Pointwise decay of solutions and of higher derivatives to Navier-Stokes equations

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Abstract

In this paper we study the space-time asymptotic behavior of the solutions and derivatives to the incompressible Navier-Stokes equations. Using moment estimates we obtain that strong solutions to the Navier-Stokes equations which decay in L^2 at the rate of $\|u(t)\|_2 \leq C(t+1)^{-\mu}$ will have the following pointwise space-time decay

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$$|D^\alpha u(x,t)| \leq C_{k,m} \frac{1}{(t+1)^{\rho_o}(1+|x|^2)^{k/2}}$$

where $\rho_o = (1-2k/n)(m/2+\mu) + 3/4(1-2k/n)$, and $|\alpha| = m$. The dimension n is $2 \leq n \leq 5$ and $0 \leq k \leq n$ and $\mu \geq n/4$

1 Introduction

In this paper we study the space-time decay of the solutions and derivatives to the incompressible Navier-Stokes equations in the whole space $\mathbf{R}^n$,

$$\begin{aligned} u_t + u \cdot \nabla u + \nabla p &= \triangle u \\ div u &= 0 \\ u(x,0) = u_o(x) \quad &\in \quad \mathbf{X} \end{aligned} \tag{1}$$

Where $\mathbf{X}$ will be specified below. Using moment techniques we show that strong solutions and derivatives of all order decay pointwise at an algebraic rate as $|x| \to \infty$ and $t \to \infty$.

Questions of decay of solutions to the Navier-Stokes equations in different norms have been studied, among others, by R. Kajikiya and T. Miyakawa [?], T. Kato [?], H. Kozono [?], H. Kozono and T. Ogawa [?], M.E. Schonbek [?], [?], M. Wiegner [?], and Zhang-Linghai [?]. Of particular interest in the direction of the present paper are the results by Takahashi [?]. In this paper Takahashi studies the pointwise decay of the solutions and first derivatives of solutions to Navier-Stokes equations with zero initial data and an external force which decays at algebraic rate in both space and time. In order to get his estimates, Takahashi, uses a weighted equation approach. Specifically in [?], Takahashi, gets pointwise decay rates both in time and space for solutions bounded in some weighted $L^{q,s}$ norms,with n/p + 2/p' =1 (the limiting Serrin class), where $L^{q,s}$ stands for the space time norm

$$\{\int_0^\infty (\int_{\mathbf{R}^n} |u(x,t)|^q dx)^{s/q} dt\}^{1/s}.$$

We consider that our results complement and extend Takahashi's results in the sense that in our case we have non zero initial data but zero external force.

Moreover we are able to obtain the results for derivatives of all orders.We note that since we are obtaining decay result for derivatives we will work directly with strong solutions. The results could be applied to weak solutions provided we start with a sufficiently large time. Since in this case we are already in the regime where the solution is smooth we prefer to simplify notation to work directly with smooth solution.
We also would like to refer the reader to [?] to get a very detailed outline of what other author in the field have done on related questions.

It is already clear at the level of the heat equation that their is a relation between the time decay and space decay. This kind of balance will be also found for solutions to the Navier-Stokes equations. In particular the balance relation we obtain between the decay in space and in time coincides with the relation for the Heat equation when we consider the solutions itself.

The plan of the paper is the following : We first have a section with notation. The next section first recall some essential estimates on the moments of approximation to the solutions and derivatives. This approximations solve a linearization of the Navier-Stokes equations. Passing to the limit they converge to a solution to Navier-Stokes. Moreover by the new uniqueness results this solutions are the ones Kato constructed in [?] Then we show that these bounds are also valid for the solutions and derivatives itself. The bounds we first obtain are valid for all time and depend on time . This will not suffice to yield a uniform time decay. By the results of [?] we have already uniform bounds for the moments but not for the moments of the derivatives. The next section is dedicated to show that these moments are also bounded independently of time. The last section deals with the space -time pointwise decay of the solution. The proof here will follow from the uniform bound of the moments and an appropriate form of the Gagliardo-Nirenberg inequality.

2 Notation and Assumptions

Let

$$\alpha = (\alpha_1, \ldots, \alpha_n), \tag{2}$$

be a multi-index with $\alpha_i \geq 0$. We will use the notation,

(3) $$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}$$

where

(4) $$|\alpha| = \alpha_1 + \ldots + \alpha_n,$$

(5) $$D_i = \frac{\partial}{\partial x_i}$$

For any integer $m \geq 0$, we set

$$D^m f(x) = \left(\sum_{|\alpha|=m} |D^\alpha f(x)|^2 \right)^{1/2},$$

where $x = (x_1, \ldots, x_n)$. The L^2 norm (or energy norm) will be denoted by

(6) $$\|u\| = \|u(.,t)\|_2 = [\int_{I\!R^n} |u(x,t)|^2 dx]^{1/2},$$

where $dx = dx_1 \ldots dx_n$. More generally we denote the L^p norm for $1 \leq p < \infty$ by

