Functional Analysis

Introduction to Spectral Theory in Hilbert Space - Script

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Introduction to Spectral Theory in Hilbert Space

The aim of this course is to give a very modest introduction to the extremely rich and well-developed theory of Hilbert spaces, an introduction that hopefully will provide the students with a knowledge of some of the fundamental results of the theory and will make them familiar with everything needed in order to understand, believe and apply the spectral theorem for selfadjoint operators in Hilbert space. This implies that the course will have to give answers to such questions as

- What is a Hilbert space?
- What is a bounded operator in Hilbert space?
- What is a selfadjoint operator in Hilbert space?
- What is the spectrum of such an operator?
- What is meant by a spectral decomposition of such an operator?

LITERATURE:

- *English:*
  - G. Helmberg: Introduction to Spectral Theory in Hilbert space
    (North-Holland Publishing Comp., Amsterdam-London)
  - R. Larsen: Functional Analysis, an introduction
    (Marcel Dekker Inc., New York)
  - M. Reed and B. Simon: Methods of Modern Mathematical Physics I: Functional Analysis
    (Academic Press, New York-London)
- *German:*
  - H. Heuser: Funktionalanalysis, Theorie und Anwendung
    (B.G. Teubner-Verlag, Stuttgart)

Chapter 1: Hilbert spaces

Finite dimensional linear spaces (=vector spaces) are usually studied in a course called Linear Algebra or Analytic Geometry, some geometric properties of these spaces may also have been studied, properties which follow from the notion of an angle being implicit in the definition of an inner product. We shall begin with some basic facts about Hilbert spaces including such results as the Cauchy-Schwarz inequality and the parallelogram and polarization identity

§1 Basic definitions and results

(1.1) **Definition:** A linear space E over $\mathbf{K} \in \{\mathbf{R},\mathbf{C}\}$ is called an **inner product space** (or a **pre-Hilbert space**) over **K** that if there is a mapping $(\ |\ ): E\times E \to \mathbf{K}$ that satisfies the following conditions:

(S1) $(x\,|\,x)\geq 0$ and $(x\,|\,x)=0$ if and only if $x=0$

(S2) $(x+y\,|\,z)=(x\,|\,z)+(y\,|\,z)$

(S3) $(\alpha x\,|\,y)=\alpha(x\,|\,y)$

(S4) $(x\,|\,y)=\overline{(y|x)}$

The mapping $(\ |\ ): E\times E \to \mathbf{K}$ is called an **inner product**.

(S1) – (S4) imply $(x\,|\,\alpha y)=\overline{(\alpha y|x)}=\alpha\overline{(y|x)}=\overline{\alpha}(x|y)$ and

$(x\,|\,y+z)=(x\,|\,y)+(x\,|\,z)$

Examples:

(1) $\mathbf{R^n}$ $x=(x_1, ..., x_n) \in \mathbf{R^n}$ $y=(y_1, ..., y_n) \in \mathbf{R^n}$

$$(x\,|\,y):=x_1y_1+x_2y_2+ ... +x_ny_n=\sum_{j=1}^{n} x_j y_j$$

(2) $\mathbf{C^n}$ $x=(x_1, ..., x_n) \in \mathbf{C^n}$ $y=(y_1, ..., y_n) \in \mathbf{C^n}$

$$(x\,|\,y):=\sum_{j=1}^{n} x_j \overline{y_j}$$

The Cauchy-Schwarz inequality will show that $(x\,|\,y) \in \mathbf{C}$

(3) $x, y \in l_2$ $x=(x_j)_j$, $y=(y_j)_j$ $x_j, y_j \in \mathbf{C}$

$$(x\,|\,y):=\sum_{j=1}^{\infty} x_j \overline{y_j}$$

$$x \in l_2 \Leftrightarrow \sum_{j=1}^{\infty} |x_j|^2 < \infty$$

(4) Let $C_0[a,b]$ denote the set of all continuous functions $f:[a,b]\to\mathbf{C}$

$$(f\,|\,g):=\int_a^b f(t)\overline{g(t)}\,dt$$

Here also the Cauchy-Schwarz inequality will make sure that $(f\,|\,g)$ *is a complex number. An inner product space* E *can be made into a normed linear space with the induced Norm* $\|x\|_2:=\sqrt{(x|x)}$ *for* $x \in E$. *In order to prove this, however, we need a fundamental inequality:*

(1.2) **Theorem:** (**Cauchy-Schwarz-inequality**):
Let $(E,(\ |\ ))$ be an inner product space over $\mathbf{K} \in \{\mathbf{R},\mathbf{C}\}$.
Then $|(x|y)|^2 \leq (x|x)\,(y|y)$ (or $|(x|y)| \leq \|x\|\,\|y\|$), all $x,y \in E$.
Moreover, given $x,y \in E$ $|(x|y)|^2 = (x|x)\,(y|y)$ if and only if x and y are linearly dependent.

Proof:

Case (1): if x=0, the inequality obviously is valid.
Case (2): for $x\neq 0$, $y \in E$, $\alpha \in \mathbf{C}$ we have:

$$0 \leq (y-\alpha x|y-\alpha x) = (y|y)-(\alpha x|y)-(y|\alpha x)+(\alpha x|\alpha x)$$

$$\Leftrightarrow 0 \leq (y|y)-(\alpha x|y)-\overline{\alpha}(y|x)+\alpha\overline{\alpha}(x|x)$$

Choose $\alpha := \frac{(y|x)}{(x|x)}$, then

$$0 \leq (y|y)-\frac{(y|x)}{(x|x)}(x|y)-\frac{(x|y)}{(x|x)}(y|x)+\frac{(x|y)(y|x)}{(x|x)(x|x)}(x|x)$$

$$= (y|y)-\frac{|(x|y)|^2}{(x|x)} \Leftrightarrow |(x|y)|^2 \leq (x|x)\,(y|y) \qquad \text{q.e.d.}$$

The inequality still remains valid if in the definition of an „inner product“ the condition „$(x|x)=0$ if and only if $x=0$“ is omitted

(1.3) **Corollary:** Let $(E,(\ |\ ))$ be an inner product space (over $\mathbf{K}$).
$\|x\| := \sqrt{(x|x)}$ for $x \in E$ is a norm on E.

Proof:

To show:
(N1) $\|x\|\geq 0$ and $\|x\|=0$ if and only if $x=0$
(N2) $\|\alpha x\| = |\alpha|\cdot\|x\|$ for $\alpha \in \mathbf{K}$
(N3) $\|x+y\| \leq \|x\|+\|y\|$

We only show (N3), (N1) and (N2) are easy to prove.

(N3)
$$\|x+y\|^2 = (x+y|x+y) = (x|x)+(y|x)+(x|y)+(y|y)$$
$$= \|x\|^2+2\cdot\mathrm{Re}(x|y)+\|y\|^2 \leq \|x\|^2+\|y\|^2+2\cdot|\mathrm{Re}(x|y)|$$
$$\leq \|x\|^2+2\cdot|(x|y)|+\|y\|^2 \leq \|x\|^2+2\cdot\|x\|\cdot\|y\|+\|y\|^2 = (\|x\|+\|y\|)^2 \qquad \text{q.e.d.}$$

A linear space E *over* $\mathbf{K} \in \{\mathbf{R};\mathbf{C}\}$ *is called a* <u>*normed linear space*</u> *of* $\mathbf{K}$ *if there is a mapping* $\|.\|: E\to\mathbf{R}$ *satisfying conditiions (N1) to (N3).* $\|.\|$ *is called a norm on* E. $(E,\|.\|)$ *is called a* <u>*Banachspace*</u> *if every Cauchy sequence in* E *converges to some element in* E. *i.e. for every sequence* $(x_n)_n \subseteq (E,\|.\|)$ *with* $\lim_{n,m\to\infty}\|x_n-x_m\|=0$ *there exists* $x \in E$ *with* $\lim_{n\to\infty}\|x_n-x\|=0$

(1.4) **Corollary:** Let $(E,(\ |\ ))$ be an inner product space, let $\|x\| = (x|x)^{½}$. Given $x,y \in E$ we have $\|x+y\|=\|x\|+\|y\|$ if and only if $y=0$ or $x=\lambda\cdot y$ for some $\lambda\geq 0$.

Proof:

If $\|x+y\|=\|x\|+\|y\|$ and $y\neq 0$ then ((1.3)) $\mathrm{Re}(x\,|\,y)=|(x\,|\,y)|=\|x\|\cdot\|y\| \Rightarrow (x\,|\,y)=\|x\|\cdot\|y\|$.
Theorem (1.2) $\Rightarrow$ $\alpha x+\beta y=0$ with $|\alpha|+|\beta|>0 \Rightarrow$ x and y are linearly dependent.
$y\neq 0$ implies $\alpha\neq 0$ and $x=-\dfrac{\beta}{\alpha}y$. From $|(x\,|\,y)| = (x\,|\,y) = -\dfrac{\beta}{\alpha}(y\,|\,y)$ we conclude $\lambda = -\dfrac{\beta}{\alpha} \geq 0$.

q.e.d.

$$\begin{array}{lll} (x_n)_n \subseteq E & \|x_n-x\|\to 0 & \\ (y_n)_n \subseteq E & \|y_n-y\|\to 0 & \Rightarrow \quad (x_n\,|\,y_n)\to (x\,|\,y) \end{array}$$

(1.5) **Corollary:** The inner product (|) of an inner product space is a **K**-valued continuous mapping on $E\times E$, where E is taken with the norm topology determined by the inner product.

Proof:

$|(x\,|\,y)\to (x_0\,|\,y_0)| \leq |(x\,|\,y)-(x_0\,|\,y)|+|(x\,|\,y_0)-(x_0\,|\,y_0)|$
$= |(x\,|\,y-y_0)|+|(x-x_0\,|\,y_0)| \leq \|x\|\cdot\|y-y_0\|+\|x-x_0\|\cdot\|y_0\|$
$= 2\|x-x_0\|\,\|y-y_0\|+\|x_0\|\,\|y-y_0\|+\|y_0\|\,\|x-x_0\|$ q.e.d.

(1.6) **Corollary:**
$\|x\|=\sup\limits_{\|y\|=1}|(x\,|\,y)|=\sup\limits_{\|y\|\leq 1}|(x\,|\,y)|$, if $x\in$ (E,(|)).

We now examine two fundamental identities in inner product spaces: the <u>parallelogram identity</u> and the <u>polarization identity</u>. We shall use the former identity to give a characterization of those normed linear spaces that are inner product spaces

(1.7) **Theorem:** (**parallelogram identity**):
Let (E,(|)) be an inner product space. Then $\|x+y\|^2+\|x-y\|^2 = 2\cdot\|x\|^2+2\cdot\|y\|^2$; $x,y\in E$.

Proof:

$$\begin{aligned}\|x+y\|^2+\|x-y\|^2 &= (x+y\,|\,x+y) + (x-y\,|\,x-y)\\ &= (x\,|\,x) + (x\,|\,y) + (y\,|\,x) + (y\,|\,y) + (x\,|\,x) - (x\,|\,y) - (y\,|\,x) + (y\,|\,y)\\ &= 2\cdot(x\,|\,x) + 2\cdot(y\,|\,y) = 2\cdot\|x\|^2 + 2\cdot\|y\|^2\end{aligned}$$ q.e.d.

{Geometrically the parallelogram identity says that the sum of the squares of the lengths of a parallelogram's diagonals equals the sum of the squares of the length of ots sides.
A similar direct computation also establishes the polarization identity which allows one to express the inner product in terms of the norm}

(1.8) **Theorem:** (**polarization identity**):
Let **(**E,(|)**)** be an inner product space over **K** ∈ {**R**,**C**}.

$$\text{Then } (x\,|\,y) = \begin{cases} \left\|\frac{x+y}{2}\right\|^2 - \left\|\frac{x-y}{2}\right\|^2 & \text{if } K = \mathbf{R} \\ \left\|\frac{x+y}{2}\right\|^2 - \left\|\frac{x-y}{2}\right\|^2 + i\cdot\left(\left\|\frac{x+i\cdot y}{2}\right\|^2 - \left\|\frac{x-i\cdot y}{2}\right\|^2\right) & \text{if } K = \mathbf{C} \end{cases}$$

Proof by simple computation

The next result characterizes those normed linear spaces whose norm is induced by an inner product.

(1.9) **Theorem:** If (E,‖.‖) is a normed linear space over **K** ∈ {**R**,**C**} such that $\|x+y\|^2 + \|x-y\|^2 = 2\cdot\|x\|^2 + 2\cdot\|y\|^2$ for every x,y ∈ E, then there exists an inner product (|) on E with $(x\,|\,x)^{1/2} = \|x\|$ for x ∈ E.

Proof: for **K**=**C**:

$$\text{Define } (x\,|\,y) := \left\|\frac{x+y}{2}\right\|^2 - \left\|\frac{x-y}{2}\right\|^2 + i\cdot\left(\left\|\frac{x+i\cdot y}{2}\right\|^2 - \left\|\frac{x-i\cdot y}{2}\right\|^2\right)$$

To prove that (x | y) is an inner product.

a) $$(x\,|\,x) := \|x\|^2 + i\cdot\left(\left\|\frac{x+i\cdot x}{2}\right\|^2 - \left\|\frac{x-i\cdot x}{2}\right\|^2\right) = \|x\|^2 + i\cdot\left(\left\|\frac{(1+i)\cdot x}{2}\right\|^2 - \left\|\frac{(1-i)\cdot x}{2}\right\|^2\right)$$

$$= \|x\|^2 + \frac{i}{4}\cdot\left(\|x\|^2\left(\underbrace{|1+i|^2 - |1-i|^2}_{=0}\right)\right) = \|x\|^2$$

b) $(x\,|\,y) = \overline{(y\,|\,x)}$ (easy to check!)

c) $(x+y\,|\,z) = (x\,|\,z) + (y\,|\,z)$

$$\mathrm{Re}\,(x\,|\,z) + \mathrm{Re}\,(y\,|\,z) = \left\|\frac{x+z}{2}\right\|^2 - \left\|\frac{x-z}{2}\right\|^2 + \left\|\frac{y+z}{2}\right\|^2 + \left\|\frac{y-z}{2}\right\|^2$$

$$= \frac{1}{4}\left(\|x+z\|^2 + \|y+z\|^2\right) - \left(\|x-z\|^2 + \|y-z\|^2\right)$$

$$\overset{\text{parallelogram}}{\underset{\text{identity}}{=}} \frac{1}{4}\left(\frac{1}{2}\|x+z+y+z\|^2 + \frac{1}{2}\|x-z-y-z\|^2\right) - \frac{1}{4}\left(\frac{1}{2}\|x-z+y-z\|^2 + \frac{1}{2}\|x+z-y+z\|^2\right)$$

$$= \frac{1}{2}\left(\left\|\frac{x+y}{2}+z\right\|^2 + \left\|\frac{x-y}{2}\right\|^2 - \left\|\frac{x+y}{2}-z\right\|^2 - \left\|\frac{x-y}{2}\right\|^2\right)$$

$$\Rightarrow \mathrm{Re}\,(x\,|\,z) + \mathrm{Re}\,(y\,|\,z) = \frac{1}{2}\mathrm{Re}\left(\frac{x+y}{2}\,\middle|\,z\right)$$

Put y=0: $\quad \mathrm{Re}\,(x\,|\,z) = \frac{1}{2}\mathrm{Re}\left(\frac{x}{2}\middle|\,z\right)$ for all $x \in E$.

Replace x by x+y:

$$\Rightarrow\ \mathrm{Re}\,(x+y|z) = \frac{1}{2}\mathrm{Re}\left(\frac{x+y}{2}\middle|z\right) = \mathrm{Re}\,(x|z) + \mathrm{Re}\,(y|z)$$

The same way: $\mathrm{Im}(x+y\,|\,z) = \mathrm{Im}(x\,|\,z) + \mathrm{Im}(y\,|\,z)$

$\Rightarrow (x+y\,|\,z) = (x\,|\,z) + (y\,|\,z)$

d) to prove $(\alpha\cdot x\,|\,y) = \alpha\cdot(x\,|\,y)$ for $a \in \mathbf{C}$:

$(2\cdot x\,|\,y) \underset{c)}{=} (x\,|\,y) + (x\,|\,y) = 2\cdot(x\,|\,y) \underset{induction}{\Rightarrow} (m\cdot x\,|\,y) = m\cdot (x\,|\,y);\ m \in \mathbf{N}$

$(-x\,|\,y) = \ ... = -\,(x\,|\,y)$ by using the definition

$\Rightarrow (m\cdot x\,|\,y) = m\cdot(x\,|\,y);\ m \in \mathbf{Z}$

$(x\,|\,y) = (\,n\cdot\frac{1}{n}\cdot x\,|\,y) = n\cdot(\frac{1}{n}\cdot x\,|\,y) \Rightarrow \frac{1}{n}(x\,|\,y) = (\frac{1}{n}\cdot x\,|\,y)$

$\Rightarrow (q\cdot x\,|\,y) = q\cdot(x\,|\,y);\ q \in \mathbf{Q}$

If $\alpha \in \mathbf{R}$, $q \in \mathbf{Q}$

$$|\,\mathrm{Re}\,(\alpha\cdot x\,|\,y) - \mathrm{Re}\,(q\cdot x\,|\,y)\,| \le \left|\left\|\frac{\alpha\cdot x+y}{2}\right\|^2 - \left\|\frac{\alpha\cdot x-y}{2}\right\|^2\right| + \left|\left\|\frac{q\cdot x+y}{2}\right\|^2 - \left\|\frac{q\cdot x-y}{2}\right\|^2\right|$$

$\mathrm{Re}\,(\alpha\cdot x\,|\,y) = \lim\limits_{q\to\alpha} \mathrm{Re}\,(q\cdot x\,|\,y) = \lim\limits_{q\to\alpha} q\cdot\mathrm{Re}\,(x\,|\,y) = \alpha\cdot\mathrm{Re}\,(x\,|\,y)$

Similarly $\mathrm{Im}\,(a\cdot x\,|\,y) = \alpha\cdot\mathrm{Im}\,(x\,|\,y)$

Finally: $(i\cdot x\,|\,y) = ... = i\cdot(x\,|\,y)$ q.e.d.

This theorem asserts that a normed linear space is an inner product space if and only if the norm satisfies the parallelogram identity. The next corollary is an immediate consequence of this fact

(1.10) **Corollary:** Let $(E,\|.\|)$ be a normed linear space over $\mathbf{K} \in \{\mathbf{R},\mathbf{C}\}$. If every two-dimensional subspace of E is an inner product space over **K**, then E is an inner product space over **K**.

If $(E,\|.\|)$ *is an inner product space the inner product induces a norm on* E. *We thus have the notions of convergence, completeness and density. In particular, we can always complete* E *to a normed linear space* $\tilde{E}$ *in which* E *is isometrically embedded as a dense subset. In fact* $\tilde{E}$ *is also an inner product space since the inner product can be extended from* E *to* $\tilde{E}$ *by continuity*

(1.11) **Definition:** Let **(**E,(|)**)** be an inner product space over $\mathbf{K} \in \{\mathbf{R},\mathbf{C}\}$. E is called a **Hilbert space**, if E is a complete normed linear space (= Banach space) with respect to $\|x\| := (x\,|\,x)^{1/2}$, $x \in E$.

§ 2 Orthogonality and orthonormal bases

In this section we study some geometric properties of an inner product space, properties which are connected with the notion of orthogonality

(1.12) **Definition:** Let $(E,(\ |\ ))$ be an inner product space, $x,y \in E$, let $M,N \subseteq E$ be subsets.
1) x and y are called **orthogonal** if and only if $(x\,|\,y)=0$ $(x \perp y)$
2) x and y are called **orthonormal** if and only if $\|x\|=\|y\|=1$ and $x \perp y$
3) M and N are called **orthogonal**, $M \perp N$, if $(x\,|\,y)=0$ for $x \in M$, $y \in N$
4) M is called an **orthonormal set** if $\|x\|=1$ for $x \in M$ and $(x\,|\,y)=0$ for $x \neq y$, $y \in M$
5) N is called an **orthogonal set** if $(x\,|\,y)=0$ for any $x,y \in N$, $x \neq y$

Facts: 1) $M \perp N \Rightarrow M \cap N \subseteq \{0\}$
2) $x=0$ is the only element orthogonal to every $y \in E$
3) $0 \notin M$ if M is an orthonormal set

A criterion for orthogonality is given by the following theorem

(1.13) **Theorem:** (**Pythagoras**):
Let $(E,(\ |\ ))$ be an inner product space over $\mathbf{K} \in \{\mathbf{R},\mathbf{C}\}$. Let $x,y \in E$.
1) If $\mathbf{K}=\mathbf{R}$ then $x \perp y$ if and only if $\|x+y\|^2 = \|x\|^2 + \|y\|^2$
2) If $\mathbf{K}=\mathbf{C}$ then a) $(x\,|\,y) \in \mathbf{R}$ if and only if $\|x+i\cdot y\|^2 = \|x\|^2 + \|y\|^2$
b) $x \perp y$ if and only if $(x\,|\,y) \in \mathbf{R}$ and $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

<u>Proof:</u>

ad 1) $\|x+y\|^2 = (x+y\,|\,x+y) = (x\,|\,x) + (y\,|\,x) + (x\,|\,y) + (y\,|\,y) = \|x\|^2 + 2\cdot(x\,|\,y) + \|y\|^2$

ad 2) a) if $(x\,|\,y) \in \mathbf{R} \Rightarrow \|x+i\cdot y\|^2 = (x+i\cdot y\,|\,x+i\cdot y) = (x\,|\,x) + i\cdot(y\,|\,x) - i\cdot(x\,|\,y) + (i\cdot y\,|\,i\cdot y)$
$= \|x\|^2 + i\cdot(y\,|\,x) - i\cdot(x\,|\,y) - i^2\cdot\|y\|^2 = \|x\|^2 + \|y\|^2$

b) if $(x\,|\,y) \in \mathbf{R}$ and $\|x+y\|^2 = \|x\|^2 + \|y\|^2 \underset{1)}{\Rightarrow} (x\,|\,y)=0$

if $(x\,|\,y)=0 \Rightarrow$ routine computation q.e.d.

(1.14) **Definition:** Let $(E,(\ |\ ))$ be an inner product space, let $M \subseteq E$ be a subset. Then the set $M^\perp := \{x \in E: (x\,|\,y)=0 \text{ for all } y \in M\}$ is called an **orthogonal complement** of M.

