathcal{H}_0(\mathcal{N}\times\overline{A_{\text{ext}}})} \\ &= \inf_{U\in\mathcal{H}_0} \|H - (WFT)_{\Phi_\rho}(U)\|_{\mathcal{H}_0(\mathcal{N}\times\overline{A_{\text{ext}}})}. \end{aligned} \qquad (3)$$

Moreover, if $H \in \mathcal{G}$, then (2) reduces to the reconstruction formula. In the context of oversampling a signal this means that the tendency for correcting errors is expressed in the least-squares property of the windowed Fourier transform. Although the oversampling of a potential (signal) might seem inefficient, such redundancy has certain advantages: it can detect and correct errors, which is impossible when only minimal information is given. Although the shape of the window may vary depending on the space width ρ the uncertainty principle (cf. [6,7,10]) gives a restriction in space and frequency. This relation is optimal (cf. [7)] when Φ_ρ is a Gaussian, in which case the windowed Fourier transform is called the *Gabor transform* (cf. [14]).

An essential problem with the windowed Fourier transform is that it poorly resolves phenomena of spatial extension shorter than the a priori chosen (fixed) window. Moreover, shortening the window to increase spatial resolution can result in unacceptable increases in computational effort.

The *wavelet transform* acts as a space and frequency localization operator in the following way. Roughly speaking, if $\{\Phi_\rho^{(2)}\}$, $\rho \in (0,\infty)$, is a harmonic scaling function and ρ is a small positive value, then $\Phi_\rho^{(2)}(y,\cdot)$, $y \in A$, is highly concentrated about the point y. Moreover, as ρ tends to $+\infty$, $\Phi_\rho^{(2)}(y,\cdot)$ becomes more and more frequency localized. Correspondingly, the uncertainty principle states that the space localization of $\Phi_\rho^{(2)}(y,\cdot)$ becomes more and more decreasing. In conclusion, the products $y \mapsto \Phi_\rho^{(2)}(x,y)U(y)$, $y \in A$, $x \in \overline{A_{\text{ext}}}$, for each fixed value ρ, display information in $U \in \mathcal{H}_0$ at various levels of spatial resolution or frequency bands. In conclusion, as ρ approaches $+\infty$, the convolution integrals

$$\begin{aligned}\Phi_\rho^{(2)} * U = \Phi_\rho * \Phi_\rho * U &= \int_A \Phi_\rho(\cdot, z) \int_A \Phi_\rho(z,y)U(y)\ d\omega(y)\ d\omega(z) \\ &= \int_A \Phi_\rho^{(2)}(\cdot, y)U(y)\ d\omega(y)\end{aligned}$$

display coarser, lower–frequency features. As ρ approaches 0, the integrals give sharper and sharper spatial resolution. In other words, like a windowed Fourier transform, the convolution integrals can measure the space–frequency variations of spectral components, but they have a different space–frequency resolution.

Each scale–space approximation $\Phi_\rho^{(2)} * U = \Phi_\rho * \Phi_\rho * U$ of a potential $U \in \mathcal{H}_0$ must be made directly by computing the relevant convolution integrals. In doing so, however, it is efficient to use information from the approximation $\Phi_\rho^{(2)} * U$ within the computation of $\Phi_{\rho'}^{(2)} * U$ with $(\rho' < \rho)$. In fact the construction of wavelets begins by a *multiresolution analysis*, i.e. a completely recursive method which is therefore ideal for computation. In this context we observe that

$$\begin{aligned}&\int_R^\infty \int_A \Psi_\rho^{(2)}(\cdot, y)U(y)\ d\omega(y) \frac{d\sigma}{\sigma} \\ &= \int_R^\infty \int_A \Psi_\rho(\cdot, z) \int_A \Psi_\rho(z,y)U(y)\ d\omega(y)\ d\omega(z)\end{aligned}$$

tends to $U \in \mathcal{H}_0$ as R tends to 0 provided that

$$\Psi_\rho^{(2)}(x,y) = \sum_{n=0}^{\infty} (\psi_\rho(n))^2 \sum_{j=1}^{2n+1} H_{n,j}(\alpha;x)H_{n,j}(\alpha;y), \quad (x,y) \in \overline{A_{\text{ext}}} \times \overline{A_{\text{ext}}}\ .$$

is given such that

$$(\psi_\rho(n))^2 = -\rho \frac{d}{d\rho} (\varphi_\rho(n))^2 \tag{4}$$

for $n = 0, 1, \ldots$ and all $\rho \in (0, \infty)$. Conventionally, the family $\{\Psi_\rho\}$, $\rho \in (0,\infty)$, is called a (scale continuous) wavelet. The wavelet transform $WT : \mathcal{H}_0 \to \mathcal{H}_0((0,\infty) \times \overline{A_{\text{ext}}})$ is defined by

$$\begin{aligned}(WT)(U)(\rho; x) &= (\Psi_\rho(x,\cdot), U)_{\mathcal{H}_0} = (\Psi_\rho * U)(x) \\ &= \int_A \Psi_\rho(x,y)U(y)\ d\omega(y)\ .\end{aligned}$$

In other words, the wavelet transform is defined as the $\mathcal{H}_0$–inner product of $U \in \mathcal{H}_0$ with a set of 'shifts' and 'dilations' of U. From the Parseval identity in terms of outer harmonics it follows that

$$\int_A \int_0^\infty (\Psi_\rho(\cdot, y), U)^2_{\mathcal{H}_0} \frac{d\rho}{\rho} \, d\omega(y) = (U, U)^2_{\mathcal{H}_0} \ .$$

i.e. WT converts a potential U of one variable into a function of two variables $x \in \overline{A_{\text{ext}}}$ and $\rho \in (0, \infty)$. It follows that the continuous wavelet transform WT is invertible on $\mathcal{H}_0$, i.e.

$$U = \int_A \int_0^\infty (WT)(U)(\rho; y) \Psi_\rho(\cdot, y) \frac{d\rho}{\rho} \, d\omega(y)$$

in the sense of $\| \cdot \|_{\mathcal{H}_0}$.

In terms of filtering $\{\Phi_\rho^{(2)}\}$ and $\{\Psi_\rho^{(2)}\}$, $\rho \in (0, \infty)$, may be interpreted as low–pass filter and band–pass filter, respectively. Correspondingly, the convolutions operators are given by

$$\begin{aligned} P_\rho(U) &= \Phi_\rho * \Phi_\rho * U, \quad U \in \mathcal{H}_0 \\ R_\rho(U) &= \Psi_\rho * \Psi_\rho * U, \quad U \in \mathcal{H}_0 \ . \end{aligned}$$

The Fourier coefficients read as follows:

$$\begin{aligned} (FT)(P_\rho(U))(n, j) &= (FT)(U)(n, j)(\varphi_\rho(n))^2 \\ (FT)(R_\rho(U))(n, j) &= (FT)(U)(n, j)(\psi_\rho(n))^2 \ . \end{aligned}$$

These formulas provide the transition from Fourier to wavelet transform, and vica versa.

The scale spaces $\mathcal{V}_\rho = P_\rho(\mathcal{H}_0)$ satisfy the *continuous multiresolution analysis*

$$\mathcal{V}_\rho \subset \mathcal{V}_{\rho'} \subset \mathcal{H}_0, \qquad 0 < \rho' < \rho,$$

$$\overline{\{U \in \mathcal{H}_0 \mid U \in \mathcal{V}_\rho \text{ for some } \rho \in (0, \infty)\}}^{\| \cdot \|_{\mathcal{H}_0}} = \mathcal{H}_0 \ .$$

Just as the windowed Fourier transform uses modulation in the space domain to shift the 'window' in frequency, the wavelet transform makes use of scaling in the space domain to scale the 'window' in frequency.

Since all scales ρ are used, the reconstruction is highly redundant. Of course, the redundancy leads us to a question which is of particular importance in data analysis:

– Given an arbitrary $H \in \mathcal{H}_0((0, \infty) \times \overline{A_{\text{ext}}})$, how can we know whether $H \in (WT)(U)$ for some potential $U \in \mathcal{H}_0$.

In analogy to the case of windowed Fourier transform the answer to the question amounts to finding the range of the wavelet transform $WT : \mathcal{H}_0 \to \mathcal{H}_0((0, \infty) \times \overline{A_{\text{ext}}})$, i.e. the subspace

$$\mathcal{W} = (WT)(\mathcal{H}_0) \subset \mathcal{H}_0((0, \infty) \times \overline{A_{\text{ext}}}).$$

The tendency for correcting errors by use of the wavelet transform is again expressed in the least–squares approximation property:

Let H be an arbitrary element of $\mathcal{H}_0((0,\infty)\times\overline{A_{\text{ext}}})$. Then the unique function $U_H \in \mathcal{H}_0$ which satisfies the property

$$\|H - (WT)(U_H)\|_{\mathcal{H}_0((0,\infty)\times\overline{A_{\text{ext}}})} = \inf_{U\in\mathcal{H}_0} \|H - (WT)(U)\|_{\mathcal{H}_0((0,\infty)\times\overline{A_{\text{ext}}})}$$

is given by

$$U_H = \int_0^\infty \int_A H(\rho;y)\Psi_\rho(\cdot,y)\, d\omega(y) \frac{d\rho}{\rho} \ .$$

$(WT)(U_H)$ is indeed the orthogonal projection of H onto $\mathcal{W}$.

The layout of the paper is as follows: Chapter 2 presents the basic material about outer harmonics to be needed for our harmonic variants of both windowed Fourier transform and wavelet transform. Chapter 3 deals with the Sobolev space structure of $\mathcal{H}_0$, while Chapter 4 gives the definition of product kernels in $\mathcal{H}_0$. Central for our considerations is the notation of harmonic scaling functions which will be discussed in Chapter 5 in mathematically rigorous way. The windowed Fourier transform will be introduced in Chapter 6, least–squares approximation will be shown to be an essential property. Chapter 7 is concerned with the scale continuous wavelet transform and its least–squares approximation property. Chapter 8 shows what happens in scale–discrete wavelet analysis. Chapter 9 lists a collection of wavelets which are of particular significance for data analysis. Finally, a variant of fully discrete wavelet transform is illustrated by use of approximation integration rules.

2 Preliminaries

We begin with some basic facts to be needed for our introduction of harmonic windowed Fourier and wavelet theory.

2.1 Regular Surfaces

$\Sigma \subset \mathbb{R}^3$ is called a *regular surface* if it satisfies the following properties:

- Σ divides three-dimensional Euclidean space $\mathbb{R}^3$ into the bounded region Σ_{int} (*inner space*) and the unbounded region Σ_{ext} (*outer space*) defined by $\Sigma_{\text{ext}} = \mathbb{R}^3\backslash\overline{\Sigma_{\text{int}}}$, $\overline{\Sigma_{\text{int}}} = \Sigma_{\text{int}} \cup \Sigma$.
- Σ_{int} contains the origin 0.
- Σ is a closed and compact surface free of double points.
- Σ has a continuously differentiable normal field ν (pointing into the outer space Σ_{ext}).

Examples are sphere, ellipsoid, spheroid, and (actual) earth's surface.

Given a regular surface Σ then there exists a positive constant α such that

$$\alpha < \sigma^{\inf} = \inf_{x\in\Sigma} |x| \leq \sup_{x\in\Sigma} |x| = \sigma^{sup} . \tag{5}$$

A (resp. $\Sigma^{\inf}$) denotes the sphere around the origin with radius α (resp. $\sigma^{\inf}$). A_{int} (resp. A_{ext}) denotes the inner (resp. outer) space of the sphere A. Correspondingly, $\Sigma_{\text{int}}^{\inf}$ (resp. $\Sigma_{\text{ext}}^{\inf}$) is the inner (resp. outer) space of $\Sigma^{\inf}$. Obviously,

$$\overline{\Sigma_{\text{ext}}} \subset \overline{\Sigma_{\text{ext}}^{\inf}} \subset \overline{A_{\text{ext}}} \quad .$$

Bild

The set

$$\Sigma(\tau) = \{x \in \mathbb{R}^3 | x = y + \tau\nu(y), y \in \Sigma\}$$

generates a parallel surface which is exterior to Σ for $\tau > 0$ and interior for $\tau < 0$. For sufficiently small $|\tau|$ the regularity of Σ implies the regularity of $\Sigma(\tau)$, and the normal to one parallel surface is a normal to the other (cf., e.g., [?]). Furthermore it is easily seen that

$$\inf_{x,y\in\Sigma} |x + \tau\nu(x) - (y + \sigma\nu(y))\,| = |\tau - \sigma|$$

provided that $|\tau|$, $|\sigma|$ are sufficiently small.

In what follows we use the following abbreviation:

$$U_{\Sigma}^{+}(x) = \lim_{\substack{s\to 0\\ s>0}} U(x + s\nu(x)),\ x \in \Sigma,\ U \in C(\overline{\Sigma_{\text{ext}}}) \quad .$$

2.2 Outer Harmonics

Let $\{Y_{n,j}\}$, $n = 0, 1, \ldots,$ $j = 1, \ldots, 2n + 1$ be an $\mathcal{L}^2$-orthonormal system of (surface) spherical harmonics, i.e.,

$$\int_{|x|=1} Y_{n,j}(x)\overline{Y_{k,l}(x)}d\omega(x) = \delta_{n,k}\delta_{j,l} \quad .$$

Then the system $\{H_{n,j}(\alpha;\cdot)\}$, $n = 0, 1, \ldots, j = 1, \ldots, 2n + 1$ of *"outer harmonics"*, defined by

$$H_{n,j}(\alpha; x) = \frac{1}{\alpha}\left(\frac{\alpha}{|x|}\right)^{n+1} Y_{n,j}\left(\frac{x}{|x|}\right),\ x \in \overline{A_{\text{ext}}}$$

satisfies the following conditions:

- $H_{n,j}(\alpha;\cdot)$ is of class $C^{(\infty)}(\overline{A_{\text{ext}}})$
- $H_{n,j}(\alpha;\cdot)$ satisfies the Laplace equation $\Delta_x H_{n,j}(x) = 0$, $x \in A_{\text{ext}}$
- $H_{n,j}(\alpha;\cdot)|_A = \frac{1}{\alpha} Y_{n,j}$
- $\int\limits_A H_{n,j}(\alpha;x) H_{k,l}(\alpha;x) d\omega(x) = \delta_{n,k}\delta_{j,l}$.

The *addition theorem of outer harmonics* (cf., e.g., [?]) reads as follows

$$\sum_{j=1}^{2n+1} H_{n,j}(\alpha;x)H_{n,j}(\alpha;y) = \frac{2n+1}{4\pi\alpha^2}\left(\frac{\alpha}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)$$

for all $(x,y) \in \overline{A_{\text{ext}}} \times \overline{A_{\text{ext}}}$, where P_n is the Legendre polynomial of degree n.

$\text{Harm}_n(\overline{A_{\text{ext}}})$ denotes the space of all outer harmonics of non-negative order n:

$$\text{Harm}_n(\overline{A_{\text{ext}}}) = span_{j=1,\dots,2n+1}(H_{n,j}(\alpha;\cdot)).$$

It is well-known that $dim(\text{Harm}_n(\overline{A_{\text{ext}}})) = 2n+1$. We set

$$\text{Harm}_{0,\dots,m}(\overline{A_{\text{ext}}}) = span_{\substack{n=0,\dots,m \\ j=1,\dots,2n+1}}(H_{n,j}(\alpha;\cdot)).$$

Of course,

$$\text{Harm}_{0,\dots,m}(\overline{A_{\text{ext}}}) = \bigoplus_{n=0}^{m} \text{Harm}_n(\overline{A_{\text{ext}}})$$

(in the sense of $\|\cdot\|_{\mathcal{L}^2(A)}$) so that

$$dim(\text{Harm}_{0,\dots,m}(\overline{A_{\text{ext}}})) = \sum_{n=0}^{m}(2n+1) = (m+1)^2.$$

For brevity, we let $\text{Harm}_{0,\dots,m}(\Xi) = \text{Harm}_{0,\dots m}(\overline{A_{\text{ext}}})|_{\Xi}$ whenever Ξ is a subset of $\overline{A_{\text{ext}}}$.

The kernel $K_{\text{Harm}_{0,\dots,m}(\overline{A_{\text{ext}}})}(\cdot,\cdot) : \overline{A_{\text{ext}}} \times \overline{A_{\text{ext}}} \to \mathbb{C}$ defined by

$$K_{\text{Harm}_{0,\dots,m}(\overline{A_{\text{ext}}})}(x,y) = \sum_{n=0}^{m} \frac{2n+1}{4\pi\alpha^2}\left(\frac{\alpha^2}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)$$

is the reproducing kernel of the space $\text{Harm}_{0,\dots,m}(\overline{A_{\text{ext}}})$ with respect to $(\cdot,\cdot)_{\mathcal{L}^2(A)}$. Observing the recursion relation for Legendre polynomials

$$(n+1)(P_{n+1}(t)-P_n(t)) - n(P_n(t)-P_{n-1}(t)) = (2n+1)(t-1)P_n(t),\ t \in [-1,+1]$$

we find for $(x,y) \in A \times A$

$$\begin{aligned} & K_{\text{Harm}_{0,\dots,m}(\overline{A_{\text{ext}}})}(x,y)\left(\frac{x}{|x|}\cdot\frac{y}{|y|} - 1\right) \\ &= \frac{m+1}{4\pi\alpha^2}\left(P_{m+1}\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right) - P_m\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)\right). \end{aligned}$$

Later on the last identity turns out to be a useful formula in numerical calculation.

2.3 Fourier Approximation by Outer Harmonics

$Pot(\Sigma_{\mathrm{ext}})$ denotes the space of all $U : \overline{\Sigma_{\mathrm{ext}}} \to \mathbb{C}$ with the following properties:

- U is a member of class $C^{(2)}(\Sigma_{\mathrm{ext}})$
- U is harmonic in Σ_{ext}, i.e. U satisfies the Laplace equation

$$\Delta U = 0 \quad \text{in} \quad \Sigma_{\mathrm{ext}},$$

- U satisfies the *regularity condition at infinity*

$$|U(x)| = O(|x|^{-1}), \ |(\nabla U)(x)| = O(|x|^{-2}), |x| \to \infty \ .$$

For $q = 0, 1, \ldots$ we let

$$Pot^{(q)}\left(\overline{\Sigma_{\mathrm{ext}}}\right) = Pot\left(\Sigma_{\mathrm{ext}}\right) \cap C^{(q)}\left(\overline{\Sigma_{\mathrm{ext}}}\right).$$

From the maximum/minimum principle of potential theory (see e.g. [17,19]) we know that

$$\sup_{x \in \overline{\Sigma_{\mathrm{ext}}}} |U(x)| \leq \sup_{x \in \Sigma} |U_{\Sigma}^{+}(x)|, \quad U \in Pot^{(0)}\left(\overline{\Sigma_{\mathrm{ext}}}\right) \ .$$

Moreover, using arguments of the theory of integral equations (cf. [3]) we have

$$\sup_{x \in K} |U(x)| \leq D_{\mathcal{L}^2}(\Sigma) \left(\int_{\Sigma} |U_{\Sigma}^{+}(x)|^2 d\omega(x) \right)^{1/2}, \ U \in Pot^{(0)}(\overline{\Sigma_{\mathrm{ext}}}), \tag{6}$$

where $D_{\mathcal{L}^2}$ is a positive constant (dependent on Σ) and K is a subset of Σ_{ext} with $dist(\overline{K}, \Sigma) > 0$.