(7) $$\|u(.,t)\|_p = [\int_{I\!R^n} |u(x,t)|^p dx]^{1/p},$$

and the L^∞ norm by

(8) $$\|u(.,t)\|_\infty = \text{ess sup}_x |u(x,t)|.$$

The H^m norm is defined by

(9) $$\|u(.,t)\|_{H^m} = [\int_{R^n} \sum_{|\alpha|\leq m} |D^\alpha u(x,t)|^2 dx]^{1/2},$$

In what follows we assume that $u = u(x,t) = (u_1(x,t), \ldots, u_n(x,t))$ is a global solution of the Navier-Stokes equations with the following decay: there exist constants

$C, \mu > n/4$ such that

(10) $$\|u(t)\|_2 \leq C(t+1)^{-\mu}$$

for $t \geq 0$. Under the following conditions, it is proved in [?] that the decay given by (??) generalizes to

$$\|D^m u(t)\|_2 \leq C(t+1)^{-\mu-m/2} \quad \text{for } t \geq 0\,. \tag{11}$$

Then recall Gagliardo-Nirenberg's inequality for $f \in H^m$:

$$\|D^j f\|_\infty \leq C\|f\|_2^{1-a}\|D^m f\|_2^a$$

with $a = a_{jm} = \dfrac{j+\frac{n}{2}}{m}$, as long as $j + (n/2) < m$. Taking m large enough (assuming we can do this) we get from equations (??) and (??)

$$\|D^j u(t)\|_\infty \leq C(t+1)^{-\mu-j/2-n/4} \quad \text{for } j = 0, 1, \ldots\,. \tag{12}$$

Since we are interested in decay of derivatives and hence in smooth solutions, we are going to work with solutions that start with small data or the results we shall establish will only be valid for large t.
The main idea in order to obtain pointwise decay, is to prove decay of the moments of the solutions and derivatives and then combine this with an appropiate Galiardo-Nirenberg inequality. It will yield decay in

L^∞, whence the pointwise decay. With this in mind we introduce the following weighted spaces.

$$f \in L_\nu^{r_1} \quad \text{iff} \quad \left(\int_{\mathbf{R}^n} |x|^{\nu r_1}|f|^{r_1}\, dx\right)^{1/r_1} < \infty. \tag{13}$$

Then for $s = 0, 1, 2, \ldots$, we define the (s, α) moments

$$M_{s,\alpha}(t) = \int_{\mathbf{R}^n} |x|^s |D^\alpha u(x,t)|^2\, dx,$$

and in particular for $s \geq 0$, $t \geq 0$, we define the moment of order s of u by

$$M_s((u)(t)) = M_{s,0}(t) = \int_{\mathbf{R}^n} |x|^s |u(t)|^2\, dx = \left(\|u(t)\|_{L^2_{s/2}}\right)^2.$$

Finally define for $s, m = 0, 1, 2, \ldots$,

$$\tilde{M}_{s,m}(t) = \sum_{|\alpha|=m} M_{s,\alpha}(t) = \int_{\mathbf{R}^n} |x|^s |D^m u(x,t)|^2\, dx.$$

3 Preliminaries

To start our calculations we need to recall some estimates obtained for weighted norms of approximating solutions to the Navier-Stokes equations, [?]. These solutions satisfy a "linearized Navier-Stokes equation" , in which both the convective and the pressure term are linearized in "explicit form. To this purpose pressure is expressed in terms of product of Riesz transforms. Specifically a sequence $\{u^\ell\}$ of approximating solutions is constructed. That is let $v = u^{\ell+1}$ satisfy

$$
\begin{aligned}
v_t - \Delta v + u^\ell \cdot \nabla v + \nabla P(u^\ell, v) &= 0, \\
\operatorname{div} v &= 0, \\
v(0) &= u_0
\end{aligned}
\tag{14}
$$

with u_0 in an appropriate space. Solutions of (??) are constructed locally by fixed point arguments and tehn the existence is extended by appriori estimates. Such solutions are unique by construction. The bilinear operator P is defined by

$$P(u, v) = \sum_{j,k} R_j R_k (u_j v_k)$$

if $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$ are functions from $\mathbf{R}^n$ to $\mathbf{R}^n$ and where R_j denotes the Riesz transforms,

$$[R_j f]\hat{}(\xi) = -i \frac{\xi_j}{|\xi|} \hat{f}(\xi)$$

for $j = 1, \ldots, n$. When $u^\ell = v$ we recuperate the Navier-Stokes equations since for Navier-Stokes