(1.15) **Theorem:** Let $M \subseteq (E,(\ |\ ))$, $(E,(\ |\ ))$ inner product space. Then
1) $M^\perp$ is a closed linear subspace of E
2) $M \subseteq (M^\perp)^\perp = M^{\perp\perp}$
3) If M is a linear subspace of E, then $M \cap M^\perp = \{0\}$

If $(E,\|.\|)$ *is a normed linear space,* $x \in E$, $M \subseteq E$ *a finite dimensional linear subspace then there exists a uniquely determined element* $y_0 \in E$ *such that* $\|x-y_0\| = \inf_{y \in M} \|x-y\|$. y_0 *is usually* *called* <u>*the element of best approximation in* M</u> *with respect to* x. *The following result generalizes this fact in a certain sense*

(1.16) **Theorem:** Let **(**E,(|)**)** be an inner product space. Let $M \subseteq E$ be a non-empty subset. Let $x \in E$. If

1) M is complete, i.e. every Cauchy sequence $(x_n)_{n \in \mathbf{N}} \subseteq M$ has a limit $x_0 \in M$
2) M is convex, i.e. $\lambda \cdot x + (1-\lambda) \cdot y \in M$ for $x,y \in M$, $\lambda \in [0,1]$

then there exists a uniquely determined element $y_0 \in M$ such that $\|x-y_0\| = \inf_{y \in M} \|x-y\|$

<u>Proof:</u>

If $x \in M$, nothing is to prove.
If $x \notin M$, define $\delta := \inf_{y \in M} \|x-y\|$, then there exists $(z_n)_{n \in \mathbf{N}} \subseteq M$ such that $\lim_{n \to \infty} \|x-z_n\| := \delta$.
If $(y_n)_{n \in \mathbf{N}} \subseteq M$ is a sequence with $\lim_{n \to \infty} \|x-y_n\| = \delta$ then we show that $(y_n)_{n \in \mathbf{N}}$ is a Cauchy sequence.

$$y_n, y_m \in M \Rightarrow \frac{1}{2}(y_n+y_m) \in M \Rightarrow \delta \le \|x - \frac{1}{2}(y_n+y_m)\| = \frac{1}{2}\|x-y_n+x-y_m)\|$$

$$\le \frac{1}{2}\|x-y_n\| + \frac{1}{2}\|x-y_m\| \underset{n,m\to\infty}{\longrightarrow} \delta$$

$\Rightarrow \lim_{n,m \to \infty} \|x - \frac{1}{2}(y_n+y_m)\| = \delta$. Using parallelogram identity we see

$$2\cdot\|x-y_n\|^2 + 2\cdot\|x-y_m\|^2 = \|(x-y_n)+(x-y_m)\|^2 + \|(x-y_n)-(x-y_m)\|^2 = \|2x-(y_n+y_m)\|^2 + \|y_n-y_m\|^2$$

$$= 4\cdot\|x - \frac{y_n - y_m}{2}\|^2 + \|y_n-y_m\|^2$$

$\Rightarrow (n,m \to \infty)$: $4\cdot\delta^2 = 4\cdot\delta^2 + \lim_{n,m\to\infty} \|y_n-y_m\|^2 \quad \Rightarrow \quad \lim_{n,m\to\infty}\|y_n-y_m\| = 0$

$\Rightarrow (y_n)_n$ Cauchy sequence in M

$\Rightarrow$ since M complete there exists $y_0 \in M$ such that $\lim_{n\to\infty} \|y_n-y_m\| = 0$

$$\delta \le \|x-y_0\| \le \|x-y_n\| + \|y_n-y_0\| \underset{n\to\infty}{\longrightarrow} \delta$$

$\Rightarrow \|x-y_0\| = \delta$. Suppose, there are elements $y_1, y_2 \in M$ with $\|x-y_1\| = \|x-y_2\| = \delta$.
We consider the Cauchy sequence $y_1, y_2, y_1, y_2, \ldots$. From this we conclude $y_1 = y_2$. q.e.d.

Since every linear subspace of a linear space is convex, we get

(1.17) **Corollary:** Let **(**E,(|)**)** be an inner product space, $M \subseteq E$ be a non-empty complete subspace, $x \in E$, then there exists a unique element $y_0 \in M$ with $\|x-y_0\| = \inf_{y \in M} \|x-y\|$.

(1.18) **Corollary:** Let **(**E,(|)**)** be a Hilbert space, $\emptyset \ne M \subseteq E$ be a closed convex set, $x \in E$, then there exists a unique element y_0 of best approximation: $\|x-y_0\| = \inf_{y \in M} \|x-y\|$.

The element of best approximation in a complete subspace of an inner product space

(E,(|)**)** *can be characterized as follows*

(1.19) **Theorem:** Let **(**E,(|)**)** be an inner product space, let $x \in E$, $M \subseteq E$ be a complete linear subspace in E. Let $y_0 \in M$. Then $\|x-y_0\| = \inf_{y \in M} \|x-y\|$ if and only if $x-y_0 \in M^\perp$.

Proof:

„⇒“:

Suppose $y_0 \in M$ with $\|x-y_0\| = \inf_{y \in M} \|x-y\|$.

To show: $(x-y_0 \mid y)=0$ for every $y \in M$.

Suppose $y \in M$, $y \neq 0$ and $\alpha=(x-y_0 \mid y) \neq 0$.

Consider $y_1 := y_0 + \alpha \cdot \frac{y}{(y|y)} \Rightarrow y_1 \in M$ and $\|x-y_1\|^2 = \left(x- y_0-\alpha\cdot\frac{y}{(y|y)} \;\middle|\; x- y_0-\alpha\cdot\frac{y}{(y|y)}\right)$

$= \|x-y_0\|^2 - \frac{|\alpha|^2}{(y|y)} < \|x-y_0\|^2$ contrary to the fact that $\|x-y_0\| = \inf_{y \in M} \|x-y\| \Rightarrow x-y_0 \in M^\perp$.

„⇐“:

$x-y_0 \in M^\perp$, $y \in M$ $\quad \Rightarrow \left\| \underbrace{x - y_0}_{\in M^\perp} + \underbrace{y_0 - y}_{\in M} \right\| \overset{(1.13)}{=} \|x-y_0\|^2 + \|y_0-y\|^2 \geq \|x-y_0\|^2$

$\Rightarrow \|x-y_0\| = \inf_{y \in M} \|x-y\|$ q.e.d.

(1.20) **Corollary:** Let **(**E,(|)**)** be an inner product space, $\emptyset \neq M \subseteq E$ be a complete subspace. Let $x \in E$. Then there exist two uniquely determined elements $x_M \in M$, $x_{M^\perp} \in M^\perp$ such that $x = x_M + x_{M^\perp}$.

(1.21) **Definition:** Let E be a linear space over **K**={**R**,**C**}, let F and G be linear subspaces of E. E is the **direct sum** of F and G if

1) for each $x \in E$ we find $x_F \in F$ and $x_G \in G$ with $x=x_F+x_G$
2) $F \cap G=\{0\}$

In this case we write $E=F \oplus G$.

It follows easily from the definition that the decomposition $x=y+z$ *is unique, if* $E=F \oplus G$

(1.22) **Corollary:** (**orthogonal decomposition theorem**):
Let H be a Hilbert space, $M \subseteq H$ be closed subspace. Then $H=M \oplus M^\perp$.

Proof:

$M^\perp$ is a closed subspace. Suppose $h \in M \oplus M^\perp$, then $(h \mid h)=0$ hence $h=0$. Thus $M \cap M^\perp = \{0\}$. (1.20) completes the proof.

It should be remarked that the hypothesis of the preceding corollary cannot be weakened, i.e., the corollary may fail if either M *is not closed or* H *is not complete.*
It is apparent from the orthogonal decomposition theorem that, given a closed linear subspace M *of a Hilbert space* H, *there exists precisely one linear subspace* N *of* H *so that* $H=M\oplus N$ *and* $M \perp N$, *namely* $N=M^{\perp}$, *and that this subspace* N *is closed. If, however, we drop the orthogonality requirement, then there may exist many linear subspaces* N *with* $H=M\oplus N$.
We now study orthonormal sets and in particular orthonormal bases in a Hilbert space

(1.23) **Theorem:** (**Bessel inequality**):
Let **(**E,(|)**)** be an inner product space, I be a finite or at most countable set of integers. Let $(y_j)_{j \in I}$ be an orthonormal set in E.
Then for each $x \in E$ $\sum_{j\in I} \left|(x|y_j)\right|^2 \le \|x\|^2$.

Proof:

For any finite subset $I_0 \subseteq I$ we have

$$0 \le \left\| x - \sum_{j\in I_0} (x|y_j)y_j \right\|^2 = \left(x - \sum_{j\in I_0} (x|y_j)y_j \;\middle|\; x - \sum_{j\in I_0} (x|y_j)y_j \right) = \|x\|^2 - \left|\sum_{j\in I_0} (x|y_j)\right|^2 \qquad \text{q.e.d.}$$

(1.24) **Corollary:** Let **(**E,(|)**)** be an inner product space, $M \subseteq E$ be an orthonormal set. Then for each $x \in E$ there exist at most countably many elements $y \in M$ such that $(x\,|\,y) \neq 0$.

Proof:

Fix $x \in E$, let $\varepsilon>0$, then there exists $m \in \mathbf{N}$ so that $m \le \frac{\|x\|^2}{\varepsilon^2} \le m+1$.
Suppose there are m+1 elements $y_1, ..., y_{m+1} \in M$ so that $|(x\,|\,y_j)| \ge \varepsilon$ for $j \in \{1, ..., m+1\}$, then $\sum_{j=1}^{m+1} \left|(x|y_j)\right|^2 \ge (m+1)\cdot\varepsilon^2 > \|x\|^2$ is contrary to Bessel inequality. For every $x \in E$ you find at most a finite number of elements $y \in M$, so that $|(x\,|\,y)| \ge \varepsilon$. If one choses $\varepsilon = \frac{1}{k}$, $k \in \mathbf{N}$, the proof is done, $|(x\,|\,y)| \neq 0$.

(1.25) **Lemma:** Let **(**H,(|)**)** be a Hilbert space, let $(x_j)_{j \in \mathbf{N}}$ be an orthonormal system in H and let $(\alpha_j)_{j \in \mathbf{N}} \subseteq \mathbf{K}$, then

1) $\sum_{j=1}^{\infty} \alpha_j \cdot x_j$ converges in H if and only if $\sum_{j=1}^{\infty} \left|\alpha_j\right|^2 < \infty$

2) $\left\| \sum_{j=1}^{\infty} \alpha_j \cdot x_j \right\|^2 = \sum_{j=1}^{\infty} \left|\alpha_j\right|^2$

3) If the sum $\sum_{j=1}^{\infty} \alpha_j \cdot x_j$ converges in H then this convergence is independent of order

Proof:

ad 1)

Consider $y_m := \sum_{j=1}^{m} \alpha_j \cdot x_j$: since

$$\|y_n - y_m\|^2 = \left\| \sum_{j=1}^{n} \alpha_j \cdot x_j - \sum_{j=1}^{m} \alpha_j \cdot x_j \right\|^2 = \left\| \sum_{j=m+1}^{n} \alpha_j \cdot x_j \right\|^2$$
$$= \left(\sum_{j=m+1}^{n} \alpha_j \cdot x_j \middle| \sum_{j=m+1}^{n} \alpha_j \cdot x_j \right) = \sum_{j=m+1}^{n} \sum_{k=m+1}^{n} \alpha_j \cdot \overline{\alpha_k} \cdot \left(x_j \middle| x_k \right) = \sum_{j=m+1}^{n} |\alpha_j|^2$$

$(y_n)_n$ is a Cauchy sequence in H if and only if $(a_m)_m$ with $a_m := \sum_{j=1}^{m} |\alpha_j|^2$ is a Cauchy sequence in **K**.

ad 3)

$$\sum_{j=1}^{\infty} |\alpha_j|^2 < \infty \Rightarrow \sum_{j=1}^{\infty} |\alpha_j|^2 = \sum_{j=1}^{\infty} |\alpha_{\pi(j)}|^2$$, π: **N**→**N** permutation. q.e.d.

(1.26) **Theorem:** Let **(**H,(|)**)** be a Hilbert space, M⊆H be an orthonormal set. Then

1) for every x ∈ H the sum $\sum_{y \in M} (x|y)y$ converges in H where the sum is taken over all y ∈ M with (x | y)≠0
2) x = $\sum_{y \in M} (x|y)y$ if x ∈ M
3) x = $\sum_{y \in M} (x|y)y$ if x ∈ $\overline{\text{span}(M)}$

Proof:

ad 1)

Given x ∈ H, there are at most countably many y ∈ M with (x | y)≠0. Bessel's inequality implies: $\sum_{y \in M} |(x|y)|^2 \leq \|x\|^2$

Lemma (1.25) shows that $\sum_{y \in M} (x|y)y$ is convergent in H to some element x_0 ∈ H.

ad 2)

If x ∈ M ⇒ x=(x | x)x= $\sum_{y \in M} (x|y)y = x_0$

ad 3)

If $x \in \text{span}(M)$ then there are $y_1, ..., y_m \subset M$ with $x = \sum_{j=1}^{m} \alpha_j \cdot y_j$

$$\Rightarrow \sum_{y\in M} (x|y)y = \sum_{y\in M} \left(\sum_{j=1}^{m} \alpha_j \cdot y_j \cdot y \right) y = \sum_{y\in M} \sum_{j=1}^{m} \alpha_j \cdot (y_j|y)y = \sum_{j=1}^{m} \alpha_j \cdot (y_j|y_j)y_j = \sum_{j=1}^{m} \alpha_j \cdot y_j = x$$

Consider the linear operator T: $H \to H$ with $x \mapsto \sum_{y\in M} (x|y)y$, T is continuous since

$$\|T(x_1 - x_2)\|^2 = \left\| \sum_{y\in M} (x_1 - x_2|y)y \right\|^2 = \sum_{y\in M} |(x_1 - x_2|y)|^2 \underset{Bessel}{\leq} \|x_1 - x_2\|^2$$

From this one can easily deduce that $x = \sum_{y\in M} (x|y)y$ if $x \in \overline{\text{span}(M)}$. q.e.d.

We now can make a meaningful definition of orthonormal basis

(1.27) **Definition:** Let **(**E,(|)**)** be an inner product space. A subset $M \subseteq E$ is called an **orthonormal basis for E** if
1) M is an orthonormal set in E
2) for each $x \in E$ we have $x = \sum_{y\in M} (x|y)y$

We don't need to mention the linear independence in the definition explicitly since an orthonormal set of elements is always linearly independent. If E *is a Hilbert space, then an orthonormal set* $M \subseteq E$ *is an orthonormal basis for* E *if and only if* $E = \overline{\text{span}(M)}$.
Orthonormal bases in a Hilbert space can be characterized as follows

(1.28) **Theorem:** Let **(**H,(|)**)** be a Hilbert space, $M \subseteq H$ be an orthonormal set. Then the following statements are equivalent:
1) M is an orthonormal basis for H
2) For each $x \in H$ we have $x = \sum_{y\in M} (x|y)y$ (*Fourier expansion*)
3) For each $x \in H$ we have $\|x\|^2 = \sum_{y\in M} |(x|y)|^2$ (*Parseval's relation*)
4) For each $x, x' \in H$ we have $(x \mid x') = \sum_{y\in M} (x|y)(y|x')$ (*Parseval's identity*)
5) $(x \mid y) = 0$ for all $y \in M$ implies $x=0$
6) M is maximal orthonormal set

<u>Proof:</u>

1) $\Rightarrow$ 2) is evident
2) $\Rightarrow$ 4) consequence of continuity
4) $\Rightarrow$ 3) take $x=x'$
3) $\Rightarrow$ 5) if $x \in H$ with $(x \mid y)=0$ for all $y \in M \Rightarrow \|x\|^2 = \sum_{y\in M} (x|y)y = 0 \Rightarrow x=0$

5) $\Rightarrow$ 6) Suppose M_0 is an orthonormal set with $M \subseteq M_0$, suppose $0 \neq x_0 \subset M_0$, $x_0 \notin M$
$\Rightarrow (x_0 \mid y)=0$ for all $y \in M_0$ hence $(x \mid y)=0$ for all $y \in M \subseteq M_0 \Rightarrow x_0=0 \Rightarrow$ contradiction
6) $\Rightarrow$ 1) Suppose $\overline{\text{span}}(M) \neq H$, then $x_0 \notin \overline{\text{span}}(M)$, $x_0 \neq 0$, $x_0 \in H$, $\overline{\text{span}}(M)$ is a proper closed subspace of H. $x_0 \in \overline{\text{span}}(M)^{\perp} \Rightarrow (x_0 \mid y)=0$ for all $y \in M$: contrary to the fact that M is maximal. q.e.d.

(1.29) **Definition:** Let **(**E,(|)**)** be an inner product space, $M \subseteq E$ be an orthonormal set, let $x \in E$. Then the sum $\sum_{y \in M} (x|y)y$ is called the **Fourier series of x with respect to M**. The numbers $(x \mid y)$ are called the **Fourier coefficients of x with respect to M**.

The question whether a Hilbert space has an orthonormal basis, is answered by

(1.30) **Theorem:** Every Hilbert space has an orthonormal basis.

Proof:

Consider the collection $\mathscr{C}$ of orthonormal sets in H. We order $\mathscr{C}$ by inclusion, i.e. we say $M_1 < M_2$ if $M_1 \subseteq M_2$, $M_1, M_2 \in \mathscr{C}$. $\mathscr{C}$ is partially ordered; it is also non-empty since if $x \in H$ is any element of H, $x \neq 0$, the set M_0 consisting only of $\frac{x}{\|x\|}$ is a orthonormal set. Now let $(M_\alpha)_\alpha$ be a linearly ordered subset of $\mathscr{C}$. Then $\bigcup_\alpha M_\alpha$ is an orthonormal set which contains each M_α and is thus an upper bound, for $(M_\alpha)_\alpha$. Since every linearly ordered subset of $\mathscr{C}$ has an upper bound, we can apply Zorn's lemma and conclude that $\mathscr{C}$ has a maximal element, i.e. an orthonormal set not properly contained in any other orthonormal set.
q.e.d.

(1.31) **Theorem:** Any two orthonormal basis M and N of a Hilbert space have the same cardinality.

Proof:

If one of the cardinalities $|M|$ or $|N|$ is finite, then H is a finite dinmensional Hilbert space. So suppose $|M| = \infty$ and $|N| = \infty$. For $x \in M$ the set $S_x := \{y \in N: (x \mid y) \neq 0\}$ is an at most countable set. Hence $\bigcup_{x \in M} S_x = N$, if not, there would exist $z \in N$ with $z \in \bigcup_{x \in M} S_x$, i.e. $(z \mid x)=0$ for all $x \in M$ and thus $z=0$ contrary to the fact that the zero element does not belong to a basis. Since $N = \bigcup_{x \in M} S_x$ we have $|N| = |\bigcup_{x \in M} S_x| \leq |M|\,|N| = |M|$. The same argument gives $|M| \leq |N|$, which implies equality. q.e.d.

(1.32) **Definition:** Let **(**H,(|)**)** be a Hilbert space. If $M = \{x_j : j \in I\} \subseteq H$ (I = *index set*) is an orthonormal basis of H, then the **dimension of H** is defined to be the cardinality of I and is denoted by dim(H).

Examples:

1) $l_2=\{(\xi_j)_{j\in\mathbf{N}} : \sum_{j\in N}|\xi_j|^2<\infty\}$ is a Hilbert space

$(x\,|\,y):=\sum_{j\in N}\xi_j\cdot\overline{\eta_j}$ $\qquad e_k=\delta_{jk}$

2) $L_2=\{f: [a,b]\to\mathbf{C}: \int|f(t)|^2\,dt<\infty\}$ $\qquad (f\cdot g):=\int f(t)\cdot\overline{g(t)}\,dt$

One useful result involving this concept is the following theorem

(1.33) **Theorem:** Let **(**H,(|)**)** be a Hilbert space. Then the following statements are equivalent:

1) H is separable (i.e. there exists a countable set $N\subseteq H$ with $\overline{N}=H$)
2) H has a countable orthonormal basis M, i.e. $\dim(H)=|N|=\chi_0$.

<u>Proof:</u>

Suppose H is separable and let N be a countable set with $H=\overline{N}$. By throwing out some of the elements of N we can get a subcollection N_0 of independent vectors whose span (finite linear combinations) is the same as N.
This gives $H=\overline{N}=\overline{\text{span } N}=\overline{\text{span } N_0}$. Applying the Gram-Schmidt procedure of this subcollection N_0 we obtain a countable orthonormal basis of H. Conversely if $M=\{y_j,\ j\in\mathbf{N}\}$ is an orthonormal basis for H then it follows from theorem (1.28) that the set of finite linear combinations of the y_j with <u>rational</u> coefficients is dense in H. Since this set is countable, H is separable. q.e.d.

§ 3 Isomorphisms

Most Hilbert spaces that arise in practise have a countable orthonormal basis. We will show that such an infinite-dimensional Hilbert space is just a disguised copy of the sequence space l_2.*To some extent this has already been done in theorem (1.28)*

(1.34) **Theorem:** Let $(H,(\ |\ ))$ be a separable Hilbert space. If $\dim(H)=\infty$ (if $\dim(H)=n$) then there exists a one-to-one (=injective) mapping $U: H\to l_2$ $\left(U: H\to(\mathbf{K}^n,(\ |\ ))\right)$

$\left[(x|y)_2 := \sum_{j=1}^{n} x_j \cdot \overline{y_j} \right.$ für $x = (x_1, ..., x_n)$, $\left. y = (y_1, ..., y_n)\right]$ with the following properties:

1) $U(x+y) = Ux+Uy$
   $U(\lambda x) = \lambda\cdot Ux$ for all $x,y \in H$, $\lambda \in \mathbf{K}$
2) $(Ux\,|\,Uy)_2 = (x\,|\,y)$, in particular $\|Ux\|_2=\|x\|$ for $x,y \in H$

Proof:

Suppose: $\dim(H)=\infty$, let $(y_j)_{j\in\mathbf{N}}$ be an orthonormal basis of H. Take $(e_j)_{j\in\mathbf{N}}\subseteq l_2$ $(e_1=(1,0,...,0)$, $e_j=(0,...,0,1,0,...,0)$ to be the canonical basis in l_2. Take $x \in H$,

$x=\sum_{j=1}^{\infty}(x|y_j)\cdot y_j$, define $U: H\to l_2$ to be $Ux:=\sum_{j=1}^{\infty}(x|y_j)\cdot e_j$, $Uy_k=e_k$

Since $\|x\|^2=\sum_{j=1}^{\infty}\left|(x|y_j)\right|^2 <\infty$, $\|Ux\|^2 = (Ux\,|\,Ux)=\left(\sum_{j=1}^{\infty}(x|y_j)\cdot e_j \,\middle|\, \sum_{j=1}^{\infty}(x|y_j)\cdot e_j\right)$

$$= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(x|y_j)\cdot\overline{(x|y_k)}\cdot(e_j,e_k)_2$$

$$=\sum_{j=1}^{\infty}(x|y_j)\cdot\overline{(x|y_j)} = \sum_{j=1}^{\infty}\left|(x|y_j)\right|^2 = \|x\|^2$$

$$\Rightarrow \|Ux\|=\|x\|$$

$$(Ux\,|\,Uy) = \left(\sum_{j=1}^{\infty}(x|y_j)\cdot e_j \,\middle|\, \sum_{k=1}^{\infty}(y|y_k)\cdot e_k\right) = \sum_{j=1}^{\infty}(x|y_j)\cdot\overline{(y|y_j)} = \sum_{j=1}^{\infty}(x|y_j)\cdot(y_j|y) \overset{(1.28)}{=} (x|y)$$

$\Rightarrow$ 2) for $\dim(H) = \infty$

U is one-to-one (=injective) since Ux=Uy implies
$\|x-y\|^2 = (x-y|x-y) = \left(U(x-y)|U(x-y)\right)_2 = \|U(x-y)\|_{l_2}^2 = \|Ux-Uy\|_{l_2}^2 = 0$ hence x=y

U is onto (=surjective), take $(\alpha_j)_{j\in\mathbf{N}} \in l_2$, define $x = \sum_{j=1}^{\infty}\alpha_j\cdot y_j$, $x \in H$

$\Rightarrow$ $Ux = \sum_{j=1}^{\infty}\alpha_j\cdot e_j = (\alpha_j)_{j\in\mathbf{N}}$;

also $(x|y_k) = \sum_{j=1}^{\infty}(\alpha_j\cdot y_j|y_k) = \sum_{j=1}^{\infty}\alpha_j\cdot(y_j|y_k) = \alpha_k$ q.e.d.