In our nomenclature the classical formulation of the *Dirichlet boundary-value problem of the Laplace equation*, i.e. the representation of a harmonic function corresponding to continuous boundary values on Σ reads as follows:

Exterior Dirichlet Problem (EDP): Given $F \in C(\Sigma)$, find a member $U \in Pot^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$ such that $U_{\Sigma}^{+} = F$.

We recall those results of the solution theory of Dirichlets problem (cf. [17]) that are of interest for our purposes below:

EDP is well-posed in the sense that its solution exists, is unique, and depends continuously on the boundary values. We have

$$C(\Sigma) = \mathcal{D}_{\Sigma}^{+} = \{U_{\Sigma}^{+} | U \in Pot^{(0)}(\overline{\Sigma_{\mathrm{ext}}})\}$$

so that

$$\overline{\mathcal{D}_{\Sigma}^{+}}^{\|\cdot\|_{\mathcal{L}^2(\Sigma)}} = \mathcal{L}^2(\Sigma).$$

From [3,4,9] we know that

$$C(\Sigma) = \mathcal{D}_{\Sigma}^{+} = \overline{span_{\substack{n=0,1,\ldots \\ j=1,\ldots,2n+1}} (H_{n,j}(\alpha; \cdot)_{\Sigma}^{+})}^{\|\cdot\|_{C(\Sigma)}} \ . \tag{7}$$

Moreover,

$$\mathcal{L}^2(\Sigma) = \overline{\mathcal{D}_\Sigma^+}^{\|\cdot\|_{\mathcal{L}^2(\Sigma)}} = \overline{span_{\substack{n=0,1,\dots \\ j=1,\dots,2n+1}} (H_{n,j}(\alpha;\cdot)_\Sigma^+)}^{\|\cdot\|_{\mathcal{L}^2(\Sigma)}} . \tag{8}$$

Eq. (8) equivalently means that the space $\mathcal{L}^2(\Sigma)$ is the completion of the set span$_{\substack{n=0,1,\dots \\ j=1,\dots,2n+1}} (H_{n,j}(\alpha;\cdot)_\Sigma^+)$ of all finite linear combinations of functions $H_{n,j}(\alpha;\cdot)_\Sigma^+$ with respect to the $\|\cdot\|_{\mathcal{L}^2(\Sigma)}$–topology.

2.4 Outer Harmonic Fourier Expansions on Regular Surfaces

In order to make use of (8) the last identity in constructive approximation we have to orthonormalize the system $\{H_{n,j}(\alpha;\cdot)\}_{\substack{n=0,1,\dots, \\ j=1,\dots,2n+1}}$ (for example, by virtue of the well-known Gram–Schmidt process) with respect to the $\|\cdot\|_{\mathcal{L}^2(\Sigma)}$–topology obtaining a system

$$\{K^*_{n,j}\}_{\substack{n=0,1,\dots, \\ j=1,\dots,2n+1}} \subset \mathrm{Pot}^{(0)}(\overline{\Sigma_{\text{ext}}})$$

with the following properties (cf. [4]):

(i) each $K^*_{n,j}$ is the unique solution of the boundary–value problem $K^*_{n,j} \subset \mathrm{Pot}^{(0)}(\overline{\Sigma_{\text{ext}}})$ corresponding to the boundary data

$$\left(K^*_{n,j}\right)_\Sigma^+ = H^*_{n,j}, \qquad n = 0,1,\dots;\ j = 1,\dots,2n+1,$$

(ii) $\left\{H^*_{n,j}\right\}_{\substack{n=0,1,\dots \\ j=1,\dots,2n+1}}$ defined by $H^*_{n,j} = (K^*_{n,j})_\Sigma^+$ is a complete orthonormal system in the Hilbert space $(\mathcal{L}^2(\Sigma)(\cdot,\cdot)_{\mathcal{L}^2(\Sigma)})$ (obtainable by $\mathcal{L}^2(\Sigma)$–orthonormalization from $\{H_{n,j}(\alpha;\cdot)_\Sigma^+\}$).

Let U be the uniquely determined solution of the boundary–value problem EDP:

$$U \in \mathrm{Pot}^{(0)}\left(\overline{\Sigma_{\text{ext}}}\right), \quad U_\Sigma^+ = F.$$

Then, in accordance with our construction (cf. [Frd]), the $\mathcal{L}^2(\Sigma)$–convergence of the orthogonal expansion

$$\sum_{n=0}^{N} \sum_{j=1}^{2n+1} \left(F, H^*_{n,j}\right)_{\mathcal{L}^2(\Sigma)} H^*_{n,j} \tag{9}$$

to the function F (in $\|\cdot\|_{\mathcal{L}^2(\Sigma)}$–sense) implies ordinary pointwise convergence of the sum

$$\sum_{n=0}^{N} \sum_{j=1}^{2n+1} \left(F, H^*_{n,j}\right)_{\mathcal{L}^2(\Sigma)} K^*_{n,j} \tag{10}$$

to the potential U as $N \to \infty$ for every point $x \in K$ with $K \subset \Sigma_{\text{ext}}$ and $\mathrm{dist}(\overline{K}, \Sigma) \geq \rho > 0$. For every compact subset $K \subset \Sigma_{\text{ext}}$ the convergence is uniform. More explicitly,

$$\lim_{N\to\infty} \left\| F - \sum_{n=0}^{N} \sum_{j=1}^{2n+1} \left(F, H^*_{n,j}\right)_{\mathcal{L}^2(\Sigma)} H^*_{n,j} \right\|_{\mathcal{L}^2(\Sigma)} = 0 \tag{11}$$

implies

$$\lim_{N\to\infty} \sup_{x\in\overline{K}} \left| U(x) - \sum_{n=0}^{N} \sum_{j=1}^{2n+1} \left(F, H^*_{n,j}\right)_{\mathcal{L}^2(\Sigma)} K^*_{n,j}(x) \right| = 0 \tag{12}$$

provided that $K \subset \Sigma_{\text{ext}}$ is a set with $\text{dist}(\overline{K}, \Sigma) \geq \rho > 0$.

Truncated orthogonal (Fourier) expansions (11), (12) have the following least–squares property

$$\begin{aligned} &\|U - \sum_{n=0}^{N} \sum_{j=1}^{2n+1} \left(U, H^*_{n,j}\right)_{\mathcal{L}^2(\Sigma)} H^*_{n,j}\|_{\mathcal{L}^2(\Sigma)} \\ &= \inf_{V \in \underset{\substack{n=0,\ldots,N \\ j=1,\ldots,2n+1}}{\text{span}} (K^*_{n,j})} \|U - V\|_{\mathcal{L}^2(\Sigma)} \end{aligned}$$

i.e. the problem of finding a linear combination in *span* $_{\substack{n=0,\ldots,N \\ j=1,\ldots,2n+1}} (K^*_{n,j})$, which is minimal in the $\mathcal{L}^2(\Sigma)$–norm, is solved by the orthogonal projection of U on the *span* $_{\substack{n=0,\ldots,N \\ j=1,\ldots,2n+1}} (H^*_{n,j})$. More explicitly, we have

$$\begin{aligned} &\left\| U - \sum_{n=0}^{N} \sum_{j=1}^{2n+1} \left(U, H^*_{n,j}\right)_{\mathcal{L}^2(\Sigma)} H^*_{n,j} \right\|^2_{\mathcal{L}^2(\Sigma)} \\ &\qquad = (U, U)_{\mathcal{L}^2(\Sigma)} - \sum_{n=0}^{N} \sum_{j=1}^{2n+1} (U, H^*_{n,j})^2_{\mathcal{L}^2(\Sigma)} \ . \end{aligned}$$

In particular, for $U \in \text{Pot}^{(0)}(\overline{A_{\text{ext}}})$ with $U^+_A = F$,

$$\lim_{N\to\infty} \left\| F - \sum_{n=0}^{N} \sum_{j=1}^{2n+1} (F, H_{n,j}(\alpha;\cdot))_{\mathcal{L}^2(A)} H_{n,j}(\alpha;\cdot) \right\|_{\mathcal{L}^2(A)} = 0 \tag{13}$$

implies

$$\lim_{N\to\infty} \sup_{x\in\overline{\Sigma_{\text{ext}}}} \left| U(x) - \sum_{n=0}^{N} \sum_{j=1}^{2n+1} (F, H_{n,j}(\alpha;\cdot))_{\mathcal{L}^2(A)} H_{n,j}(\alpha;x) \right| = 0 \tag{14}$$

Eq. (13) indicates the conventional approach to modelling the earth‘s gravitational potential in (today‘s spherical) geodesy. (Particularly important spherical outer harmonic models are GRIM4 (cf. [22]), OSU91A (cf. [21]) and EGM96 (cf. [18])). Non–spherical outer harmonic models (11), (12) corresponding to the actual earth‘s surface Σ (which today may be supposed to be known from GPS) are a challenge for future (low–wavelength) approximation.

3 Sobolev Spaces

Let $\{A_n\}_{n\in\mathbb{N}_0}$ be a complex sequence. The sequence $\{A_n\}_{n\in\mathbb{N}_0}$ is called $\{B_n\}$-*summable* if $|A_n| \neq 0$ for all n and the sum

$$\Sigma(\{B_n\},\{A_n\}) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi\alpha^2} \frac{|B_n|^2}{|A_n|^2}$$

is finite. A $\{1\}$-summable sequence is simply called *summable*, i.e.

$$\Sigma\left(\{A_n\}\right) = \Sigma\left(\{1\},\{A_n\}\right) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi\alpha^2} \frac{1}{A_n^2} < \infty.$$

3.1 Definition

For a given sequence $\{A_n\}$ with $|A_n| \neq 0$ for all n, consider the linear space $\mathcal{E} = \mathcal{E}(\{A_n\}, \overline{A_{\text{ext}}}) \subset Pot^{(\infty)}(\overline{A_{\text{ext}}})$ of all potentials U of the form

$$U = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left(U, H_{n,j}(\alpha;\cdot)\right)_{\mathcal{L}^2(A)} (\alpha;\cdot) \tag{15}$$

with

$$\left(U, H_{n,j}(\alpha;\cdot)\right)_{\mathcal{L}^2(A)} \left(U, H_{n,j}(\alpha;\cdot)\right)_{\mathcal{L}^2(A)} = \int_A U(y) H_{n,j}(\alpha; y)\, d\omega(y)$$

satisfying

$$\sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} |A_n|^2 \left|\left(U, H_{n,j}(\alpha;\cdot)\right)_{\mathcal{L}^2(A)}\right|^2 < \infty.$$

From the Cauchy-Schwarz inequality it follows that

$$\left|\sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left(U, H_{n,j}(\alpha;\cdot)\right)_{\mathcal{L}^2(A)} \left(V, H_{n,j}(\alpha;\cdot)\right)_{\mathcal{L}^2(A)}\right| \tag{16}$$

$$\leq \left(\sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} |A_n|^2 \left(U, H_{n,j}(\alpha;\cdot)\right)_{\mathcal{L}^2(A)}\right)^{1/2} \left(\sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} |A_n|^2 \ \left(V, H_{n,j}(\alpha;\cdot)\right)_{\mathcal{L}^2(A)}\right)^{1/2}$$

for all $U, V \in \mathcal{E}$. In other words, the left hand side of (16) is finite whenever each member of the right hand side is finite. Therefore we are able to impose on $\mathcal{E}$ an inner product $(\cdot,\cdot)_{\mathcal{H}(\{A_n\};\overline{A_{\text{ext}}})}$ by letting

$$(U,V)_{\mathcal{H}(\{A_n\};\overline{A_{\text{ext}}})} = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} |A_n|^2 \left(U, H_{n,j}(\alpha;\cdot)\right)_{\mathcal{L}^2(A)} \left(V, H_{n,j}(\alpha;\cdot)\right)_{\mathcal{L}^2(A)} \ .$$

The associated norm is given by $\|\cdot\|_{\mathcal{H}(\{A_n\};\overline{A_{\text{ext}}})} = \sqrt{(\cdot,\cdot)_{\mathcal{H}(\{A_n\};\overline{A_{\text{ext}}})}}$.

Definition 1. *Let $\{A_n\}$ be a real sequence such that $|A_n| \neq 0$ for all n. Then the Sobolev space $\mathcal{H}$ (more accurately: $\mathcal{H}(\{A_n\};\overline{A_{\text{ext}}})$) is the completion of $\mathcal{E}$ under the norm $\|\cdot\|_{\mathcal{H}(\{A_n\};\overline{A_{\text{ext}}})}$:*

$$\mathcal{H}(\{A_n\};\overline{A_{\text{ext}}}) = \overline{\mathcal{E}(\{A_n\};\overline{A_{\text{ext}}})}^{\|\cdot\|_{\mathcal{H}(\{A_n\};\overline{A_{\text{ext}}})}}.$$

$\mathcal{H}$ equipped with the inner product $(\cdot,\cdot)_{\mathcal{H}(\{A_n\};\overline{A_{\text{ext}}})}$ is a Hilbert space.

From the Cauchy-Schwarz inequality it follows that $(U,V)_{\mathcal{H}(\{1\};\overline{A_{\text{ext}}})}$ exists if $U \in \mathcal{H}(\{A_n\};\overline{A_{\text{ext}}})$ and $V \in \mathcal{H}(\{A_n^{-1}\},\overline{A_{\text{ext}}})$. Moreover,

$$|(U,V)_{\mathcal{H}(\{1\};\overline{A_{\text{ext}}})}| \leq \|U\|_{\mathcal{H}(\{A_n\};\overline{A_{\text{ext}}})} \|V\|_{\mathcal{H}(\{A_n^{-1}\};\overline{A_{\text{ext}}})}.$$

Hence, $(\cdot,\cdot)_{\mathcal{H}(\{A_n\};\overline{A_{\text{ext}}})}$ defines a duality of $\mathcal{H}(\{A_n\};\overline{A_{\text{ext}}})$ and $\mathcal{H}(\{A_n^{-1}\};\overline{A_{\text{ext}}})$.

For brevity, we let

$$\mathcal{H}_s(\overline{A_{\text{ext}}}) = \mathcal{H}(\{(n+1/2)^s\};\overline{A_{\text{ext}}})$$

for each real value of s. In particular,

$$\mathcal{H}_0(\overline{A_{\text{ext}}}) = \mathcal{H}\left(\{1\};\overline{A_{\text{ext}}}\right) \quad .$$

Remark 1. In what follows we simply write $\mathcal{H}_0$ (instead of $\mathcal{H}_0(\overline{A_{\text{ext}}})$) when confusion is not likely to arise.

3.2 Sobolev Lemma

If we associate to U the series (15) it is of fundamental importance to know when the series (15) converges uniformly on $\overline{A_{\text{ext}}}$. The answer is provided by the following lemma.

Lemma 1. *(Sobolev) Let the sequence $\{A_n\}$ be $\{B_n\}$-summable with ($|B_n| \neq 0$ for all n). Then each $U \in \mathcal{H}(\{A_n\};\overline{A_{\text{ext}}})$ corresponds to a potential of class $Pot^{(0)}(\overline{A_{\text{ext}}})$.*

Proof. For each sufficiently large N, we have

$$\begin{aligned}&\left|\sum_{n=0}^{N} B_n A_n^{-1} H_{n,j}(\alpha;x) A_n B_n^{-1}\,(U, H_{n,j}(\alpha;\cdot))_{\mathcal{L}^2(A)}\right| \\ &\leq \Sigma(\{B_n\};\{A_n\})\|U\|_{\mathcal{H}(\{B_n^{-1}A_n\};\overline{A_{\text{ext}}})}.\end{aligned}$$

This proves Lemma 1. □

By similar arguments we obtain the following results (cf. [5,6])

Lemma 2. *If $U \in \mathcal{H}_s(\overline{A_{\text{ext}}})$, $s > k+1$, then U corresponds to a potential of class* $\mathrm{Pot}^{(k)}(\overline{A_{\text{ext}}})$.

Furthermore, we have (cf. [6,11])

Lemma 3. *Suppose that U is of class $\mathcal{H}_s(\overline{A_{\text{ext}}})$, $s > [l] + 1$. Then*

$$\sup_{x \in \overline{A_{\text{ext}}}} \left\| \left(\nabla^l U\right)(x) - \sum_{n=0}^{N} \sum_{j=1}^{2n+1} (U, H_{n,j}(\alpha;\cdot))_{\mathcal{L}^2(A)} \left(\nabla^l H_{n,j}\right)(\alpha; x) \right\| \leq C N^{[l]-s} \|U\|_{\mathcal{H}_s(\overline{A_{\text{ext}}})}$$

holds for all positive integers N (with $\nabla^l = \partial^{[l]}/(\partial x_1)^{l_1}(\partial x_2)^{l_2}(\partial x_3)^{l_3}$, l_i: non-negative integers, $l_1 + l_2 + l_3 = [l]$), where C is a positive constant independent of U.

3.3 Outer Harmonic Fourier Expansions in $\mathcal{H}_0$

The *outer harmonic Fourier transform* $FT : U \mapsto (FT)(U)$, $U \in \mathcal{H}_0$, is defined by

$$(FT)(U)(n,j) = (U, H_{n,j}(\alpha;\cdot))_{\mathcal{H}_0} = \int_A U(x) H_{n,j}(\alpha; x)\, d\omega(x) \ .$$

The Fourier transform FT forms a mapping from $\mathcal{H}_0$ into the space $\mathcal{H}_0(\mathcal{N})$ of sequences $\{V(n,j)\}_{(n,j)\in\mathcal{N}}$ with $V(n,j) = (V, H_{n,j}(\alpha;\cdot))_{\mathcal{H}_0}$, $V \in \mathcal{H}_0$, satisfying

$$\sum_{(n,j)\in\mathcal{N}} |V(n,j)|^2 = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} |V(n,j)|^2 < \infty \ .$$

Any potential $U \in \mathcal{H}_0$ is characterized by its 'amplitude spectrum'

$$\{(U, H_{n,j}(\alpha;\cdot))_{\mathcal{H}_0}\} \ .$$

More explicitly, for $U, V \in \mathcal{H}_0$,

$$\lim_{N\to\infty} \left\| U - \sum_{n=0}^{N} \sum_{j=1}^{2n+1} (V, H_{n,j}(\alpha;\cdot))_{\mathcal{H}_0} H_{n,j}(\alpha;\cdot) \right\|_{\mathcal{H}_0} = 0$$

we have $U = V$ (in sense of $\|\cdot\|_{\mathcal{H}_0}$). In addition, for $U \in \mathcal{H}_0$,

$$\lim_{N\to\infty} \left\| U - \sum_{n=0}^{N} \sum_{j=1}^{2n+1} (U, H_{n,j}(\alpha;\cdot))_{\mathcal{H}_0} H_{n,j}(\alpha;\cdot) \right\|_{\mathcal{H}_0} = 0$$

implies

$$\lim_{N\to\infty} \sup_{x\in\overline{K}} \left| U(x) - \sum_{n=0}^{N} \sum_{j=1}^{2n+1} (U, H_{n,j}(\alpha;\cdot))_{\mathcal{H}_0} H_{n,j}(\alpha; x) \right| = 0$$

for all $\overline{K} \subset A_{ext}$ with $\operatorname{dist}(\overline{K}, A) > 0$.

