The Riemann-Siegel Integral Formula for Dirichlet Series Associated to Cusp Forms

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0. Introduction. One of the most powerful tools for the study of Dirichlet series are suitable integrals representing them. In many cases such integrals allow the deduction of analytic (e.g. functional equation) or asymptotic properties of the corresponding series. In the case of the Riemann zeta function, which naturally has received the most attention, it is the celebrated *Riemann-Siegel integral formula* serving both purposes. Riemann discovered this particular formula in the middle of the 19th century, but did not publish it. Afterwards it fell into oblivion until it was rediscovered in 1926 by Bessel-Hagen in Riemann's *Nachlaß* and published a few years later by Carl Ludwig Siegel [7]. The formula reads

$$\zeta(s) = \int_{0\searrow 1} \frac{x^{-s} e^{-\pi i x^2}}{e^{\pi i x} - e^{-\pi i x}} \mathrm{d}x + \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \int_{0\swarrow 1} \frac{x^{s-1} e^{\pi i x^2}}{e^{\pi i x} - e^{-\pi i x}} \mathrm{d}x.$$

Here $0 \searrow 1$ (and similarly $0 \swarrow 1$) denotes a straight line from $-\infty e^{-\pi i/4}$ to $\infty e^{-\pi i/4}$ cutting the real axis between 0 and 1. From this representation the functional equation of the zeta function follows immediately. Moreover, employing the saddle point method, the integrals can be evaluated very precisely for $t \to \infty$, thus giving the asymptotic expansion of $\zeta(s)$ (*Riemann-Siegel formula*).

Note that the above equation can also be written in the form

$$\zeta(s) = f(s) + X(s)\overline{f(1-\overline{s})}, \quad X(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})},$$

where

$$f(s) = \int_{0\searrow 1} \frac{x^{-s} e^{-\pi i x^2}}{e^{\pi i x} - e^{-\pi i x}} \mathrm{d}x. \tag{0}$$

In view of the numerous applications of the Riemann-Siegel integral formula [3] one may ask whether other types of Dirichlet series admit similar integral representations. In the case of Dirichlet L functions Siegel's proof can directly be generalized [8]. Apart from these no other formulas of this kind have been found up to now.

The purpose of the present paper is to show how to obtain an analogue for Dirichlet series φ associated to cusp forms. The main idea is to modify the Mellin integral for φ so as to exploit the behaviour of the underlying cusp form in the vicinity of a cusp of the modular group. The essential part of the argument is given in Section 2 in its simplest form in order to show the principle most clearly. In Section 3 we shall obtain a large generalization, extending the method to cusp forms for any conguence subgroup of $SL_2(\mathbf{Z})$ and an arbitrary cusp. The final section contains a similar formula for the function $\zeta(s)\zeta(s+1)$ which may be thought of as corresponding to a modular form of weight 1. But in this case the presence of a pole at $s=1$ and $s=2$ causes additional difficulties which can, however, be overcome with a modified method.

One of the main application of the Riemann-Siegel integral formula is the famous Riemann-Siegel formula mentioned above. In the present case one expects to be able to derive similar results for cusp forms. This is indeed possible [2], and extends the class of Dirichlet series admitting this type of asymptotic expansion. Up to now the only examples known are $\zeta(s)$ [7], Dirichlet L functions $L(s,\chi)$ [8] and $\zeta^2(s)$ [5] (compare also Jutila's results [4]).

1. Cusp Forms for the Modular Group. Let k be an even integer and denote by S_k the $\mathbf{C}$ vector space of $SL_2(\mathbf{Z})$ cusp forms of weight k, i.e. $f \in S_k$ if, and only if, the following conditions are satisfied:

I) $f:\mathcal{H}\to\mathbf{C}$ is holomorphic, where $\mathcal{H}$ denotes the upper half plane.
II) $f(\frac{az+b}{cz+d})=(cz+d)^k f(z)$ for $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in SL_2(\mathbf{Z})$.
III) f has a Fourier expansion

$$f(z)=\sum_{n=1}^{\infty} a(n)e^{2\pi i n z},\quad \mathrm{Im}(z)>0.$$

The most well known example is the discriminant

$$\Delta(z)=e^{2\pi i z}\prod_{n=1}^{\infty}(1-e^{2\pi i n z})^{24}=\sum_{n=1}^{\infty}\tau(n)e^{2\pi i n z}$$

of weight 12, whose Fourier coefficients are given by Ramanujan's tau function.