$$\Delta p = -\sum_{j,k} \frac{\partial^2}{\partial x_j \partial x_k} (u_j u_k),$$

hence

$$\hat{p}(\xi, t) = -\sum_{j,k} \frac{\xi_j \xi_k}{|\xi|^2} \widehat{u_j u_k}(\xi, t),$$

thus

$$p = \sum_{j,k} R_j R_k (u_j u_k) = P(u, u).$$

The linearization (??) is of the type used by Caffarelli, Kohn and Nirenberg in [?], by Kajikiya and Miyakawa in [?], by Leray in [?], and by Sohr, von Wahl and Wiegner in [?]. The advantage of making the linearization explicit is that for the approximations we can use well known properties of the Riesz transforms, such as their boundedness in L^p-spaces (cf. [?]) and in weighted L^p spaces satisfying the Muckenhoupt condition (cf. [?], [?]) to obtain bounds for the solutions of the Navier-Stokes equations and their moments. We expect that our proofs to establish bounds in weighted L^p-spaces, with some modifications, could be used for the approximating solutions constructed by Caffarelli, Kohn and Nirenberg [?], Kajikiya and Miyakawa [?], Sohr, von Wahl and Wiegner [?].

In [?] we constructed solutions to (??) via fixed point methods. We showed that the sequence $\{u_\ell\}$ converges in $C([0,T):, L^2 \cap L^r)$ to a weak solution of Navier-Stokes, provided the data in $L^2 \cap L^r$ and $r > n$. If the data is also in H^1 and sufficiently small the solution will be smooth. This are the solutions we will be interested in. Moreover we note that due to the new uniqueness results [?] and [?] the solutions we are considering are the solutions that Kato constructed in [?]

One essential result we need to proceed is given by Lemma (2.2) of [?] which will be stated below. For this reason we need to introduce the following real numbers ν, q, r and r_1 which satisfy the relations

$$0 \le \nu < n, \quad 2 \le r_1 \le r, \quad 1 \le q \le \infty, \quad r > n \tag{15}$$

$$\frac{1}{q} < \frac{\nu}{2} - \frac{n}{2r} + \frac{1}{2}, \quad \frac{1}{r} \le \frac{1}{r_1} + \frac{\nu}{n} < 1 - \frac{1}{r}; \tag{16}$$

The next lemma (lemma (2.2) of [?]) is quite technical. It states that if the linearized equation (??) is obtained via a function which is sufficiently smooth and the data is in an appropriate weighted space then so is the solution.

In order to state the next lemma we need to recall some notation from [?]. Let

$$F(x,t) = F(t)(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$$

be the fundamental solution of the heat equation in n space variables. If v solves (??) , set

$$H(u,v) = u \cdot \nabla v + \nabla P(u,v),$$

then

$$v(t) = F(t) * u_0 - \int_0^t F(t-s) * H(u^\ell, v)(s)\, ds. \tag{17}$$

Let $\varphi \in L^2([0,T], H^1(\mathbf{R}^n, \mathbf{R}^n))$, set

$$\begin{aligned} \mathcal{M}\varphi(t) &= \int_0^t F(t-s) * \left[u^\ell \cdot \nabla\varphi(s) + \nabla P(u^\ell, \varphi)(s)\right] ds \\ &= \int_0^t F(t-s) * H(u^\ell, \varphi)(s)\, ds, \end{aligned} \tag{18}$$

and

$$\mathcal{L}\varphi(t) = F(t) * u_0 - \mathcal{M}\varphi(t). \tag{19}$$

Then the integral version (**??**) of the (LNS) with respect to u becomes

$$v = \mathcal{L}v.$$

That is the solution to the linearized Navier-Stokes can be obtained as a fixed point of the operator $\mathcal{L}$, (see [**?**]).

Lemma 3.1 *Assume (**??**)-(**??**) and assume the function u satisfies*

$$u \in C([0,T], W^{m,r}(\mathbf{R}^n)^n \cap L^q([0,T], W^{m,r_1}(\mathbf{R}^n)^n).$$

There exists a constant $K(T,u)$ of the form

$$K(T,u) = C(T)\left(\|u\|_{C_T(W^{m,r})} + \|u\|_{L^q_T(W^{m,r_1})}\right) \tag{20}$$

with $C(T)$ independent of u and depends only on T, n and the exponents r, r_1, ν, q. Let $D^\alpha u_0 \in L^{r_1}_\nu \cap L^r(\mathbf{R}^n, \mathbf{R}^n)$ for $|\alpha| \le m$, then the fixed point v of $\mathcal{L}$ satisfies $D^\alpha v \in C([0,T], L^{r_1}_\nu(\mathbf{R}^n, \mathbf{R}^n))$ for $|\alpha| \le m$ and

$$\begin{aligned} \|D^\alpha v(t)\|_{L^{r_1}_\nu} \ &\le \ C(T)\left(\|u_0\|_{W^{m,r_1}} + \sum_{\beta \le m} \|D^\beta u_0\|_{L^{r_1}_\nu}\right) \\ &\quad + K(T,u)\left(\|u_0\|_{W^{m,r}} + \|u_0\|_{W^{m,r_1}}\right). \end{aligned}$$