4 Product Kernels

Of particular importance for our considerations below are *product kernels* of the form

$$K(x,y) = \sum_{n=0}^{\infty} \kappa(n) \sum_{j=1}^{2n+1} H_{n,j}(\alpha;x) H_{n,j}(\alpha;y), \quad (x,y) \in \overline{A_{\text{ext}}} \times \overline{A_{\text{ext}}}, \tag{17}$$

where $\kappa(n)$ are real numbers for $n = 0, 1, \ldots$. Notice that $K(x,y) = K(y,x)$. By virtue of the addition theorem (see e.g. [7,20]) the product kernel K may be rewritten as follows:

$$K(x,y) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi\alpha^2} \kappa(n) \left(\frac{\alpha^2}{|x||y|} \right)^{n+1} P_n \left(\frac{x}{|x|} \cdot \frac{y}{|y|} \right), \quad (x,y) \in \overline{A_{\text{ext}}} \times \overline{A_{\text{ext}}}.$$

The sequence $\{K^\wedge(n)\}_{n=0,1,\ldots}$, with

$$K^\wedge(n) = \kappa(n), \quad n = 0, 1, \ldots$$

is called the *symbol of the product kernel* K.

A product kernel K of the form (17) is called an $\mathcal{H}_0$*-kernel* if $\{(K^\wedge(n))^{-1}\}$ is summable, i.e.

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi\alpha^2} (K^\wedge(n))^2 < \infty \ .$$

Let K be an $\mathcal{H}_0$-kernel. Suppose that U is of class $\mathcal{H}_0$. Then we understand the *convolution* of U and K to be the function of class $\mathcal{H}_0$ given by

$$(U * K)(x) = (U, K(\cdot, x))_{\mathcal{H}_0} = \int_A U(y) K(y,x) d\omega(y), \quad x \in \overline{A_{\text{ext}}}.$$

Obviously, $U * K$ is a member of class $\mathcal{H}_0$. Furthermore, we have

$$(U * K)^\wedge(n,j) = U^\wedge(n,j) K^\wedge(n), \quad n = 0, 1, \ldots, j = 1, \ldots, 2n+1.$$

If L is another $\mathcal{H}_0$-kernel, then $L * K$ is defined by

$$(L * K)(x,y) = \int_A L(z,y) K(z,x) d\omega(z), \quad (x,y) \in \overline{A_{\text{ext}}} \times \overline{A_{\text{ext}}}.$$

It is readily seen that

$$(L * K)^\wedge(n) = L^\wedge(n) K^\wedge(n), \quad n = 0, 1, \ldots.$$

We usually write $K^{(2)} = K * K$ to indicate the convolution of a kernel with itself. $K^{(2)}$ is said to be the *iterated kernel* of K. Obviously,

$$(K^{(2)})^\wedge(n) = (K^\wedge(n))^2$$

for $n = 0, 1, \ldots$.

4.1 $\mathcal{H}_0$–kernel Fourier Expansions on Regular Surfaces

Let K be an $\mathcal{H}_0$-kernel. Then the following results are known (cf. [3,5,6,9]):

(EDP) Let $\{x_k\}_{k=1,2,\ldots}$ be a countable dense system of points x_k on Σ. Then

$$C(\Sigma) = \mathcal{D}_\Sigma^+ = \overline{span_{k=1,2,\ldots}(K(\cdot, x_k)_\Sigma^+)}^{\|\cdot\|_{C(\Sigma)}}$$

and

$$\mathcal{L}^2(\Sigma) = \overline{\mathcal{D}_\Sigma^+}^{\|\cdot\|_{\mathcal{L}^2(\Sigma)}} = \overline{span_{k=1,2,\ldots}(K(\cdot, x_k)_\Sigma^+)}^{\|\cdot\|_{\mathcal{L}^2(\Sigma)}}.$$

For purposes of constructive approximation we again have to orthonormalize the system $\{K(\cdot, x_k)\}_{k=1,2,\ldots}$ with respect to the $\|\cdot\|_{\mathcal{L}^2(\Sigma)}$–topology obtaining a system

$$\{K_k^*\}_{k=1,2,\ldots} \subset \mathrm{Pot}^{(0)}\left(\overline{\Sigma_{\mathrm{ext}}}\right)$$

with the following properties:

(i) each K_k^* is the unique solution of the boundary–value problem $K_k^* \subset \mathrm{Pot}^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$ corresponding to the boundary data $(K_k^*)_\Sigma^+ = L_k, k = 1, 2, \ldots,$

(ii) $\{L_k^*\}_{k=1,2,\ldots}$ defined by $(K_k^*)_\Sigma^+ = L_k$ is a complete $\mathcal{L}^2(\Sigma)$–orthonormal system in $\left(\mathcal{L}^2(\Sigma), (\cdot,\cdot)_{\mathcal{L}^2(\Sigma)}\right)$.

Given $U \in \mathrm{Pot}^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$ with $U_\Sigma^+ = F$, then

$$\lim_{N\to\infty} \left\| F - \sum_{k=1}^{N} (F, L_k^*)_{\mathcal{L}^2(\Sigma)} L_k^* \right\|_{\mathcal{L}^2(\Sigma)} = 0$$

implies

$$\lim_{N\to\infty} \sup_{x\in\overline{\Sigma_{\mathrm{ext}}}} \left| U(x) - \sum_{k=1}^{N} (F, L_k^*)_{\mathcal{L}^2(\Sigma)} K_k^*(x) \right| = 0$$

for all subsets $K \subset \Sigma_{\mathrm{ext}}$ with $dist(\overline{K}, \Sigma) > 0$.

5 Harmonic Scaling Functions

The wavelet approach presented now is an extension of ideas developed in spherical theory (cf. [7,12,13]). Starting point is a "continuous version φ of a symbol" $\{\Phi^\wedge(n)\}_{n=0,1,\ldots}$ associated to an "$\mathcal{H}_0$-kernel"

$$\Phi(x, y) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi\alpha^2} \varphi(n) \left(\frac{\alpha^2}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right), \ (x, y) \in \overline{A_{\mathrm{ext}}} \times \overline{A_{\mathrm{ext}}} \ . \quad (18)$$

i.e.,

$$\Phi^\wedge(n) = \varphi(n), \quad n = 0, 1, \ldots \ .$$

5.1 Scaling Function

Definition 2. *A piecewise continuous function $\gamma : [0,\infty) \to \mathbb{R}$ is called admissible, if*

$$\sum_{n=0}^{\infty} \left(\sup_{x\in[n,n+1)} |\gamma(x)| \right)^2 < +\infty \ . \tag{19}$$

Lemma 4. *Let $\gamma : [0,\infty) \to \mathbb{R}$ be piecewise continuous. Furthermore, assume that there exists $\varepsilon > 0$ such that*

$$\gamma(t) = O(t^{-1-\varepsilon}), \qquad t \to \infty \ . \tag{20}$$

Then γ is admissible.

Proof. As $\frac{\gamma(t)}{t^{-1-\varepsilon}}$ is bounded as $t \to \infty$, we are able to introduce

$$M = \sup_{t\in(0,\infty)} \left| \frac{\gamma(t)}{t^{-1-\varepsilon}} \right| (< +\infty) \ .$$

Hence

$$\begin{aligned} &\sum_{n=0}^{\infty} \left(\sup_{x\in[n,n+1)} |\gamma(t)| \right)^2 \\ &= \sum_{n=0}^{\infty} \left(\sup_{x\in[n,n+1)} \left| \frac{\gamma(t)}{t^{-1-\varepsilon}} t^{-1-\varepsilon} \right| \right)^2 \\ &= M \cdot \sum_{n=0}^{\infty} \left(\sup_{x\in[n,n+1)} t^{-1-\varepsilon} \right)^2 \\ &= M \cdot \sum_{n=0}^{\infty} \left(\frac{1}{n^{1+\varepsilon}} \right) < +\infty \end{aligned}$$

This shows Lemma 4. □

Note that, on the other hand, the function γ given by

$$\gamma(t) = \begin{cases} (t \log t)^{-1} & \text{for } x > 1 \\ 1 & \text{for } 0 \le t \le 1 \end{cases}$$

is admissible, as

$$|\gamma(t)|^2 = \left| \frac{1}{t \log t} \right| \le \frac{1}{t^2},$$

if $x \ge e$. However, γ does not satisfy (??). Assume that there exists a value $\varepsilon > 0$, such that

$$\gamma(t) = O(t^{-1-\varepsilon})$$

as $t \to \infty$. Then there exists $M \in \mathbb{R}$, such that

$$\left|\frac{\gamma(t)}{t^{-1-\varepsilon}}\right| \leq M$$

for all $t \in (0,\infty)$. In particular,

$$\left|\frac{(t\log t)^{-1}}{t^{-1-\varepsilon}}\right| \leq M$$

for all $t > 1$. But

$$\lim_{\substack{t\to t\\ t<1}} (t\log t)^{-1} = +\infty,$$

whereas

$$\lim_{\substack{t\to t\\ t>1}} t^{-1-\varepsilon} = 1 \ .$$

This is a contradiction. (cf. Beth). Hence, the implication of Lemma 4 is not true in the opposite direction.

An immediate consequence of Definition 2 is that a kernel Φ with $\Phi^\wedge(n) = \gamma_1(n)$ for $n = 0, 1, \ldots$, where γ_1 is admissible, is an $\mathcal{H}_0$–kernel. Using an admissible generator γ_1 we can define a dilated generator $\gamma_\rho : \mathcal{H}_0(0,\infty) \to \mathbb{R}$ by

$$\gamma_\rho(t) = D_\rho\gamma_1(t) = \gamma_1(\rho t), \quad t \in (0,\infty)$$

(cf. [12]).

We are now able to verify to the admissibility for dilated functions.

Lemma 5. *Let $\gamma_1 : (0,\infty) \to \mathbb{R}$ be admissible and $\rho \in (0,1)$ be a given number. Then the dilated function γ_ρ is admissible.*

Proof. We use the denotations $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ for rounding real numbers: $\lfloor t\rfloor := \max\{n \in \mathbb{Z} \mid n \leq t\}$, $\lceil t\rceil = \min\{n \in \mathbb{Z} \mid n \geq t\}$, where $t \in \mathbb{R}$. We obtain

$$\begin{aligned}
&\sum_{n=0}^{N} \sup_{t\in[n,n+1)} (\gamma_1(\rho t))^2 \\
&= \sum_{n=0}^{N} \sup_{s\in[\rho n,\rho(n+1))} (\gamma_1(s))^2 \\
&\leq \sum_{n=0}^{N} \sup_{s\in[\lfloor\rho n\rfloor,\lceil\rho(n+1)\rceil)} (\gamma_1(s))^2 \\
&\leq \sum_{n=0}^{N} \left(\sup_{s\in[\lfloor\rho n\rfloor,\lceil\rho n\rceil)} (\gamma_1(s))^2 + \sup_{s\in[\lceil\rho n\rceil,\lceil\rho(n+1)\rceil)} (\gamma_1(s))^2 \right) \ .
\end{aligned} \tag{21}$$

As $0 < \rho < 1$, every interval in the last line is either empty or has the form $[p, p+1)$,where $p \in \mathbb{N}$. But some intervals can accur several times. There are at most $\lceil \frac{1}{\rho} \rceil + 1$ equal intervals of the kind $[\lfloor \rho n \rfloor, \lceil \rho n \rceil)$, as $\lfloor \rho n \rfloor = \lfloor \rho m \rfloor (n, m \in \mathbb{N})$ implies

$$\rho n = p + \alpha, \quad \rho m = p + \beta,$$

where $p = \lfloor \rho n \rfloor \in \mathbb{N}$ and $\alpha, \beta \in [0, 1)$. Without loss of generality we assume that $\alpha \leq \beta$, i.e. $n \leq m$. Thus,

$$\beta - \alpha = \rho(m - n)$$

implies

$$m - n = \frac{\beta - \alpha}{\rho} \leq \frac{1}{\rho} \ .$$

Analogously, we see that there are at most $\lceil \frac{1}{\rho} \rceil + 1$ equal intervals of the form $[\lceil \rho n \rceil, \lceil \rho(n+1) \rceil)$.

Furthermore, the largest values that we obtain for s in [19] are the intervals $[\lfloor \rho N \rfloor, \lceil \rho N \rceil)$ and $[\lceil \rho N \rceil, \lceil \rho(N+1) \rceil)$, where $\lceil \rho(N+1) \rceil = \lceil \rho N + \rho \rceil \leq \lceil \rho N \rceil + 1$. Hence, we obtain

$$\begin{aligned}
&\sum_{n=0}^{N} \sup_{x \in [n, n+1)} (\gamma_1(\rho t))^2 \\
&\leq \sum_{\rho=0}^{\lceil \rho N \rceil} 2 \cdot \left(\lceil \frac{1}{\rho} \rceil + 1 \right) \cdot \sup_{\rho \in [p, p+1[} (\gamma_1(s))^2 \\
&\leq 2 \cdot \left(\lceil \frac{1}{\rho} + 1 \right) \cdot \sum_{n=0}^{\infty} \sup_{t \in [n, n+1)} (\gamma_1(t))^2 < +\infty \ .
\end{aligned}$$

□

This proves Lemma 5.

Lemma 6. *Let $\gamma_1 : (0, \infty) \to \mathbb{R}$ be admissible and $\rho \in (1, \infty)$ be a given number. Then the dilated function γ_ρ is admissible.*

Proof. Note that

$$\lceil \rho n + \rho \rceil \leq \lceil \lceil \rho n \rceil + \lceil \rho \rceil \rceil = \lceil \rho n \rceil + \lceil \rho \rceil \leq \lfloor \rho n \rfloor + 1 + \lceil \rho \rceil \leq \lfloor \rho n \rfloor + \lfloor \rho \rfloor + 2.$$

We obtain for an arbitrary but fixed number $N \in \mathbb{N}$

$$\begin{aligned}
&\sum_{n=0}^{N} \sup_{s \in [\rho n, \rho(n+1))} (\gamma_1(s))^2 \\
&\leq \sum_{n=0}^{N} \sup_{s \in [\lfloor \rho n \rfloor \lceil \rho n + \rho \rceil)} (\gamma_1(s))^2 \\
&\leq \sum_{n=0}^{N} \left(\sup_{s \in [\lfloor \rho n \rfloor, \lfloor \rho n \rfloor + 1)} (\gamma_1(y))^2 + \ldots + \sup_{s \in [\lfloor \rho n \rfloor + \lfloor \rho \rfloor + 1, \lfloor \rho n \rfloor + \lfloor \rho \rfloor + 2)} (\gamma_1(s))^2 \right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{N} \left(\sum_{m=0}^{\lfloor\rho\rfloor+1} \sup_{s\in[\lfloor\rho n\rfloor+m,\lfloor\rho n\rfloor+m+1)} (\gamma_1(s))^2 \right) \\
&= \sum_{m=0}^{\lfloor\rho\rfloor+1} \left(\sum_{n=0}^{N} \sup_{s\in[\lfloor\rho n\rfloor+m,\lfloor\rho n\rfloor+m+1)} (\gamma_1(s))^2 \right) \quad .
\end{aligned}
$$

Let us keep m fixed for a moment. We see that

$$
\lfloor\rho n\rfloor + m = \lfloor\rho\nu\rfloor + m; \quad n,\nu\in\mathbb{N};
$$

is equivalent to

$$
\begin{aligned}
&\rho n = p + \alpha \\
&\rho\nu = p + \beta; \qquad p = \lfloor\rho n\rfloor \in \mathbb{N}; \qquad \alpha,\beta\in[0,1) \quad .
\end{aligned}
$$

Hence,

$$
n = \frac{p+\alpha}{\rho} = \frac{p+\beta+\alpha-\beta}{\rho} = \nu + \frac{\alpha-\beta}{\rho} \quad .
$$

As $\alpha-\beta\in(-1,1)$ and $\rho>1$, we see that $\alpha-\beta=0$ and consequently $n=\nu$ must hold. Thus, for fixed m, the intervals used are disjoint. Consequently, we obtain

$$
\begin{aligned}
&\sum_{n=0}^{N} \sup_{s\in[\rho n,\rho(n+1))} (\gamma_1(s))^2 \\
&\le \sum_{m=0}^{\rfloor\rho\lfloor+1} \sum_{k=0}^{\lfloor\rho N\rfloor+m} \sup_{s\in[k,k+1)} (\gamma_1(s))^2 \\
&\le \sum_{m=0}^{\lfloor\rho\rfloor+1} \sum_{k=0}^{\lfloor\rho N\rfloor+\lfloor\rho\rfloor+1} \sup_{s\in[k,k+1)} (\gamma_1(s))^2 \\
&\le (\lfloor\rho\rfloor+2)\cdot \sum_{k=0}^{\infty} \sup_{s\in[k,k+1)} (\gamma_1(s))^2 .
\end{aligned}
$$

Hence, γ_ρ is admissible, as required. □

Definition 3. *An admissible function $\varphi:[0,\infty)\to\mathbb{R}$ is called $\mathcal{H}_0$-generator of the kernel $\Phi:\overline{A_{\text{ext}}}\times\overline{A_{\text{ext}}}\to\mathbb{R}$ given by (18) if $\Phi^\wedge(n)=\varphi(n)$ for all $n=0,1,\ldots$.*

From Definition 3.1 it is clear that Φ is an $\mathcal{H}_0$-kernel provided that φ is an admissible generator of Φ.