This theorem clarifies what is meant by „disguised copy". Intuitively, it says that by means of the mapping U *we may identify the elements of* H *and* l_2 *in such a way that each of these Hilbert spaces appears (algebraically and topologically) as a perfect copy of the other*

Example:

$L_2=\{f: \int |f|^2\, dx \text{ exists}\}$

$(f\,|\,g):=\int f\cdot \overline{g}\, dx$ $\qquad$ $L_2\,|_N$ with $N=\{f: \|f\|_2=0\}$ is a Hilbert space

(1.35) **Definition:** Let H_1, H_2 be Hilbert spaces over **K**. Let $D\subseteq H$ be a linear subspace.
A mapping $A: D\to H_2$ is called

1) **linear**, if $A(\lambda\cdot x+\mu\cdot y)=\lambda\cdot Ax+\mu\cdot Ay$
2) **isometric** (or an **isometry**) if $(Ax\,|\,Ay)_{H_2} = (x\,|\,y)_{H_1}$ for all $x,y\in D$
3) **an isometric isomorphism** of H_1 onto H_2, if $D=H_1$, $A(H_1)=H_2$, A is linear and isometric
4) **an automorphism** if $H_1=H_2$ and A is an isometric isomorphism

H_1 and H_2 are called **isometric isomorphic** if there exists an isometric isomorphism $T: H_1\to H_2$.

Obvious observations:

$A: H_1\to H_2$ linear, isometric, then A is one-to-one:

$\|x-y\|^2 = (x-y\,|\,x-y) = \big(A(x-y)\,|\,A(x-y)\big) = (Ax-Ay\,|\,Ax-Ay) = \|Ax-Ay\|^2$

$A: H_1\to H_2$ isometric isomorphism, then $A^{-1}: H_2\to H_1$ is an isometric isomorphism.

(1.36) **Corollary:** Two separable Hilbert spaces are isometric isomorphic if and only if they have the same dimension.

The statement of this corollary remains if the word „separable" is omitted

Chapter 2: Bounded linear operators

In this chapter we will study mappings of some subset D *of a Hilbert space* $H \neq \emptyset$ *into some other Hilbert space* H'. *In this context we get confronted with two familiar aspects of such mappings: the algebraic aspect is well taken care of if the mapping* A *in question is linear. In order that this requirement should make sense it is necessary that the subset* $D \subseteq H$ *on which* A *is defined be a linear subspace of* H.
In order to be able to take care of the topological aspect we study two concepts for linear mappings which will turn out to be closely related with each other: boundedness and continuity

§1 Bounded linear mappings

(2.1) **Definition:** A linear mapping $A: D \subseteq H_1 \to H_2$ (H_1, H_2 Hilbert spaces, $D \subseteq H_1$ linear subspace) is called **bounded** if there is $M>0$ so that $\|Ax\| \leq M \cdot \|x\|$ for all $x \in D$.
If $A: D \to H_2$ is bounded and $D \neq \emptyset$, then the non-negative number

$$\|A\| := \sup_{\substack{x \in D \\ x \neq 0}} \frac{\|Ax\|}{\|x\|}$$ is called the norm of A.

$\mathcal{L}_b(D, H_2)$ denotes the set of all bounded linear operators $A: D \subseteq H_1 \to H_2$

{The definition of a bounded linear mapping can easily be extended to the case where H_1, H_2 *are normed linear spaces}*

(2.2) **Lemma:** If $A: D \subseteq H_1 \to H_2$ is a bounded linear mapping, then

1) $\|A\| = \sup\limits_{\substack{\|x\|=1 \\ x \in D}} \|Ax\| = \sup\limits_{\substack{\|x\| \leq 1 \\ x \in D}} \|Ax\|$

2) $\|Ax\| \leq \|A\| \cdot \|x\|$ for all $x \in D$

Examples:

1) $U: H_1 \to H_2$ be an isometric isomorphism, U is bounded
$\|U\| = 1$ and $(Ux \mid Uy) = (x \mid y)$

2) Let $M \subseteq H$ be a closed linear subspace of the Hilbert space H. Given $x \in H$, define $P_M: H \to M$, $x \mapsto P_M x$ ($P_M x$ is the unique element of best approximation in M), then $P_M x = x$ if $x \in M$. $P_M^2 = P_M$
P_M is linear: $x \in H \Rightarrow x = x_M + x_{M^\perp}$

$$x + y = x_M + x_{M^\perp} + y_M + y_{M^\perp} = (x_M + y_M) + (x_{M^\perp} + y_{M^\perp})$$

$$\Rightarrow P_M(x+y) = x_M + y_M = P_M x + P_M y$$

also: $P_M(\lambda x) = \lambda \cdot P_M x$

$$\|x\|^2 = \|x_M + x_{M^\perp}\|^2 = \|x_M\|^2 + \|x_{M^\perp}\|^2 \geq \|x_M\|^2 = \|P_M x\|^2 \Rightarrow \|P_M\| \leq 1$$

$\Rightarrow \|P_M\| = 1$ since $P_M x_M = x_M$; $x_M \in M$

P_M is linear and bounded, $P_M^2 = P_M$, $P_M: H \to M$ is onto. Also $Id - P_M: H \to M^\perp$ is linear, $(Id-P_M)^2 = Id - P_M$, $\|Id-P_M\|=1$

(2.3) **Definition:** Let H_1, H_2 be inner product spaces, let $D \subseteq H_1$ be a subset.
A mapping $A: D \to H_2$ is
1) said to be **continuous at $x_0 \in D$** if for every $\varepsilon>0$ there exists $\delta>0$ such that $\|A\cdot x_0 - A\cdot x\| < \varepsilon$ for all $x \in D$ with $\|x-x_0\|<\delta$
2) said to be **continuous on D** if A is continuous at every point of D

This definition can also be extended to the case where H_1, H_2 *are normed linear spaces. The same is true for the following characterizations in case of linear mappings*

(2.4) **Theorem:** Let H_1, H_2 be inner product spaces, let $D \subseteq H_1$ be a linear subspace. For a linear mapping $A: D \to H_2$ the following statements are equivalent:
1) A is bounded on D
2) A is continuous on D
3) A is continuous at $x_0=0$
4) For every sequence $(x_n)_n \subseteq D$ with $\lim\limits_{n\to\infty} \|x_n\|=0$ we have $\lim\limits_{n\to\infty} \|A\cdot x_n\|=0$
5) For every sequence $(x_n)_n \subseteq D$ converging to some $x_0 \in D$ we have $\lim\limits_{n\to\infty} \|A\cdot x_n - A\cdot x_0\|=0$

<u>Proof:</u>

1) $\Rightarrow$ 2):
A bounded implies the existence of M>0 with $\|A\cdot x\| \le M\cdot\|x\|$ for all $x \in D$, hence $\|A\cdot x - A\cdot y\| = \|A\cdot(x-y)\| \le M\cdot\|x-y\|$ implies 2)

2) $\Rightarrow$ 3)
obvious, since $x_0=0 \in D$

3) $\Rightarrow$ 4)
Given a sequence $(x_n)_n \subseteq D$ with $\lim\limits_{n\to\infty} \|x_n\|=0$; given $\varepsilon>0$, there exists $n_\varepsilon \in \mathbf{N}$ so that $\|x_n\|<\delta$ for $n \ge n_\varepsilon$, δ chosen as in definition (2.2).
Then $\|Ax_n\| < \varepsilon$ for all n_δ which gives $\lim\limits_{n\to\infty} \|Ax_n\|=0$

4) $\Rightarrow$ 5)
simple, if one considers $(x_n - x)_{n \in \mathbf{N}}$

5) $\Rightarrow$ 1)
Suppose A is not bounded, then for every $n \in \mathbf{N}$ one can find $y_n \in D$, $y_n \neq 0$ with $\|A \cdot y_n\| > n^2 \cdot \|y_n\|$, define $z_n := \frac{y_n}{\|y_n\|}$ $\Rightarrow$ $\|A \cdot z_n\| > n^2$, consider $x_n := \frac{z_n}{n}$

$\Rightarrow \lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \frac{\|z_n\|}{n} = 0$, but $\|A \cdot x_n\| > n$. q.e.d.

As soon as one thinks of a linear mapping one also has to think of its particular domain. The following example indicates that this may have something to do with unboundedness of the mapping in question.

Examples:

$H = l_2$, $D := \{(x_k)_{k \in \mathbf{N}} \in l_2 : x_k \neq 0 \text{ for at most finitely many } k \in \mathbf{N}\}$ is a linear subspace
$A: D \to l_2$ $A(x_k) := (k \cdot x_k)_k$, A is linear,
take $e_j = (\delta_{jk})_{k \in \mathbf{N}}$ with $\delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$
$\left\|A_{e_j}\right\| = \|(k \cdot \delta_{jk})_{k \in \mathbf{N}}\| = j$ A is unbounded

On the other hand, any bounded linear mapping $A: D \subseteq H_1 \to H_2$ *can be extended to a bounded linear mapping* $\overline{A} : \overline{D} \to H_2$*, so the domain of bounded linear mappings can always be assumed to be a Hilbert space*

(2.5) **Theorem:** Let H_1, H_2 be Hilbert spaces, let $D \subseteq H_1$ be a linear subspace. For every bounded linear operator $A: D \to H_2$ there exists a unique bounded linear operator $\overline{A} : \overline{D} \to H_2$ with
1) $A \cdot x = \overline{A} \cdot x$ for $x \in D$ and
2) $\|\overline{A}\| = \|A\|$

Proof:

Let $\overline{x} \in \overline{D}$, then there exists a sequence $(x_n)_{n \in \mathbf{N}} \subseteq D$ with $\lim_{n\to\infty} \|x_n - \overline{x}\| = 0$. The sequence $(A \cdot x_n)_{n \in \mathbf{N}}$ is a Cauchy sequence in H_2 since $\|A \cdot x_n - A \cdot x_m\| = \|A \cdot (x_n - x_m)\| \leq \|A\| \cdot \|x_n - x_m\|$.
Since H_2 is complete, $(A \cdot x_n)_{n \in \mathbf{N}}$ converges to some element $y_{\overline{x}}$ in H_2. If $(y_n) \subseteq D$ with $\lim_{n\to\infty} \|y_n - x_n\|$, then $\|A \cdot y_n - A \cdot x_n\| \leq \|A\| \cdot \|y_n - x_n\| \leq \|A\| \cdot (\|y_n - x\| + \|x_n - x\|) \xrightarrow[n\to\infty]{} 0$.

Define $\overline{A} : \overline{D} \to H_2$ by $\overline{A} \cdot \overline{x} := \lim_{n\to\infty} A \cdot x_n$ if $\lim_{n\to\infty} \|x_n - \overline{x}\| = 0$, $\overline{A}$ is linear (easy to verify!)
$\overline{A}$ is bounded since $\|A \cdot x_n\| \leq \|A\| \cdot \|x_n\|$ and $\lim_{n\to\infty} |\,\|x_n\| - \|\overline{x}\|\,| \leq \lim_{n\to\infty} \|x_n - \overline{x}\| = 0$ imply $\|\overline{A} \cdot \overline{x}\| \leq \|A\| \cdot \|\overline{x}\|$ which gives $\|\overline{A}\| \leq \|A\|$.

On the other hand $\|\overline{A}\| = \sup_{x\in\overline{D}} \|\overline{A}\cdot x\| \geq \sup_{\substack{x\in D\\ \|x\|=1}} \|A\cdot x\| = \|A\|$

$\Rightarrow$ 1) and 2)

Uniqueness of $\overline{A}$:
If B: $\overline{D} \to H_2$ is a bounded linear operator on $\overline{D}$ with $B\cdot x = A\cdot x$ for $x \in D$, take $\overline{x} \in \overline{D}$, choose sequence $(x_n)_{n\in\mathbf{N}} \subseteq D$ with $\lim_{n\to\infty} \|x_n - \overline{x}\| = 0$,
then $\|B\cdot\overline{x} - \overline{A}\cdot\overline{x}\| \leq \|B\cdot\overline{x} - B\cdot x_n\| + \|B\cdot x_n - \overline{A}\cdot\overline{x}\| = \|B\cdot\overline{x} - B\cdot x_n\| + \|A\cdot x_n - \overline{A}\cdot\overline{x}\|$
$\leq \|B\|\cdot\|\overline{x} - x_n\| + \|A\|\cdot\|\overline{x} - x_n\| \underset{n\to\infty}{\longrightarrow} 0.$ q.e.d.

(2.6) **Lemma:** Let H_1,H_2 be inner product spaces, let $D_A\subseteq H_1$, $D_B\subseteq H_1$ be linear subspaces. If A: $D_A\to H_2$ and B: $D_B\to H_2$ are linear, then A+B: $D_A\cap D_B \to H_2$ defined by $(A+B)x := Ax+Bx$ and $\lambda\cdot A$: $D_A\to H_2$ defined by $(\lambda A)(x) := \lambda\cdot Ax$ are linear. Let $D_C\subseteq H_2$ be a linear subspace, C: $D_C\to H_3$ be a linear operator if H_3 is another inner product space, then the operator $C\cdot A$ defined by $(C\cdot A)\cdot x := C\cdot(A\cdot x)$ for all $x \in D_A$ with $A\cdot x \in D_C$ is linear also.

This theorem shows that in general case we ought to be careful about the domains of these operators

(2.7) **Definition:** Let H_1,H_2 be inner product spaces. $D_A\subseteq H_1$, $D_B\subseteq H_1$ be linear subspaces. A: $D_A\to H_2$ and B: $D_B\to H_2$ are said to be **equal** if $D_A=D_B$ and $A\cdot x=B\cdot x$ for $x \in D_A=D_B$.

(2.8) **Theorem:** Let H_1,H_2,H_3 be inner product spaces, $D_A\subseteq H_1$, $D_B\subseteq H_1$, $D_C\subseteq H_2$ be linear subspaces. Let A: $D_A\to H_2$, B: $D_B\to H_2$ and C: $D_C\to H_3$ be bounded linear operators then
1) $\|A+B\| \leq \|A\|+\|B\|$
2) $\|\lambda\cdot A\| = |\lambda|\cdot\|A\|$
3) $\|C\cdot A\| \leq \|C\|\cdot\|A\|$

Proof:

For $x \in D_A \cap D_B$ we have $\|(A+B)x\| \leq \|Ax\|+\|Bx\| \leq (\|A\|+\|B\|)x$.
For $x \in D_A$ we have $\|(\lambda A)x\| = |\lambda|\cdot\|Ax\| \leq |\lambda|\cdot\|A\|\cdot\|x\|$
For $x \in D_A$ with $Ax \in D_C$ one has $\|CAx\| \leq \|C\|\cdot\|Ax\| \leq \|C\|\cdot\|A\|\cdot\|x\|$ q.e.d.

(2.9) **Theorem:** Let H_1 be an inner product space, H_2 be a Hilbert space, $D\subseteq H_1$ be linear subspace. The set $\mathfrak{L}_b(D,H_2)$ of all bounded linear operators on D is complete (Banach space) with respect to $\|A\|$ for $A \in \mathfrak{L}_b(D,H_2)$.

Proof:

Let $(A_n)_{n \in \mathbf{N}} \subseteq 2_b(D,H_2)$ be a Cauchy sequence, i.e. $\lim\limits_{n,m\to\infty} \|A_n-A_m\|=0$. Consider $(A_n\cdot x)_{n \in \mathbf{N}}$, $x \in D$, $(A_n\cdot x)_n$ is a Cauchy sequence in H_2 since $\|A_n\cdot x-A_m\cdot x\|=\|(A_n-A_m)\cdot x\|\leq\|A_n-A_m\|\cdot\|x\|$ which converges in H_2.

Define A: $D\to H_2$ by $A\cdot x:=\lim\limits_{n\to\infty} A_n\cdot x$ for $x \in D$. A is linear and bounded, because of

$\|A\cdot x\|=\lim\limits_{n\to\infty}\|A_n\cdot x\|\leq\lim\limits_{n\to\infty}\|A_n\|\cdot\|x\|\leq M\cdot\|x\|$

It remains to show that $\lim\limits_{n\to\infty}\|A_n-A\|=0$:

Given $\varepsilon>0$ we can find $n_\varepsilon \in \mathbf{N}$ such that $\|A_n-A_m\|<\varepsilon$ for all $n,m \geq n_\varepsilon$

$\Rightarrow x \in D$: $\|A_n\cdot x-A_m\cdot x\|\leq\|A_n-A_m\|\cdot\|x\|<\varepsilon\cdot\|x\|$

$\underset{m\to\infty}{\Rightarrow}$ $\|A\cdot x_n-A\cdot x\|\leq\varepsilon\cdot\|x\|$ for all $n\geq n_\varepsilon$. q.e.d.

Two special cases: 1) $2_b(H,H)$

2) $2_b(H,\mathbf{K})$=:H‘ (dual of H)

(2.10) **Definition:** Let H be a Hilbert space, $D\subseteq H$ be a subset. A mapping A: $D\to H$ is called

1) **an operator *in* H**
2) **an operator *on* H, if D=H**

(2.11) **Theorem:** Let H be a Hilbert space. Let $(A_n)_{n \in \mathbf{N}}$ and $(B_n)_{n \in \mathbf{N}}$ be sequences of bounded linear operators on H with $\lim\limits_{n\to\infty} A_n=A$ (i.e. $\lim\limits_{n\to\infty}\|A_n-A\|=0$) and $\lim\limits_{n\to\infty} B_n=B$.

Then $\lim\limits_{n\to\infty} A_n\cdot B_n=A\cdot B$.

Proof:

$\|A\cdot B-A_n\cdot B_n\|\leq\|A\cdot B-A_n\cdot B\|+\|A_n\cdot B-A_n\cdot B_n\|\leq\|A-A_n\|\cdot\|B\|+\|A_n\|\cdot\|B-B_n\|$ and
$|\,\|A_n\|-\|A\|\,|\leq\|A_n-A\|$. q.e.d.

(2.12) **Definition:** Let H_1,H_2 be Hilbert spaces. An operator $A \in 2_b(H_1,H_2)$ is called **invertible** if there exists an operator $B \in 2_b(H_2,H_1)$ such that $A\cdot B=Id_{H_2}$ and $B\cdot A=Id_{H_1}$. We denote the inverse of A by A^{-1}.

(2.13) **Theorem:** Let H_1,H_2 be Hilbert spaces. An invertible bounded linear operator A: $H_1\to H_2$ is one-to-one and maps H_1 onto H_2. The inverse of A is unique.

Proof:

Suppose A^{-1} and B are inverses of A. Then we conclude
$Bx = Id_H Bx = (A^{-1}A)Bx = A^{-1}(AB)x = A^{-1}Id_H x = A^{-1}x$ for all $x \in H_2$
From $AA^{-1} = Id_{H_2}$ we conclude that A maps H_1 onto H_2; for every $y \in H_2$ we have
$y = (AA^{-1})y = A(A^{-1}y)$. From $AA^{-1} = Id_{H_1}$ we deduce that A is one-to-one:
for $Ax_1=Ax_2$ we get $x_1 = (A^{-1}A)x_1 = A^{-1}(Ax_1) = A^{-1}(Ax_2) = (A^{-1}A)x_2 = x_2$ q.e.d.

Remark:

Consider $l_p=\{(x_n)_{n\in\mathbf{N}}: \sum_{k=0}^{\infty}|x_k|^p < \infty\}$, define

$\|x\|_p := \left(\sum_{k=0}^{\infty}|x_k|^p\right)^{\frac{1}{p}}$ for $1<p<\infty$, then $l_p' = l_q$ is the **dual** of l_p

$l_p'' = l_q' = l_p$ $\qquad \frac{1}{p}+\frac{1}{q}=1$

$l_1' = l_\infty$ but $c_0' = l_1$. We will show H‘=H thus H‘‘=H‘=H

(2.14) **Definition:** Let H be an inner product space over **K**={**R**,**C**}. A linear mapping of H into **K** is called a **linear functional on H**.
The set of all bounded linear functionals on H, $\mathfrak{L}_b(H,\mathbf{K})$, will be denoted H‘ and is called the **dual of H**.

$T: X\to Y \qquad T': Y'\to X'$

$y' \mapsto T'y'$ with $T'y'(x)=\langle x,T'y'\rangle=\langle Tx,y'\rangle$

Theorem (2.8) in particular states that H‘ *is a Banach space. The following theorem shows that there is a one-to-one correspondence between a Hilbert space and its dual*

(2.15) **Theorem:** (**Riesz-representation theorem**):
Let H be a Hilbert space over **K**={**R**,**C**}.
1) If $x \in H$, then $f_x: H\to\mathbf{K}$ defined by $f_x(y) := (y\,|\,x)$ is a bounded linear functional on H with $\|f_x\|=\|x\|$
2) Given $x' \in H' = \mathfrak{L}_b(H,\mathbf{K})$. Then there exists a unique element $x \in H$ such that $f_x = x'$, i.e. $(y\,|\,x)=x'(y)$ for all $y \in H$. Also $\|x\|=\|x'\|$.

Proof:

ad 1)
$f_x(\alpha_1\cdot y_1+\alpha_2\cdot y_2)=(\alpha_1\cdot y_1+\alpha_2\cdot y_2\,|\,x)=\alpha_1(y_1\,|\,x)+\alpha_2(y_2\,|\,x)=\alpha_1 f_x(y_1)+\alpha_2 f_x(y_2) \Rightarrow f_x$ is linear.

$|f_x(y)| = |(y|x)| \le \|y\|\cdot\|x\| \Rightarrow \|f_x\| = \sup_{\|y\|=1} |f_x(y)| \le \|x\|$

and also $\|x\|^2 = (x|x) = |f_x(x)|$

ad 2)

We consider $N=\{y \in H: x'(y)=0\}$, N is a linear subspace of H, N is closed since for any sequence $(y_n)_{n\in\mathbf{N}}$ in N with $\lim_{n\to\infty} \|y_n-y\|=0$ we have

$|x'(y)-x'(y_n)| = |x'(y-y_n)| = \|x'\|\cdot\|y-y_n\| \to 0 \quad ; n \to \infty.$

$x'(y)=\lim_{n\to\infty} x'(y_n)=0.$

If N=H, i.e. $x'(y)=0$ for *all* $y \in H$, then take x=0. $x'(y)=0=(y|0)$.

If $N\neq 0$, then there exists $x_0 \in H$, $x_0 \notin N$, $\|x_0\|=1$, $x_0 \perp N$. As a consequence we have $x'(x_0)\neq 0$.

Define $x:=\overline{x'(x_0)}\cdot x_0$, then $x \perp N$. For every $y \in N$ we have $y=\underbrace{y - \frac{x'(y)}{|x'(x_0)|^2}x}_{\in N} + \underbrace{\frac{x'(y)}{|x'(x_0)|^2}x}_{\in N^\perp}$

since $x'\left(y-\frac{x'(y)}{|x'(x_0)|^2}\cdot x\right) = x'(y) - \frac{x'(y)}{|x'(x_0)|^2}\cdot x'(x) = x'(y) - \frac{x'(y)}{|x'(x_0)|^2}\cdot\overline{x'(x_0)}\cdot x'(x_0) = 0.$

$\Rightarrow (y|x)=$

$$\left(y-\frac{x'(y)}{|x'(x_0)|^2}\cdot x \,\middle|\, x\right) + \frac{x'(y)}{|x'(x_0)|^2}\cdot(x|x) = 0 + \frac{x'(y)}{|x'(x_0)|^2}\cdot(x|x) = \frac{x'(y)}{|x'(x_0)|^2}\cdot\overline{x'(x_0)}\cdot x'(x_0)\cdot(x_0|x_0)$$

$= x'(y)$ for $y \in H$.