Proof. To establish the estimate of this lemma one proceeds as follows . First write the solution in the implicit form as a solution to the integral equation as mentioned above (**??**) ie, $v(t) = F(t) * u_0 - \int_0^t F(t-s) * H(u,v)(s)\,ds$. The process now consist in bounding $F(t) * u_0$ and $F(t-s) * H(u,v)(s)$ in the appropriate weighted L^p spaces . The main tools are estimates on the heat semi-group and the boundedness of the Riesz operator (for the pressure term) in L^p spaces and weighted L^p spaces.
For details of the proof see [**?**]. In this proof $C(T)$ denotes a constant depending only on T, n and the exponents r, r_1, ν, q.

■

To obtain decay results we need that our estimates are independent of time. For this one proceeds in several steps. First the results of the above lemma are extended to strong solutions of the Navier-Stokes. That is we can suppose in what follows that the initial data u_o satisfies [**?**]

$$u_o \in L^n \cap L^2, \quad \|u_o\|_{L^r} \text{ sufficiently small} \tag{21}$$

where $r > n$. We note that Kato [**?**] has obtained smooth solutions with small data in $L^2 \cap L^n$. The reason why we do not use his construction solutions is that we want to insure that the solutions also lie in the appropriate weighted space provided we add that the data is in that space too. We note that effectively we could also obtain the bounds for weak solutions for a sufficiently large time . But this would put us in the regime where the solutions have become regular. Thus instead of mentioning each time that we are working for sufficiently large time we suppose we are working with smooth solutions. The estimates for strong solutions follow by lemma (**??**) and the the estimates on the derivatives obtained in [**?**]. Specifically we have

Lemma 3.2 *Assume (**??**)-(**??**) . Let $u_o \in W^{m,r} \cap W^{m,r_1} \cap H^1(\mathbf{R}^n)$. Suppose there exists u a strong solution for the Navier-Stokes equations with data u_o. Then there exists a constant $K(T, u_o)$ of the form*

$$K(T, u_o)) \leq C(t)\|D^m u_o\|_{L^r}, \|D^m u_o\|_{L^{r_1}} \tag{22}$$

with $C(T)$ independent of u, such that if $D^\alpha u_0 \in L^{r_1}_\nu \cap L^r(\mathbf{R}^n, \mathbf{R}^n)$ for $|\alpha| \leq m$, then the solution u satisfies $D^\alpha u \in C([0,T], L^{r_1}_\nu(\mathbf{R}^n, \mathbf{R}^n))$ for $|\alpha| \leq m$

and

$$\|D^\alpha u(t)\|_{L^{r_1}_\nu} \leq C(T)\left(\|u_0\|_{W^{m,r_1}} + \sum_{\beta\leq m}\|D^\beta u_0\|_{L^{r_1}_\nu}\right)$$
$$(23) \qquad +K(T,u_o)\left(\|u_0\|_{W^{m,r}} + \|u_0\|_{W^{m,r_1}}\right).$$

Proof.
We note that in order to have strong solutions with bounded derivatives it is enough to suppose that the solutions have small data in H^1 for n=3 (If we are in $n > 3$ then it suffices to require that the initial data is small in H^2. Moreover it is shown in [?] that such solutions (for $2 \leq n \leq 5$) decay in the $W^{p,m}$ norms . Hence the above bounds are immediate consequence of (**??**) and the bounds in $W^{p,m}$ for $p = r$ and $p = r_1$. spaces obtained in [?]. Hence the $K(T, u_k)$ of last lemma (where $u = u_k$ of the approximating sequence) is bounded uniformly in k for all approximating solutions in terms of norms of the data. Since the $\{u_k\}$ converge strongly to the solution u of Navier-Stokes (see [?]). Hence by standard methods the (**??**) follows.
Note that for the decay for derivatives we only need that the L^2 norm of the solution is bounded ie. μ could be zero.

■

As an immediate consequence we obtain.

Corollary 3.3 *Under the hypothesis of the Lemma (* **??***) it follows that*

$$(24) \qquad \|D^\alpha u(t)\|_{L^{r_1}_\nu} \leq C(T)C_o$$

where C_o depends only on the appropriate norms of the data.