For an admissible generator $\varphi=\varphi_1$ in the above sense we now introduce the functions $\varphi_\rho:[0,\infty)\to\mathbb{R}$ by letting

$$
\varphi_\rho(t) = D_\rho\varphi_1(t) = \varphi_1(\rho t) = \varphi(\rho t),\ t\in[0,\infty), \tag{22}
$$

for $\rho \in (0,\infty)$. It is easily seen that each function $\varphi_\rho, \rho \in (0,\infty)$, defined by (21) is an $\mathcal{H}_0$-generator of the kernel Φ_ρ via $\Phi_\rho^\wedge(n) = \varphi_\rho(n)$, $n = 0, 1, \ldots$. But this enables us to write $\Phi_\rho = D_\rho \Phi_1$. Note that

$$\Phi_{\rho\rho'} = D_\rho \Phi_{\rho'} = D_{\rho\rho'} \Phi_1 \quad .$$

D_ρ is called *dilation operator of level* ρ. We are also able to introduce the *inverse of* D_ρ denoted by $D_{\rho^{-1}}$, $\rho \in (0,\infty)$. To be more specific,

$$\Phi_{\rho^{-1}}(x,y) = D_{\rho^{-1}}\Phi(x,y) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi\alpha^2} \Phi^\wedge(\rho^{-1}n) \left(\frac{\alpha^2}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)$$

$(x,y) \in \overline{A_{\mathrm{ext}}} \times \overline{A_{\mathrm{ext}}}$, whenever Φ is an $\mathcal{H}_0$-kernel of the form (18) with $\Phi^\wedge(n) = \varphi(n)$, $n = 0, 1, \ldots$.

We now introduce those $\mathcal{H}_0$-generators which define scaling functions.

Definition 4. *An admissible function $\varphi_1 : [0,\infty] \to \mathbb{R}$ is called $\mathcal{H}_0$-generator of a scaling function if it satisfies the following properties: (i) φ_1 is monotonically decreasing on $[0,\infty)$, (ii) φ_1 is continuous at 0 with value $\varphi_1(0) = 1$.*

Indeed, if φ_1 satisfies the assumptions of an $\mathcal{H}_0$-generator of a scaling function, then φ_1 and its dilates φ_ρ generate the *scaling function* $\{\Phi_\rho\}$, $\Phi_\rho \in \mathcal{H}_0$, $\rho \in (0,\infty)$ via $\Phi_\rho^\wedge(n) = \varphi_\rho(n)$. It is easily seen that $\varphi_1(t) \geq 0$ for all $t \in [0,\infty)$. Furthermore, for each $t \in [0,\infty)$, we find

$$\lim_{\substack{\rho\to 0\\ \rho>0}} \varphi_\rho(t) = \lim_{\substack{\rho\to 0\\ \rho>0}} \varphi_1(\rho t) = \varphi_1(0) = 1,$$

since φ_1 is continuous at 0. Moreover, the monotonicity of φ_1 on $[0,\infty)$ and the definition of φ_ρ imply the monotonicity of the sequence $\{\varphi_\rho(t)\}$ for each $t \in [0,\infty)$.

Our considerations now enable us to verify an *approximate convolution identity.*

Theorem 1. *Let φ_1 be a generator of a scaling function $\{\Phi_\rho\}, \rho \in (0,\infty)$. Then*

$$\lim_{\substack{\rho\to 0\\ \rho>0}} ||U - \Phi_\rho * \Phi_\rho * U||_{\mathcal{H}_0} = 0$$

holds for every $U \in \mathcal{H}_0$.

Proof. Observing the Parseval identity we obtain

$$||U - \Phi_\rho * \Phi_\rho * U||_{\mathcal{H}_0} = \left(\sum_{n=0}^{\infty} (1 - (\varphi_\rho(n))^2)^2 (U, H_{n,j}(\alpha;\cdot))_{\mathcal{H}_0}|^2\right)^{1/2} \quad .$$

Letting ρ tend to 0 we obtain the desired result. $\square$

6 Windowed Fourier Transform

We begin with the definition of the windowed Fourier transform.

Definition 5. *For arbitrary but fixed $\rho \in (0,\infty)$, let ϕ_ρ be a member of a scaling function $\{\Phi_\rho\}$. Assume that U is of class $\mathcal{H}_0$. Then the windowed Fourier transform is defined by*

$$(WFT)_{\phi_\rho}(U)(n,j;x) = \left(\phi_\rho^{(2)}(1)\right)^{-1/2} (U, \phi_\rho(x,\cdot)H_{n,j}(\alpha;\cdot))_{\mathcal{H}_0}$$
$$= \left(\phi_\rho^{(2)}(1)\right)^{-1/2} \int_A U(y)\phi_\rho(x,y)H_{n,j}(\alpha;y)\ d\omega(y)$$

for $(n,j) \in J$ and $x \in \overline{A_{\text{ext}}}$, where $\Phi_\rho^{(2)}(1)$ is a normalization constant given by

$$\phi_\rho^{(2)}(1) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi\alpha^2}(\varphi_\rho(n))^2 \ . \tag{23}$$

The windowed Fourier transform converts a potential $U \in \mathcal{H}_0$ of one space–variable into a potential $(WFT)_{\Phi_\rho}(U)(n,j;x)$ of the two variables $(n,j) \in J$ and $x \in \overline{A_{\text{ext}}}$. The windowed Fourier transform is generated by the '*y–shift operator*' S_y and the (n,j)-*modulation operator* $M_{n,j}$ defined by

$$S_y : \phi_\rho(x,\cdot) \mapsto S_y\phi_\rho(x,\cdot) = \phi_\rho(x,y), \quad (x,y) \in \overline{A_{\text{ext}}} \times \overline{A_{\text{ext}}}$$
$$M_{n,j} : \phi_\rho(x,\cdot) \mapsto M_{n,j}\phi_\rho(x,\cdot) = \phi_\rho(x,\cdot)H_{n,j}(\alpha;\cdot), (n,j) \in \mathcal{N}, \quad x \in \overline{A_{\text{ext}}},$$

respectively. In other words,

$$(WFT)_{\phi_\rho}(U)(n,j;x) = \left(\phi_\rho^{(2)}(1)\right)^{-1/2} (U, M_{n,j}S_x\phi_\rho(\cdot,\cdot))_{\mathcal{H}_0}, \quad U \in \mathcal{H}_0.$$

6.1 Reconstruction Formula

Denote by $\mathcal{H}_0(\mathcal{N} \times \overline{A_{\text{ext}}})$ the space of all functions $G : \mathcal{N} \times \overline{A_{\text{ext}}} \to \mathbb{R}$ such that $G(n,j;\cdot) \in \mathcal{H}_0$ for all $(n,j) \in \mathcal{N}$ and

$$\sum_{(n,j)\in\mathcal{N}} ||G(n,j;\cdot)||^2_{\mathcal{H}_0} = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} ||G(n,j;\cdot)||^2_{\mathcal{H}_0} < \infty \ .$$

On the space $\mathcal{H}_0(\mathcal{N}\times\overline{A_{\text{ext}}})$ we are able to impose an inner product $(\cdot,\cdot)_{\mathcal{H}_0(\mathcal{N}\times\overline{A_{\text{ext}}})}$ corresponding to the norm

$$||G||_{\mathcal{H}_0(\mathcal{N}\times\overline{A_{\text{ext}}})} = \left(\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} ||G(n,j;\cdot)||^2_{\mathcal{H}_0}\right)^{1/2} .$$

This enables us to formulate the following theorem.

Theorem 2. *Let U be of class $\mathcal{H}_0$. The Gabor transform forms a mapping from the space $\mathcal{H}_0$ into $\mathcal{H}_0(\mathcal{N} \times \overline{A_{\text{ext}}})$, i.e. $(WFT)_{\Phi_\rho} : \mathcal{H}_0 \to \mathcal{H}_0(\mathcal{N} \times \overline{A_{\text{ext}}})$, and we have*

$$\begin{aligned} \|U\|^2_{\mathcal{H}_0} &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \| (WFT)_{\Phi_\rho} (n, j; \cdot) \|^2_{\mathcal{H}_0} \\ &= \| (U, \Phi_\rho(\cdot, \cdot) H_{\cdot,\cdot}(\alpha; \cdot))_{\mathcal{H}_0} \|^2_{\mathcal{H}_0(\mathcal{N} \times \overline{A_{\text{ext}}})} \ . \end{aligned}$$

Proof. The Parseval indentity of the theory of outer harmonics shows us that

$$\begin{aligned} &\int_A \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left|(WFT)_{\Phi_\rho}(U)(n, j, x)\right|^2 \ d\omega(x) \\ &= \left(\Phi_\rho^{(2)}(1)\right)^{-1} \int_A \left(\int_A |U(y)|^2 \, |\Phi_\rho(x, y)|^2 \ d\omega(x) \right) \ d\omega(y) \ . \end{aligned}$$

Now we observe that, for $y \in A$,

$$\int_A |\Phi_\rho(x, y)|^2 \ d\omega(x) = \Phi_\rho^{(2)}(y, y) = \Phi_\rho^{(2)}(1) \ .$$

But this yields the desired result. □

Theorem 2 is equivalent to the statement that any potential $U \in \mathcal{H}_0$ can be recovered by its Gabor expansion

$$\left(\Phi_\rho^{(2)}(1)\right)^{-1/2} \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \int_A (GT)_{\Phi_\rho} (U)(n, j; y) \Phi_\rho(\cdot, y) \ d\omega(y) H_{n,j}(\alpha; \cdot)$$

(relative to the $\mathcal{H}_0$–kernel Φ_ρ). To be more specific, we have

$$U = \left(\Phi_\rho^{(2)}(1)\right)^{-1/2} \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \int_A (GT)_{\Phi_\rho}(U)(n, j; x) \Phi_\rho(x, \cdot) \ d\omega(x) H_{n,j}(\alpha; \cdot)$$

in the sense of $\| \cdot \|_{\mathcal{H}_0}$. In particular, for every subset $K \subset A_{\text{ext}}$ with $\text{dist}(\overline{K}, A) > 0$, the convergence is uniform.

6.2 Least–squares Property

By virtue of the Cauchy–Schwarz inequality we obtain for $U \in \mathcal{H}$, $x \in \overline{A_{\text{ext}}}$, and arbitrary but fixed $\rho \in (0, \infty)$

$$\begin{aligned} |(WFT)(U)(n, j; x)| &\leq \left(\Phi_\rho^{(2)}(1)\right)^{-1/2} \left| (U, \Phi_\rho(x, \cdot) H_{n,j}(\alpha; \cdot))_{\mathcal{H}_0} \right| \\ &\leq \left(\Phi_\rho^{(2)}(1)\right)^{-1/2} \|U\|_{\mathcal{H}_0} \|\Phi_\rho(x, \cdot) H_{n,j}(\alpha; \cdot)\|_{\mathcal{H}_0} \ . \end{aligned}$$

In other words, $U \in \mathcal{H}_0$ implies that $(WFT)_{\Phi_\rho}(U) \in \mathcal{H}_0(\mathcal{J} \times \overline{A_{\text{ext}}})$ is bounded. But this transform $(WFT)_{\Phi_\rho}$ is not surjective on $\mathcal{H}_0(\mathcal{N} \times \overline{A_{\text{ext}}})$ (note that $\mathcal{H}_0(\mathcal{N} \times \overline{A_{\text{ext}}})$ contains unbounded elements). Therefore

$$\mathcal{G} = (WFT)_{\Phi_\rho}(\mathcal{H}_0)$$

is a proper subspace of $\mathcal{H}_0(\mathcal{N} \times \overline{A_{\text{ext}}})$:

$$\mathcal{G} \subsetneqq \mathcal{H}_0\left(\mathcal{N} \times \overline{A_{\text{ext}}}\right) \quad .$$

Hence we are led to the question of how to characterize $\mathcal{G}$ within the framework of $\mathcal{H}_0(\mathcal{N} \times \overline{A_{\text{ext}}})$.

For the purpose we consider the operator P: $\mathcal{H}_0(\mathcal{N} \times \overline{A_{\text{ext}}}) \to \mathcal{G}$ given by

$$(PH)(n,j;x) = \sum_{p=0}^{\infty} \sum_{q=1}^{2n+1} \int_{\Omega} K_\rho(n,j;x \mid p,q;y) H(p,q;y)\, d\omega(y), \tag{24}$$

where

$$K_\rho(n,j;x \mid p,q;y) = \left(\Phi_\rho^{(2)}(1)\right)^{-1} \int\limits_{A} \Phi_\rho(x,y) H_{n,j}(\alpha,z) \Phi_\rho(z,y) H_{p,q}(\alpha;z)\, d\omega(z). \tag{25}$$

Our aim is to show that P defines a projection operator.

Lemma 7. *The operator $P : \mathcal{H}_0(\mathcal{N} \times \overline{A_{\text{ext}}}) \to \mathcal{G}$ defined by (23), (24) is a projection operator.*

Proof. Assume that $H = \tilde{U} = (WFT)_{\Phi_\rho}(U) \in \mathcal{G}$. Then we obtain

$$\begin{aligned}
&(PH)(n,j;x) \\
&= \left(\Phi_\rho^{(2)}(1)\right)^{-1} \int\limits_{A} \Phi_\rho(x,z) H_{n,j}(\alpha;z) \\
&\quad \left(\sum_{p=0}^{\infty} \sum_{q=1}^{2n+1} \int\limits_{A} H(p,q;y) \Phi_\rho(z,y) H_{p,q}(\alpha;z)\, d\omega(y) \right) d\omega(z) \\
&= \left(\Phi_\rho^{(2)}(1)\right)^{-1/2} \int\limits_{A} U(\zeta) \Phi_\rho(x,z) H_{n,j}(\alpha;z)\, d\omega(z) \\
&= \tilde{U}(n,j;x) \\
&= H(n,j;x),
\end{aligned}$$

hence, $PH = H$ for all $H \in \mathcal{G}$. □

Next, we show, that for all $H^\perp \in \mathcal{G}^\perp$ we have $PH^\perp = 0$. Assume therefore that $H^\perp \in \mathcal{G}^\perp$, i.e. for all $F \in \mathcal{H}_0$

$$\left(H^\perp, (WFT)_{\Phi_\rho}(U)\right)_{\mathcal{H}_0(\mathcal{J} \times \overline{A_{\text{ext}}})} = 0 \quad . \tag{26}$$

If $x \in \overline{A_{\text{ext}}}$, $(n,j) \in \mathcal{N}$, then it follows from (25) with the special choice

$$U = \left(\Phi_\rho^{(2)}(1)\right)^{-1/2} \Phi_\rho(x,\cdot)H_{n,j}(\alpha;\cdot)$$

that

$$\begin{aligned}
0 &= \left(H^\perp, (GT)\Phi_\rho\left(\left(\Phi_\rho^{(2)}(1)\right)^{-(1/2)} \Phi_\rho(x,\cdot)H_{n,j}(\alpha;\cdot)\right)\right)_{\mathcal{H}_0(\mathcal{J}\times\overline{A_{\text{ext}}})} \\
&= \sum_{p=0}^{\infty}\sum_{q=1}^{2p+1}\left(\int_A H^\perp(p,q;y)\right. \\
&\quad \left.\left(\Phi_\rho^{(2)}(1)\right)^{-1/2}\left((WFT)_{\Phi_\rho}\left(\Phi_\rho(x,\cdot)H_{n,j}(\alpha;x)\right)\right)(p,q;y)\ d\omega(y)\right) \\
&= \sum_{p=0}^{\infty}\sum_{q=1}^{2p+1}\int_A H^\perp(p,q;y) \\
&\quad \left(\Phi_\rho^{(2)}(1)\right)^{-1}\int_A \Phi_\rho(x,z)H_{n,j}(\alpha;z)\Phi_\rho(y,z)H_{p,q}(\alpha;z)\ d\omega(z)\ d\omega(y) \\
&= \sum_{p=0}^{\infty}\sum_{q=1}^{2p+1}\int_A H^\perp(p,q;y)K_\rho(n,j;x\mid p,q;y)\ d\omega(y) \\
&= (PH^\perp)(n,j;x)\ .
\end{aligned}$$

Hence it is clear that $PH^\perp = 0$ for all $H^\perp \in \mathcal{G}^\perp$.

Summarizing our results we therefore obtain $P(\mathcal{H}_0(\mathcal{N}\times\overline{A_{\text{ext}}})) = \mathcal{G}$, $P\mathcal{G}^\perp = 0$, $P^2 = P$.

From our investigations we are therefore able to deduce that $\mathcal{G}$ is characterized as follows:

Lemma 8. *$H \in \mathcal{G}$ if and only if*

$$H(n,j;x) = \left(\Phi_\rho^{(2)}(1)\right)^{-1/2}\sum_{p=0}^{\infty}\sum_{q=1}^{2p+1}\int_A K_\rho(n,j;x\mid p,q;y)H(p,q;y)\ d\omega(y)\ . \tag{27}$$

In windowed Fourier theory (26) is known as *consistency condition* associated with the 'kernel' Φ_ρ (cf. [15, 16]). From the consistency conditions it follows that not any function $H \in \mathcal{H}_0(\mathcal{N}\times\overline{A_{\text{ext}}})$ can be the windowed Fourier transform of a potential $U \in \mathcal{H}_0$. In fact, if the consistency were not valid, then we could design space–dependent potentials with arbitrary space–momentum propertiy and thus violate the uncertainty principle.

It is not difficult to see that $K_\rho(n,j;y\mid\cdot,\cdot;\cdot) \in \mathcal{G}$ and $K_\rho(\cdot,\cdot;\cdot\mid p,q;y) \in \mathcal{G}$. The kernel $(n,j;x\mid p,q;y) \mapsto K_{h_0}(n,j,x\mid p,q;,y)$ $(n,j)\in\mathcal{N},(p,q)\in\mathcal{N},(x,y)\in\overline{A_{\text{ext}}}\times\overline{A_{\text{ext}}}$, is the reproducing kernel in $\mathcal{G}$.

Next we prove the following theorem.

Theorem 3. *Let H be an arbitrary element of $\mathcal{H}_0(\mathbb{R}\mathcal{N}\times\overline{A_{\text{ext}}})$. Then the unique function $U_H \in \mathcal{H}_0$ satisfying*

$$\|H - \tilde{U}_H\|_{\mathcal{H}_0(\mathcal{J}\times\overline{A_{\text{ext}}})} = \inf_{U\in\mathcal{H}_0} \|H - \tilde{U}\|_{\mathcal{H}_0(\mathcal{J}\times\overline{A_{\text{ext}}})}$$

(with $\tilde{U}_H = (WFT)_{\Phi_\rho}(U_H)$) is given by

$$U_H(x) = \left(\Phi_\rho^{(2)}(1)\right)^{-1/2} \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \int_A H(n,j;y)\Phi_\rho(x,y)\; d\omega(y) H_{n,j}(\alpha;x) \qquad (28)$$

Proof. We know that $\tilde{U}_{\mathcal{H}}$ is the orthogonal projection of H onto $\mathcal{G}$. This proves Theorem 2. $\square$

Our considerations have shown that the coefficients in $\mathcal{H}_0(\mathcal{N}\in\overline{A_{\text{ext}}})$ for reconstructing a function $U \in \mathcal{H}_0$ are not unique. This can be immediately seen from the identity

$$U = \left(\Phi_\rho^{(2)}(1)\right)^{-1/2} \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \int_\Omega \left(\tilde{U}(n,j;y) + \tilde{U}^\perp(n,j;y)\right) \Phi_\rho(\cdot,y)\; d\omega(y) H_{n,j}(\alpha;\cdot),$$

where $\tilde{U} = (WFT)_{\Phi_\rho}(U)$ and $U^\perp$ is an arbitrary member of $\mathcal{H}^\perp$.