$\Rightarrow f_x(y) = (y|x) = x'(y) \in N \Rightarrow f_x=x'$.

<u>Uniqueness:</u>

If $\hat{x} \in H$ such that $(y|\hat{x})=x'(y)$ for $y \in H \Rightarrow (y|\hat{x}-x) = (y|\hat{x}) - (y|x) = x'(y)-x'(y) = 0$

$\Rightarrow x=\hat{x}$. q.e.d.

§ 2 Adjoint operators

Not every bounded linear operator A *on a Hilbert space* H *has an inverse, but* A *always has a twin brother* A* *of some other sort, connected with* A *by the equation* $(Ax\,|\,y) = (x\,|\,A^*y)$ *for* $x,y \in H$.
To see this, let H *be a Hilbert space over* $\mathbf{K} \in \{\mathbf{R},\mathbf{C}\}$ *and* A *a bounded linear operator on* H.
For given $x \in H$ *we define* x‘ *by* $x'(y):=(Ay\,|\,x)$ *for* $y \in H$. *Clearly* x‘ *is bounded linear functional on* H, *i.e.* $x' \in H'$, *since for all* $y \in H$ $|x'(y)| = |(Ay\,|\,x)| \le \|Ay\|\,\|x\| \le \|A\|\,\|y\|\,\|x\|$ *hence* $\|x'\| \le \|A\|\,\|x\|$. *Thus by the Riesz representation theorem (2.14) there exists a unique* $z_x \in H$ *such that* $(y\,|\,z_x) = x'(y) = (Ay\,|\,x)$ *for all* $y \in H$.
This leads to the following definition

(2.16) **Definition:** Let H be a Hilbert space. Let $A \in \mathfrak{L}_b(H,H)$. Then the **Hilbert space adjoint** A* of A is the mapping $A^*: H \to H$ defined by $A^*y:=z_y$, $y \in H$, where z_y is the unique element in H, so that $(Ax\,|\,y)=(x\,|\,z_y)$ $(= (x\,|\,A^*y))$.

(2.17) **Theorem:** Let $A,B \in \mathfrak{L}_b(H,H)$, $\lambda \in \mathbf{K}$. Then
1) A* is a bounded linear operator on H with $\|A^*\|=\|A\|$
2) $(A^*x\,|\,y)=(x\,|\,Ay)$
3) $A^{**}:=(A^*)^*=A$
4) $(\lambda\cdot A)^*=\overline{\lambda}\cdot A^*$
5) $(A+b)^*=A^*+B^*$
6) $(AB)^*=B^*A^*$
7) if A is invertible then so is A* with $(A^*)^{-1} = (A^{-1})^*$
8) $\|A^*A\|=\|A\|^2$

(2.18) **Definition:** Let H be a Hilbert space over $\mathbf{K} \in \{\mathbf{R},\mathbf{C}\}$. A bounded linear operator A on H is called
1) **selfadjoint** or **hermitesch**, if A=A*
2) **unitary** (or **orthogonal** if **K**=**R**) if $A\cdot A^* = A^*\cdot A = Id_H$
3) **normal** if $A^*\cdot A = A\cdot A^*$

Obviously selfadjoint bounded linear operators as well as unitary ones are normal

Examples:

1) $A=2\cdot i\cdot Id_H \Rightarrow A^*\cdot A = (2\cdot i\cdot Id_H)^*\cdot(2\cdot i\cdot Id_H) = -2\cdot i\cdot Id_H\cdot 2\cdot i\cdot Id_H = (2\cdot i\cdot Id_H)\cdot(-2\cdot i\cdot Id_H) = A\cdot A^*$
   $\Rightarrow$ A is normal
   A is neither selfadjoint nor unitary
2) $A_r: l_2 \to l_2$ with $(\alpha_1,\alpha_2,\alpha_3,...) \mapsto (0,\alpha_1,\alpha_2,\alpha_3,...)$ *right shift operator*
   $A_l: l_2 \to l_2$ with $(\alpha_1,\alpha_2,\alpha_3,...) \mapsto (\alpha_2,\alpha_3,\alpha_4,...)$ *left shift operator*
   $a=(\alpha_j)_j$ $b=(\beta_j)_j$
   $$(A_r\cdot a\,|\,b) = ((0,\alpha_1,\alpha_2,...)\,|\,(\beta_1,\beta_2,...)) = \sum_{j=1}^{\infty}\alpha_j\cdot\overline{\beta_{j+1}} = (a\,|\,A_l\cdot b) \Rightarrow A_r^*=A_l$$
   $$A_r^*\cdot A_r = A_l\cdot A_r = Id_{l_2}$$
   $$A_r\cdot A_r^*(\alpha_1,\alpha_2,\alpha_3,...) = A_r\cdot A_l = A_r\cdot(\alpha_2,\alpha_3,...) = (0,\alpha_2,\alpha_3,...) \neq Id_{l_2}$$

Our first general results about <u>selfadjoint operators</u> are given in

(2.19) **Theorem:** Let H be a Hilbert space over $\mathbf{K}=\{\mathbf{R},\mathbf{C}\}$. Let $A \in \mathcal{L}_b(H,H)$.
1) If A is selfadjoint, then $(Ax\,|\,x) \in \mathbf{R}$
2) If $\mathbf{K}=\mathbf{C}$ and if $(Ax\,|\,x) \in \mathbf{R}$ for all $x \in H$, then A is selfadjoint

<u>Proof:</u>

ad 1)
$A=A^* \Rightarrow (Ax\,|\,x) = (x\,|\,A^*x) = (x\,|\,Ax) = \overline{(Ax\,|\,x)} \Rightarrow (Ax\,|\,x) \in \mathbf{R}$

ad 2)
$\mathbf{K}=\mathbf{C}$ and $(Ax\,|\,x) \in \mathbf{R}$
$(A^*x\,|\,x) = \overline{(x\,|\,A^*x)} = \overline{(Ax\,|\,x)} = (Ax\,|\,x) \in \mathbf{R}$ for all $x \in H$

Consequently $((A-A^*)x\,|\,x) = (x\,|\,(A^*-A)x) = \overline{((A^*-A)x\,|\,x)} = \overline{(A^*x\,|\,x)} - \overline{(Ax\,|\,x)}$
$= (A^*x\,|\,x) - (Ax\,|\,x) = -((A-A^*)x\,|\,x)$
$\Rightarrow ((A^*-A)x\,|\,x)=0$ for all $x \in H$
$\Rightarrow A^*=A$ by the following result q.e.d.

(2.20) **Theorem:** Let H be a *complex* Hilbert space, let $S,T \in \mathcal{L}_b(H,H)$. Then
1) $(Tx\,|\,x)=0$ for all $x \in H$ implies $T=0$
2) $(Tx\,|\,x)=(Sx\,|\,x)$ for all $x \in H$ implies $S=T$

<u>Proof:</u>

ad 1)
To show: $(Tx\,|\,x)=0$ implies $(Tx\,|\,y)=0$ for all $x,y \in H$. Let $x,y \in H$:
$0=(T(x+y)\,|\,x+y)=\underbrace{(Tx\,|\,x)}_{=0}+(Ty\,|\,x)+(Tx\,|\,y)+\underbrace{(Ty\,|\,y)}_{=0}=(Ty\,|\,x)+(Tx\,|\,y)$
$\Rightarrow (Ty\,|\,x) = -(Tx\,|\,y)$
$\Rightarrow (T(iy)\,|\,x) = -(Tx\,|\,iy)$
$\Rightarrow 0 = (T(iy)\,|\,x) + (Tx\,|\,iy) = i\cdot(Ty\,|\,x) - i\cdot(Tx\,|\,y)$
$\Rightarrow (Ty\,|\,x) = (Tx\,|\,y)$
$\Rightarrow (Tx\,|\,y) = -(Ty\,|\,x) = -(Tx\,|\,y)$
$\Rightarrow (Tx\,|\,y) = 0$ for all $x,y \in H$
$\Rightarrow 0 = (Tx\,|\,Tx) = \|Tx\|^2 \Rightarrow Tx=0;\ x \in H$

ad 2)
simple q.e.d.

$(Tx\,|\,x)$ is called the **quadratic form** of T

(2.21) **Theorem:** Let H be a Hilbert space over $\mathbf{K} \in \{\mathbf{R},\mathbf{C}\}$. Let A be a selfadjoint operator on H. Then $\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=1} |(Ax \mid x)|$

Proof:

$|(Ax \mid x)| \le \|Ax\| \cdot \|x\| \le \|A\| \cdot \|x\|^2$
$\Rightarrow \sup_{\|x\|=1} |(Ax \mid x)| \le \|A\|$

Define n(A):= $\sup_{\|x\|=1} |(Ax \mid x)|$, then n(A)< ∞; let λ>0 then

$$4\cdot\|Ax\|^2 = \left(A\left(\lambda\cdot x - \frac{1}{\lambda} A\cdot x\right) \middle| \lambda\cdot x + \frac{1}{\lambda}\cdot A\cdot x \right) - \left(A\left(\lambda\cdot x - \frac{1}{\lambda} A\cdot x\right) \middle| \lambda\cdot x - \frac{1}{\lambda}\cdot A\cdot x \right)$$
$$\le n(A)\cdot\left[\left\|\lambda\cdot x + \frac{1}{\lambda}\cdot A\cdot x\right\|^2 + \left\|\lambda\cdot x - \frac{1}{\lambda}\cdot A\cdot x\right\|^2\right] \underset{\text{identity}}{\overset{\text{parallelogram}}{\le}} 2\cdot n(A)\cdot\left[\lambda\cdot\|x\|^2 + \frac{1}{\lambda^2}\cdot\|A\cdot x\|\right]$$

case 1: $\|Ax\|=0 \Rightarrow 0 \le n(A)\cdot\|x\|^2$
$\Rightarrow \|Ax\| \le n(A)$ for all $x \in H$, $\|x\|=1$

case 2: $\|Ax\| \neq 0$, take $\lambda = \frac{\|Ax\|}{\|x\|}$
$\Rightarrow 4\cdot\|Ax\|^2 \le 2\cdot n(A)\cdot[\|Ax\|\cdot\|x\| + \|Ax\|\cdot\|x\|] = 4\cdot n(A)\cdot\|Ax\|\cdot\|x\|$
$\Rightarrow \|Ax\| \le n(A)\cdot\|x\| \Rightarrow \|A\| \le n(A)$ q.e.d.

This theorem shows that the quadratic form of a selfadjoint operator on a Hilbert space determines the norm of this operator.
A final simple, but useful result about selfadjoint operators is given in

(2.22) **Theorem:** Let H be a *complex* Hilbertspace. Let $T \in \mathfrak{L}_b(H,H)$. Then there exist uniquely determined (bounded) selfadjoint opeartors $A \in \mathfrak{L}_b(H,H)$, $B \in \mathfrak{L}_b(H,H)$, so that $T=A+i\cdot B$. The operator T is normal if and only if AB=BA.

Proof:

Define $A := \frac{1}{2}\cdot(T+T^*)$, $B := \frac{1}{2\cdot i}\cdot(T - T^*) = -\frac{i}{2}\cdot(T - T^*)$
Then A=A* and B=B*

If $T=C+D\cdot i \Rightarrow T^*=C^*-i\cdot D^*=C-i\cdot D$
$\Rightarrow T+T^*=2\cdot C \Rightarrow C=\frac{1}{2}\cdot(T+T^*) = A$
$T-T^*=2\cdot i\cdot D \Rightarrow D=\frac{1}{2\cdot i}\cdot(T - T^*) = B$

If T is normal

$$\Rightarrow A\cdot B=\frac{1}{2}\cdot(T+T^*)\cdot\frac{1}{2\cdot i}\cdot(T-T^*)=\frac{1}{4\cdot i}\cdot\left(T^2+T^*\cdot T-T\cdot T^*-(T^*)^2\right)=\frac{1}{4\cdot i}\cdot\left(T^2-(T^*)^2\right)$$

$$B\cdot A=\frac{1}{2\cdot i}\cdot(T-T^*)\cdot\frac{1}{2}\cdot(T+T^*)=\frac{1}{4\cdot i}\cdot\left(T^2-T^*\cdot T+T\cdot T^*-(T^*)^2\right)=\frac{1}{4\cdot i}\cdot\left(T^2-(T^*)^2\right)$$

$\Rightarrow A\cdot B=B\cdot A$

If $A\cdot B=B\cdot A$

$\Rightarrow T\cdot T^*=(A+i\cdot B)\cdot(A^*-i\cdot B^*)=(A+i\cdot B)\cdot(A-i\cdot B)=A^2+i\cdot B\cdot A-i\cdot B\cdot A+B^2=\ldots=T^*\cdot T$ q.e.d.

Let us pay our attention to <u>normal operators</u>. A result similar to (2.19) is given by

(2.23) **Theorem:** (*compare 2.19*):

Let H be a Hilbert space over $\mathbf{K}\in\{\mathbf{R},\mathbf{C}\}$. Let $A\in\mathcal{L}_b(H,H)$

1) If A is normal then $\|Ax\|=\|A^*x\|$ for all $x\in H$
2) If $\mathbf{K}=\mathbf{C}$ and if $\|Ax\|=\|A^*x\|$ for all $x\in H$, then A is normal

<u>Proof</u>:

ad 1)

$A^*A=AA^*$ implies

$\|Ax\|^2=(Ax\,|\,Ax)=(x\,|\,A^*Ax)=(x\,|\,AA^*x)=(x\,|\,A^{**}A^*x)=(A^*x\,|\,A^*x)=\|A^*x\|^2$

ad 2)

If $\mathbf{K}=\mathbf{C}$ and $\|Ax\|=\|A^*x\|$ for all $x\in H$, then $\|Ax\|^2=(Ax\,|\,Ax)=(A^*Ax\,|\,x)$ and $\|A^*x\|^2=(A^*x\,|\,A^*x)=(AA^*x\,|\,x)$, hence $(AA^*x\,|\,x)=(A^*Ax\,|\,x)$ for $x\in H$.

By theorem (2.19) we have $AA^*=A^*A$ q.e.d.

(2.24) **Theorem:** Let H be a Hilbert space over $\mathbf{K}\in\{\mathbf{R},\mathbf{C}\}$. Let $A\in\mathcal{L}_b(H,H)$. Then the following statements are equivalent:

1) A is unitary
2) A is surjective (=onto) and $\|Ax\|=\|x\|$, $x\in H$
3) A is onto and one-to-one, and $A^{-1}=A^*$

<u>Proof:</u>

$1)\Rightarrow 2)$

$\|Ax\|^2=(Ax\,|\,Ax)=(A^*Ax\,|\,x)=(x\,|\,x)=\|x\|^2$

$AA^*=Id=A^*A$ given „onto“

$2) \Rightarrow 3)$
$x,y \in H$

$$2) \Rightarrow \left\|\frac{Ax+Ay}{2}\right\|^2 - \left\|\frac{Ax-Ay}{2}\right\|^2 = \left\|A\cdot\left(\frac{x+y}{2}\right)\right\|^2 - \left\|A\cdot\left(\frac{x-y}{2}\right)\right\|^2 = \left\|\left(\frac{x+y}{2}\right)\right\|^2 - \left\|\left(\frac{x-y}{2}\right)\right\|^2$$

$\Rightarrow (Ax\,|\,Ay) = (x\,|\,y)$ by using polarization identity
$\Rightarrow (A^*Ax\,|\,y) = (Ax\,|\,Ay) = (x\,|\,y) \Rightarrow A^*A = Id_H$

$\|Ax\| = \|x\|$ implies that A is one-to-one
$\Rightarrow A^{-1}$ exists (since A is onto)
$\Rightarrow A^{-1} = A^*$

$3) \Rightarrow 1)$
$A^{-1} = A^*$
$\Rightarrow Id_H = A^{-1}A = A^*A$ and $Id_H = AA^{-1} = AA^*$ q.e.d.

It is perhaps worth to mention explicitly that a unitary operator preserves inner products, i.e. $(Ax\,|\,Ay) = (x\,|\,y)$, $x,y \in H$. *In particular, every unitary operator is an isometry.*
Note that in defining a unitary operator it is not enough to require only that either $A^*A=Id_H$ or $AA^*=Id_H$. *For instance, the shift operator on* l_2 *is an operator such that* $A^*A=Id_H$. *However it is not unitary since it is not onto (=surjective).*
The general reason for each of the equations $A^*A=Id_H$ *or* $AA^*=Id_H$ *alone not being enough to imply* A *to be unitary is as follows: for infinite-dimensional Hilbert spaces* H *a bounded linear operator* A *on* H *must be bijective (=onto and one-to-one) in order to be sure that* A^{-1} *exists and is a bounded linear operator on* H. *If* H *is a finite dimensional Hilbert space then injectivity and surjectivity of linear operators are equivalent.*
Why should selfadjoint and normal operators be so interesting? There is a large class of very special and simple bounded selfadjoint operators, namely the projection operators which will be studied in the next section. Every bounded selfadjoint operator on H *(in fact even every unbounded selfadjoint operator on* H*) can be built up in some sense with the help of projections. This is the central result of spectral theory. The point here is that every bounded linear operator* A *on* H *is a linear combination of two bounded selfadjoint operators* B *and* C *an* H *which even commute, i.e. satisfy* BC=CB, *if* A *is normal (theorem (2.21))*

§ 3 Projection operators

An important class of operators on Hilbert spaces is that of the orthogonal projections

(2.25) **Definition:** Let H be a Hilbert space over $\mathbf{K} \in \{\mathbf{R},\mathbf{C}\}$. A bounded linear operator P on H is called
1) a **projection** if $P^2 = P$
2) an **orthogonal projection** if $P^2 = P$ and $P^* = P$

Note that the range Ran(P)=P(H) *of a projection on a Hilbert space* H *always is a closed linear subspace of* H *on which* P *acts like the identity. If in addition* P *is orthogonal, then* P *acts like the zero operator on* $(\mathrm{Ran}(P))^\perp$. *If* x=y+z *with* $y \in \mathrm{Ran}(P)$ *and* $z \in \mathrm{Ran}(P)^\perp$, *is the decomposition guaranteed by the projection theorem, then* Px=y. *Thus the projection theorem sets up a one-to-one correspondence between orthogonal projections and closed linear subspaces. This correspondence will be clarified in the following theorem*

(2.26) **Theorem:** Let H be a Hilbert space $M \subseteq H$ be a subset. Then the following statements are equivalent:
1) M is a closed linear subspace
2) There exists a unique orthogonal projection $P \in \mathcal{2}_b(H,H)$ with Ran P = P(H) = M

Proof:

1) $\Rightarrow$ 2)
Define $P_M: H \to M$ by $P_M x := x_M$, where x_M is the unique element of best approximation in M with respect to $x \in H$. P_M is linear and $P_M(x)=x$ for $x \in M \Rightarrow P_M^2(x)=P_M(x)$, i.e. $P_M^2=P_M$, also $\|P_M\|=1$;
take $x \in H$, $y \in H$, $x = x_M + x_{M^\perp}$, $y = y_M + y_{M^\perp}$.
To show: $(P_M x \mid y) = (x \mid P_M y)$
$(P_M x \mid y) = \big(P_M(x_M + x_{M^\perp}) \mid y_M + y_{M^\perp}\big) = (x_M \mid y_M + y_{M^\perp}) = (x_M \mid y_M) + (x_M \mid y_{M^\perp})$
$= (x_M \mid y_M) = (x_M \mid y_M) + (x_{M^\perp} \mid y_M) = (x_M + x_{M^\perp} \mid P_M y) = (x \mid P_M y)$
$\Rightarrow P_M^* = P_M$.

Uniqueness: take $Q \in \mathcal{2}_b(H,H)$ with $Q^2 = Q = Q^*$ and Ran Q = M. To show: $P_M = Q$.
1) $\{x \in H: P_M x = x\} = \mathrm{Ran}\, P_M = M = \mathrm{Ran}\, Q = \{x \in H: Qx=x\}$
since $\{x \in H: P_M x=x\} \subseteq \mathrm{Ran}\, P_M = M$ and if $x \in \mathrm{Ran}\, P_M$ then $x = P_M z$ (for some element z)
$\Rightarrow P_M x = P_M^2 z = P_M z = x$
$\Rightarrow \mathrm{Ran}\, P_M \subseteq \{x \in H: P_M x=x\}$
2) $\{x \in H: P_M x=0\} = (\mathrm{Ran}\, P_M)^\perp = M^\perp = (\mathrm{Ran}\, Q)^\perp = \{x \in H: Qx=0\}$
$P_M x=0 \Rightarrow (P_M y \mid x) = (y \mid P_M x) = 0$ for all $y \in H \Rightarrow x \in (\mathrm{Ran}\, P_M)^\perp$
if $x \in M^\perp = (\mathrm{Ran}\, P_M)^\perp \Rightarrow (y \mid P_M x) = (P_M y \mid x) = 0$ for al $y \in H$ put $y := P_M x$
$\Rightarrow 0 = (P_M x \mid P_M x) = \|P_M x\|^2 \Rightarrow P_M x=0$
Let $x \in H$, $x = x_M + x_{M^\perp} \Rightarrow P_M x = P_M(x_M + x_{M^\perp}) = x_M = Q(x_M + x_{M^\perp}) = Qx$

2) $\Rightarrow$ 1)
Ran P = M is a linear subspace; to show: M is closed by showing $M=N^\perp$ for some set $N\subseteq H$ (theorem 1.15)
P is bounded , linear $\Rightarrow$ Id_H-P is linear and bounded,
define $N:=(Id_H-P)(H) = \text{Ran}(Id-P)$.
If $x \in M = \text{Ran } P \Rightarrow x = Pu$ for some $u \in H$; if $y \in N = \text{Ran}(Id-P) \Rightarrow y=(Id-P)v$ for some $v \in H$

$$\Rightarrow (x\,|\,y) = \big(Pu\,|\,(Id-P)v\big) \overset{P=P^*}{=} \big(u\,|\,P(Id-P)v\big) = (u\,|\,Pv-P^2v) \overset{P=P^2}{=} (u\,|\,Pv-Pv) = 0$$

$M\subseteq N^\perp$: Conversely if $x \in N^\perp$, then for any $y \in H$: $(Id-P)y \in N$
$\Rightarrow 0 = \big(x\,|\,(Id-P)y\big) = (x\,|\,y-Py) = (x\,|\,y) - (x\,|\,Py) = (x\,|\,y) - (Px\,|\,y) = \big((Id-P)x\,|\,y\big)$
in particular: $y = (Id-P)x \Rightarrow \|(Id-P)x\|=0$
$\Rightarrow x=Px \in \text{Ran } P = M \Rightarrow M=N^\perp$. q.e.d.