4 Decay of Moments of derivatives.

In order to obtain the decay of moments of the derivatives we will first need to establish uniform bounds. Once these are obtained the decay will follow by a Holder inequality between the (n,s)moments and the L^2 norm of the derivates. Specifically we can show that

Theorem 4.1 *Let u_o be as Lemma (??). Let u be a strong solution with data u_o, satisfying*

$$\|u(t)\|_2 \leq C(t+1)^{-\mu} \tag{25}$$

where $\mu \geq n/4$. Then

$$\tilde{M}_{s,m}(t) \leq C(t+1)^{-(2\mu+m)(1-\frac{s}{n})}$$

for $m = 0, 1, 2, \ldots,\ s = 0, 1, \ldots, n$.

Proof. As before we note that if the data is sufficiently small then such solutions u exists.In particular if $u \in H^2 \cap L^\infty$ Then all the derivatives of higher order are in L^2 (see [?].) Moreover the decay in L^2 yields as noted before ([?])

$$\|D^m u(t)\|_2 \leq C(t+1)^{-\mu-m/2} \tag{26}$$

Thus as we pointed out before Gagliardo-Nirenberg yields

$$\|D^j u(t)\|_\infty \leq C(t+1)^{-\mu-j/2-n/4} \tag{27}$$

for $j = 0, 1, \ldots$

The proof follows the steps of Theorem 4.1 in [?].The proof is by induction on m. We notice first that the case $m = 0$ is Theorem 4.1 of [?]. That is from Theorem(4.1) ([?]) we have

$$M_k(u)(t) \leq C(t+1)^{-2\mu(1-\frac{k}{n})} \tag{28}$$

for all $t \geq 0$, $k = 0, 1, \ldots, n$.

We note also that the case $s = 0$ is covered by (??). Let $m > 0$. As in Theorem (4.1) [?] the estimate now for $0 < s < n$ will follow from the estimates for $s = 0$ and $s = n$ by Hölder interpolation.

The theorem then follows by Hölder interpolation. In fact, with $1/p = (n-s)/n$, $1/p' = s/n$. Let $|\alpha| = m$ and write $|D^\alpha u|^2 = (|D^\alpha u|^2)^{s/n}(|D^\alpha u|^2)^{(n-s)/n}$, then it easey to see

$$\begin{aligned} M_{s,\alpha}(t) &= \int_{\mathbf{R}^n} |x|^s |D^\alpha u|^2\,dx \leq \left(\int_{\mathbf{R}^n} |D^\alpha u|^2\,dx\right)^{1/p} \left(\int_{\mathbf{R}^n} |x|^n D^\alpha |u|^2\,dx\right)^{1/p'} \\ &= M_o(D^\alpha u)(t)^{1-\frac{k}{n}} M_n(u)(t)^{k/n} \leq C(t+1)^{-(2\mu+m)(1-\frac{s}{n})} M_n(u)(t)^{s/n}. \end{aligned}$$

Thus we have if $M_n(u)(t)$ is uniformly bounded that

$$\tilde{M}_{s,m}(t) \leq C(t+1)^{-(2\mu+m)(1-\frac{s}{n})}$$

It thus suffices to prove the estimate for $s = n$, which merely says that $\tilde{M}_{n,m}(t)$ is bounded uniformly in $t > 0$. In proving this, we assume m is a positive integer and the estimates have been proved up to $m-1$.

Let α be a multi-index with $|\alpha| = m$. For a function g and a multi-index β, we set $g_\beta = D^\beta g$. By Leibniz' product formula,

$$u_{\alpha t} = \Delta u_\alpha - \sum_{\beta+\gamma=\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} u_\beta \cdot \nabla u_\gamma - \nabla p_\alpha;$$

dot multiplying by $|x|^n u_\alpha$ and using that $\operatorname{div} u = 0$, $\operatorname{div} u_\alpha = 0$ we get, after some technical but straightforward manipulations,

$$\begin{aligned} |x|^n u_{\alpha t} \cdot u_\alpha &= -|x|^n |\nabla u|^2 + \frac{1}{2} n(2n-2)|x|^{n-2}|u_\alpha|^2 + \frac{1}{2} n |x|^{n-2}(x \cdot u)|u_\alpha|^2 \\ &\quad -|x|^n \sum_{\beta+\gamma=\alpha, \beta\neq 0} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (u_\beta \cdot \nabla u_\gamma) \cdot u_\alpha + n|x|^{n-2}(x \cdot u_\alpha) p_\alpha + \operatorname{div} E_{n,\alpha} \end{aligned}$$

where

$$E_{n,\alpha} = \frac{|x|^n}{2} \nabla(|u_\alpha|^2) - \frac{n}{2}|x|^{n-2}|u_\alpha|^2 x - \frac{|x|^n}{2}|u_\alpha|^2 u - |x|^n u_\alpha p_\alpha.$$

One can prove now, as in Lemma 6.1, Appendix B of [?], that

$$\liminf_{R\to\infty} \int_{|x|=R} |E_{n,\alpha}| \, dS = 0.$$

More precisely the proof is a repetition of the arguments in the above mentioned Lemma where we replace u by u_α using the appropriate estimates for the derivatives obtained in [?].