But we are able to formulate the following result.

Theorem 4. *For arbitrary $F\in\mathcal{H}_0$ the coefficient function $\tilde{U} = (WFT)_{\Phi_\rho}(U) \in \mathcal{G}$ is the unqiue element in $\mathcal{H}_0(\mathcal{N}\times\overline{A_{\text{ext}}})$ which satisfies the minimum norm condition*

$$\|\tilde{U}\|_{\mathcal{H}_0(\mathcal{N}\times\overline{A_{\text{ext}}})} = \inf_{\substack{H\in\mathcal{H}_0(\mathcal{N}\times\overline{A_{\text{ext}}})\\ (WFT)^{-1}_{\Phi_\rho}(H)=U}} \|H\|_{\mathcal{H}_0(\mathcal{N}\times\overline{A_{\text{ext}}})} \;.$$

Proof. We know already that $H = \tilde{U} + \tilde{U}^\perp$. Thus we are able to deduce that

$$\begin{aligned} \|H\|^2_{\mathcal{H}_0(\mathcal{N}\times\overline{A_{\text{ext}}})} &= \|\tilde{U} + \tilde{U}^\perp\|^2_{\mathcal{H}_0(\mathcal{N}\times\overline{A_{\text{ext}}})} \\ &= \|\tilde{U}\|^2_{\mathcal{H}_0(\mathcal{N}\times\overline{A_{\text{ext}}})} + \|\tilde{U}\|^2_{\mathcal{H}_0}\left(\mathcal{N}\times\overline{A_{\text{ext}}}\right) \\ &\geq \|\tilde{U}\|^2_{\mathcal{H}_0(\mathcal{N}\times\overline{A_{\text{ext}}})}, \end{aligned}$$

as required. $\square$

As mentioned in our introduction, the windowed Fourier transform works by first dividing a 'signal' $U\in\mathcal{H}_0$ into short consecutive segments of fixed size by use of a 'cutoff kernel' (window function) Φ_ρ and then computing the Fourier coefficients of each segment. In other words, the windowed Fourier transform maps local changes of the function being represented to local changes of the coefficients in the expansion and thereby also reduces the computational complexity. However, there is still a defect in reconstructing a function using a sole, fixed 'window parameter' $\rho \in (0,\infty)$. In poorly resolves phenomena shorter than the window which leads to non-optimal computational costs in many circumstances. This can be remedied by kernels with decreasing window diameters (i.e. $\rho \to 0$) exhibiting the so–called 'zooming-in' property.

The meaning of Theorem 6 may be explained as follows (confer the arguments in Euclidean wavelet theory due to [16]): Suppose we want a potential with certain specified properties in frequency (momentum) and in space. In other words, we are interested in a potential $U \in \mathcal{H}_0$ such that $(WFT)_{\Phi_\rho}(U)(n,j;x) = H(n,j;x)$, where $H \in \mathcal{H}_0(\mathcal{N} \times \overline{A_{\text{ext}}})$ is given. Lemma 8 informs us that no potential can exist unless H satisfies the consistency condition. The function U_H introduced above is closest in the sence that the '$\mathcal{H}_0(\mathcal{N} \times \overline{A_{\text{ext}}})$–distance' of its windowed Fourier transform $\tilde{U}_H$ to H is a minimum. U_H is called the *least–squares approximation* to the desired potential $U \in \mathcal{H}_0$. In the case that $H \in \mathcal{G}$, Eq. (27) reduces to the reconstruction formula.

The least–squares approximation may be used to process potentials simultaneously in frequency and in space. More explicitly, given a potential $U \in \mathcal{H}_0$, we may first compute $(WFT)_{\Phi_\rho}(U)(n,j;x)$ and then modify it in any desirable way (such as by suppressing some frequencies and amplifying others while simultaneously localizing in space). Of course, the modified expression $H(n,j;x)$ is generally no longer the windowed Fourier transform of any (space–dependent) potential $U \in \mathcal{H}_0$, but its least-squares approximation U_H comes closest to being such a potential, in the above topology.

Another essential aspect of the least–squares approximation is that even when we do not purposefully tamper with $(WFT)_{\Phi_\rho}(U)(n,j;x)$, 'noise' is introduced in it. Hence, by the position we are ready to reconstruct $U \in \mathcal{H}_0$, the resulting expression $H(n,j;x)$ will no longer belong to $\mathcal{G}$. Hence, any randon change is almost certain to take $H \in \mathcal{H}_0(J \times \overline{A_{\text{ext}}})$ out of $\mathcal{G}$. The 'reconstruction formula' in the form (27) then automatically yields the least–squares approximation to the original signal, given the incomplete or erroneous information at hand. This is a kind of built–in stability of the windowed Fourier reconstruction related to oversampling.

7 Continuous Wavelet Transform

With the definitions of (Chapter 5) in mind, we are now interested in introducing the wavelet transform (WT). In a consistent setup scale continuous as well as scale discrete wavelets are discussed. It turns out that the relation between scaling function and scale continuous wavelet is characterized by a differential equation. This assumes the piecewise differentiability of the scaling function under consideration.

Definition 6. *Let $\varphi_1 : [0,\infty)$ be a piecewise differentiable $\mathcal{H}_0$-generator of a scaling function. Then the function $\psi_1 : [0,\infty) \to \mathbb{R}$ is said to be the $\mathcal{H}_0$-*

generator of the mother wavelet kernel Ψ_1 *given by*

$$\Psi_1(x,y) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi\alpha^2}\psi_1(n)\left(\frac{\alpha^2}{|x||y|}\right)^{n+1} P_n(\frac{x}{|x|}\cdot\frac{y}{|y|}),\ (x,y)\in\overline{A_{\text{ext}}}\times\overline{A_{\text{ext}}},$$

if ψ_1 *is admissible and satisfies, in addition, the differential equation*

$$\psi_1(t) = \left(-t\frac{d}{dt}(\varphi_1(t))^2\right)^{1/2} = (-2t\varphi_1(t)\varphi_1'(t))^{1/2}.$$

It is not difficult to show that the generator ψ_1 and its dilates $\psi_\rho = D_\rho\psi_1$

$$\psi_\rho(t) = \psi_1(\rho t) = (-2\rho t\varphi_1(\rho t)\varphi_1'(\rho t))^{1/2} = (-\rho\frac{d}{d\rho}(\varphi_\rho(t))^2)^{1/2}$$

satisfy the following properties:

- $\psi_\rho(0) = 0\ ,\ \rho\in(0,\infty)$
- $\varphi_\rho(t) = \left(\int_\rho^\infty (\psi_\gamma(t))^2\frac{d\gamma}{\gamma}\right)^{1/2}\quad ,\quad t\in(0,\infty), \rho\in(0,\infty),$
- $\lim_{\substack{\rho\to 0\\ \rho>0}} \psi_\rho(t) = 0,\ t\in(0,\infty),$
- $\left(\int_0^\infty (\psi_\rho(t))^2\frac{d\rho}{\rho}\right)^{1/2} = 1,\quad t\in(0,\infty),$
- $\sum_{n=1}^{\infty}\frac{2n+1}{4\pi\alpha^2}\int_\rho^\infty(\psi_\gamma(n))^2\frac{dx}{\gamma} < \infty,\quad \rho\in(0,\infty).$

The first condition justifies the name *wavelet* (i.e., "small wave"). The last condition is of later significance in that it essentially assures the reconstruction formula of our scale continuous wavelet theory. The intermediate properties are straightforward consequences of our definition of the mother wavelet kernel.

Definition 7. *The family* $\{\Psi_\rho\}, \rho\in(0,\infty)$ *of* $\mathcal{H}_0$*-kernels corresponding to the mother wavelet* Ψ_1 *defined via* $\Psi_\rho^\wedge(n) = \psi_\rho(n), n = 0,1,\ldots$ *is called a scale continuous harmonic wavelet.*

Let $\Psi_{\rho;y}$ be defined as follows

$$\Psi_{\rho;y} : x\mapsto \Psi_{\rho;y}(x) = \Psi_\rho(x,y) = S_y D_\rho \Psi_1(x,\cdot),\quad x\in\overline{A_{\text{ext}}},$$

where the *'y-shift operator'* S_y and the *'ρ-dilation operator'* D_ρ are given by

$$\begin{aligned} S_y &: \Psi_1(x,\cdot)\mapsto S_y\Psi_1(x,\cdot) = \Psi_1(x,y),\quad (x,y)\in\overline{A_{\text{ext}}}\times\overline{A_{\text{ext}}},\\ D_\rho &: \Psi_1(x,\cdot)\mapsto D_\rho\Psi_1(x,\cdot) = \Psi_\rho(x,\cdot),\quad x\in\overline{A_{\text{ext}}},\end{aligned}$$

respectively.

Definition 8. *Let $\{\Psi_\rho\}, \rho \in (0,\infty)$, be a scale continuous wavelet as defined above. Then the scale continuous harmonic wavelet transform (WT) of scale $\rho \in (0,\infty)$ and position $y \in \overline{A_{\text{ext}}}$ is defined by*

$$(WT)(U)(\rho; y) = (U, \Psi_{\rho;y})_{\mathcal{H}_0} = \int_A U(x)\Psi_{\rho;y}(x)d\omega(x)$$

for all $U \in \mathcal{H}_0$.

Consequently, as in the case of the windowed Fourier transform, the (continuous) wavelet transform converts a potential $U \in \mathcal{H}_0$ into an expression of two variables, namely scale and position.

7.1 Reconstruction Formula

The scale continuous wavelet transform admits an inverse on the space of functions $U \in \mathcal{H}_0$ satisfying $(U, H_{0,1}(\alpha;\cdot))_{\mathcal{H}_0} = 0$.

Theorem 5. *(Reconstruction formula). Let $\{\Psi_\rho\}$, $\rho \in (0,\infty)$ be a wavelet. Suppose that $U \in \mathcal{H}_0$ satisfies $(U, H_{0,1}(\alpha;\cdot))_{\mathcal{H}_0} = 0$. Then*

$$\lim_{\substack{R\to 0\\ R>0}} \left\| U - \int_A \int_R^\infty (WT)(U)(\rho;y)\Psi_{\rho;y}(\cdot)\frac{d\rho}{\rho}d\omega(y) \right\|_{\mathcal{H}_0} = 0 \ .$$

Proof. Choose an arbitrary $R > 0$. Then we have

$$\begin{aligned}
&\int_A \int_R^\infty (WT)(U)(\rho;y)\Psi_{\rho;y}(x)\frac{d\rho}{\rho}d\omega(y) \\
&= \int_A \int_R^\infty \left(\int_A U(z)\Psi_\rho(z,y)d\omega(z) \right) \Psi_\rho(x,y)\frac{d\rho}{\rho}d\omega(y) \\
&= \int_A U(z) \left(\int_R^\infty \sum_{n=1}^\infty \frac{2n+1}{4\pi\alpha^2}(\Psi_\rho^\wedge(n))^2 \left(\frac{\alpha^2}{|x||z|}\right)^{n+1} P_n\left(\frac{x}{|x|}\cdot\frac{z}{|z|}\right)\frac{d\rho}{\rho} \right) d\omega(z) \\
&= \int_A U(z)\ \Phi_R^{(2)}(z,x)d\omega(z) \\
&= \left(\Phi_R^{(2)} * U\right)(x)
\end{aligned}$$

for every $x \in \overline{A_{\text{ext}}}$. Now we know that $\lim_{\substack{R\to 0\\ R>0}} U * \Phi_R^{(2)} = U$ in the sense of $\|\cdot\|_{\mathcal{H}_0}$. But this is the desired result. □

In connection with (6) we obtain the following result.

Corollary 1. *Under the assumptions of Theorem 5*

$$\lim_{\substack{R\to 0\\ R>0}} \sup_{x\in\overline{\Sigma_{\text{ext}}}} \left| U(x) - \int_A \int_R (WT)(U)(\rho;y)\Psi_{\rho;y}(x)\frac{d\rho}{\rho} d\omega(y) \right| = 0 \ .$$

In other words, a constructive approximation by wavelets defined on $\overline{A_{\text{ext}}}$ is found to approximate the solution of the Dirichlet boundary-value problem for the Laplace equation on $\overline{\Sigma_{\text{ext}}}$.

7.2 Least–squares Property

Denote by $\mathcal{H}_0\left((0,\infty)\times\overline{A_{\text{ext}}}\right)$ the space of all functions $U : (0,\infty)\times\overline{A_{\text{ext}}} \to \mathbb{R}$ such that $U(\rho;\cdot)\in\mathcal{H}_0$ for every $\rho\in(0,\infty)$ and

$$\int_0^\infty \|U(\rho;\cdot)\|^2_{\mathcal{H}_0} \frac{d\rho}{\rho} = \int_0^\infty \int_A |U(\rho;y)|^2 \ d\omega(y)\frac{d\rho}{\rho} < \infty \ . \tag{29}$$

On the space $\mathcal{H}_0((0,\infty)\times\overline{A_{\text{ext}}})$ we are able to impose an inner product $(\cdot,\cdot)_{\mathcal{H}_0((0,\infty)\times\overline{A_{\text{ext}}})}$ by letting

$$\begin{aligned}(U(\cdot,\cdot),V(\cdot,\cdot))_{\mathcal{H}_0((0,\infty)\times\overline{A_{\text{ext}}})} &= \int_0^\infty \int_A U(\rho;y)V(\rho;y) \ d\omega(y)\frac{d\rho}{\rho} \\ &= \int_0^\infty (U(\rho;\cdot), \ V(\rho,\cdot))_{\mathcal{H}_0} \frac{d\rho}{\rho}\end{aligned}$$

for $U,V\in\mathcal{H}_0((0,\infty)\times\overline{A_{\text{ext}}})$. Correspondingly,

$$\begin{aligned}\|U(\cdot,\cdot)\|_{\mathcal{H}_0((0,\infty)\times\overline{A_{\text{ext}}})} &= \left(\int_0^\infty \int_A |U(\rho;y)|^2 \ d\omega(y)\frac{d\rho}{\rho}\right) \\ &= \left(\int_0^\infty \|U(\rho,\cdot)\|^2 \frac{d\rho}{\rho}\right)^{1/2} .\end{aligned}$$

From Theorem 2 we obtain the following result telling us that the wavelet transform does not change the total energy.

Lemma 9. *Let $\{\Psi_\rho\}$, $\rho\in(0,\infty)$, be a wavelet. Suppose that U,V are of class $\mathcal{H}_0$. Then*

$$\int_0^\infty \int_A (U,\Psi_{\rho;\cdot}(y))_{\mathcal{H}_0} (V,\Psi_{\rho;\cdot}(y))_{\mathcal{H}_0} \ d\omega(y)\frac{d\rho}{\rho} = (U,V)_{\mathcal{H}_0} \ . \tag{30}$$

As we have seen, WT is a transform form the one–parameter space $\mathcal{H}_0$ into the two–parameter space $\mathcal{H}_0((0,\infty)\times\overline{A_{\rm ext}})$. But the transform WT is not surjective on $\mathcal{H}_0((0,\infty)\times\overline{A_{\rm ext}})$ (note that $\mathcal{H}_0((0,\infty)\times\overline{A_{\rm ext}})$ contains unbounded elements). That means that

$$\mathcal{W} = (WT)(\mathcal{H}_0) \tag{31}$$

is a proper subspace of $\mathcal{H}_0((0,\infty)\times\overline{A_{\rm ext}})$:

$$\mathcal{W} \subsetneqq \mathcal{H}_0((0,\infty)\times\overline{A_{\rm ext}}).$$

Therefore, one may ask the question of how to characterize $\mathcal{W}$ within the framework of $\mathcal{H}_0((0,\infty)\times\overline{A_{\rm ext}})$.

For that purpose we consider the operator

$$P : \mathcal{H}_0\left((0,\infty)\times\overline{\Sigma_{\rm ext}}\right) \to \mathcal{W} \tag{32}$$

defined by

$$P(U)(\rho\cdot y) = \int\limits_0^\infty \int\limits_A K\left(\rho; y \mid \rho; y\right) U\left(\rho; y\right)\ d\omega(y)\frac{d\rho}{\rho},\ \rho\in(0,\infty),\ y\in\overline{A_{\rm ext}}, \tag{33}$$

where we have introduced the kernel

$$K(\rho'; y' \mid \rho; y) = \int\limits_A \Psi_{\rho';y'}(x)\Psi_{\rho;y}(x)\ d\omega(x) = \left(\Psi_{\rho';y'}(\cdot), \Psi_{\rho;y}(\cdot)\right)_{\mathcal{H}_0}\ .$$

First our purpose is to verify the following lemma.

Lemma 10. *The operator $P : \mathcal{H}_0((0,\infty)\times\overline{A_{\rm ext}}) \to \mathcal{W}$ defined by (31), (32) is a projection operator.*

Proof. Assume that $H = \tilde{U} = (WT)(U) \in \mathcal{W}$. Then it is not difficult to see that for $x \in \overline{A_{\rm ext}}$

$$\begin{aligned} P(H)(\rho; x) &= \int\limits_0^\infty \int\limits_A K(\rho; x \mid \sigma; y)(WT)(U)(\sigma; y)\ d\omega(y)\frac{d\sigma}{\sigma} \\ &= \tilde{U}(\rho; x) \\ &= (WT)(U)(x). \end{aligned}$$

Consequently, $P(H)(\cdot,\cdot) = H(\cdot,\cdot)$ for all $H(\cdot,\cdot)\in\mathcal{W}$.