It is because of this characterization that one speaks of <u>*the orthogonal projection* P *onto the closed linear subspace* M⊆H</u>.
It is easily verified that a bounded linear operator P *on a Hilbert space is an orthogonal projection if and only if* Id_H-P *is an orthogonal projection, and that for an orthogonal projection* P *always* $\text{Ran}(P)=\big(\text{Ran}(Id_H-P)\big)^\perp$ *and* $\big(\text{Ran}(Id_H-P)\big)=\big(\text{Ran}(P)\big)^\perp$. *These observations combined with the orthogonal decomposition theorem (1.22) immediately yield*

(2.27) **Theorem:** If $P \in \mathcal{L}_b(H,H)$ is an orthogonal projection on H, then $H = \text{Ran}P \oplus (\text{Ran } P)^\perp = \text{Ran } P \oplus \text{Ran}(Id-P)$ and $\text{Ran}(Id-P) \perp \text{Ran } P$.

We now study the question under what circumstances the product, sum and difference of two orthogonal projections again is an orthogonal projection. We start with the <u>*product of two orthogonal projections*</u>

(2.28) **Theorem:** Let P and Q be orthogonal projections on H. Then the following statements are equivalent:
1) PQ = QP
2) PQ is an orthogonal projection
3) QP is an orthogonal projection

<u>Proof:</u>

1) $\Rightarrow$ 2)
$(PQ)^* = Q^*P^* = QP = PQ$ and $PQPQ = PPQQ = P^2Q^2 = PQ$

2) $\Rightarrow$ 1)
$QP = Q^*P^* = (PQ)^* = PQ$

the same with 3) q.e.d.

(2.29) **Corollary:** If PQ is an orthogonal projection, then PQ is an orthogonal projection onto

$M \cap N$ if $M=P(H)$, $N=Q(H)$.

Let us turn to the *<u>sum of two orthogonal projections</u>*

(2.30) **Theorem:** Let P and Q be orthogonal projections on H with $M:=P(H) = \text{Ran } P$ and $N:=Q(H) = \text{Ran } Q$. Then the following statements are equivalent:
1) $M \perp N$
2) $P(N) = \{0\}$
3) $Q(M) = \{0\}$
4) $PQ = 0$
5) $QP = 0$
6) P+Q is an orthogonal projection

<u>Proof:</u>

$1) \Rightarrow 2)$
$N \perp M \Rightarrow N \subseteq M^\perp \Rightarrow P(N) = \{0\}$ (see 2.27)
$2) \Rightarrow 4)$
$x \in H \Rightarrow Qx \in N \Rightarrow PQx=0$
$4) \Rightarrow 6)$
$(P+Q)^* = P^*+Q^* = P+Q$ and $(P+Q)^2 = (P+Q)(P+Q) = P^2+QP+PQ+Q^2 = P^2+Q^2 = P+Q$
$6) \Rightarrow 1)$
$$x \in M \Rightarrow \|x\|^2 \geq \|(P+Q)x\|^2 = \big((P+Q)x \,|\, (P+Q)x\big) = \big((P+Q)^2 x \,|\, x\big) + \big((P+Q)x \,|\, x\big)$$
$$= (Px \,|\, x)(Qx \,|\, x) \overset{x \in M}{=} (x \,|\, x) + (Qx \,|\, x) = \|x\|^2 + (Q^2 x \,|\, x) = \|x\|^2 + (Qx \,|\, Qx)$$
$$= \|x\|^2 + \|Qx\|^2 \Rightarrow \|Qx\|=0$$
$\Rightarrow$ for $y \in N$, $x \in M$ $(x \,|\, y) = (x \,|\, Qy) = (Qx \,|\, y) = 0 \Rightarrow M \perp N$ q.e.d.

(2.31) **Corollary:** If P+Q is an orthogonal projection, then P+Q is an orthogonal projection onto M+N with $M=P(H)=\text{Ran } P$ and $N=Q(H)=\text{Ran } Q$.

In view of the equivalence of (1), (4) and (6), the projections P *and* Q *are called orthogonal to each other (denoted by* $P \perp Q$*) if* $PQ=0$, *or, equivalently*, $QP=0$. *Finally we come to the* *<u>dfference of two orthogonal projections</u>* *which in particular takes care of the orthogonal projection* Id-P *if* P *is an orthogonal projection.*

(2.32) **Theorem:** Let P and Q be orthogonal projection on H with $M=P(H)=\text{Ran } P$ and $N=Q(H)=\text{Ran } Q$. Then the following statements are equivalent:
1) $M \subseteq N$
2) $QP = P$
3) $PQ = P$
4) $\|Px\| \leq \|Qx\|$ for all $x \in H$
5) $\big((Q-P)x \,|\, x\big) \geq 0$ for all $x \in H$
6) $(Qx \,|\, x) \geq (Px \,|\, x)$ for $x \in H$
7) Q-P is an orthogonal projection

<u>Proof:</u>

1) $\Rightarrow$ 2)
If $x \in H$ then $Px \in M \subseteq N$ and $QPx=Px$
2) $\Rightarrow$ 3)
$PQ = P^*Q^* = (QP)^* = P^* = P$
3) $\Rightarrow$ 7)
trivial: $(Q-P)^* = Q-P$ and $(Q-P)^2 = (Q-P)$
7) $\Rightarrow$ 5) ($\Leftrightarrow$ 6)
$\big((Q-P)x \mid x\big) = \big((Q-P)^2x \mid x\big) = \big((Q-P)x \mid (Q-P)x\big) = \|(Q-P)x\|^2 \geq 0$
5) $\Rightarrow$ 4)
$$\|Qx\|^2 - \|Px\|^2 = (Qx \mid Qx) - (Px \mid Px) = (Q^2x \mid x) - (P^2x \mid x) = (Qx \mid x) - (Px \mid x) = \big((Q-P)x \mid x\big) \geq 0$$
4) $\Rightarrow$ 1)
$x \in M \Rightarrow \|x\| = \|Px\| \leq \|Qx\| = \|x\| \Rightarrow \|x\|=\|Qx\|$ if $x\neq 0$
then $\|x\|^2 = \| x_N \|^2 + \| x_{N^\perp} \|^2 = \|Qx\|^2 + \| x_{N^\perp} \|^2$
$\Rightarrow \| x_{N^\perp} \| = 0 \Rightarrow x_{N^\perp} = 0$
$\Rightarrow x = x_N \in N$ q.e.d.

(2.33) **Corollary:** If Q-P is an orthogonal projection, then Q-P is an orthogonal projection onto $N-M := N \cap M^\perp$.

Proof:

$Q-P = Q-QP = Q(Id-P)$ is the orthogonal projection onto $N \cap M^\perp$ by corollary (2.29) q.e.d.

§ 4 Baire's Category Theorem and Banach-Steinhaus-Theorem

There probably is no theorem in functional analysis which is more boring on one side but more powerful on the other side. In order to be able to present Baire's category theorem in a general form we need some notation

(2.34) **Definition:**

1) Let $X \neq \emptyset$ be a set. A mapping $d: X \times X \to \mathbf{R}$ is called a **metric** if it has the following properties:
   (M1) $d(x,y) \geq 0$ for $x,y \in X$
   (M2) $d(x,y)=0$ if and only if $x=y$
   (M3) $d(x,y)=d(y,x)$
   (M4) $d(x,y) \leq d(x,z)+d(z,y)$
   (X,d) is called a **metric space**
2) A metric space (X,d) is called **complete** if every sequence $(x_n)_{n \in \mathbf{N}} \subseteq X$ with $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$ converges, i.e.: there is an element $x \in X$ so that $\lim\limits_{n\to\infty} d(x_n, x) = 0$
3) Let M be a subset of (X,d). A point $x \in X$ is called a **limit point of M**, if for every $r>0$ $B_r(x) \cap (M \setminus \{x\}) \neq \emptyset$ where $B_r(x):=\{y \in (X,d): d(x,y)<r\}$
4) $M \subseteq (X,d)$ is called **closed** if M contains all its limit points ($\overline{M} = M$)

If $(X,\|.\|)$ is given, then $d(x,y):=\|x-y\|$ is a metric on X.

(2.35) **Theorem:** (**Baire's Category Theorem**):
Let (X,d) be a complete metric space, let $F_n \subseteq X$ be closed subsets of X, $n \in \mathbf{N}$, so that $X = \bigcup\limits_{n\in \mathbf{N}} F_n$. Then at least one F_n contains a closed ball (hence an open ball).

Proof:

Idea: Suppose $X = \bigcup\limits_{n\in \mathbf{N}} F_n$, $F_n = \overline{F_n}$ and each subset F_n contains no closed ball.

Let $n \in \mathbf{N}$, if $B_r(x_0)=\{y \in X: d(x_0,y) \leq r\}$, then $\overline{B}_{r/2}(x_0)$ contains an element $x_n \notin F_n$; since F_n is closed, there exists $0<r_n<\frac{r}{2}$ so that $\overline{B}_{r_n}(x_n) \cap F_n = \emptyset$ and $\overline{B}_{r_n}(x_n) \subseteq \overline{B}_r(x_0)$ since

$d(y,x_0) \leq d(y,x_n) + d(x_n,x_0) \leq r_n + \frac{r}{2} < r$ for $y \in \overline{B}_{r_n}(x_n)$.

We start with $\overline{B}_1(x_0)$ an arbitrary closed ball of radius 1. We find $\overline{B}_{r_1}(x_1) \subseteq \overline{B}_1(x_0)$ with $\overline{B}_{r_1}(x_1) \cap F_1 = \emptyset$ and $r_1<\frac{1}{2}$.

We also find $\overline{B}_{r_2}(x_2) \subseteq \overline{B}_{r_1}(x_1)$ with $\overline{B}_{r_2}(x_2) \cap F_2 = \emptyset$ and $r_2<\frac{1}{2^2}$. By induction we find a sequence $(\overline{B}_{r_j}(x_j))_{j \in \mathbf{N}}$ of closed balls with $\overline{B}_{r_{j+1}}(x_{j+1}) \subseteq \overline{B}_{r_j}(x_j)$, $\overline{B}_{r_j}(x_j) \cap F_j = \emptyset$, $r_j<\frac{1}{2^j}$.

Now $(x_j)_j$ is a Cauchy sequence in X since for $n,m \geq N$ we have $x_n,x_m \in \overline{B}_{r_N}(x_N)$, $d(x_n,x_m) \leq 2^{-N} + 2^{-N} = 2^{1-N} \to 0$ for $N \to \infty$.

Since (X,d) is complete, we find $x \in X$ with $x=\lim_{j\to\infty} x_j$ $\left(\lim_{j\to\infty} d(x,x_j)=0\right)$. Since $x_n \in B_{r_N}(x_N)$ for $n\geq N$, we have $x \in \overline{B}_{r_N}(x_N) \subseteq \overline{B}_{r_{N-1}}(x_{N-1})$

$\Rightarrow x \notin F_{N-1}$ for any N, contradicting $X=\bigcup_{j\in \mathbf{N}} F_j$. q.e.d.

One of the most important consequences is the principle of uniform boundedness which we state in a special form

(2.36) **Theorem:** (**Banach-Steinhaus, principle of uniform boundedness**):
Let X be a Banch space, Y be a normed linear space. Let $(A_j)_j$ be a sequence of bounded linear operators, $A_j \in 2_b(X,Y)$. If for each $x \in X$ there exists M with $\sup_j \|A_j x\| \leq M_X$, then there exists M>0 so that $\sup_j \|A_j\| \leq M$.

Proof:

$n \in \mathbf{N}$, define $F_n:=\{x \in X: \sup_j \|A_j x\| \leq n\}$, then $X=\bigcup_{n\in\mathbf{N}} F_n$. Since A_n is bounded, F_n is closed.
At least one set F_n contains a closed (and hence open) ball. There exist $N \in \mathbf{N}$, $y \in F_N$ and $\varepsilon>0$, so that $\|x-y\|< \varepsilon$ implies $x \in F_n$.
-y has the same property. Since F_N is convex, every $n \in X$ with $\|u\|\leq \varepsilon$ implies

$$u = \left(\frac{1}{2}(u+y)+\frac{1}{2}(u-y)\right) \in \frac{1}{2}(F_N + F_N) \subseteq F_N,$$ i.e. $\|A_j u\|\leq N$ for all $j \in \mathbf{N}$

$\Rightarrow \sup_{j\in\mathbf{N}} \|A_j\| \leq \frac{N}{\varepsilon}$. q.e.d.

(2.37) **Corollary:** Let X be a Banach-space, Y be a normed linear space. If $(A_j)_j$ is a sequence of operators $A_j \in 2_b(X,Y)$, so that for each $x \in X$ there exists $y_x \in Y$ with $\lim_{j\to\infty} A_j x = y_x$, then the formula $A: X\to Y$ with $Ax:=y_x$ defines a bounded linear operator on X with $\|A\|\leq \liminf_{j\to\infty} \|A_j\|$.

Proof:

A is linear (easy to verify), $(A_j x)_j$ is a Cauchy sequence in Y for each $x \in X$. By the triangle inequality for the norm, $(\|A_j x\|)_j$ is a Cauchy sequence in $\mathbf{R}$ which converges to $\|y_x\|=\|Ax\|$.
Therefore there exists M>0 with $\sup_j \|A_j\| \leq M$.

Hence $\|Ax\|=\lim_{j\to\infty}\|A_j x\| \leq \liminf_{j\to\infty}\|A_j\| \cdot \|x\| \leq M \cdot \|x\|$ q.e.d.

Chapter 3: Spectral analysis of bounded linear operators

A linear operator A *from a finite-dimensional linear space* E *into itself can always be represented by some matrix* $(a_{jk})_{j,k}$ *and the coefficient vectors of the elements of* E*; the matrix* (a_{jk}) *and the coefficient vectors, of course, depend on the choice of a basis in* E*. For certain types of such operators one can choose the basis in* E *in such a way that the matrix* (a_{ij}) *is a diagonal matrix. This means that for example in the case where* E *is* $\mathbf{C^n}$*, one can write*

$Ax = \sum_{j=1}^{m} (x \mid e_j)\lambda_j e_j$ *for* $x \in \mathbf{C^n}$ *where* $e_1, e_2, \ldots, e_m$ *are certain linearly independent orthonormal vectors in* $\mathbf{C^n}$*,* $\lambda_1, \ldots, \lambda_m$ *are the (not necessarily distinct) non-zero diagonal entries of the matrix* (a_{ij})*, and* $m \le n$*. This is the case, for instance, whenever the matrix corresponding to* A *is equal to its conjugate transpose, i.e. whenever* A *is selfadjoint.*
In this chapter a counterpart of this result for certain selfadjoint linear operators on an arbitrary (infinite-dimensional) Hilbert space over **C** *will be established. Throughout this chapter any Hilbert space considered is supposed to be a* <u>*complex*</u> *Hilbert space*

§1 The order relation for bounded selfadjoint operators

Recall that a bounded linear operator A *on a Hilbert space* H *is selfadjoint if and only if* $(Ax \mid x) \in \mathbf{R}$ *for all* $x \in H$ *(theorem (2.19)) and that for a selfadjoint operator* A

$\|A\| = \sup_{\substack{x \in H \\ \|x\|=1}} (Ax \mid x) = \sup_{\|x\|=\|y\|=1} \left|(Ax \mid y)\right|$ *(theorem (2.21))*

With the help of the quadratic form $(Ax \mid x)$ *we introduce an order relation on the set of all bounded selfadjoint operators on* H.

(3.1) **Definition:** Let A be a bounded linear selfadjoint operator on H. A is called
1) **positive** if $(Ax \mid x) \ge 0$ for each $x \in H$ (we write $A \ge 0$)
2) **larger or equal to B** if A-B is positive, i.e. $(Ax \mid x) \ge (Bx \mid x)$ (we write $A \ge B$)

Examples:

1) P_M: $H \to M$ orthogonal projection, then $0 \le P_M \le Id_H$ since
$0 \le \|P_M x\|^2 = (P_M x \mid P_M x) = (P_M x \mid x) \le \|x\|^2 = (x \mid x) = (Id_H x \mid x)$.
$P_M \le Q_N \Leftrightarrow M \subseteq N$
2) $A \in \mathfrak{L}_b(H,H) \Rightarrow A^*A$ and AA^* are positive, A^*A and AA^* are not comparable

The following theorem justifies the use of the familiar notation $\le$ *for the relation between bounded selfadjoint operators*

(3.2) **Theorem:** Let A,B and C be bounded selfadjoint operators on a Hilbert space H. Let $\alpha,\beta \in \mathbf{R}$, then

1) if $A\leq B$ and $B\leq C$, then $A\leq C$
2) if $A\leq B$ and $B\leq A$, then $A=B$
3) if $A\leq B$ and $\alpha\geq 0$, then $A+C\leq B+C$ and $\alpha\cdot A\leq\alpha\cdot B$
4) $A\leq A$
5) If $\alpha\leq\beta$, then $\alpha\cdot A\leq\beta\cdot A$

$[x\,|\,y]=(Ax\,|\,y)$ with $A\geq 0$ is a **semi inner product** (conditions (S2) – (S4) of (1.1), but not (S1))

Because of Cauchy-Schwarz:

(3.3) **Theorem:** Let A be a positive bounded selfadjoint operator on H. Then

1) $([x\,|\,y])^2 \leq [x\,|\,x]\cdot[y\,|\,y]$, i.e. $(Ax\,|\,y)^2 \leq (Ax\,|\,x)\cdot(Ay\,|\,y)$
2) $\|Ax\|^2 \leq \|A\|\cdot (Ax\,|\,x)$

Proof:

ad 2)
$\|Ax\|^4 = (Ax\,|\,Ax)^2 \leq (Ax\,|\,x)\cdot(A^2x\,|\,Ax) \leq (Ax\,|\,x)\cdot\|A\|\cdot\|Ax\|^2$. Divided by $\|Ax\|^2$ the statement is proved. q.e.d.

We want to prove the existence of a special decomposition for operators on a Hilbert space which is analogous to the decomposition $z=|z|\cdot e^{i\cdot\arg(z)}$ *for complex numbers.*
First we must describe a suitable analogue of the positive numbers

(3.4) **Theorem:** Let $(A_j)_j$ and B be bounded selfadjoint operators on H with $A_1 \leq A_2 \leq \ldots \leq A_n \leq \ldots \leq B$. Then there exists a bounded selfadjoint operator A on H, so that $A_n \leq A \leq B$ for all $n \in \mathbf{N}$ and $Ax = \lim_{j\to\infty} A_jx$.

Proof:

$A_1 \leq A_2 \leq \ldots \leq A_n \leq \ldots \leq B \Rightarrow\ B-A_n \geq A_n-A_m \geq 0\ (n\geq m\geq 1)$

$\Rightarrow \|A_n-A_m\| = \sup\limits_{\|x\|\leq 1}\left|((A_n - A_m)x\,|\,x)\right| \leq \sup\limits_{\|x\|\leq 1}\left|((B - A_1)x\,|\,x)\right| = \|B-A_1\|$

$\Rightarrow ((A_jx\,|\,x))_{j\in\mathbf{N}}$ is increasing and has an upper bound, i.e. $((A_jx\,|\,x))_j$ converges for every $x \in H$

$\Rightarrow \|(A_n-A_m)x\|^2 \overset{(3.3)}{\leq} \|(A_n-A_m)\|\cdot((A_n-A_m)x\,|\,x) \leq \|B-A_1\|\cdot((A_n-A_m)x\,|\,x) \underset{n\geq m\geq 1}{\to} 0\ (n,m\to\infty,$ $n\geq m)$

$\Rightarrow (A_jx)_j$ is a Cauchy sequence in H for every $x \in H$, A_jx converges to a unique element y_x for every $x \in H$.

Define A: H→H by $Ax := y_x = \lim_{j\to\infty} A_j x$.

A is linear. Banach-Steinhaus implies, that A is bounded.

$x,y \in H$: $(Ax\,|\,y) = \lim_{j\to\infty}(A_j x\,|\,y) = \lim_{j\to\infty}(x\,|\,A_j y) = (x\,|\,Ay)$

$\Rightarrow$ A is a bounded selfadjoint operator. With $(A_m x\,|\,x) \le \lim_{\substack{n\to\infty\\ n\ge m}}(A_n x\,|\,x) = (Ax\,|\,x) \le (Bx\,|\,x)$

q.e.d.

(3.5) **Lemma:** Let A and B be selfadjoint operators on H. If A and B are positive, then
1) A+B is positive
2) AB is positive if AB=BA

Proof:

ad 2)

AB=BA $\Rightarrow$ AB is selfadjoint (see (2.17)). To show: $AB\ge 0$. Define a sequence $(B_n)_{n\in\mathbf{N}}$ of linear operators by $B_1 := \frac{B}{\|B\|}$, $B_2 := B_1 - B_1^2$, ..., $B_{n+1} := B_n - B_n^2$; $n \in \mathbf{N}$.

$\Rightarrow B_n$ are bounded and selfadjoint (by induction)

$(B_n x\,|\,x) = (B_{n+1}x\,|\,x) + (B_n^2 x\,|\,x) = (B_{n+1}x\,|\,x) + (B_n x\,|\,B_n x) > (B_{n+1}x\,|\,x) \Rightarrow B_n \ge B_{n+1}$

$\Rightarrow B_1 - B_n \le B_1 - B_{n+1}$, to show: $0 \le B_n \le \mathrm{Id}$

$0\le B \Rightarrow 0 \le (Bx\,|\,x) = \|B\|\cdot(B_1 x\,|\,x) \Rightarrow B_1 \ge 0$, $0 \le (Bx\,|\,x) \le \|B\|\cdot\|x\|^2 = \|B\|\cdot(x\,|\,x)$

$\Rightarrow B_1 \le \mathrm{Id}$.

If $0 \le B_k \le \mathrm{Id}$ then $0 \le \big((\mathrm{Id}-B_k)B_k x\,|\,B_k x\big) = \big(B_k^2(\mathrm{Id}-B_k)x\,|\,x\big) \Rightarrow 0 \le B_k^2(\mathrm{Id}-B_k)$

Similarly $(\mathrm{Id}-B_k)^2 B_k \ge 0 \Rightarrow 0 \le B_k^2(\mathrm{Id}-B_k) + (\mathrm{Id}-B_k)^2 B_k = \ldots = B_k - B_k^2 = B_{k+1}$,

also: $0 \le (\mathrm{Id}-B_k) + B_k^2 = \mathrm{Id} - B_{k+1} \Rightarrow 0 \le B_{k+1} \le \mathrm{Id}$.

Since $\sum_{j=1}^{n} B_j^2 = \sum_{j=1}^{n} B_j - B_{j+1} = B_1 - B_{n+1} \le B_1$

$$\Rightarrow \sum_{j=1}^{n}\|B_j x\|^2 = \sum_{j=1}^{n}\big(B_j x\,\big|\,B_j x\big) = \sum_{j=1}^{n}\big(B_j^2 x\,\big|\,x\big) \le \big(B_1 x\,\big|\,x\big)$$

$\Rightarrow \lim_{j\to\infty}\|B_n x\| = 0$ for every x and $\lim_{j\to\infty} B_n x = 0$

$\Rightarrow B_1 x = \sum_{j=1}^{\infty} B_j^2 x$ for all $x \in H$.

$$(ABx\,|\,x) = \|B\|\cdot(AB_1 x\,|\,x) = \|B\|\cdot\sum_{j=1}^{\infty}\big(AB_j^2 x\,\big|\,x\big) \overset{AB_j = B_j A}{=} \|B\|\cdot\sum_{j=1}^{\infty}\big(B_j A B_j x\,\big|\,x\big)$$

$$= \|B\|\cdot\sum_{j=1}^{\infty}\big(AB_j x\,\big|\,AB_j x\big) \ge 0$$

q.e.d.