It follows that integrating with respect to x over $\mathbf{R}^n$ the divergence term integrates to 0 and we get

$$\frac{1}{2}\frac{d}{dt} M_{n,\alpha}(t) = A(t) + B(t) + C(t) + D(t), \tag{29}$$

where

$$
\begin{aligned}
A(t) &= -\int_{\mathbf{R}^n} |x|^n |\nabla u_\alpha|^2 \, dx + \frac{n}{2}(2n-2) M_{n-2,\alpha}(t) \le \frac{n}{2}(2n-2) M_{n-2,\alpha}(t), \\
B(t) &= \frac{n}{2} \int_{\mathbf{R}^n} |x|^{n-2} (x \cdot u) |u_\alpha|^2 \, dx, \\
C(t) &= -\sum_{\beta+\gamma=\alpha, \beta\neq 0} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \int_{\mathbf{R}^n} |x|^n (u_\beta \cdot \nabla u_\gamma) \cdot u_\alpha \, dx, \\
D(t) &= n \int_{\mathbf{R}^n} |x|^{n-2} (x \cdot u_\alpha) p_\alpha \, dx.
\end{aligned}
$$

We have to first obtain a time dependent bound on $M_{n,m}(t)$. This bound follows by induction on the order of the moment ie we let $0 \le k \le n$ For $k = 0$. The estimate is immediate. The induction step follows using energy estimates . These are quite straightforward and as such will be omitted (for details of a proof of similar type please see [?]). To obtain the necessary uniform bound we proceed as follows

Bound for A(t)

Notice first that if $n = 2$ then

$$A(t) \le 2M_{o,\alpha} \le C_o(1+t)^{2\mu} \tag{30}$$

Where $2\mu > n/2 = 1$ Suppose now $3 \le n \le 5$ by Hölder and by (**??**),

$$M_{n-2,\alpha}(t) \le M_{n,\alpha}(t)^{(n-2)/n} \|u_\alpha\|^{4/n} \le C(1+t)^{-\rho} M_{n,\alpha}(t)^{(n-2)/n},$$

with $\rho = (4/n)(\mu + m/2) > 1$ since $m \ge 1$ and $\mu \ge n/4$. For $m = 0$ please see [?]. In general, from now on, ρ denotes a constant > 1, not the same one in all equations. By the definition of $A(t)$, using also

$$(1+t)^{-\rho} M_{n,\alpha}(t)^{(n-2)/n} \le \frac{2}{n}(1+t)^{-\rho} + \frac{n-2}{n}(1+t)^{-\rho} M_{n,\alpha}(t),$$

we proved

$$|A(t)| \le C(1+t)^{-\rho} \left(1 + \tilde{M}_{n,m}(t)\right) \tag{31}$$

Bound for B(t)

$$
\begin{aligned}
|B(t)| &= \left|\frac{n}{2}\int_{\mathbf{R}^n} |x|^{n-2}(x\cdot u)|u_\alpha|^2\,dx\right| \le \frac{n}{2}\int |x|^{n-1}|u||u_\alpha|^2\,dx \\
&\le \frac{n}{2}\|u_\alpha\|_2^{2/n}\|u\|_\infty \left(\tilde{M}_{n,m}(t)\right)^{(n-1)/n}
\end{aligned}
$$

so that by (??) and (??),

$$
\begin{aligned}
|B(t)| &\le C(1+t)^{-\rho}\left(\tilde{M}_{n,m}(t)\right)^{(n-1)/n} \\
&\le C(1+t)^{-\rho}\left(1+\tilde{M}_{n,m}(t)\right),
\end{aligned}
$$

where this time $\rho = (2/n)(\mu + m/2) + \mu + n/4 > 1$.

Bound for C(t)
Note that C(t) is a sum in terms of α and β, where $|\beta| + |\gamma| = |\alpha|$. We will consider the terms when $\beta = 0$ first.In this case we get the following bound

$$
\left|\int_{\mathbf{R}^n} |x|^n \left|(u\cdot\nabla u_\alpha)\cdot u_\alpha\right|\,dx\right| \le \|D^{m+1}|u\|_\infty \int_{\mathbf{R}^n} |x|^n u D^\alpha u dx \le
$$
$$
\|D^{m+1}|u\|_\infty \tilde{M}_{n,m}(t)^{1/2}) M_{n,o}(t)^{1/2} \le (1+t)^\rho M_{n,o}(t)^{1/2}\tilde{M}_{n,m}(t)^{1/2}
$$

where $\rho = \frac{(m+1)}{2} + \mu + n/4 > 1$ and we recall (see [?]) $M_{n,o}(t)^{1/2} \le C_o$. Thus

$$
\left|\int_{\mathbf{R}^n} |x|^n \left|(u\cdot\nabla u_\alpha)\cdot u_\alpha\right|, dx\right| \le C_o((1+t)^\rho + 1)\tilde{M}_{n,m}(t)
$$