Next we want to show that for all $H^\perp(\cdot,\cdot)\in\mathcal{W}^\perp$ we have $P(H^\perp(\cdot,\cdot)) = 0$. For that purpose, consider an element $H^\perp(\cdot,\cdot)$ of $W^\perp$. Then, for all $U\in\mathcal{H}_0$ we have

$$\left(H^\perp(\cdot,\cdot), (WT)(U)(\cdot,\cdot)\right)_{H_0((0,\infty)\times\overline{A_{\rm ext}})} = 0\ . \tag{34}$$

Now, for $\rho \in (0,\infty)$ and $x \in \overline{A_{\text{ext}}}$, we obtain under the special choice $U = \Psi_{\rho;x}(\cdot)$

$$
\begin{aligned}
0 &= \left(H^{\perp}(\cdot,\cdot), (WT)(\Psi_{\rho;x})(\cdot))\right)_{\mathcal{H}_0((0,\infty)\times\overline{A_{\text{ext}}})} \\
&= \int_0^\infty \int_A \left(H^{\perp}(\sigma;\cdot), \Psi_{\sigma;y}(\cdot)\right)_{\mathcal{H}_0} ((WT)(\Psi_{\rho;x}(\cdot), \Psi_{\sigma;y}(\cdot)))_{\mathcal{H}} \ d\omega(y) \frac{d\sigma}{\sigma} \\
&= \int_0^\infty \int_A K(\rho;x \mid \sigma;y) H^{\perp}(\sigma;y) \ d\omega(y) \frac{d\sigma}{\sigma} \\
&= P(H^{\perp})(\rho;x) \ .
\end{aligned}
$$

In other words, $P(H^{\perp}(\cdot,\cdot)) = 0$ for all $H^{\perp}(\cdot,\cdot) \in \mathcal{W}^{\perp}$. Therefore we get

$$
\begin{aligned}
&P\left(\mathcal{H}_0((0,\rho)\times\overline{A_{\text{ext}}})\right) = \mathcal{W}, \\
&P(W^{\perp}(\cdot,\cdot)) = 0,
\end{aligned}
$$

$P^2 = P$, as desired. □

$\mathcal{W} = (WT)(\mathcal{H}_0)$ is characterized as follows:

Lemma 11. *$H \in \mathcal{W}$ if and only if the 'consistency condition'*

$$
H(\rho';y') = \int_0^\infty \int_A K(\rho';y' \mid \rho;x) H(\rho;x) \ d\omega(x) \frac{d\rho}{\rho}
$$

is satisfied.

Obviously,

$$
\begin{aligned}
K(\rho',y' \mid \cdot;\cdot) \in \mathcal{W}, &\quad \rho' \in (0,\infty), y' \in \overline{A_{\text{ext}}}, \\
K(\cdot;\cdot \mid \rho,y) \in \mathcal{W}, &\quad \rho' \in (0,\infty), y' \in \overline{A_{\text{ext}}},
\end{aligned}
$$

i.e.:

$$
(\rho';y' \mid \rho;y) \mapsto K(\rho',y' \mid \rho;y)
$$

is the (uniquely determined) reproducing kernel in $\mathcal{W}$.

Summarizing our results we therefore obtain the following theorem.

Theorem 6. *Let H be an arbitrary element of $\mathcal{H}_0((0,\infty)\times\overline{A_{\text{ext}}})$. Then the unique function $U_H \in \mathcal{H}_0$ satisfying the property*

$$
||H - \tilde{U}_H||_{\mathcal{H}_0((0,\infty)\times\overline{A_{\text{ext}}})} = \inf_{U\in\mathcal{H}_0} ||H - \tilde{U}||_{\mathcal{H}_0((0,\infty)\times\overline{A_{\text{ext}}})}
$$

is given by

$$
U_H(x) = \int_0^\infty \int_A \Psi_{\rho;x}(y) H(\rho;y) \ d\omega(y) \frac{d\rho}{\rho} \ . \tag{35}
$$

Theorem 6 means that U_H defined above comes closest in the sense of the '$\mathcal{H}((0,\infty)\times\overline{A_{\rm ext}})$–distance' of its wavelet transform $\tilde{U}_H$ to H assumes a minimum. In analogy to the windowed Fourier theory we call U_H the *least–squares approximation* to the desired potential $U\in\mathcal{H}_0$. Of course,, for $H\in\mathcal{W}$, Eq. (34) reduces to the reconstruction formula. All aspects of least–squares approximation discussed earlier for the windowed Fourier transform remain valid in the same way. The coefficients in $\mathcal{H}_0((0,\infty)\times\overline{A_{\rm ext}})$ for reconstructing a potential $U\in\mathcal{H}_0$ are not unique. This can be readily developed from the following identity

$$U=\int\limits_0^\infty\int\limits_A\left(\tilde{U}(\rho;y)+\tilde{U}^\perp(\rho;y)\right)\,d\omega(y)\frac{d\rho}{\rho},$$

where $\tilde{U}=(WT)(U)$ is a member of W and $\tilde{U}^\perp$ is an arbitrary member of $\mathcal{W}^\perp$.

But our considerations enable us to formulate the following minimum norm representation:

Theorem 7. *For arbitrary $U\in\mathcal{H}_0$ the function $\tilde{U}=(WT)(U)\in\mathcal{W}$ is the unqiue element in $\mathcal{H}((0,\infty)\times\overline{A_{\rm ext}})$ satisfying*

$$\|\tilde{U}\|_{\mathcal{H}((0,\infty)\times\overline{A_{\rm ext}})}=\inf_{\substack{H\in\mathcal{H}_0((0,\infty)\times\overline{A_{\rm ext}})\\(WT)^{-1}(H)=U}}\|H\|_{\mathcal{H}((0,\infty)\times\overline{A_{\rm ext}})}\ .$$

8 Scale Discrete Wavelet Transform

Until now emphasis has been put on the whole scale interval. In what follows, however, scale discrete wavelets will be discussed. We start from a strictly decreasing sequence $\{\rho_j\}$, $j\in\mathbb{Z}$ such that $\lim_{j\to\infty}\rho_j=0$ and $\lim_{j\to-\infty}\rho_j=\infty$. For reasons of simplicity, we choose $\rho_j=2^{-j}$, $j\in\mathbb{Z}$, throughout this paper.

Definition 9. *Let $\varphi_0^D=\varphi_{\rho_0}$ be the generator of a scaling function (as defined above). Then the piecewise continuous function $\psi_0^D:[0,\infty)\to\mathbb{R}$ is said to be the $\mathcal{H}_0$-generator of the mother wavelet kernel Ψ_0^D (of a scale discrete harmonic wavelet) if it is admissible and satisfies, in addition, the difference equation*

$$(\psi_0^D(t))^2=\left(\varphi_0^D(t/2)\right)^2-\left(\varphi_0^D(t)\right)^2,\qquad t\in[0,\infty)\ .$$

For φ_0^D resp. ψ_0^D we introduce functions φ_j^D resp. ψ_j^D in the canonical way

$$\begin{aligned}\varphi_j^D(t)&=D_j^D\varphi_0^D(t)=\varphi_0^D(2^{-j}t),\qquad t\in[0,\infty),\\ \psi_j^D(t)&=D_j^D\psi_0^D(t)=\psi_0^D(2^{-j}t),\qquad t\in[0,\infty)\ .\end{aligned}$$

Then, each function φ_j^D, resp. ψ_j^D, $j\in\mathbb{Z}$ is admissible. This enables us to write $\psi_j^D=D_1\psi_{j-1}^D$, $j\in\mathbb{Z}$ whenever φ_0^D is admissible. Correspondingly, for the $\mathcal{H}_0$-kernel Ψ_j^D, $j\in\mathbb{Z}$, generated by ψ_j^D via

$$(\Psi_j^D)^\wedge(n)=\psi_j^D(n),n\in\mathbb{N}_0,$$

we let $\Psi_j^D=D_1\Psi_{j-1}^D$, $j\in\mathbb{Z}$.

Definition 10. *The subfamily* $\{\Psi_j^D\}$, $j \in \mathbb{Z}$ *of the space* $\mathcal{H}_0$ *generated by* Ψ_0^D *via* $(\Psi_j^D)^\wedge(n) = \psi_j^D(n)$, $n = 0, 1, \ldots$ *is called a scale discrete harmonic wavelet.*

The generator $\psi_0^D : [0, \infty) \to \mathbb{R}$ and its dilates $\psi_j^D = D_j^D \psi_0^D$ satisfy the following properties:

$$\psi_j^D(0) = 0, \quad j \in \mathbb{Z},$$

$$(\psi_j^D(t))^2 = (\varphi_{j+1}^D(t))^2 - (\varphi_j^D(t))^2, \quad j \in \mathbb{Z}, t \in [0, \infty),$$

$$(\varphi_0^D(t))^2 + \sum_{j=0}^{J} (\psi_j^D(t))^2 = (\varphi_{J+1}^D(t))^2, \quad J \in \mathbb{Z}, t \in [0, \infty) \ . \tag{36}$$

It is natural to apply the operator D_j^D directly to the mother wavelet. In connection with the "shifting operator" S_y, this will lead us to the definition of the kernel $\Psi_{j;y}$. More explicitly, we have

$$\Psi_j^D = D_j^D \Psi_0^D, \quad j \in \mathbb{Z}, \tag{37}$$

and

$$(S_y \Psi_j^D)(x) = \Psi_{j;y}^D(x) = \Psi_j^D(x, y) = \Psi_{\rho_j}(x, y), \quad (x, y) \in \overline{A_{\mathrm{ext}}} \times \overline{A_{\mathrm{ext}}} \ . \tag{38}$$

Putting together (36) and (37) we therefore obtain for $(x, y) \in \overline{A_{\mathrm{ext}}} \times \overline{A_{\mathrm{ext}}}$,

$$\Psi_{j;y}^D(x) = (S_y D_j^D \Psi_0^D)(x) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi\alpha^2} (\Psi_0^D)^\wedge(2^{-j}n) \left(\frac{\alpha^2}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right) \ .$$

Definition 11. *Let the* $\mathcal{H}_0$*-kernel* Ψ_0^D *be a mother wavelet kernel corresponding to a scaling function* $\Phi_0^D = \Phi_{\rho_0}$. *Then, the scale discrete wavelet transform at scale* $j \in \mathbb{Z}$ *and position* $y \in \overline{A_{\mathrm{ext}}}$ *is defined by*

$$(WT)^D(U)(j; y) = (U, \Psi_{j;y}^D(\cdot))_{\mathcal{H}_0}, \quad U \in \mathcal{H}_0 \ .$$

It should be mentioned that each scale continuous wavelet $\{\Psi_\rho\}$, $\rho \in (0, \infty)$ implies a scale discrete wavelet $\{\Psi_j^D\}$, $j \in \mathbb{Z}$ by letting

$$((\Psi_j^D)^\wedge(n))^2 = ((\Phi_{j+1}^D)^\wedge(n))^2 - ((\Phi_j^D)^\wedge(n))^2, \ n \in \mathbb{N}_0$$

where

$$((\Phi_j^D)^\wedge(n))^2 = \int_{\rho_j}^{\infty} (\Psi_\rho^\wedge(n))^2 \frac{d\rho}{\rho},$$

i.e.,

$$(\Psi_j^D)^\wedge(n) = \left(\int_{\rho_{j+1}}^{\rho_j} (\Psi_\rho^\wedge(n))^2 \frac{d\rho}{\rho} \right)^{1/2} \ .$$

Note that this construction leads to a *partition of unity* in the following sense

$$\int_0^\infty (\Psi_\rho^\wedge(n))^2 \frac{d\rho}{\rho} = \sum_{j=-\infty}^{+\infty} ((\Psi_j^D)^\wedge(n))^2 = ((\Phi_0^D)^\wedge(n))^2 + \sum_{j=0}^{\infty}((\Psi_j^D)^\wedge(n))^2 = 1$$

for $n \in \mathbb{N}$.

Our investigations now enable us to reconstruct a potential $U \in \mathcal{H}_0$ from its discrete wavelet transform as follows.

Theorem 8. *(Reconstruction Formula). Any potential $U \in \mathcal{H}_0$ can be approximated by its J-level scale discrete wavelet approximation*

$$U_J = \int_A (U, \Phi_{0;y}^D)_{\mathcal{H}_0} \Phi_{0;y}^D(\cdot) d\omega(y) + \sum_{j=0}^{J} \int_A (WT)^D(U)(j;y) \Psi_{j;y}^D(\cdot) d\omega(y) \tag{39}$$

in the sense that

$$\lim_{J\to\infty} \|U - U_J\|_{\mathcal{H}_0} = 0 \ .$$

Proof. Let U be a member of class $\mathcal{H}_0$. From (35) it follows that

$$\int_A (U, \Phi_{0;y}^D)_{\mathcal{H}_0} \Phi_{0;y}^D(\cdot) d\omega(y) + \sum_{j=0}^{J} \int_A (WT)^D(j;y) \Psi_{j;y}^D(\cdot) d\omega(y)$$
$$\int_A (U, \Phi_{J+1;y}^D)_{\mathcal{H}_0} \Phi_{J+1;y}^D(\cdot) d\omega(y) \ .$$

Letting J tend to infinity the result follows easily from Theorem 1. □

As an immediate consequence we obtain

Corollary 2. *Let Σ be a regular surface. Under the assumptions of Theorem 8 we have*

$$\lim_{J\to\infty} \sup_{x\in\overline{\Sigma_{\mathrm{ext}}}} |U(x) - U_J(x)| = 0 \ .$$

8.1 Multiresolution

Next we come to the concept of multiresolution by means of scale discrete harmonic wavelets. For $U \in \mathcal{H}_0$ denote by R_j^D (*band-pass filters*), P_j^D (*low-pass filters*), the convolution operators given by

$$R_j^D(U) = \Psi_j^D * \Psi_j^D * U, \qquad U \in \mathcal{H}_0$$
$$P_j^D(U) = \Phi_j^D * \Phi_j^D * U, \qquad U \in \mathcal{H}_0$$

respectively. The *scale spaces* $\mathcal{V}_j^D$ and the *detail spaces* $\mathcal{W}_j^D$ are defined by

$$\mathcal{V}_j^D = P_j^D(\mathcal{H}_0),$$
$$\mathcal{W}_j^D = R_j^D(\mathcal{H}_0),$$

respectively. The collection $\{\mathcal{V}_j^D\}$ of all spaces $\mathcal{V}_j^D$, $j \in \mathbb{Z}$ is called *multiresolution analysis* of $\mathcal{H}_0$.

Loosely spoken, $\mathcal{V}_j^D$ contains all j-scale smooth functions of $\mathcal{H}_0$. The notion "detail space" means that $\mathcal{W}_j^D$ contains the "detail" information needed to go from an approximation at resolution j to an approximation at resolution $j+1$. To be more concrete, $\mathcal{W}_j^D$ denotes the space complementary to $\mathcal{V}_j^D$ in

$$\mathcal{V}_{j+1}^D : \mathcal{V}_{j+1}^D = \mathcal{V}_j^D + \mathcal{W}_j^D \quad .$$

Note that

$$\mathcal{V}_0^D + \sum_{j=0}^{J} \mathcal{W}_j^D = \mathcal{V}_{J+1}^D \quad .$$

Any potential $U \in \mathcal{H}_0$ can be decomposed in the following way: Starting from $P_0^D(U)$ we have

$$P_{J+1}^D(U) = P_0^D + \sum_{j=0}^{J} R_j^D(U) \quad .$$

The partial reconstruction $R_{J+1}^D(U)$ is nothing else than the difference of two "smoothings" $P_{J+1}^D(U) - P_J^D(U)$ at consecutive scales

$$R_J^D(U) = P_{J+1}^D(U) - P_j^D(U) \quad .$$

Moreover, in spectral language we have

$$\begin{array}{l} (P_j^D(U), H_{n,l}(\alpha;\cdot))_{\mathcal{H}_0} = (U, H_{n,l}(\alpha;\cdot))_{\mathcal{H}_0}((\Phi_j^D)^\wedge(n))^2, \\ (R_j^D(U), H_{n,l}(\alpha;\cdot))_{\mathcal{H}_0} = (U, H_{n,l}(\alpha;\cdot))_{\mathcal{H}_0}((\Psi_j^D)^\wedge(n))^2. \end{array} \tag{40}$$

The formulas (39) give (scale discrete) wavelet decomposition an interpretation in terms of Fourier analysis by means of outer harmonics by explaining how the frequency spectrum of a potential $U \in \mathcal{V}_j^D$ is divided up between the space $\mathcal{V}_{j-1}^D$ and $\mathcal{W}_{j-1}^D$.

The multiresolution can be illustrated by the following scheme:

$$\begin{array}{ccccccccccccc}
P_0^D(U) & & & P_1^D(U)\ldots & & & P_j^D(U) & & P_{j+1}^D(U)\ldots \to U & & \\
\rotatebox{90}{$\in$} & & & \rotatebox{90}{$\in$} & & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
\mathcal{V}_0^D & \subset & & \mathcal{V}_1^D & \ldots & \subset & \mathcal{V}_j^D & \subset & \mathcal{V}_{j+1}^D & \ldots = \mathcal{H}_0 & \\
\mathcal{V}_0^D & + & \mathcal{W}_0^D & + & \ldots + \mathcal{W}_{j-1}^D & + & \mathcal{W}_j^D & + & \ldots = \mathcal{H}_0 & & \\
\rotatebox{90}{$\ni$} & & \rotatebox{90}{$\ni$} & & & & \rotatebox{90}{$\ni$} & & \rotatebox{90}{$\ni$} & & \rotatebox{90}{$\ni$} \\
P_0^D(U) & + & R_0^D(U) & + & \ldots + R_{j-1}^D(U) & + & R_j^D(U) & + & \ldots = U & &
\end{array}$$

8.2 Least–squares Approximation

For notational convenience, we set

$$N_{-1} = \mathbb{N}_0 \cup \{-1\} \ .$$

Moreover, we let

$$\varphi_o(n) = \psi_{-1}(n),\ n \in \mathbb{N}_0,\ \Phi_0^D = \Psi_{-1},\ \mathcal{V}_0^D = \mathcal{W}_{-1}^D,$$

etc. Accordingly we have

$$R_{-1}^D = U * \Phi_0^D * \Phi_0^D = U * \Psi_{-1}^D * \Psi_{-1}^D \ .$$

Then it follows that

$$\sum_{n=-1}^{\infty} \left(\left(\Psi_j^D\right)^\wedge (n) \right)^2 = 1 \ .$$

The *reconstruction formula* (Theorem 3) may be rewritten as follows:

$$\lim_{J\to\infty} \|U - U_J\|_{\mathcal{H}_0} = 0, \quad U \in \mathcal{H}_0,$$

where the J–level scale discrete wavelet approximation now reads in shorthand notation as follows:

$$U_J = \sum_{j=-1}^{J} \int_A (WT)^D(U)(j;y)\Psi_{j;y}^D(\cdot)\ d\omega(y) \ .$$

As in the continuous case we can make use of the projection property in the scale discrete case. We know already that $(WT)^D : \mathcal{H}_0 \to \mathcal{H}_0(\mathbb{N}_{-1} \times \overline{A_{\mathrm{ext}}})$, where $\mathcal{H}_0(\mathbb{N}_{-1} \times \overline{A_{\mathrm{ext}}})$ is the space of all functions $U : (j;x) \mapsto U(j;x)$ with $U(j;\cdot) \in \mathcal{H}_0$ for every $j \in \mathbb{N}_{-1}$ and

$$\sum_{j=-1}^{\infty} \|U(j;\cdot)\|_{\mathcal{H}_0}^2 = \sum_{j=-1}^{\infty} \int_A |U(j;x)|^2\ d\omega(x) < \infty \ .$$

It is not hard to see that

$$\mathcal{W}^D = (WT)^D(\mathcal{H}_0) \subsetneqq \mathcal{H}_0(\mathbb{N}_{-1} \times \overline{A_{\mathrm{ext}}}).$$

Hence, we are able to define the projection operator $P^D : \mathcal{H}_0(\mathbb{N}_{-1} \times \overline{A_{\mathrm{ext}}}) \to \mathcal{W}^D$ by

$$P^D(U)(j;y) = \sum_{j=-1}^{\infty} \int_A K^D(j;y \mid j;y)U(j;y)\ d\omega(y),$$

where

$$K^D(j,y \mid j;y) = \int_A \Psi_{j;y}^D(x)\Psi_{j;y}^D(x)\ d\omega(x) = \left(\Psi_{j;y}^D(\cdot), \Psi_{j;y}^D(\cdot)\right)_{\mathcal{H}_0} \ . \tag{41}$$

In similarity to results of the scale continuous case it can be deduced that P^D is a projection operator. Therefore we have the following characterization of $\mathcal{W}^D$:

Lemma 12. *$U(\cdot,\cdot) \in \mathcal{W}^D$ if and only if the consistency condition*

$$U(j;y) = \sum_{j=-1}^{\infty} \int_A K^D(j;y \mid j;y) U(j;y)\ d\omega(y)$$
$$= \sum_{j=-1}^{\infty} \left(K^D(j;y \mid j;\cdot) U(j;\cdot)\right)_{\mathcal{H}_0} \ .$$

is satisfied.