We now come to a result which is important for the decomposition of a bounded linear operator as a product of a positive (selfadjoint bounded linear) operator and a unitary operator

(3.6) **Lemma:** (**Square root lemma**):
Let $A \in 2_b(H,H)$ be positive (and selfadjoint). Then there exists a unique $B \in 2_b(H,H)$ which is positive (and selfadjoint) and $A=B^2$. Furthermore, any bounded linear operator C with AC=CA commutes with B also.

<u>Proof:</u>

$0 \le (Ax\,|\,x) \le \|A\|\cdot(x\,|\,x) \Rightarrow 0 \le A \le \|A\|\cdot Id$, we assume $\|A\|\le 1$ and $0 \le A \le Id$.

Consider $(B_n)_{n \in \mathbf{N}}$ defined by $B_0=0$, $B_{n+1}=B_n+\frac{1}{2}\cdot(A-B_n^2)$; $n \in \mathbf{N}$.

1) $C \in 2_b(H,H)$ with $CA=AC \Rightarrow B_0C=CB_0$ and if $B_kC=CB_k$:
$B_{k+1}C = B_k+\frac{1}{2}\cdot(A-B_k^2)C = \ldots = CB_{k+1}$
$\Rightarrow B_nC=CB_n$ for all $n \in \mathbf{N_0}$ in particular: $B_nA=AB_n$ for all $n \in \mathbf{N_0}$.

$m \in \mathbf{N}$: $B_mB_0=B_0B_m$ if $B_mB_k=B_kB_m \Rightarrow B_mB_{k+1} = B_m\cdot\left(B_k+\frac{1}{2}\cdot(A-B_k^2)\right) = \ldots = B_{k+1}B_m$
$\Rightarrow B_nB_m = B_mB_n$ for all $n,m \in \mathbf{N_0}$.

2) A and B_0 are selfadjoint. If B_k is selfadjoint
$\Rightarrow (B_{k+1}x\,|\,y) = (B_kx\,|\,y) + \frac{1}{2}\cdot\left((Ax\,|\,y)-(B_k^2x\,|\,y)\right) = (x\,|\,B_{k+1}y) \Rightarrow B_n$ is selfadjoint for all $n \in \mathbf{N_0}$.

3) $0 \le \frac{1}{2}\cdot(Id-B_n)^2+\frac{1}{2}\cdot(Id-A) = Id-B_n-\frac{1}{2}\cdot(A-B_n^2) = Id-B_{n+1}$
$\Rightarrow B_{n+1} \le Id$ for all $n \in \mathbf{N_0}$.
$B_{n-1}B_n = B_nB_{n-1} \Rightarrow \left((Id-B_{n+1})+(Id-B_n)\right)\cdot(B_n-B_{n-1}) = B_{n+1}-B_n \underset{\text{induction}}{\overset{\text{by}}{\Rightarrow}} B_n \le B_{n+1}$ and $B_1 \le Id$
$\Rightarrow 0 \le B_0 \le B_1 \le B_2 \le \ldots \le B_n \le \ldots \le Id$

By (3.4) there exists $B \in 2_b(H,H)$ with $\lim\limits_{n\to\infty} B_nx$ and $B_n \le B \le Id$ for all $n \in \mathbf{N_0}$.
$B_{n+1}=B_n+\frac{1}{2}\cdot(A-B_n^2) \overset{n\to\infty}{\to} B=B+\frac{1}{2}\cdot(A-B^2) \Rightarrow A=B^2$ with AC=CA implying BC=CB

4) *Uniqueness of B*:
$\hat{B} \in 2_b(H,H)$ be positive and $\hat{B}^2 = A$ and . B_n are positive and commute with B
$\Rightarrow B_n$ commute with $\hat{B}$. Take $x \in H$, define $y:= (B-\hat{B})x$
$\Rightarrow (By\,|\,y) + (\hat{B}y\,|\,y) = \left((B+\hat{B})\,y\,|\,y\right) = \left((B^2-\hat{B}^2)x\,|\,y\right) = 0$ since $(By\,|\,y)\ge 0$ and $(\hat{B}y\,|\,y)\ge 0 \Rightarrow (By\,|\,y) = (\hat{B}y\,|\,y) = 0$

We can find positive R with RB=BR and $B=R^2$

$\Rightarrow 0 = (By \mid y) = (R^2 y \mid y) = \|Ry\|^2 \Rightarrow Ry=0$

$\Rightarrow 0 = R^2 y = By$; also $\hat{B}y = 0$

B , $\hat{B}$ and $B-\hat{B}$ are selfadjoint

$\Rightarrow \|Bx-\hat{B}x\|^2 = \big((B-\hat{B})\,x \mid (B-\hat{B})\,x\big) = \big((B-\hat{B})\,x \mid y\big) = \big(x \mid (B-\hat{B})\,y\big) = 0$

$\Rightarrow Bx = \hat{B}x$ q.e.d.

(3.7) **Corollary:** Let A be a bounded linear operator on a Hilbert space H. For A*A there exists a unique positive operator $|A| \in 2_b(H,H)$ so that $|A|^2 = AA^*$

$|A|$ is often called the **absolute value of A** $\in 2_b(H,H)$, we also write $|A| := \sqrt{A^*A}$

In definition (1.35) we defined what is meant by a linear isometric operator in a Hilbert space. If $A: D \subseteq H \to H$ *is an isometrix linear operator, then in particular* $\|Ax\|=\|x\|$ *for all* $x \in D$. *This implies that* $\ker(A):=\{y \in H: Ay=0\}=\{0\}$ *and hence* $\overline{D} = (\ker(A))^{\perp}$. *Also an isometric linear operator* A *in* H *does not have to be onto. With these remarks in mind we define*

(3.8) **Definition:** A bounded linear operator $U \in 2_b(H,H)$ on a Hilbert space H is called a **partial isometry** if $\|Ux\|=\|x\|$ for all $x \in (\mathrm{Ker}(U))^{\perp}$.

If U is a partial isometry $\Rightarrow H = \mathrm{Ker}(U) \oplus (\mathrm{Ker}(U))^{\perp} = \mathrm{Ran}(U) \oplus (\mathrm{Ran}(U))^{\perp}$

$U: (\mathrm{Ker}(U))^{\perp} \to \mathrm{Ran}(U)$ is a unitary operator

U* is a partial isometry: $U^*: \mathrm{Ran}(U) \to (\mathrm{Ker}(U))^{\perp}$

We now come to the announced decomposition

(3.9) **Theorem:** (**polar decomposition**):
Let $A \in 2_b(H,H)$ be an operator on a Hilbert space H. Then there exists a partial isometry U with the following properties:

1) $A=U|A|$
2) $U: \overline{\mathrm{Ran}(|A|)} \to \overline{\mathrm{Ran}(A)}$
3) U is unitary and an isometry on $\overline{\mathrm{Ran}(|A|)}$
4) $\mathrm{Ker}(A) = \mathrm{Ker}(U)$ and $\mathrm{Ran}(U) = \overline{\mathrm{Ran}(A)}$
5) $|A| = U^*A$
6) $A^* = U^*|A^*|$
7) $|A^*| = U|A|U^*$
8) U is uniquely determined in the following sense: for any positive operator $B \in 2_b(H,H)$ and any unitary operator $V: \overline{\mathrm{Ran}(B)} \to \mathrm{Ran}(A)$ with $A=VB$ we have $V=U$ and $B=|A|$

<u>Proof:</u>

ad 1) – 3)

$\| |A|x\|^2 = (|A|x \,|\, |A|x) = (|A|^2 x \,|\, x) = (A^*Ax|x) = (Ax|Ax) = \|Ax\|^2$

$\Rightarrow \| |A|x\| = \|Ax\|$; $x \in H$

Define U_0: Ran$|A| \to$ Ran(A) by $U_0(|A|x) := Ax$ for $x \in$ Ran(A), then U_0 is linear and onto (=surjective)

also: $\|U_0(|A|x)\| = \|Ax\| = \| |A|x\|$ for $x \in H$.

If $|A|x = |A|y \Rightarrow 0 = |A|^2 (x-y) = A^*A\,(x-y) \Rightarrow Ax=Ay$

If $Ax = Ay \Rightarrow 0 = A^*A\,(x-y) = |A|^2 (x-y) \Rightarrow 0 = (|A|^2 (x-y) \,|\, x-y) = (|A|(x-y) \,|\, |A|(x-y))$

$= \| |A|(x-y)\|$

U_0 is one-to-one (=injective) since if $z \in$ Ran$|A|$ with $0 = U_0 z$, then $z = |A|x$ for some $x \in H$

$\Rightarrow 0 = U_0 z = U_0|A|x = Ax \Rightarrow x \in$ Ker(A) = Ker$|A| \Rightarrow z = |A|x = 0$.

Thus: U_0: Ran$|A| \to$ Ran(A) is an isomorphism.

Extend U_0 to the closure $\overline{U_0} : \overline{\mathrm{Ran}|A|} \to \overline{\mathrm{Ran}(A)}$ which is an isomorphism, too.

Now define U: $H \to H$ by

$$Ux := \begin{cases} \overline{U_0}x & \text{if } x \in \overline{\mathrm{Ran}|A|} \\ 0 & \text{otherwise} \end{cases}$$

Then $A = U|A|$ and U is a partial isometry. Since $|A|$ is selfadjoint we have

$(U^*U|A|x \,|\, |A|y) = (U|A|x \,|\, U|A|y) = (Ax|Ay) = (A^*Ax|y) = (|A|^2 x \,|\, y)$

$= (|A|x \,|\, |A|y)$

$\Rightarrow$ for all $v,w \in$ Ran$|A|$: $0 = ((U^*U-Id_H v|w)$, in particular with $w = (U^*U-Id_H)v$ we have $U^*U = Id_H \Rightarrow$ U is unitary.

ad 4)

Since Ker(A) = Ker$|A|$ and Ker$|A| = (\mathrm{Ran}|A|)^\perp$

$\Rightarrow$ Ker(A) = $(\mathrm{Ran}|A|)^\perp$ = Ker(U)

Similarly $\overline{\mathrm{Ran}(A)}$ = Ran(U)

ad 5)

$A = U|A| \Rightarrow U^*A = U^*U|A| = |A|$

ad 7)

$|A| \geq 0 \Rightarrow 0 \leq (|A|U^*x \,|\, U^*x) = (U|A|U^*x \,|\, x)$ for $x \in$ Ran(A)

$\Rightarrow U|A|U^*$ is positive since $(U|A|U^*)^2 = U|A|U^*U|A|U^* = U|A|^2U^*$

$= (U|A|)(|A|U^*) = AA^* = |A^*|^2$

$\Rightarrow |A^*| = U|A|U^*$

ad 6)

7) $\Rightarrow U^*|A^*| = U^*U|A|U^* = |A|U^* = A^*$

ad 8)
if $A=VB$ with $B\geq 0$, V unitary
$\Rightarrow A^*=B^*V^* \Rightarrow |A|^2 = A^*A = BV^*VB = B^2$
$\Rightarrow |A| = B$
$\Rightarrow U=V$ q.e.d.

In the decomposition $A=|U|A$ *of a bounded linear operator on a Hilbert space the operator* $|A|$ *corresponds to the absolute value of a complex number* z *written in the form* $z=|z|\cdot e^{i\cdot \arg(z)}$ *while* U *is the analogue of the complex number* $e^{i\cdot \arg(z)}$ *whi ch is of modulus one. One might expect that the unitary operators would be the analogue of the complex numbers of modulus one. The following example shows that this is not the case*

Example:
Let A_r: $l_2\rightarrow l_2$ be the right shift operator on l_2 with $(\alpha_1,\alpha_2,...)\mapsto(0,\alpha_1,\alpha_2,...)$.
Then $A_r^*=A_l$ and $A_r^*A_r = A_lA_r = Id_H$, thus $|A_r| = Id_H$. If we write $A_r = U|A|$ we have $A_r=U$.
However, A_r is not unitary since the sequence $e_1=(1,0,...,0)$ is not in its range.

§ 2 *Compact operators on a Hilbert space*

For the case of a one-dimensional Hilbert space H *the well known theorem of Bolzano-Weierstraß asserts that a bounded sequence of vectors in* H *contains a converging subsequence. Combining this with some simple topological facts one arrives at the statement, well-known from elementary analysis, that a subset of the complex plane is compact if and only if it is closed and bounded. We first want to study the connections between compactness, closure and boundedness in case of a subset of a Hilbert space* H *in general, and later on we want to use the results founded to characterize a special class of bounded linear operators, the so-called compact operators*

(3.10) **Definition:** Let H be a Hilbert space. A subset $M \subseteq H$ is called

1) **relatively compact** if every sequence $(x_j)_j \subseteq M$ contains a subsequence which converges to some element $x \in H$
2) **compact** if every sequence $(x_j)_j \subseteq M$ contains a subsequence which converges to some $x \in M$
3) **bounded** if there exists $\gamma > 0$ so that $\|x\| \leq \gamma$ for all $x \in M$

compact $\Rightarrow$ relatively compact

Before going on let us describe a very useful „diagonalization lemma“

(3.11) **Theorem:** Let $(e_k)_k$ be a sequence in a Hilbert space H. Let $(h_n)_n \subseteq H$ be a sequence so that for every $k \in \mathbf{N}$ $|(h_n | e_k)| \leq \delta_k$ for all n ($\delta_k \geq 0$). Then there exists a subsequence $(h_{nj})_j \subseteq (h_n)_{n \in \mathbf{N}} \subseteq H$, so that the limit $\gamma_k := \lim\limits_{j\to\infty} |(h_{nj} | e_k)|$ exists for every $k \in \mathbf{N}$ and $|\gamma_k| \leq \delta_k$.

(3.12) **Theorem:** A subset $M \subseteq H$, H Hilbert space, is relatively compact if and only if $\overline{M}$ is compact.

<u>Proof:</u>

If M is relatively compact and if $(x_n)_n$ is any sequence in $\overline{M}$, then for every $n \in \mathbf{N}$ we may choose $y_n \in M$ such that $\|x_n - y_n\| \leq \frac{1}{n}$. If $(y_{nk})_k \subseteq (y_n)_n$ is a subsequence converging to some element $y_0 \in \overline{M}$, then we also have $\|y_0 - x_{nk}\| \leq \|y_0 - y_{nk}\| + \|y_{nk} - x_{nk}\| \leq \|y_0 - y_{nk}\| + \frac{1}{n_k}$ and hence $\lim\limits_{k\to\infty} \|y_0 - x_{nk}\| = 0$. Thus $\overline{M}$ is compact. If $\overline{M}$ is compact then obviously M is relatively compact. q.e.d.

(3.13) **Corollary:** A closed relatively compact set $M \subseteq H$, H Hilbert space, is compact.

(3.14) **Theorem:** A compact set $M \subseteq H$, H Hilbert space, is closed and bounded.

The following theorem shows that in contrast to what happens in finite-dimensional Euclidean spaces, a closed and bounded subset of a Hilbert space does not have to be compact in general

(3.15) **Theorem:** The closed unit ball $\overline{B_1(0)}=\{x \in H: \|x\|_2 \leq 1\}$ in a Hilbert space is compact if and only if $\dim(H) < \infty$.

Proof:

If $\dim(H)=:m < \infty$ then $H \leftrightarrow \mathbf{C}^m$; let $(e_k)_{1 \leq k \leq m}$ be an orthonormal basis of H. Let $(h_k)_k$ be a sequence in $\overline{B_1(0)}$, then with (3.11) there exists a subsequence $(h_{jk})_{j \in \mathbf{N}} \subseteq (h_k)$ so that $\gamma_n := \lim_{j\to\infty} (h_{jk} \mid e_n)$ with $\|\gamma_n\| \leq 1$, $n \in \{1,...,n\}$

Define $x = \sum_{\mu=1}^{m} \gamma_n e_\mu \in H \Rightarrow \lim_{j\to\infty} \|x-h_{kj}\|^2 = \lim_{j\to\infty} \sum_{\mu=1}^{m} \left|\gamma_n \left(h_{kj} \middle| e_\mu\right)\right|^2 = 0$ and $\lim_{j\to\infty} \|h_{kj}\| = \|x\|$

$\Rightarrow \|x\| \leq 1$

If H is infinite-dimensional $\Rightarrow$ let $(e_j)_j$ be an orthonormal sequence

$\Rightarrow \|e_j-e_k\|^2 = (e_j-e_k \mid e_j-e_k) = 2$ for $j \neq k$

Therefore there is no converging subsequence. q.e.d.

(3.16) **Corollary:** A bounded set of a finite dimensional Hilbert space is relatively compact.

We now come to a special class of bounded linear operators, the so-called compact operators

(3.17) **Definition:** A linear operator A on a Hilbert space H is called **compact** (=**completely continuous**) if it maps the closed unit ball $\overline{B_1(0)}$ into a relatively compact set $A(\overline{B_1(0)})$.

The following two results to some extent motivate the alternative terminology „completely continuous“ for a compact linear operator

(3.18) **Theorem:** Let A be a compact operator on a Hilbert space H. Then

1) A is bounded (hence continuous)
2) A maps every bounded set into a relatively compact set

Proof:

ad 2)
Let $M \subseteq H$ be bounded, i.e. there exists $\alpha > 0$ with $\|x\| \leq \alpha$ for all $x \in M$, let $(x_j)_j \subseteq M$

$\Rightarrow \left(\frac{x_j}{\alpha}\right)_j \subseteq \overline{B_1(0)}$

$\Rightarrow \left(A\left(\frac{x_j}{\alpha}\right)\right)_{j \in \mathbf{N}}$ has a converging subsequence

$\Rightarrow (Ax_{jk})_k$ converges also

$\Rightarrow A(M)$ is relatively compact. q.e.d.

We exhibit a special class of compact linear operators which will turn out later to reflect quite typically the structure of compact linear operators in general.

(3.19) **Definition:** A bounded linear operator T on a Hilbert space H is said to be of **finite rank** if Ran(T):=T(H) is a finite-dimensional subspace of H.

(3.20) **Theorem:** Let $T \in \mathfrak{L}_b(H,H)$ be an operator on H. Then

1) T is a finite rank operator if and only if there exist $x_1,\dots,x_m,y_1,\dots,y_m$ with

$$Tx = \sum_{j=1}^{m} (x|x_j) y_j$$

2) T is compact if T is of finite rank

Proof:

ad 1)
T finite rank operator $\Rightarrow n := \dim(T(H)) < \infty$

Choose an orthonormal basis $y_1,\dots,y_n$ in T(H)

$\Rightarrow Tx = \sum_{j=1}^{n} (Tx|y_j) y_j = \sum_{j=1}^{n} (x \,|\, \underbrace{T^* y_j}_{x_j}) y_j$ q.e.d.

Compact linear operators can be characterized in another way. This new characterization will not only reveal some further interesting features of the individual compact linear operators but it will also help us to gain some insight in the algebraic and topological structure of the set of all compact linear operators or a given Hilbert space H. *For this purpose we have to introduce the concept of a weakly convergent sequence in* H.

(3.21) **Definition:** A sequence $(x_j)_{j \in \mathbf{N}}$ in a Hilbert space H is said

1) to **converge weakly** if for every $y \in H$ the sequence $((x_j | y))_{j \in \mathbf{N}}$ converges (in **C**)
2) to **converge weakly to x** if for every $y \in H$ one has $\lim_{j \to \infty} |(x_j | y) - (x | y)| = 0$

Example:

$(x_j)_j$ be an orthonormal basis in H. Take $y \in H$

$\|y\|^2 = \sum_{j=1}^{\infty} |(y|x_j)|^2 \Rightarrow \lim_{j\to\infty} |(y|x_j)| = 0$

$\Rightarrow (x_j)_j$ converges weakly to 0

But $\lim_{j\to\infty} \|x_j - 0\| = \lim_{j\to\infty} \|x_j\| = 1$

$\Rightarrow (x_j)_j$ does not converge to 0 in the usual norm sense.

If $x_0 \in H$ with $\lim_{j\to\infty} \|x_j - x_0\| = 0$

$\Rightarrow 0 \le \lim_{j\to\infty} |\,\|x_0\| - \|x_j\|\,| \le \lim_{j\to\infty} \|x_j - x_0\| = 0$

$\Rightarrow \|x_0\| = \lim_{j\to\infty} \|x_j\| = 1$

$\Rightarrow 0 = \lim_{j\to\infty} \|x_j - x_0\|^2 = \lim_{j\to\infty} (x_j - x_0 | x_j - x_0) = \lim_{j\to\infty} \left(\|x_j\|^2 - (x_0|x_j) - (x_j|x_0) + \|x_0\|^2\right)$

$= \lim_{j\to\infty} \left(\|x_j\|^2 - 2\cdot\mathrm{Re}((x_j|x_0)) + \|x_0\|^2\right) = 2 - 2\cdot \lim_{j\to\infty} \mathrm{Re}(x_j|x_0) = 2$ contradiction !

(3.22) **Theorem:** Let $(x_j)_{j\in\mathbf{N}}$ be a sequence in a Hilbert space with $\lim_{j\to\infty} \|x_j - x_0\| = 0$ for some $x_0 \in H$. Then $\lim_{j\to\infty} (x_j - x_0 | y) = 0$ for every $y \in H$.

Proof:

$\lim_{n\to\infty} |(x_n - x_0 | y)| \le \lim_{n\to\infty} \|y\|\,\|x_n - x_0\| = 0,\ y \in H.$ q.e.d.

(3.23) **Theorem:** Let $(x_j)_j$ be a weakly convergent sequence in a Hilbert space H. Then
1) $(x_j)_j$ is bounded
2) there exists a unique element $x_0 \in H$ with $\lim_{j\to\infty} (x_j - x | y)$ for all $y \in H$

Proof:

ad 1)

$(x_j)_j$ converges weakly $\Rightarrow ((x_j|y))_j$ converges in **C** and is bounded. Apply Banach-Steinhaus, then $(\|x_j\|)_j$ is bounded

ad 2)

By 1) there exists $\gamma > 0$ with $\|x_j\| < \gamma$, $j \in \mathbf{N}$. Define f: $H \to \mathbf{K}$ by $f(y) := \lim_{j\to\infty} \overline{(x_j|y)} = \lim_{j\to\infty} (y|x_j)$

f is a linear functional with $|f(y)| \le \lim_{j\to\infty} |(x_j|y)| \le \lim_{j\to\infty} \|x_j\|\cdot\|y\| < \gamma\cdot\|y\|$

$\Rightarrow$ f is continuous and linear.

By Riesz representation theorem there exists a unique $x_f \in H$ with $f(y) = (y|x_f)$

$\Rightarrow \overline{(y|x_f)} = \overline{f(y)} = \lim_{j\to\infty} (x_j|y)$ q.e.d.

The following result ought to be compared with theorem (3.15) that shows that in the closed unit ball $\overline{B_1(0)}$ *of a Hilbert space every sequence contains a subsequence which converges in* $\overline{B_1(0)}$ *(in the usual norm sense) if and only if H is finite-dimensional*

(3.24) **Theorem:** Let H be a Hilbert space with the closed unit ball $\overline{B_1(0)}$. Every sequence in $\overline{B_1(0)}$ contains a subsequence which converges weakly to some element in $\overline{B_1(0)}$.