The general term when $\beta > 0$ in $C(t)$ can be estimated by

$$
\left|\int_{\mathbf{R}^n} |x|^n \left|(u_\beta\cdot\nabla u_\gamma)\cdot u_\alpha\right|, dx\right| \le \|D^j u\|_\infty \tilde{M}_{n,\ell}(t)^{1/2}\tilde{M}_{n,m}(t)^{1/2},
$$

where $j = \min(|\beta|, |\gamma|+1)$, $\ell = \max(|\beta|, |\gamma|+1)$, so that $0 \le j \le [m/2]$, $[(m+1)/2] \le \ell \le m$ and $j + \ell = m+1$. For $\ell = m$, so $j = 1$, equations (??) imply a bound of the form

$$
C(1+t)^{-(\mu+n/4+1)}\tilde{M}_{n,m}(t).
$$

The terms with $\ell < m$ are bounded, by the induction hypothesis and by (??), by

$$
C(1+t)^{-(\mu+n/4+j/2)}\tilde{M}_{n,m}(t)^{1/2},
$$

and we obtain again an estimate of the form

$$|C(t)| \leq C(1+t)^{-\rho}\left(1+\tilde{M}_{n,m}(t)\right). \tag{32}$$

where $\rho > 1$.

Bound for D(t)

Because the Riesz transforms are bounded in L^2_ν with $\nu = (n-2)/2$ and D^α commutes with the Riesz transforms,

$$p_\alpha = D^\alpha p = \sum_{j,k} R_j R_k [D^\alpha(u_j u_k)] = \sum_{k,j,\beta+\gamma=\alpha} \left(\begin{array}{c}\alpha\\ \beta\end{array}\right) R_j R_k(u_{\beta,j} u_{\gamma,k}),$$

and we have

$$\begin{aligned}
|D(t)| &= \left| n \int_{\mathbf{R}^n} |x|^{n-2}(x\cdot u_\alpha) p_\alpha \, dx \right| \\
&\leq C \int_{\mathbf{R}^n} |x|^{n-1} |u_\alpha||p_\alpha| \, dx \leq C\tilde{M}_{n,m}(t)^{1/2} \|p_\alpha\|_{L^2_\nu} \\
&\leq C\tilde{M}_{n,m}(t)^{1/2} \sum_{\beta+\gamma=\alpha} \| |u_\beta||u_\gamma| \|_{L^2_\nu}.
\end{aligned}$$

By Hölder,

$$\begin{aligned}
\| |u_\beta||u_\gamma| \|_{L^2_\nu} &= \left(\int_{\mathbf{R}^n} |x|^{n-2} |D^\beta u|^2 |D^\gamma u|^2 \, dx \right)^{1/2} \\
&\leq C \|D^j u\|_\infty \|D^\ell u\|_2^{1/n} \tilde{M}_{n,\ell}(t)^{(n-2)/2n},
\end{aligned}$$

with $j = \min(\beta,\gamma)$, $\ell = \max(\beta,\gamma)$ (so $0 \leq j \leq m/2$). Once more we apply (**??**), (**??**) to get $\|D^j u\|_\infty \|D^\ell u\|_2^{1/n} \leq C(1+t)^{-\rho}$ with $\rho = (1/n)(\mu + \ell/2) + \mu + n/4 + j/2 > 1$. By the induction hypothesis $\tilde{M}_{n,\ell}$ is bounded uniformly in t if $\ell < m$, so all terms with $\ell < m$ in the last estimate for D can be bounded by $C(1+t)^{-\rho}$, the remaining term is bounded by

$$C(1+t)^{-\rho} \tilde{M}_{n,m}^{(n-2)/2n} \leq C(1+t)^{-\rho}\left(1+\tilde{M}_{n,m}\right),$$

so that $D(t)$ has a bound of the same type as $A(t)$, $B(t)$, $C(t)$. Combining all the estimates, we proved

$$\frac{d}{dt}\tilde{M}_{n,m}(t) \leq C(1+t)^{-\rho} + C(1+t)^{-\rho}\tilde{M}_{n,m}(t)$$

where $\rho > 1$, hence integrating

$$\tilde{M}_{n,m}(t) \leq \left(\tilde{M}_{n,m}(0) + \frac{C}{\rho - 1}\right) + C \int_0^t (s+1)^{-\rho} \tilde{M}_{n,m}(s)\, ds.$$

By Gronwall's lemma,

$$\tilde{M}_{n,m}(t) \leq \left(\tilde{M}_{n,m}(0) + \frac{C}{\rho - 1}\right) e^{c/(\rho-1)}$$

proving $\tilde{M}_{n,m}(t)$ is bounded in t for $t > 0$. ∎

NOTE: We took some pains to avoid having to bound $\|D^j u\|_\infty$ for $j > [(m+1)/2]$. In this way, bounds on the L^2-norm of derivatives of order m will give (sometimes) all the needed L^∞ bounds on the $D^j u$'s.
The next theorem will establish the spatial and time decay of strong solutions to equations for which the moments decay. we will first establish the result in n dimensions with $2 \leq n \leq 5$.