Summarizing our results we obtain the following theorem.

Theorem 9. *Let H be an arbitrary element of $\mathcal{H}_0(\mathbb{N}_{-1} \times \overline{A_{\text{ext}}})$. Then the unique function $F_H^D \in \mathcal{H}_0$ satifying the property*

$$\|H(\cdot,\cdot) - \tilde{U}_H^D(\cdot,\cdot)\|_{\mathcal{H}_0(\mathbb{N}_{-1}\times\overline{A_{\text{ext}}})} = \inf_{U\in\mathcal{H}_0} \|H(\cdot,\cdot) - \tilde{U}^D(\cdot,\cdot)\|_{\mathcal{H}_0(\mathbb{N}_{-1}\times\overline{A_{\text{ext}}})}$$

is given by

$$U_H(x) = \sum_{j=-1}^{\infty} \int_A U(j;y) \Psi_{j;y}^D(x)\ d\omega(y).$$

Moreover, we have Theorem ??. For arbitrary $U \in \mathcal{H}_0$ the function $\tilde{U} \in (WT)^D(U) \in \mathcal{W}^D$ is the unqiue element in $\mathcal{H}_0(\mathbb{N}_{-1} \times \overline{A_{\text{ext}}})$ satisfying

$$\left\|\tilde{U}^D\right\|_{\mathcal{H}(\mathbb{N}_{-1}\times\overline{A_{\text{ext}}})} = \inf_{\substack{H\in\mathcal{H}_0(\mathbb{N}_{-1}\times\overline{A_{\text{ext}}}) \\ ((WT)^D)^{-1}(H)=U}} \|H|_{\mathcal{H}(\mathbb{N}_{-1}x\times\overline{A_{\text{ext}}})} \ .$$

8.3 Fully Scale Discrete Wavelets

For $j = 0, 1, \ldots$ let b_i^{N-j}, $i = 1, \ldots, N_j$ be the generating coefficients of the approximate integration rules

$$\int_A F(y)\ d\omega(y) = \sum_{i=1}^{N_j} b_i^{N_j} F(y_i^{N_j}) + \varepsilon_j(F), \quad F \in \mathcal{V}_j,$$

corresponding to (prescribed) nodal systems $\{y_1^{N_j}, \ldots, y_{N_j}^{N_j}\} \subset \overline{A_{\text{ext}}}$ such that

$$\lim_{j\to x} |\varepsilon_j(F)| = 0, \quad F \in \mathcal{V}_j$$

(i.e. the coefficients $b_i^{N_j}$, $j = 1, \ldots, N_j$, are supposed to be determined by an a priori calculation using approximate integration, interpolation, etc.

Assume that U is a potential of class $\text{Pot}^{(0)}(\overline{A_{\text{ext}}}) \subset \mathcal{H}_0$. Then the J–level scale discrete wavelet approximation can be represented in fully discrete form as follows:

$$
\begin{aligned}
U_J &= \sum_{i=1}^{N_J} b_i^{N_J} U\left(y_i^{N_J}\right) \left(\Phi^{(2)}\right)^D_{J;y_i^{N_j}}(\cdot) + \varepsilon_j\left(\left(\Phi^{(2)}\right)^D_{J;\cdot} * U\right) \\
&= \sum_{j=-1}^{J} \sum_{i=1}^{N_J} b_i^{N_j} U\left(y_i^{N_j}\right) \left(\Psi^{(2)}\right)^D_{j;y_i^{N_j}}(\cdot) + \sum_{j=-1}^{J} \varepsilon_j\left(\left(\Psi^{(2)}\right)^D_{j;\cdot} * U\right) \ .
\end{aligned}
$$

This leads us to the following result.

Theorem 10. *Any potential* $U \in \mathrm{Pot}(\overline{A_{\mathrm{ext}}})$ *can be approximated in the form*

$$
\lim_{J\to\infty} \left(\int_A \left| U(y) - \sum_{j=-1}^{J} \sum_{i=1}^{N_j} b_i^{N_j} U\left(y_i^{N_j}\right) \left(\Psi^{(2)}\right)^D_{j;y_i^{N_j}}(y) \right|^2 d\omega(y) \right)^{1/2} = 0 \ .
$$

Moreover,

$$
\lim_{J\to\infty} \sup_{x\in\overline{K}} \left| U(x) - \sum_{j=-1}^{J} \sum_{i=1}^{N_j} b_i^{N_j} U(y_i^{N_j}) \left(\Psi^{(2)}\right)^D_{j;y_i^{N_j}}(x) \right| = 0
$$

for all subsets K *of* A_{ext} *with* $\mathrm{dist}(\overline{K}, A) > 0$.

Fast evaluation methods (tree algorithms and pyramid schemata) has been presented in Freeden (1999).

9 Examples

Now we are prepared to introduce some important examples of scaling functions and corresponding wavelets. We distinguish two types of wavelets, namely non-band-limited and band-limited wavelets. Since there are only a few conditions for a function to be a generator of a scaling function, a large number of wavelet examples may be listed. For the sake of brevity, however, we have to concentrate on a few examples.

9.1 Non-band-limited Wavelets

All wavelets discussed in this subsection share the fact that their generators have a global support.

Rational Wavelets: Rational wavelets are realized by the function $\varphi_1 : [0,\infty) \to \mathbb{R}$ given by

$$
\varphi_1(t) = (1+t)^{-s},\ t \in [0,\infty).
$$

Indeed, $\varphi_1(0) = 1$, φ_1 is monotonously decreasing, φ_1 is continuously differentiable on the interval $[0,\infty)$, and we have $\varphi_1(t) = O(|t|^{-1-\varepsilon})$, $t \to \infty$ for

$s = 1 + \varepsilon$, $\varepsilon > 0$. The (scale continuous) scaling function $\{\Phi_\rho\}, \rho \in (0, \infty)$ is given by

$$\Phi_\rho(x, y) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi\alpha^2} \frac{1}{(1+\rho n)^s} \left(\frac{\alpha^2}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right), \; (x, y) \in \overline{A_{\mathrm{ext}}} \times \overline{A_{\mathrm{ext}}} \; .$$

It is easy to see that $\psi_1(t) = \sqrt{2st}(1+t)^{-s-1/2}$ so that the scale continuous harmonic wavelets $\{\Psi_\rho\}, \rho \in (0, \infty)$, are obtained from $\psi_\rho(n) = \sqrt{2s\rho n}(1 + \rho n)^{-s-1/2}$, $s > 1$, $n \in \mathbb{N}_0$, whereas the scale discrete wavelets $\{\Psi_j^D\}$, $j \in \mathbb{Z}$, are generated by

$$\psi_j^D(n) = \left((1 + 2^{-j-1}n)^{-2s} - (1 + 2^{-j}n)^{-2s}\right)^{1/2}, \; j \in \mathbb{Z}, n \in \mathbb{N}_0 \; .$$

Exponential Wavelets. We choose $\varphi_1(t) = e^{-Rt}, R > 0, \; t \in [0, \infty)$. Then it follows that $\varphi_\rho(t) = e^{-R\rho t}, \rho \in (0, \infty)$ and $\psi_\rho(t) = \sqrt{2R\rho}e^{-Rt}$, $\rho \in (0, \infty)$. Moreover,

$$\psi_j^D = \left((e^{-2^{-j-1}Rn})^2 - (e^{-2^{-j}Rn})^2\right)^{1/2} \; , \; j \in \mathbb{Z}, n \in \mathbb{N}_0 \; .$$

9.2 Band-limited Wavelets

All wavelets discussed in this subsection are chosen in such a way that the support of their generators is compact. As a consequence the resulting wavelets are band-limited. A particular result is that the Shannon wavelets provides us with an orthogonal multiresolution.

Shannon Wavelet. The generator of the Shannon scaling function is defined by

$$\varphi_1(t) = \begin{cases} 1 \text{ for } t \in [0, 1) \\ 0 \text{ for } t \in [1, \infty) \end{cases} .$$

The scale continuous harmonic scaling function $\{\Phi_\rho\}, \rho \in (0, \infty)$ is given via

$$\varphi_\rho(n) = \begin{cases} 1 \text{ for } n \in [0, \rho^{-1}) \\ 0 \text{ for } n \in [\rho^{-1}, \infty) \end{cases} .$$

A scale continuous wavelet does not make sense. However, a scale discrete wavelet $\{\Psi_j^D\}$, $j \in \mathbb{Z}$, is available. More precisely,

$$\psi_j^D(n) = \begin{cases} 1 \; n \in [2^j, 2^{j+1}) \\ 0 \text{ elsewhere.} \end{cases}$$

But this means that the scale discrete multiresolution is orthogonal (i.e., $V_{j+1} = V_j \oplus W_j$ is an orthogonal sum for all j).

Modified Shannon Wavelet. The generator of the modified Shannon scaling function reads as follows

$$\varphi_1(t) = \begin{cases} 1 & \text{for } t \in [0, \frac{1}{e}) \\ \sqrt{-ln x} & \text{for } t \in [\frac{1}{e}, 1) \\ 0 & \text{for } t \in [1, \infty) \end{cases}$$

The scale continuous harmonic wavelets $\{\Psi_\rho\}, \rho \in (0,\infty)$, are given by

$$\psi_\rho(n) = \begin{cases} 0 \text{ for } n \in [0, \frac{1}{e}\rho^{-1}) \\ 1 \text{ for } n \in [\frac{1}{e}\rho^{-1}, \rho^{-1}) \\ 0 \text{ for } n \in [\rho^{-1}, \infty). \end{cases}$$

The scale discrete harmonic wavelets $\{\Psi_j^D\}, j \in \mathbb{Z}$ are obtainable via

$$\Psi_j^D(n) = \begin{cases} 0 & \text{for } n \in [0, 2^j \frac{1}{e}) \\ \left(1 - (ln(2^{-j}n))^2\right)^{1/2} & \text{for } n \in [2^j \frac{1}{e}, 2^{j+1}\frac{1}{e}) \\ \left((ln(2^{-j-1}n))^2 - (ln(2^{-j}n))^2\right)^{1/2} & \text{for } n \in [2^{j+1}\frac{1}{e}, 2^j) \\ -ln(2^{-j-1}n) & \text{for } n \in [2^j, 2^{j+1}) \\ 0 & \text{for } n \in [2^{j+1}, \infty). \end{cases}$$

C(ubic) P(olynomial) Wavelet (*CP*-Wavelet). In order to have a higher intensity of the smoothing effect than in the case of modified Shannon wavelets we introduce a function $\varphi_1 : [0,\infty) \to \mathbb{R}$ in such a way that $\varphi_1|_{[0,1)}$ coincides with the uniquely determined cubic polynomial $p = [0,1] \to [0,1]$ with the properties:

$$p(0) = 1 \ , \ p(1) = 0$$
$$p`(0) = 0 \ , \ p`(1) = 0 \ .$$

It is easy to see that these properties are fulfilled by

$$p(t) = (1-t)^2(1+2t), \quad t \in [0,1] \ .$$

This leads us to a function $\varphi_1 : [0,\infty] \to \mathbb{R}$ given by

$$\varphi_1(t) = \begin{cases} (1-t)^2(1+2t) & \text{for } t \in [0,1 \\ 0 & \text{for } t \in [1,\infty). \end{cases}$$

It is clear that φ_1 is a monotonically decreasing function. The (scale continuous) scaling function $\{\Phi_\rho\}$, $\rho \in (0,\infty)$, is given by

$$\varphi_\rho(n) = \phi_1(\rho n) = \begin{cases} (1-\rho n)^2(1+2\rho n) & \text{for } n \in [0,\rho^{-1}) \\ 0 & \text{for } n \in [\rho^{-1},\infty) \ . \end{cases}$$

Scaling continuous and discrete wavelets are obtainable by obvious manipulations.

10 Band–limited Runge–Walsh Wavelet Transform

Our final interest (see Theorem 5.5) is fully discrete wavelet approximation outside the real earths surface Σ (cf. 5) by use of band-limited wavelets. The keystone is that when using band-limited wavelets, we do not need the wavelet transform at all positions. It suffices to know the wavelet transform on a finite set of linear functionals (i.e., function values or normal derivatives) for each scale j. In other words, each J-level wavelet approximation (cf. Theorem 8) can be expressed exactly as a finite sum. The reason are approximate integration formulas which will be discussed under the following (non-restrictive) assumptions:

(A1) The generator $\varphi_0^D : [0,\infty) \to \mathbb{R}$ of a scale discrete scaling function satisfies $supp\ \varphi_0^D = [0,1]$ and $\varphi_0^D(1) = 0$.
(A2) The generator $\psi_0^D : [0,\infty) \to \mathbb{R}$ of the mother wavelet satisfies $supp\ \psi_0^D = [0,2]$ and $\psi_0^D(2) = 0$.

Then it follows that

$$\begin{aligned} supp\ \varphi_j^D &\subset [0, 2^j], \\ supp\ \psi_j^D &\subset [0, 2^{j+1}]. \end{aligned}$$

Hence, we have $\varphi_j^D(2^j) = \psi_j^D(2^{j+1}) = 0$, $j \in \mathbb{N}_0$. It follows that

$$\begin{aligned} \Phi_{j;y}^D &\in \mathrm{Harm}_{0,\ldots,2^j-1}(\overline{\Sigma_{\mathrm{ext}}}),\ y \in \overline{\Sigma_{\mathrm{ext}}},\ j = 0,1,\ldots, \\ \Psi_{j;y}^D &\in \mathrm{Harm}_{0,\ldots,2^{j+1}-1}(\overline{\Sigma_{\mathrm{ext}}}),\ y \in \overline{\Sigma_{\mathrm{ext}}},\ j = 0,1,\ldots. \end{aligned}$$

10.1 Runge-Walsh Approximation

In what follows we want to show that a band-limited potential can be exactly determined by band-limited harmonic wavelets. In conclusion, a constructive Runge-Walsh approximation of any (non-band-limited) potential U can be established by using exact wavelet approximation by means of band-limited wavelets. Since the set of all finite linear combinations of harmonics when restricted to $\overline{\Sigma_{\mathrm{ext}}}$ is uniformly dense in the space $Pot^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$, the space $\mathcal{H}_0|_{\overline{\Sigma_{\mathrm{ext}}}}$ is a uniformly dense subset of $Pot^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$, too. Written out in formulas this means that

$$Pot^{(0)}(\overline{\Sigma_{\mathrm{ext}}}) = \overline{\mathcal{H}_0|_{\overline{\Sigma_{\mathrm{ext}}}}}^{\|\cdot\|_{C(\overline{\Sigma_{\mathrm{ext}}})}} = \overline{span_{\substack{n=0,1,\ldots \\ j=1,\ldots,2n+1}} (H_{n,j}(\alpha;\cdot)|_{\overline{\Sigma_{\mathrm{ext}}}})}^{\|\cdot\|_{C(\overline{\Sigma_{\mathrm{ext}}})}}.$$

Suppose now that there is known from a potential V of class $Pot^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$ a set $\{v_1,\ldots,v_M\}$ of M values v_i, $i = 1,\ldots,M$ corresponding to M points $x_1,\ldots,x_M$ in $\overline{\Sigma_{\mathrm{ext}}}$. Then an extended version of Helly's theorem (cf. [25]) tells us that, corresponding to the potential $V \in Pot^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$, there exists a member U (i.e., a Runge-Walsh approximation) of class $\mathcal{H}_0$ such that $U|_{\overline{\Sigma_{\mathrm{ext}}}}$ is in an $(\varepsilon/2)$- neigbourhood to V (understood in $C(\overline{\Sigma_{\mathrm{ext}}})$-topology) and $U(x_i) = v_i$, $i = 1,\ldots,M$ (note that we occasionally write $U_{0,\ldots,\infty}$ instead of U to indicate that all Harm-spaces generally contribute to U). Moreover, there exists an element $U_{0,\ldots,m}$ (i.e., a band-limited approximation to the Runge-Walsh approximation) of class $\mathrm{Harm}_{0,\ldots,m}(\overline{\Sigma_{\mathrm{ext}}})$ such that $U_{0,\ldots,m}|_{\overline{\Sigma_{\mathrm{ext}}}}$ may be considered to be in $(\varepsilon/2)$-accuracy to $U|_{\overline{\Sigma_{\mathrm{ext}}}}$ uniformly on $\overline{\Sigma_{\mathrm{ext}}}$ and, in addition, $U_{0,\ldots,m}(x_i) = U(x_i) = v_i$, $i = 1,\ldots,M$. In other words, corresponding to the potential $V \in Pot^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$ there exists in ε-accuracy on $\overline{\Sigma_{\mathrm{ext}}}$ a band-limited potential in $\mathcal{H}_0$, (namely $U_{0,\ldots,m} \in \mathrm{Harm}_{0,\ldots,m}(\overline{\Sigma_{\mathrm{ext}}})$) consistent with the original data (i.e., $v_i = U(x_i) = U_{0,\ldots,m}(x_i), i = 1,\ldots,M$). This is the reason why we are interested below in wavelet approximations of potentials $U_{0,\ldots,m}$ of class $\mathrm{Harm}_{0,\ldots,m}(\overline{\Sigma_{\mathrm{ext}}})$ uniformly on $\overline{\Sigma_{\mathrm{ext}}}$ from a finite set of function values.