Proof:

Let $(x_n)_{n\in\mathbf{N}}\subseteq \overline{B_1(0)}$ be a sequence and let H_0:=span$\{x_j: j \in \mathbf{N}\}$.
If H_0 is finite-dimensional then applying theorem (3.15) we find a subsequence $(x_{nj})_j\subseteq (x_n)_n\subseteq \overline{B_1(0)}$ with $\lim\limits_{j\to\infty}\|x_{nj}-x_0\|=0$ for some element $x_0 \in \overline{B_1(0)}$.
With (3.22) $\Rightarrow$ $(x_{nj})_j$ converges weakly to x_0. If H_0 is infinite-dimensional we choose an orthonormal basis $(e_k)_k$ for H_0. Since $(x_n)_n \subseteq \overline{B_1(0)}$ we have $|(x_n\,|\,e_k)|\le 1$ for all n $\in$ **N**, k $\in$ **N**.
By theorem (3.11) there exists a subsequence $(x_{nj})_j\subseteq (x_n)_n \subseteq \overline{B_1(0)}$ such that the limit $\lim\limits_{j\to\infty}(x_{nj}\,|\,e_k)$ exists for every k $\in$ **N**.

$\Rightarrow \lim\limits_{j\to\infty}(x_{nj}\,|\,\sum\limits_{m=1}^{l}\alpha_m\cdot e_m) = \sum\limits_{m=1}^{l}\overline{\alpha_m}\lim\limits_{j\to\infty}(x_{nj}\,|\,e_m)$ exists for every linear combination $\sum\limits_{m=1}^{l}\alpha_m\cdot e_m$.
Let y $\in$ H be arbitrary, suppose $y=y_1+y_2$ with $y_1 \in H_0$ and $y_2 \in H_0^{\perp}$.
Let $\varepsilon>0$ be given and l $\in$ **N** be chosen such that $z_l=\sum\limits_{m=1}^{l}(y_1|e_m)e_m \in H_0$ and $\|y_1-z_1\|<\frac{\varepsilon}{4}$.
Then for sufficiently large $n_0=n_0(\varepsilon)$ and $n_i\ge n_0$, $n_j\ge n_0$ we have

$$\begin{aligned}|(x_{ni})\,|\,y)|-(x_{nj}\,|\,y)| &= |(x_{ni}-x_{nj}\,|\,y)| = |(x_{ni}-x_{nj}\,|\,y_1+y_2)| = |(x_{ni}-x_{nj}\,|\,y_1)|\\ &\le |(x_{ni}-x_{nj}\,|\,z_1)| + |(x_{ni}-x_{nj}\,|\,y_1-z_1)| < \frac{\varepsilon}{2}+\|x_{ni}-x_{nj}\|\,\|y_1-z_1\|<\frac{\varepsilon}{2}+2\cdot\frac{\varepsilon}{4}=\varepsilon\end{aligned}$$

$\Rightarrow$ $(x_{nj})_j$ converges weakly.
By theorem (3.23) there exists a unique element $x_0 \in$ H such that $(x_0\,|\,y)=\lim\limits_{j\to\infty}(x_{nj}\,|\,y)$ and $|(x_0\,|\,y)|=\lim\limits_{j\to\infty}|(x_{nj}\,|\,y)|\le\|y\|$ for every y $\in$ H.
We conclude $\|x_0\|=\sup\limits_{\|y\|\le 1}|(x_0\,|\,y)|\le 1$
$\Rightarrow x_0 \in \overline{B_1(0)}$ q.e.d.

(3.25) **Corollary:** Every bounded sequence in a Hilbert space H contains a weakly converging subsequence.

(3.26) **Corollary:** Let A be a bounded linear operator on a Hilbert space H and let the sequence $(x_n)_n\subseteq$H converge weakly to x $\in$ H. Then the sequence $(Ax_n)_n$ converges weakly to Ax.

If we apply a compact linear operator to a weakly converging sequence then something happens which even serves to characterize compactness of the operator in question

(3.27) **Theorem:** A bounded linear operator on a Hilbert space H is compact if and only if it maps every weakly converging sequence into a sequence converging in the usual norm sense.

Proof:

„⇒“:

Let $A \in \mathfrak{L}_b(H,H)$ be a compact linear operator, let $(x_n)_n \subseteq H$ be a weakly converging sequence. Then $(x_n)_n$ is bounded by theorem (3.23) and the set $M:=\{Ax_n: n \in \mathbf{N}\}$ is relatively compact by theorem (3.18). If the sequence $(Ax_n)_n$ did not converge, then by the relative compactness of M it would have to contain at least two subsequences $(Ax_{nj})_j$ and $(Ax_{mj})_j$ converging (in the usual $\|\,\|$ sense) to different elements $\hat{x}_1$ and $\hat{x}_2$ respectively.

We then obtain $(\hat{x}_1 \mid y) = \lim_{j\to\infty}(Ax_{nj} \mid y) = \lim_{j\to\infty}(x_{nj} \mid A^*y) = \lim_{n\to\infty}(x_n \mid A^*y) = \lim_{j\to\infty}(x_{mj} \mid A^*y)$

$= \lim_{j\to\infty}(Ax_{mj} \mid y) = (\hat{x}_2 \mid y)$ for all $y \in H$

which implies $\hat{x}_1 = \hat{x}_2$ contradiction!

Therefore the sequence $(Ax_n)_n$ must converge.

„⇐“:

Suppose A is a bounded linear operator on H mapping every weakly converging sequence into a converging one.

Let $(y_n)_n$ be a sequence in $A\left(\overline{B_1(0)}\right)$. Without loss of generality we may assume $y_n=Ax_n$, $\|x_n\|\leq 1$ for all $n \in \mathbf{N}$.

By theorem (3.24) there exists a weakly converging subsequence $(x_{nj})_j \subseteq (x_n)_n \subseteq \overline{B_1(0)}$ which by hypothesis is mapped by A into the converging sequence $(Ax_{nj})_j$. Since $(Ax_{nj})_j$ is a subsequence of $(Ax_n)_n$ $(= (y_n)_n)$ we see that $A\left(\overline{B_1(0)}\right)$ is relatively compact and that A is a compact linear operator. q.e.d.

(3.28) **Corollary:** Let A be a compact linear operator on a Hilbert space H. Let the sequence $(x_n)_n \subseteq H$ converge weakly to x. Then $\lim_{n\to\infty} \|Ax_n - Ax\| = 0$.

Proof:

By theorem (3.27) there exists $y \in H$ with $\lim_{n\to\infty}\|Ax_n - y\| = 0$. Then $\lim_{n\to\infty} |(Ax_n - y \mid h)| = 0$ for every $h \in H$. On the other hand by corollary (3.26) the sequence $(Ax_n)_n$ converges weakly to Ax. Since there is only one weak limit of the sequence $(Ax_n)_n$ by theorem (3.23), we conclude y=Ax. q.e.d.

The following results are concerned with the set of all compact linear operators on a Hilbert space H

(3.29) **Theorem:** Let A and B be compact linear operators on a Hilbert space H and let C be a bounded linear operator on H. Then
1) A+B is a compact linear operator on H
2) $\lambda \cdot A$ ($\lambda \in \mathbf{C}$) is a compact linear operator on H
3) AC and CA are compact linear operators on H
4) A* is a compact linear operator on H

Proof:

1) – 3) are proved easily by using theorem (3.27) anc corollary (3.26)

ad 4)
If the sequence $(x_n)_n \subseteq H$ is weakly convergent, then it is bounded by theorem (3.23), i.e. $\|x_n\| \le \alpha$ ($\alpha > 0$) for all $n \in \mathbf{N}$.
We obtain $\|A^*x_n - A^*x_m\|^2 = \left(A^*(x_n - x_m) \mid A^*(x_n - x_m)\right) = \left(AA^*(x_n - x_m) \mid x_n - x_m\right)$
$= \|AA^*(x_n - x_m)\| \, \|x_n - x_m\| \le 2 \cdot \alpha \|AA^*x_n - AA^*x_m\| = 0$
A* therefore is compact. q.e.d.

(3.30) **Theorem:** Let $(A_n)_n$ be a sequence of compact linear operators on a Hilbert space H with $\lim_{n,m \to \infty} \|A_n - A_m\| = 0$. Then the operator $A := \lim_{n \to \infty} A_n$ is a compact linear operator on H.

Proof:

The proof of theorem (2.9) shows that A, defined by $Ax := \lim_{n \to \infty} A_n x$ for $x \in H$, is a bounded linear operator on H. Let $(x_n)_n$ be a weakly converging sequence in H and suppose $\|x_n\| \le \alpha$ ($\alpha > 0$) for all $n \in \mathbf{N}$.
Then for every $k \in \mathbf{N}$ we have
$\|Ax_n - Ax_m\| \le \|(A - A_k)x_n\| + \|(A - A_k)x_m\| + \|A_k x_n - A_k x_m\| \le 2 \cdot \alpha \|A - A_k\| + \|A_k x_n - A_k x_m\|$.
Given any $\varepsilon > 0$ we choose k so that $\|A - A_k\| < \frac{\varepsilon}{4\alpha}$. Keeping k fixed we then choose $n(\varepsilon)$ so large that $\|A_k x_n - A_k x_m\| < \frac{\varepsilon}{2}$ for all $n \ge n(\varepsilon)$, $m \ge n(\varepsilon)$
$\Rightarrow$ by theorem (3.28) $(A_k x_n)_n$ converges.
We conclude $\|Ax_n - Ax_m\| \le 2 \cdot \alpha \cdot \frac{\varepsilon}{4\alpha} + \frac{\varepsilon}{2} = \varepsilon$ for all $n \ge n(\varepsilon)$, $m \ge n(\varepsilon)$.
Thus $(Ax_n)_n$ is a Cauchy sequence in H which converges in H. By theorem (3.27) the operator A is compact. q.e.d.

This result shows, that the set C(H,H) *of all compact linear operators on a Hilbert space H is a closed linear subspace of* 2_b(H,H) *with respect to the operator norm.* C(H,H) *is also closed under passing to adjoints. This is formulated in short by saying that* C(H,H) *is symmetric. Since* C(H,H) *is also closed under addition, scalar multiplication and under left and right* *multiplication by arbitrary bounded linear operators on H, all these statements are combined in the following statement:*

C(H,H) *is a closed symmetric two-sided ideal in* 2_b(H,H).

§ 3 *Eigenvalues of compact operators*

Generalizing the concept of an eigenvalue of a matrix we say

(3.31) **Definition:** Let H be a Hilbert space, D_A be a linear subspace. Let A: $D_A \to H$ be a linear operator in H. A complex number λ is called

1) an **eigenvalue of A** in H if there exists a non-zero element $x \in D$, called the **corresponding eigenvector**, such that $Ax=\lambda\cdot x$
2) a **generalized eigenvalue of A** in H if there exists a sequence of unit vectors $(x_n)_n \subseteq D$ such that $\lim\limits_{n\to\infty}(A-\lambda\cdot Id_H)x_n = 0$
3) a **regular value of A** if the operator $A-\lambda\cdot Id_H$ is one-to-one (=injective) and $(A-\lambda\cdot Id_H)^{-1}$ is a bounded linear operator on H

The set $\sigma(A)$ of all $\lambda \in \mathbf{C}$ that are not regular values of A is called the **spectrum of A**. If $\lambda \in \mathbf{C}$ is an eigenvalue of A, then $H_\lambda:=\{y \in D: Ay=\lambda\cdot y\}$ is called the **corresponding eigenspace**.

Obviously every eigenvalue of A is a generalized eigenvalue of A and cannot be a regular value of A. More precisely we show in the next theorem.

(3.32) **Theorem:** Let A be a linear operator in a Hilbert space H. A complex number λ is an eigenvalue of A if and only if the operator $A-\lambda\cdot Id_H$ is not one-to-one (=not injective).

Proof:

If $\lambda \in \mathbf{C}$ is an eigenvalue then there exists $x\neq 0$ with $x \in D$ and $Ax=\lambda\cdot x$. Since $A\cdot 0=\lambda\cdot 0$, $A-\lambda\cdot Id_H$ is not one-to-one. If $A-\lambda\cdot Id_H$ is not one-to-one, then there exist $x_1 \in D$ and $x_2 \in D$ with $x_1\neq x_2$ and $Ax_1=\lambda\cdot x_1$ and $Ax_2=\lambda\cdot x_2$. This implies $(A-\lambda\cdot Id_H)(x_1-x_2)=0$ with $x_1-x_2\neq 0$. q.e.d.

(3.33) **Theorem:** Let A be a linear operator in a Hilbert space H. The following statements are equivalent:

1) $\lambda \in \mathbf{C}$ is an eigenvalue of A or (if λ is not an eigenvalue of A) $(A-\lambda\cdot Id_H)^{-1}$ exists and is unbounded
2) there exists a sequence$(x_n)_n \subseteq D_A$ of unit vectors such that $\lim\limits_{n\to\infty}(A-\lambda\cdot Id_H)x_n = 0$

Proof:

1) $\Rightarrow$ 2)
If λ is an eigenvalue of A and if x is a corresponding eigenvector then choose the sequence $(x_n)_n$ with $x_n=x$ for all $n \in \mathbf{N}$. If λ is not an eigenvalue and $(A-\lambda\cdot Id_H)^{-1}$ is unbounded, then there exists a sequence of unit vectors $y_n \in D_{(A-\lambda\cdot Id_H)^{-1}}$ such that $\lim\limits_{n\to\infty}\|(A-\lambda\cdot Id_H)^{-1}y_n\| = \infty$.

With $x_n:=\dfrac{(A-\lambda\cdot Id_H)^{-1}y_n}{\left\|(A-\lambda\cdot Id_H)^{-1}y_n\right\|} \in D_A$ for $n \in \mathbf{N}$, we have $\|x_n\|=1$ and

$\lim\limits_{n\to\infty}(A\text{-}\lambda\cdot Id_H)x_n = \lim\limits_{n\to\infty}\dfrac{y_n}{\left\|(A - \lambda\cdot Id_H)^{-1}y_n\right\|} = 0$, since y_n are unit vectors.

2) $\Rightarrow$ 1)

If the sequence $(x_n)_n \subseteq D_A$ has the properties mentioned in 2) and if λ is not an eigenvalue of A, then taking $y_n := \dfrac{(A - \lambda\cdot Id_H)^{-1}x_n}{\left\|(A - \lambda\cdot Id_H)^{-1}x_n\right\|} \in D_{(A\text{-}\lambda\cdot Id_H)^{-1}}$ we obtain a sequence of unit vectors $y_n \in D_{(A\text{-}\lambda\cdot Id_H)^{-1}}$ such that $\lim\limits_{n\to\infty}\|(A\text{-}\lambda\cdot Id_H)^{-1}y_n\| = \lim\limits_{n\to\infty}\dfrac{1}{\left\|(A - \lambda\cdot Id_H)x_n\right\|} = \infty$.

This shows that the operator $(A\text{-}\lambda\cdot Id_H)^{-1}$ is unbounded. q.e.d.

This theorem shows that every generalized eigenvalue and in particular every eigenvalue of A belongs to the spectrum of A. If λ *is not an eigenvalue of A, then the operator* $A\text{-}\lambda\cdot Id_H$ *is one-to-one and therefore the inverse operator* $(A\text{-}\lambda\cdot Id_H)^{-1}$ *is defined on* $(A\text{-}\lambda\cdot Id_H)D_A$ *.*
There are still two possibilities that something goes wrong with the inverse:
Its domain $(A\text{-}\lambda\cdot Id_H)D_A$ *might not yet coincide with H or* $(A\text{-}\lambda\cdot Id_H)^{-1}$ *might not be bounded.*
Thus the spectrum of an operator might contain complex numbers which are not generalized eigenvalues. However it will turn out that for the classes of selfadjoint operators, the spectrum consists entirely of generalized eigenvalues.
The special features displayed by compact operators in general also have effect upon the spectrum of such an operator. Apart from the point 0 it only consists of eigenvalues; moreover, if there are infinitely many eigenvalues then they may be arranged to a sequence converging to 0. We will study this more systematically.

(3.34) **Lemma:** Let A be compact linear operator on a Hilbert space H, let $(e_n)_n$ be an orthonormal sequence in H. Then $\lim\limits_{n\to\infty}(Ae_n \mid e_n)=0$.

Proof:

The sequence $(e_n)_n$ converges weakly to zero. The sequence $(Ae_n)_n$ then converges to zero also. By Cauchy's inequality we thus obtain $\lim\limits_{n\to\infty}|(Ae_n \mid e_n)| \le \lim\limits_{n\to\infty}\|Ae_n\|=0$. q.e.d.

(3.35) **Theorem:** Let A be a compact linear operator on a Hilbert space H. Let $\rho>0$ be given. Every family of linearly independent eigenvectors of A corresponding to eigenvalues with absolute values not smaller than ρ is finite.

Proof:

Assume that there exists an infinite sequence $(x_n)_n$ of linearly independent eigenvectors of A such that for the corresponding eigenvalues λ_n we have $|\lambda_n| \ge \rho$ for all $n \in \mathbf{N}$.
Let the orthonormal sequence $(e_n)_n$ be obtained from $(x_n)_n$ by Gram-Schmidt orthonormalization.

We obtain $(A\text{-}\lambda_n\cdot Id_H)e_n = (A\text{-}\lambda_n\cdot Id_H)\left(\sum\limits_{k=1}^{n}\alpha_{n,k}\cdot x_k\right) = \sum\limits_{k=1}^{n}\alpha_{n,k}(Ax_k - \lambda_n x_k)$

$$=\sum_{k=1}^{n}\alpha_{n,k}(\lambda_k x_k - \lambda_n x_n) = \sum_{k=1}^{n-1}\alpha_{n,k}(\lambda_k - \lambda_n)x_n =: y_n$$

with $y_n \in \text{span}\{x_1,\dots,x_{n-1}\}$.
Therefore $y_n \perp e_n$. We conclude $(Ae_n \mid e_n) = (y_n+\lambda_n e_n \mid e_n) = \lambda_k(e_n \mid e_n) = \lambda_n$ and by lemma (3.34) $\lim_{n\to\infty}\lambda_n = \lim_{n\to\infty}(Ae_n \mid e_n) = 0$ in contradiction to our assumption $|\lambda_n| \geq \rho > 0$ for $1 \leq n < \infty$.

q.e.d.

(3.36) **Corollary:** Let A be a compact linear operator on a Hilbert space H. If $\lambda \neq 0$ is an eigenvalue of A, then the corresponding eigenspace is a finite-dimensional subspace.

(3.37) **Corollary:** Let A be a compact linear opeartor on a Hilbert space H. The only possible accumulation point of the eigenvalues of A in the complex plane is 0.

Proof:

If $(\lambda_n)_n$ were a sequence of different eigenvalues converging to $\lambda \neq 0$, then a sequence of corresponding eigenvectors would be linearly independent and therefore would violate the conclusion of theorem (3.35). q.e.d.

(3.38) **Corollary:** Let A be a compact linear operator on a Hilbert space H. There exist at most countably many differrent eigenvalues of A. If A has infinitely many eigenvalues λ_n, $n \in \mathbf{N}$, then $\lim_{n\to\infty}\lambda_n = 0$.

So far nothing has been said about the rest of the spectrum of A which for bounded linear operators in general need not consist of eigenvalues only. The statement which we are looking for in the compact case – the rest of the spectrum consists at most of the point 0 – is quite satisfying but the theorem laying the ground for this statement seems somewhat technical-confusing rather than satisfying.
Asking whether $\lambda \in \mathbf{C}$ *belongs to the spectrum of A we are interested in the invertibility of* $A-\lambda\cdot Id_H$. *An inverse of* $A-\lambda\cdot Id_H$ *should assign to every* $y \in (A-\lambda\cdot Id_H)(H)$ *a unique element* $x \in H$. *Moreover, as a bounded linear operator, it should do this in such a way that* $\|x\| = \|(A-\lambda\cdot Id_H)^{-1}y\| \leq \gamma_\lambda\cdot\|y\|$ *for some constant* $\gamma_\lambda > 0$ *not depending on* y (x *being the image of* y) .
The following theorem states that we can always reverse the action of $A-\lambda\cdot Id_H$ *in a bounded way: never mind uniqueness, for every given* $y \in (A-\lambda\cdot Id_H)(H)$ *there is an element* x *associated with* y *by some sort of bounded inverse of* $A-\lambda\cdot Id_H$, *the bound being* γ_λ .

(3.39) **Theorem:** Let A be a compact linear operator on a Hilbert space H. Given $\lambda \neq 0$, $\lambda \in \mathbf{C}$, there exists a constant $\gamma_\lambda > 0$ with the following property:
For every $y \in (A-\lambda\cdot Id_H)(H)$ there exists some element x_y (depending on y) such that $(A-\lambda\cdot Id_H)x_y = y$ and $\|x_y\| \leq \gamma_\lambda\cdot\|y\|$.

Proof:

We consider the linear subspace $H_\lambda:=\{x \in H: Ax=\lambda\cdot x\}=\{x \in H: (A-\lambda\cdot Id_H)x=0\}$. If λ is an eigenvalue of A, then H_λ is the corresponding eigenspace; if λ is not an eigenvalue, then $H_\lambda=\{0\}$. H_λ is closed since for any $x \in H$ we have $Ax-\lambda x=0$ if and only if $(Ax-\lambda x\,|\,y)=0$ for all $y \in H$ if and only if $\big(x\,|\,(A^*-\overline{\lambda}\cdot Id_H)y\big)=0$ for all $y \in H$ if and only if $x \perp (A^*-\overline{\lambda}\cdot Id_H)(H)$, i.e. $H_\lambda=\big((A^*-\overline{\lambda}\cdot Id_H)H\big)^\perp$. Given any $y \in (A-\lambda\cdot Id_H)(H)$ and any x such that $(A-\lambda\cdot Id_H)x=y$, we observe that $(A-\lambda\cdot Id_H)(x-z)=y$ if and only if $z \in H_\lambda$.
If $P_\lambda: H\to H_\lambda$ is the orthogonal projection, we consider $z_x:=P_\lambda x \in H_\lambda$ and $x_y:=x-P_\lambda x$ and obtain $(A-\lambda\cdot Id_H)x_y=y$ and $\|x_y\|=\|x-P_\lambda x\|=\min\{\|x-z\|: z \in H_\lambda\}$ and therefore (putting $x-z=:x'$) $\|x_y\|=\min\{\|x'\|: (A-\lambda\cdot Id_H)x'=y\}$.
In this way we have associated with every $y \in (A-\lambda\cdot Id_H)(H)$ a unique element x_y such that $(A-\lambda\cdot Id_H)x_y=y$ holds.
We now prove the existence of a constant $\gamma_\lambda>0$ such that $\|x_y\|\le\gamma_\lambda\cdot\|y\|$ for all $y \in (A-\lambda\cdot Id_H)(H)$.

Assuming the contrary, we have $\sup\{\frac{\|x_y\|}{\|y\|}: y\neq 0,\ y \in (A-\lambda\cdot Id_H)(H)\}=\infty$.

We can therefore choose a sequence $(y_n)_n \subseteq (A-\lambda\cdot Id_H)(H)$ such that $y_n \neq 0$, $x_{y_n} \neq 0$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty}\frac{\|x_{y_n}\|}{\|y_n\|}=\infty$.

For $x_n:=\frac{x_{y_n}}{\|x_{y_n}\|}$, $z_n:=\frac{y_n}{\|x_{y_n}\|}$ we obtain $(A-\lambda\cdot Id_H)x_n=z_n$,
$\|x_n\|=\min\{\|x'\|: (A-\lambda\cdot Id_H)x'=z_n\}=1$ and $\lim_{n\to\infty} z_n=0$

$(x_n)_n$ contains a weakly convergent subsequence $(x_{nj})_j$ (corollary (3.26))
Then $(Ax_{nj})_j$ converges to some element h and, as a consequence, $\lim_{j\to\infty}\lambda\cdot x_{nj}=\lim_{j\to\infty}(Ax_{nj}-z_{nj})=h$

Because $\lambda\neq 0$, also the sequence $(x_{nj})_j$ converges to the element $\frac{h}{\lambda}$.