Theorem 4.2 . *Let $2 \leq n \leq 5$ Under the same hypothesis of Theorem 4.1 let u be a strong solution with data u_o .Let $k \leq n/2$. Then*

$$|D^\alpha u(x,t)| \leq C_{k,m} \frac{1}{(t+1)^{\rho_o}(1+|x|^2)^{k/2}} \tag{33}$$

Here $\rho_o = (\mu + m/2)(1 - 2k/n) + n/4(1 - 2k/n)$, and $|\alpha| = m$.

Proof We note first that the restriction to $2 \leq n \leq 5$ is due to that is where we have estimates for the moments. The main tools for the proof are the last theorem and Gagliardo Nirenberg's inequality. Let

$$v(x,t) = (1+|x|^2)^{k/2} D^\alpha u.$$

By Leibnitz formula that

$$D^s v = \sum_{j=o}^{s} c_j^s (1+|x|^2)^{\frac{k-j}{2}} D^{s-j} u \tag{34}$$

From where it follows by the decay of the moments of derivatives Theorem 4.1 that

$$\|D^s v\|_2 \leq C_o \sum_{j=0}^{s} (1+t)^{-(\mu+m/2+(s-j)/2)(1-2(k-j)/)n} \tag{35}$$

Since the function $f(j) = (\mu + m/2 + (s-j)/2)(1 - 2(k-j)/n)$ is increasing it has a minimum at $j = 0$. Thus we have

$$\|D^s v\|_2 \leq C_o (1+t)^{-(\mu+m/2+s/2)(1-2k/)n} \tag{36}$$

In particular when $s = o$

$$\|v\|_2 \leq C_o (1+t)^{-(\mu+m/2)(1-2k/n} \tag{37}$$

We apply Gagliardo Nirenberg's inequality

$$\|v(x,t)\|_\infty \leq \|v(x.t)\|_2^{1-a} \|D^s v\|_2^a \tag{38}$$

where

$$0 = a(1/2 - s/n) + (1-a)1/2$$

Thus $a = n/2s$ and for $a > 1$ we need $s > [n/2]$. Hence combining the Gagliardo-Nirenberg inequality with (**??**) and (**??**) yields

$$|(1+|x|^2)^{k/2} D^\alpha u| \leq ||v(t)||_\infty \leq C_o (1+t)^{\rho_o}.$$

Where

$$\begin{aligned} \rho_o &= (\mu + m/2 + s/2)(1 - 2k/n)n/2s + (1 - n/2s)(\mu + m/2) \\ &= (\mu + m/2)(1 - 2k/n) + (n/4(1 - 2k/n) \end{aligned}$$

We note that the above value of ρ_o is independent of s. Thus we could have obtained using only the s derivative with $s > [n/2]$. In particular note that when $n = 3$ than it suffices to use $s = 2$ and $\rho = (m/2 + \mu)(1 - 2k/3) + (1 - 2k/3)3/4$ The proof is now complete.

4.1 Comparison with the heat equation

It is easy to show that the Heat kernel ie $E(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$ which is the fundamental solution of the Heat equation , ie the linear part of Navier-Stokes has the following asymptotic behavior

$$|D^{\alpha} E(x,t)| \leq c_o |x|^{-a} t^{-b},$$

where $a + 2b = n + m$, with $m = |\alpha|$.It is also easy to show that there is a large class of solutions to the heat equation which will have the same type of decay. For instance solutions such that the data satisfies $u_0 \in \mathcal{K}$ where $\mathcal{K} = \{u_o(y) : u_o/geqe^{-y^2/4t_o}\}$, will have the above type of decay provided we are considering $t \geq t_o + \epsilon$ In the case of solutions to the Navier-Stokes equations the relation that holds between the decay in space in time is if we take $\mu = n/4$

$$2\rho_o + 2k = m + n - \frac{2km}{n}$$

.

We note that for $k = 0$ we recuperate the decay of the heat equation ,but this only gives decay in time If $m = 0$ we recuperate the relation $2\rho_o + 2k = m + n$ ie we have the same decay relation in space and time as for solutions to the heat equation.