To be more specific, our strategy is to represent $U_{0,\ldots,m} \in \mathrm{Harm}_{0,\ldots,m}(\overline{\Sigma_{\mathrm{ext}}})$ by a J-level approximation $(U_{0,\ldots,m})_J$ (cf. (38)) with J chosen in such a way that $m_J = 2^{J+1} - 1 \geq m$ (note that $U_{0,\ldots,m}$ coincides with $(U_{0,\ldots,m})_J$ uniformly on $\overline{\Sigma_{\mathrm{ext}}}$ in the case of orthogonal Shannon wavelets).

10.2 Integration Methods

We want to express the J-level wavelet approximation $(U_{0,\ldots,m})_J$ of $U_{0,\ldots,m}(m_J \geq m)$ *exactly* only by use of M values $v_1, \ldots, v_M$ corresponding to the points $x_1, \ldots, x_M$. To this end we observe that

$$\int_A U_{0,\ldots,m}(y)\Phi^D_{0;y}(\cdot)d\omega(y)|_A \in \mathrm{Harm}_{0,1}(A),$$

$$(WT)^D(U_{0,\ldots,m})(j;\cdot)|_A \in \mathrm{Harm}_{0,\ldots,m_j}(A),\ j = 0, \ldots, J.$$

Moreover, it is known that whenever $F, G \in \mathrm{Harm}_{0,\ldots,m}(A)$, the product FG is of class $\mathrm{Harm}_{0,\ldots,2m}(A)$.

Starting point for our intensions of exact integration in Harm-spaces are so-called fundamental systems.

Definition 12. *A set* $\{x_1^M, \ldots, x_M^M\}$, $M = (a+1)^2$ *of* M *points in* $\overline{A_{\mathrm{ext}}}$ *is called* $\mathrm{Harm}_{0,\ldots,a}$- *fundamental system in* $\overline{A_{\mathrm{ext}}}$ *if the matrix*

$$\begin{pmatrix} H_{0,1}(\alpha; x_1^M) & \ldots & H_{0,1}(\alpha; x_1^M) \\ \vdots & & \vdots \\ H_{a,2a+1}(\alpha; x_1^M) & \ldots & L_M^M H_{a,2a+1}(\alpha; x_l^M) \end{pmatrix} \tag{42}$$

is regular.

Obviously,

$$\begin{pmatrix} H_{0,1}(\alpha; x_1^M) \ldots L_1^M H_{a,2a+1}(\alpha; x_1^M) \\ \vdots \qquad\qquad \vdots \\ H_{0,1}(\alpha; x_1^M) \ldots L_M^M H_{a,2a+1}(\alpha; x_1^M) \end{pmatrix} \begin{pmatrix} H_{0,1}(\alpha; x_1^M) & \cdots & H_{0,1}(\alpha; x_1^M) \\ \vdots & & \vdots \\ H_{a,2a+1}(\alpha; x_1^M) & \cdots & H_{a,2a+1}(\alpha; x_M^M) \end{pmatrix}$$

$$= \begin{pmatrix} K_{\mathrm{Harm}_{0,\ldots,a}(\overline{\Sigma_{\mathrm{ext}}})}(x_1^M, x_1^M) & \ldots & K_{\mathrm{Harm}_{0,\ldots,a}(\overline{\Sigma_{\mathrm{ext}}})}(x_1^M, x_M^M) \\ \vdots & & \vdots \\ K_{\mathrm{Harm}_{0,\ldots,a}(\overline{\Sigma_{\mathrm{ext}}})}(x_M^M, x_1^M) & \ldots & K_{\mathrm{Harm}_{0,\ldots,a}(\overline{\Sigma_{\mathrm{ext}}})}(x_M^M, x_M^M) \end{pmatrix}. \tag{43}$$

Thus, it is clear that the regularity of the Gram matrix (42) is equivalent to the regularity of (41), and the property of $\{x_1^M, \ldots, x_M^M\}$ being a $\mathrm{Harm}_{0,\ldots,a}$-fundamental system is independent of the special choice of the basis of $\mathrm{Harm}_{0,\ldots a}$–harmonics.

The existence of fundamental systems as introduced above is guaranteed by a well-known induction procedure described in [20]. The definition of fundamental systems immediately leads us to the following integration rules on $\mathrm{Harm}_{0,\ldots,2a}$-spaces.

Lemma 13. *Let* $\{y_1^M, \ldots, y_M^M\} \subset A$, $M = (2a+1)^2$ *define a* $\mathrm{Harm}_{0,\ldots,2a}$*-fundamental system on* A*. Furthermore, suppose that* $P_{0,\ldots,a}, Q_{0,\ldots,a} \in \mathrm{Harm}_{0,\ldots,a}(A)$*. Then*

$$(P_{0,\ldots,a}, Q_{0,\ldots,a})_{\mathcal{H}_0} = \sum_{n=1}^{M} a_n P_{0,\ldots,a}(y_n^M) Q_{0,\ldots,a}(y_n^M) \tag{44}$$

holds for all weights $a_1, \ldots, a_M$ *satisfying*

$$\sum_{r=1}^{M} a_r K_{\mathrm{Harm}_{0,\ldots,2a}}(y_i^M, y_r^M) = \int_A K_{\mathrm{Harm}_{0,\ldots,2a}}(y_i^M, x)\, d\omega(x), \;\; i = 1, \ldots, M$$

Proof. The product $P_{0,\ldots,a}Q_{0,\ldots,a}$ is a member of class $\mathrm{Harm}_{0,\ldots,2a}(A)$. The formula (43), in fact, is constructed in such a way that it is exact for all members of $\mathrm{Harm}_{0,\ldots,2a}(A)$. □

Lemma 14. *Let* $\{x_1^M, \ldots, x_M^M\}$, $M = (a+1)^2$ *be a* $\mathrm{Harm}_{0,\ldots,a}$*-fundamental system in the sense of Definition 12. Furthermore, suppose that* $P_{0,\ldots,a}$ *and* $Q_{0,\ldots,a}$ *are members of* $\mathrm{Harm}_{0,\ldots,a}(\overline{\Sigma_{\mathrm{ext}}})$*. Then*

$$(P_{0,\ldots,a}, Q_{0,\ldots,a})_{\mathcal{H}_0} = \sum_{n=0}^{a} \sum_{j=1}^{2n+1} \sum_{r=1}^{M} a_r^{n,j} (Q_{0,\ldots,a}, H_{n,j}(\alpha;\cdot))_{\mathcal{H}_0}\; P_{0,\ldots,a}(x_r^M)$$

holds for all weights $a_1^{n,j}, \ldots, a_M^{n,j}$, $n = 0, \ldots, a$, $j = 1, \ldots, 2n+1$ *satisfying the linear equations*

$$\sum_{r=1}^{M} a_r^{n,j} H_{k,i}(\alpha; x_r^M) = \delta_{n,k}\delta_{j,i}, \;\; k = 0, \ldots, a, i = 1, \ldots, 2k+1.$$

Proof. Applying the Parseval identity we obtain

$$\begin{aligned}
(P_{0,\ldots,a}, Q_{0,\ldots,a})_{\mathcal{H}_0} &= \sum_{k=0}^{a} \sum_{i=1}^{2n+1} (P_{0,\ldots,a}, H_{k,i}(\alpha;\cdot))_{\mathcal{H}_0}\; (Q_{0,\ldots,a}, H_{k,i}(\alpha;\cdot))_{\mathcal{H}_0} \\
&= \sum_{k=0}^{a} \sum_{i=1}^{2k+1} (P_{0,\ldots,a}, H_{k,i}(\alpha;\cdot))_{\mathcal{H}_0} \sum_{n=0}^{a} \sum_{j=1}^{2n+1} (Q_{0,\ldots,a}, H_{n,j}(\alpha;\cdot))_{\mathcal{H}_0} \delta_{n,k}\delta_{j,i} \\
&= \sum_{k=0}^{a} \sum_{i=1}^{2k+1} (P_{0,\ldots,a}, H_{k,i}(\alpha;\cdot))_{\mathcal{H}_0} \sum_{n=0}^{a} \sum_{j=1}^{2n+1} \sum_{r=1}^{M} a_r^{n,j} (Q_{0,\ldots,a}, H_{n,j}(\alpha;\cdot))_{\mathcal{H}_0} H_{k,i}(\alpha; x_r^M) \\
&= \sum_{n=0}^{a} \sum_{j=1}^{2n+1} \sum_{r=1}^{M} a_r^{n,j} (Q_{0,\ldots,a}, H_{n,j}(\alpha;\cdot))_{\mathcal{H}_0} P_{0,\ldots,a}(x_r^M).
\end{aligned}$$

□

In order to reduce the number of integration weights in our integration rules we formulate

Theorem 11. *Under the assumptions of Lemma 14, the integration formula*

$$(P_{0,\ldots,a}, Q_{0,\ldots,a})_{\mathcal{H}_0} = \sum_{r=1}^{M} a_r L_r^M P_{0,\ldots,a} \tag{45}$$

holds for all weights $a_1, \ldots, a_M$ *satisfying the linear equations*

$$\sum_{r=1}^{M} a_r K_{\mathrm{Harm}_{0,\ldots,a}}(x_i^M, x_r^M) = \sum_{n=0}^{a} \sum_{j=1}^{2n+1} (Q_{0,\ldots,a}, H_{n,j}(\alpha;\cdot))_{\mathcal{H}_0} H_{n,j}(\alpha; x_i^M),\ i = 1, \ldots, M.$$

Proof. We set

$$a_r = \sum_{n=0}^{a} \sum_{j=1}^{2n+1} a_r^{n,j} (Q_{0,\ldots,a}, H_{n,j}(\alpha;\cdot))_{\mathcal{H}_0}$$

for $r = 1, \ldots, M$. Thus, by applying Lemma 14 the integration rule (44) holds if $a_1, \ldots, a_M$ satisfy the linear equations (in matrix form)

$$\begin{pmatrix} H_{0,1}(\alpha; x_1^M) & \ldots & H_{0,1}(\alpha; x_M^M) \\ \vdots & & \vdots \\ H_{a,2a+1}(\alpha; x_1^M) & \ldots & H_{a,2a+1}(\alpha; x_M^M) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_M \end{pmatrix} = \begin{pmatrix} (Q_{0,\ldots,a}, H_{0,1}(\alpha;\cdot))_{\mathcal{H}_0} \\ \vdots \\ (Q_{0,\ldots,a}, H_{a,2a+1}(\alpha;\cdot))_{\mathcal{H}_0} \end{pmatrix}.$$

Multiplication with the adjoint matrix yields the desired result. □

It should be mentioned that on the one hand side the number of integration weights is reduced, but on the other hand side the integration weights now depend on $Q_{0,\ldots,a}$.

10.3 Fully Discrete Runge-Walsh-Wavelet Approximation

The results on exact integration in Harm-spaces developed in Section 5.2 now enable us to develop a constructive version of the Runge-Walsh theorem by means of a J-level wavelet approximation provided that the potential U we are looking for is assumed (as proposed in our introduction) to be a member of class $\mathcal{H}_0|_{\overline{\Sigma_{\mathrm{ext}}}}$ (note that $\mathcal{H}_0|_{\overline{\Sigma_{\mathrm{ext}}}}$is a uniformly dense subset of $Pot^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$). Let the generators φ_0^D, $\psi_0^D : [0, \infty) \to \mathbb{R}$ satisfy the assumptions $(A1)$, $(A2)$ of the beginning of the last section. Then we obtain from Theorem 3.12 in connection with Lemma 5.2 and Theorem 5.4.

Theorem 12. *Let* $\{x_1^M, \ldots, x_M^M\} \subset \overline{\Sigma_{\mathrm{ext}}}$, $M = (m+1)^2$ *be a* $\mathrm{Harm}_{0,\ldots,m}$*-fundamental system on* $\overline{\Sigma_{\mathrm{ext}}}$*. Furthermore, assume that there are known the data* $U_{0,\ldots,m}(x_i^M) = v_i$, $i = 1, \ldots, M$*. Then, under the assumption of band-limited wavelets, the fully discrete* J*-level wavelet approximation of* $U_{0,\ldots,m}$ *reads as follows:*
<u>*Variant a:*</u>

$$(U_{0,\ldots,m})_J(x) = \sum_{s=1}^{M} a_s^0 v_s + \sum_{j=0}^{J} \sum_{s=1}^{M} b_s^j v_s$$

holds for each $x \in \overline{\Sigma_{\rm ext}}$, *where the weights* $a_1^0, \ldots, a_M^0$ *satisfy the linear equations*

$$\sum_{s=1}^{M} a_s^0 K_{\mathrm{Harm}_{0,\ldots,m}(\overline{A_{\rm ext}})}(x_i^M, x_s^M) = (\Phi_0^D)^{(2)}(x, x_i^M),\ i = 1, \ldots, M,$$

and the weights $b_1^j, \ldots, b_M^j$, $j = 0, \ldots, J$ *satisfy*

$$\sum_{s=1}^{M} b_s^j K_{\mathrm{Harm}_{0,\ldots,m}(\overline{A_{\rm ext}})}(x_i^M, x_s^M) = (\Psi_j^D)^{(2)}(x, x_i^M),\ i = 1, \ldots, M.$$

<u>*Variant b:*</u>

$$(U_{0,\ldots,m})_J(x) = \sum_{n=1}^{M_0} b_n^0 \sum_{s=1}^{M} \tilde{a}_s^{0,n} v_s \Phi^D_{0;y_n^{M_0}}(x) + \sum_{j=0}^{J}\sum_{n=1}^{M_j} b_n^j \sum_{s=1}^{M} a_s^{j,n} v_s \Psi^D_{j;y_n^{M_j}}(x)$$

holds for each $x \in \overline{\Sigma_{\rm ext}}$, *where the weights* $\tilde{a}_1^{0,n}, \ldots, \tilde{a}_M^{0,n}$, $n = 1, \ldots, M_0$ *satisfy the linear equations*

$$\sum_{s=1}^{M} \tilde{a}_s^{0,n} K_{\mathrm{Harm}_{0,\ldots,m}(\overline{A_{\rm ext}})}(x_i^M, x_s^M) = \Phi_0^D(y_n^{M_0}, x_i^M),\ i = 1, \ldots, M,$$

the weights $a_1^{j,n}, \ldots, a_M^{j,n}$, $j = 0, \ldots, J$, $n = 1, \ldots, M_j$ *satisfy*

$$\sum_{s=1}^{M} a_s^{j,n} K_{\mathrm{Harm}_{0,\ldots,m}(\overline{A_{\rm ext}})}(x_i^M, x_s^M) = \Psi_j^D(y_n^{M_j}, x_i^M),\ i = 1, \ldots M$$

and $b_1^j, \ldots, b_{M_j}^j$, $j = 0, \ldots, J$ *satisfy equations*

$$\sum_{n=1}^{M_j} b_n^j K_{\mathrm{Harm}_{0,\ldots,2m_j}(\overline{A_{\rm ext}})}(y_i^{M_j}, y_n^{M_j}) = \int\limits_A K_{\mathrm{Harm}_{0,\ldots,2m_j}(\overline{A_{\rm ext}})}(y_i^{M_j}, x)\, d\omega(x),\ i = 1, \ldots, M_j \tag{46}$$

provided that $\{y_1^{M_j}, \ldots, y_{M_j}^{M_j}\} \subset A$, $M_j = (2m_j + 1)^2$, $m_j = 2^{j+1} - 1$, *define* $\mathrm{Harm}_{0,\ldots,2m_j}$*-fundamental systems on* A, $j = 0, \ldots, J$.

It should be pointed out that a great number of linear systems must be solved. But if we look carefully at the linear systems we realize that we are always confronted with the same coefficient matrix. Having inverted the matrix once, all weights for numerical integration can be obtained by a matrix-vector multiplication and stored elsewhere in an a priori step for computation. In addition, it should be mentioned that the solution of the linear systems determining the weights of the reconstruction step (45) can be avoided completely if we place the wavelet coefficients for each detail step $j = 0, \ldots, J$ on a special longitude-latitude grid on the sphere A. The corresponding set of integration weights for reconstruction purposes is explicitly available without solving any linear system (for more details the reader is referred to [1]).

10.4 Non-band-limited Approximation

The wavelet representations (Theorem 12) of a band-limited potential from a given finite set of linear functionals admits a variety of applications. The list includes the *wavelet approximation* $(U_{0,\ldots,m})_J$ of the solution of the Dirichlet problem)(EDP)

$$U_{0,\ldots,m}|_{\overline{\Sigma_{\text{ext}}}} \in \text{Harm}_{0,\ldots,m}\left(\overline{\Sigma_{\text{ext}}}\right), U_{0,\ldots,m}|_{\Sigma} = F$$

under the assumption that the M boundary data

$$v_i = U_{0,\ldots,m}(x_i^M) = F(x_i^M), i = 1, \ldots, M$$

are known. According to our construction $(U_{0,\ldots,m})_J$ is the J-level wavelet approximation of the band-limited potential $U_{0,\ldots,m}$ which itself may be understood, e.g., as the m-th truncated orthogonal expansion of the Runge-Walsh approximation $U \in \mathcal{H}$. Of course, the integration rules leading to the exact representation of $(U_{0,\ldots,m})_J$ are no longer exact when applied to U_J. But they may be regarded still as approximate rules. In this respect it is worth mentioning that the error between a potential $U \in \mathcal{H}_s(\overline{A_{\text{ext}}})$, $s > 1$, and its m-th truncated orthogonal expansion in terms of Harmaharmonics can be estimated by analogous arguments as presented in Section 2.4.

Theorem 13. *Let $U_{0,\ldots,m}$ be the m-th order truncation of a potential $U \in \mathcal{H}_s(\overline{\Sigma_{\text{ext}}})$, $s > 1$. Furthermore assume that $\{x_1^M, \ldots, x_M^M\} \subset \Sigma$, $M = (m+1)^2$ defines a* $\text{Harm}_{0,\ldots,m}$*–fundamental system on Σ. Then, for any $G \in \text{Harm}_{0,\ldots,m}(\overline{\Sigma_{\text{ext}}})$,*

$$|(U, G)_{\mathcal{H}_0} - \sum_{r=1}^{M} a_r U(x_r^M)| \leq \frac{C}{m^{s-1}} \left(\sum_{r=1}^{M} |a_r| \right) ||U||_{\mathcal{H}_s},$$

where C is a constant depending on the value s and $a_1, \ldots, a_M$ are the weights of the integration rule.

Proof. We use the triangle inequality and the fact that $(U - U_{0,\ldots,m}, G)_{\mathcal{H}_0} = 0$, (note, that $\{x_1^M, \ldots, x_M^M\}$ is a $\text{Harm}_{0,\ldots,m}$–fundamental system). □

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