From $(A-\lambda\cdot Id_H)\left(\frac{h}{\lambda}\right)=\lim_{j\to\infty}(A-\lambda\cdot Id_H)x_{nj}=\lim_{j\to\infty} z_{nj}=0$ and $(A-\lambda\cdot Id_H)x_n=z_n$ we conclude

$(A-\lambda\cdot Id_H)(x_n-\frac{h}{\lambda})=z_n$

while $\lim_{j\to\infty}\|x_{nj}-\frac{h}{\lambda}\|=0$ and therefore $\|x_n-\frac{h}{\lambda}\|<1$ for infinitely many $n \in \mathbf{N}$.
This however contradicts $\|x_n\|=1$ for all $n \in \mathbf{N}$. q.e.d.

(3.40) **Theorem:** If A is a compact linear operator on a Hilbert space H and if $\lambda\neq 0$, then $(A-\lambda\cdot Id_H)(H)$ is a closed linear subspace of H.

Proof:

Suppose the sequence $(y_n)_n \subseteq (A-\lambda\cdot Id_H)(H)$ converges to some $y_0 \in H$. The sequence $(y_n)_n$ is bounded and therefore the sequence $(x_{y_n})_n$ contains a weakly converging subsequence $(x_j)_j \subseteq (x_{y_n})_n$. Then the sequence $(Ax_j)_j$ converges. Since $x_j=\frac{1}{\lambda}\cdot(Ax_j-y_{nj})$ the sequence $(x_j)_j$ converges also. For $x_0:=\lim_{j\to\infty} x_j$ we obtain $(A-\lambda\cdot Id_H)x_0 = \lim_{j\to\infty}(A-\lambda\cdot Id_H)x_j = \lim_{j\to\infty} y_{nj} = y_0$, i.e. $y_0 \in (A-\lambda\cdot Id_H)(H)$ q.e.d.

We now characterize regular values $\lambda\neq 0$ of a compact linear operator

(3.41) **Theorem:** Let A be a compact linear operator on a Hilbert space H. A complex number $\lambda\neq 0$ is a regular value of A if and only if $(A-\lambda\cdot Id_H)(H)=H$.

Proof:

„$\Rightarrow$“:
If $\lambda\neq 0$ is a regular value, then $(A-\lambda\cdot Id_H)(H)=H$ by definition.

„$\Leftarrow$“:
Suppose $(A-\lambda\cdot Id_H)(H)=H$ with $\lambda\neq 0$. Suppose λ is an eigenvalue of A and let x_1 be a corresponding eigenvector, then $(A-\lambda\cdot Id_H)x_1=0$ with $x_1\neq 0$.
Since $(A-\lambda\cdot Id_H)(H)=H$ we can find $x_2 \in H$ such that $(A-\lambda\cdot Id_H)x_2=x_1$ and inductively construct a sequence $(x_n)_n$ such that $(A-\lambda\cdot Id_H)x_n=x_{n-1}$ for $n \in \mathbf{N}$ with $x_0:=0$.
The elements x_n are linearly independent, $n \in \mathbf{N}$, which can be easily seen by induction. Let the orthonormal sequence $(e_n)_n$ be obtained from $(x_n)_n$ by Gram-Schmidt orthonormalization. As in the proof of theorem (3.35) we conclude

$$(A-\lambda\cdot Id_H)e_n = (A-\lambda\cdot Id_H)\left(\sum_{k=1}^{n}\alpha_{n,k}\cdot x_k\right)=\sum_{k=1}^{n}\alpha_{n,k}\cdot(Ax_k-\lambda_n x_k)=\sum_{k=1}^{n}\alpha_{n,k}(\lambda_k x_k-\lambda_n x_k)$$

$$=\sum_{k=1}^{n-1}\alpha_{n,k}(\lambda_k-\lambda_n)x_k =: y_n \in \text{span}\{x_1,\ldots,x_{n-1}\}$$ and therefore $y_n \perp e_n$.

We obtain $(Ae_n\,|\,e_n) = (y_n+\lambda\cdot e_n\,|\,e_n) = \lambda\cdot(e_n\,|\,e_n) = \lambda \neq 0$ for all $n \in \mathbf{N}$, which contradicts the conclusion of lemma (3.34). Hence λ cannot be an eigenvalue of A. If λ is not an eigenvalue of A, then $(A-\lambda\cdot Id_H)^{-1}$ exists and is defined on H and bounded by theorem (3.39).
Therefore λ is a regular value. q.e.d.

(3.42) **Corollary:** Let A be a compact linear operator on a Hilbert space H. A complex number $\lambda\neq 0$ is an eigenvalue of A if and only if $\overline{\lambda}$ is an eigenvalue of A*.

Proof:

If $\overline{\lambda}$ is an eigenvalue of A*, then $(A^*-\overline{\lambda}\cdot Id_H)(H)$ is a proper subspace of H by theorems (3.40) and (3.41) (remember A* is also compact by theorem (3.29)).
Then $((A^*-\overline{\lambda}\cdot Id_H)(H))^{\perp} = H_{\lambda}:=\{x \in H: Ax=\lambda\cdot x\}$ (compare proof of theorem (3.39)) and λ is an eigenvalue of A with H_{λ} as the corresponding eigenspace.
A symmetric reasoning in the other direction completes the proof. q.e.d.

(3.43) **Corollary:** Let A be a compact linear operator on a Hilbert space H. A complex number $\lambda\neq 0$ is either a regular value of A or an eigenvalue of A.

Proof:

If λ is not a regular value then $(A-\lambda\cdot Id_H)(H)$ is a proper subspace of H. By the argument used in the proof of theorem (3.39) we conclude that $\overline{\lambda}$ is an eigenvalue of A*. Then λ is an eigenvalue of A. q.e.d.

This corollary cannot be extended to a bounded linear operator on H *in general. For this consider the left shift operator* A *on* l_2 *defined by* $A(\xi_1,\xi_2,\xi_3,...):= (\xi_2,\xi_3,...)$. *Then* $\lambda=0$ *is an eigenvalue of A with the corresponding eigenvector* $e_1=(\delta_{1k})_{k \in \mathbf{N}}$.
A* *is the right shift operator on* l_2 *which does not have eigenvalues at all. Consequently* $\lambda=1$ *is an eigenvlaue of* $A+Id_H$ *but not of* $(A+Id_H)^* = A^*+Id_H$

(3.44) **Corollary:** Let A be a compact linear operator on an infinite-dimensional Hilbert space H. Then $\lambda=0$ is a generalized eigenvalue and therefore belongs to the spectrum of A.

Proof:

Let $(e_n)_n$ be an orthonormal sequence in H. The sequence $(e_n)_n$ converges weakly to 0 and therefore $(Ae_n)_n$ converges also to 0. Thus $\lambda=0$ is a generalized eigenvalue. q.e.d.

As a summary of the previous results we obtain:

(3.45) **Theorem:** (**Fredholm alternative**):
Let A be a compact linear operator on a Hilbert space H. Let $\lambda \in \mathbf{C} \setminus \{0\}$ be given.
Either the inhomogenous equations $(A-\lambda\cdot Id_H)x=y$ and $(A^*-\overline{\lambda}\cdot Id_H)x'=y'$ have solutions x and x' for every given y and y' respectively or the homogenous equations $(A-\lambda\cdot Id_H)x=0$ and $(A^*-\overline{\lambda}\cdot Id_H)x'=0$ have non-zero solutions x and x'.
In the first case the solutions x and x' are unique and depend continuously on y and y' respectively.

In the second case $(A-\lambda \cdot Id_H)x=y$ has a solution x if and only if y is orthogonal to all solutions of $(A^*-\overline{\lambda} \cdot Id_H)x'=0$. Also $(A^*-\overline{\lambda} \cdot Id_H)x'=y'$ has a solution x‘ if and only if y‘ is orthogonal to all solutions of $(A-\lambda \cdot Id_H)x=0$.

<u>Proof:</u>

Either λ is a regular value of A (and $\overline{\lambda}$ is a regular value of A*) or λ (respectively $\overline{\lambda}$) is an eigenvalue of A (A* respectively). In the first case we have $x=(A-\lambda \cdot Id_H)^{-1}y$,
$x'=(A^*-\overline{\lambda} \cdot Id_H)^{-1}y'$ where $(A-\lambda \cdot Id_H)^{-1}$ and $(A^*-\overline{\lambda} \cdot Id_H)^{-1}$ are bounded linear operators on H.
In the second case, if H_λ is the eigenspace of A corresponding to λ, we have
$(A^*-\overline{\lambda} \cdot Id_H)(H)=H_\lambda^\perp$.
$(A^*-\overline{\lambda} \cdot Id_H)x'=y'$ has a solution x‘ if and only if $y' \in (A^*-\overline{\lambda} \cdot Id_H)(H)$ or, in other words, if and only if y‘ is orthogonal to the eigenspace H_λ, which in turn consists of all solutions of $(A-\lambda \cdot Id_H)x=0$.
The remaining assertion is shown similarly. q.e.d.

§ 4 *The spectral decomposition of a compact linear operator*

With all the informations we have collected so far we shall now obtain the desired decomposition of a compact linear operator into simpler parts. We will do this with the help of the eigenvalues of compact selfadjoint operators and the corresponding eigenspaces and with the help of the polar decomposition of the compact linear operator

(3.46) **Theorem:** Let $A\neq 0$ be a compact selfadjoint operator on a Hilbert space H. Then

1) there exists an eigenvalue λ of A such that $|\lambda| = \|A\|$
2) each eigenvalue of A is real
3) the spectrum of A is real
4) the eigenspaces corresponding to eigenvalues λ and μ with $\lambda\neq 0$, $\mu\neq 0$, $\lambda\neq\mu$ are orthogonal

Proof:

ad 1)
We choose a sequence $(x_n)_n \subseteq H$ such that $\|x_n\|=1$ and $\lim\limits_{n\to\infty} \|Ax_n\| = \|A\| = \sup\limits_{\|x\|=1} \|Ax\|$.

Since A is compact we can find a subsequence $(Ax_{nj})_j$ of $(Ax_n)_n$ which converges to some element y. Since $|\,\|y\|-\|Ax_{nj}\|\,| \leq \|y-Ax_{nj}\|$ we have $\|y\| = \lim\limits_{j\to\infty} \|Ax_{nj}\| = \|A\| > 0$.

By the Cauchy-Schwarz-inequality we obtain
$\|A\|^2 = \lim\limits_{j\to\infty} \|Ax_{nj}\|^2 = \lim\limits_{j\to\infty} (Ax_{nj} \mid Ax_{nj}) = \lim\limits_{j\to\infty} (A^2x_{nj} \mid x_{nj}) \leq \lim\limits_{j\to\infty} \|A^2x_{nj}\| = \|Ay\|$ and thus
$\|A^2y\|\,\|y\| \geq (A^2y \mid y) = (Ay \mid Ay) = \|Ay\|^2 \geq \|A\|^4 = \|A\|^2\|y\|^2 \geq \|A^2\|\,\|y\|^2 \geq \|A^2y\|\,\|y\|$

This implies $\|A^2y\|\,\|y\| = (A^2y \mid y)$ and therefore $A^2y = \alpha y$ with $\alpha = \dfrac{(A^2y \mid y)}{(y \mid y)} = \dfrac{\|A\|^4}{\|A\|^2} = \|A\|^2$

Now we define $x := y + \dfrac{Ay}{\|A\|}$.

If x=0, then $Ay = -\|A\|\,\|y\|$, i.e. $\lambda := -\|A\|$ is an eigenvalue of A

If $x\neq 0$, then $Ax = Ay + \dfrac{A^2y}{\|A\|} = Ay + \dfrac{\alpha y}{\|A\|} = Ay + \dfrac{\|A\|^2 y}{\|A\|} = Ay + \|A\|y = \|A\|\left(y + \dfrac{Ay}{\|A\|}\right)$

Which shows that $Ax = \|A\|x$, i.e. $\lambda := \|A\|$ is an eigenvalue of A

ad 2)
Let λ be an eigenvalue of H with the corresponding eigenvector x. Then
$\lambda(x \mid x) = (\lambda x \mid x) = (Ax \mid x) = (x \mid Ax) = (x \mid \lambda x) = \overline{\lambda}(x \mid x)$ and hence $\lambda = \overline{\lambda}$

ad 3)
Corollaries (3.43) and (3.44) imply that $\sigma(A) = \{0\} \cup \{\lambda \in \mathbf{C} : \lambda \text{ eigenvalue of } A\}$

ad 4)
If $\lambda\neq 0$ and $\mu\neq 0$ are different eigenvalues of A and if $x \in H_\lambda$ and $y \in H_\mu$ are eigenvectors corresponding to λ and μ respectively, then
$\lambda(x \mid y) = (\lambda x \mid y) = (Ax \mid y) = (x \mid Ay) = (x \mid \mu y) = \overline{\mu}(x \mid y) = \mu(x \mid y)$, i.e. $(\lambda-\mu)(x \mid y) = 0$ and hence $(x \mid y)=0$ q.e.d.

If a compact linear operator A on a Hilbert space H *is selfadjoint, then all of its eigenvalues are real, A has at most countably many different eigenvalues* $\lambda_k \neq 0$ *with* $\lim_{k\to\infty} \lambda_k = 0$ *if* A *has infinitely many eigenvalues. If* $\lambda_j \neq 0$ *is an eigenvalue of* A *then the corresponding eigenspace is a finite-dimensional subspace of* H *. We count every non-zero eigenvalue as many times as indicated by ist multiplicity, i.e. by the dimension of the corresponding eigenspace, and obtain a sequence* $(\lambda_j)_j$ *of non-zero eigenvalues of* A *with* $\|A\| = |\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_k| > 0$ *and* $\lim_{j\to\infty} \lambda_j = 0$ *if* A *has infinitely many non-zero eigenvalues. For* $j \in I$ *we choose an orthonormal basis of the corresponding eigenspace* $H_{\lambda_j} = \{x \in H: Ax = \lambda_j x\}$*. Since* $H_{\lambda_j} \perp H_{\lambda_k}$ *for eigenvalues* $\lambda_k \neq \lambda_j$*,* $\lambda_k \neq 0$*,* $\lambda_j \neq 0$*, we obtain an orthonormal system* $(x_j)_{j\in I}$ *corresponding to the sequence* $(\lambda_j)_{j\in I}$ *; here* $I = \mathbf{N}$ *if A has infinitely many non-zero eigenvalues or* $|I| < \infty$ *if A has finitely many non-zero eigenvalues.*
The sequence $(\lambda_j, x_j)_j$ *is called an* ***eigensystem of the compact selfadjoint operator* A**

(3.47) **Theorem:** (**Spectral theorem for compact selfadjoint operators**):
Let A be a compact selfadjoint operator on a Hilbert space H, let $(\lambda_j, x_j)_{j\in I}$ be its eigensystem. Then we have

1) $Ax = \sum_{j\in I} \lambda_j (x \mid x_j) x_j$ for all $x \in H$
2) $\overline{A(H)} = \bigoplus_j H_{\lambda_j}$ where $H_{\lambda_j} = \{x \in H: Ax = \lambda_j x\}$
3) $H = N(A) \oplus \overline{A(H)}$ where $N(A) = \{x \in H: Ax = 0\}$ is the kernel of A

<u>Proof:</u>

ad 1) + 2)
Since $H_\lambda \perp H_\mu$ if $\lambda, \mu \in \sigma(A) \setminus \{0\}$ and $\lambda \neq \mu$, we have $\overline{A(H)} = \bigoplus_{\substack{\lambda\in\sigma(A)\\ \lambda\neq 0}} H_\lambda$.

Theorem (3.46) guarantees the existence of $\lambda_1 \in \sigma(A) \setminus \{0\}$ such that $|\lambda_1| = \|A\|$. Let x_1 be an eigenvector with $\|x_1\| = 1$ corresponding to λ_1. We define $H_1 := H$ and $H_2 := \{x \in H: (x \mid x_1) = 0\} = H_1 \ominus \mathrm{span}\{x_1\}$. Since A is selfadjoint we have $(Ax \mid x_1) = (x \mid Ax_1) = (x \mid \lambda_1 x_1) = \overline{\lambda_1}(x \mid x_1) = \lambda_1 (x \mid x_1) = 0$ for all $x \in H_2$ and hence $A(H_2) \subseteq H_2$, i.e. $A|_{H_2}: H_2 \to H_2$. $A|_{H_2}$ is compact and selfadjoint. If $A|_{H_2} \neq 0$ we can find $\lambda_2 \in \mathbf{C}$ with $|\lambda_2| = \|A|_{H_2}\|$ and $x_2 \in H_2$ such that $\|x_2\| = 1$ and $Ax_2 = \lambda_2 x_2$.
Obhviously $|\lambda_2| \leq |\lambda_1|$ since $\|A|_{H_2}\| \leq \|A\|$.
Continuing this way we find a system $(\lambda_1, x_1), \ldots, (\lambda_p, x_p)$ such that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_p|$, $(x_j, x_k) = \delta_{jk}$, and $Ax_j = \lambda_j x_j$ for $j,k \in \{1,\ldots,p\}$.
We also find subspaces $H_1, H_2, \ldots, H_p, H_{p+1}$ such that $H_{j+1} = \{x \in H_j: (x \mid x_j) = 0\}$ for $j \in \{1,\ldots,p\}$.
The system $(\lambda_1, x_1), \ldots, (\lambda_n, x_n)$ is finite and ends with (λ_n, x_n) and H_{n+1} if and only if $A|_{H_{n+1}} = 0$.
In this case we define $y_n := x - \sum_{j=1}^{n} (x \mid x_j) x_j$ and obtain

$(y_n \mid x_i) = (x \mid x_i) - \sum_{j=1}^{n} (x \mid x_j)(x_j \mid x_i) = (x \mid x_i) - (x \mid x_i) = 0$ for $i \in \{1,\ldots,n\}$, i.e. $y_n \in H_{n+1}$ and

hence $0 = Ay_n = Ax - \sum_{j=1}^{n} (x \mid x_j) Ax_j = Ax - \sum_{j=1}^{n} \lambda_j (x \mid x_j) x_j$.

If $A|_{H_n}\neq 0$ for all $n\in\mathbf{N}$, we again define $y_n:= x-\sum_{j=1}^{n}(x\,|\,x_j)x_j$ and obtain

$\|y_n\|^2 = \|x\|^2-\sum_{j=1}^{n}\left|(x\,|\,x_j)\right|^2 \le \|x\|^2$ and therefore

$Ay_n = Ax-\sum_{j=1}^{n}(x\,|\,x_j)\lambda_j x_j$ and $\|Ay_n\|=\|A|_{H_{n+1}}y_n\|\le|\lambda_{n+1}|\,\|y_n\|$.

Since $\lim\limits_{j\to\infty}\lambda_j=0$ we finally have $Ax=\lim\limits_{n\to\infty}\sum_{j=1}^{n}\lambda_j(x\,|\,x_j)x_j=\sum_{j=1}^{\infty}\lambda_j(x\,|\,x_j)x_j$ and therefore $(\lambda_j,x_j)_j$ is an eigensystem.

If $\lambda\neq 0$ is an eigenvalue of A and $x\neq 0$ is a corresponding eigenvector and if (λ,x) is not a member of the eigensystem $(\lambda_j,x_j)_j$, then $(x\,|\,x_j)=0$ for $j\in\mathbf{N}$ and thus

$\lambda x = Ax = \sum_{j=1}^{\infty}\lambda_j(x\,|\,x_j)x_j = 0$ contrary to $\lambda\neq 0$ and $x\neq 0$.

Therefore the system $(\lambda_j,x_j)_j$ contains every non-zero eigenvalue of A.
Suppose $x\neq 0$ and $x\in H_{\lambda_k} = N(A-\lambda_k\cdot Id_H)$, $k\in\mathbf{N}$, then $x\perp H_{\lambda_j}$ for all $j\in\mathbf{N}$, $j\neq k$.
Therefore $\lambda_k x = Ax = \sum_{\substack{j\\ \lambda_j=\lambda_k}}\lambda_j(x\,|\,x_j)x_j$ which implies $x = \sum_{\substack{j\\ \lambda_j=\lambda_k}}\lambda_j(x\,|\,x_j)x_j$

This shows that every $x\in H_{\lambda_k}$ is a linear combination of $(x_j)_j$. This implies $(x_j)_{j\in I_0}$ is a basis in H_{λ_k}, where $I_0=\{j\in\mathbf{N}: \lambda_j=\lambda_k\}$. Hence for any eigenvalue $\lambda\neq 0$ of A an orthonormal basis of $H_\lambda=N(A-\lambda\cdot Id_H)$ is part of the eigensystem $(\lambda_j,x_j)_j$

ad 3)
If $x\in A(H)^\perp$ then $0 = (x\,|\,Ay) = (Ax\,|\,y)$ for every $y\in H$ that is $Ax=0$ or $x\in N(A)$.
Conversely $x\in N(A)$ implies $x\in A(H)^\perp$. Since $A(H)^\perp = N(A)$ we have $H = N(A)\oplus A(H)$

q.e.d.

Now we apply the polar decomposition (theorem (3.9)) and obtain the main result of this chapter

(3.48) **Theorem:** Let H be a Hilbert space. Let $A\in \mathfrak{L}_b(H,H)$ be a compact linear operator. Then there exist orthonormal systems $(x_j)_j\subseteq H$ and $(y_j)_j\subseteq H$ and a sequence $(\lambda_j)_j\subseteq\mathbf{C}$ such that $|\lambda_j|\ge|\lambda_{j+1}|$, $j\in\mathbf{N}$, $\lim\limits_{j\to\infty}\lambda_j=0$ and $Ax=\sum_{j=1}^{\infty}\lambda_j(x\,|\,x_j)y_j$ for $x\in H$.

Proof:

Since A is a compact linear operator, $|A| \in \mathcal{L}_b(H,H)$ is also compact (theorem (3.29)).

$|A|$ is selfadjoint $\Rightarrow$ we have $|A|x = \sum_{j=1}^{\infty} \lambda_j (x \mid x_j) x_j$ for $x \in H$

Where $(\lambda_j, x_j)_{j \in I}$ is the eigensystem of $|A|$ obtained in theorem (3.47). If the system $(\lambda_1, x_1), (\lambda_2, x_2), \ldots, (\lambda_n, x_n)$ is finite we define $\lambda_j := 0$ for $j \in \mathbf{N}$, $j \geq n+1$.
By the polar decomposition theorem (3.9) we find a partial isometry U such that

$Ax = U|A|x = \sum_{j=1}^{\infty} \lambda_j (x \mid x_j) U x_j = \sum_{j=1}^{\infty} \lambda_j (x \mid x_j) y_j$ with $y_j := Ux_j$.

Since $(y_j \mid y_k) = (Ux_j \mid Ux_k) = (U^*Ux_j \mid x_k) = (x_j \mid x_k) = \delta_{jk}$ we obtain that $(y_j)_j$ is an orthonormal system also.
Since $0 \leq (|A| x_j \mid x_j) = (\lambda_j x_j \mid x_j) = \lambda_j$, $j \in \mathbf{N}$, we obtain the desired result. q.e.d.