Now let $f\in S_k$ be given. For our purposes it will be convenient to work with the function ψ defined by

$$\psi(x)=f(ix)=\sum_{n=1}^{\infty} a(n)e^{-2\pi n x},\quad \mathrm{Re}(x)>0, \tag{1}$$

Then ψ is holomorphic in the right half plane $\mathrm{Re}(x)>0$. The transformation II) yields with the matrix $\left(\begin{smallmatrix} 0 & 1\\ -1 & 0\end{smallmatrix}\right)$

$$f(z)=z^{-k}f\left(-\frac{1}{z}\right),\quad \psi(x)=(-1)^{\frac{k}{2}}x^{-k}\psi\left(\frac{1}{x}\right). \tag{2}$$

The associated Dirichlet series $\varphi(s)=\sum_{n=1}^{\infty}a(n)n^{-s}$ converges absolutely provided $\sigma>\frac{k}{2}+1$. This follows at once from Hecke's estimate $a(n)=O(n^{\frac{k}{2}})$. Moreover, by Mellin's formula

$$R(s):=(2\pi)^{-s}\Gamma(s)\varphi(s)=\int_0^{\infty}\psi(x)x^{s-1}\mathrm{d}x. \tag{3}$$

Since $\psi(x)$ vanishes exponentially fast for $x\to\infty$, and by (2) also for $x\to 0$, the integral in (3) converges absolutely for all complex s. Hence R, and consequently φ, are entire functions. We modify the integral as follows. Let δ be a real number, $0\le|\delta|<\frac{\pi}{2}$ and $\xi=e^{i\delta}$. Turning the line of integration about the origin and splitting at $x=\xi$, we get

$$\begin{aligned} R(s)&=\int_0^{\xi\infty}\psi(x)x^{s-1}\mathrm{d}x=\left(\int_0^{\xi}+\int_{\xi}^{\xi\infty}\right)\psi(x)x^{s-1}\mathrm{d}x\\ &=\int_{\xi}^{\xi\infty}\psi(x)x^{s-1}\mathrm{d}x+(-1)^{\frac{k}{2}}\int_{\xi^{-1}}^{\xi^{-1}\infty}\psi(x)x^{k-s-1}\mathrm{d}x.\end{aligned}$$

Here we can again rotate the path of integration so as to run in a direction parallel to the positive real axis. Hence we have proved

Theorem 1: *Let δ be real, $|\delta|<\frac{\pi}{2}$, $\xi=e^{i\delta}$. Then for all complex s except $s=k,k+1,\dots$*

$$\varphi(s)=T_0(s,\xi)+(-1)^{\frac{k}{2}}X(s)T_0(k-s,\xi^{-1}),$$

where

$$T_0(s,\xi)=(2\pi)^s\Gamma(s)^{-1}\int_{\xi}^{\xi+\infty}\psi(x)x^{s-1}\mathrm{d}x,\quad X(s)=(2\pi)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}.$$

We shall now show that it is possible to take the limit $\delta\to\frac{\pi}{2}$, i.e. $\xi\to i$, in the integral occurring above. This possibility depends on the fact that f is a cusp form and consequently $\psi(x)$ decays rapidly as x tends to i. Define temporarily

$$I(\xi)=\int_{\xi}^{\xi+\infty}\psi(x)x^{s-1}\mathrm{d}x=\left(\int_0^1+\int_1^{\infty}\right)\psi(\xi+x)(\xi+x)^{s-1}\mathrm{d}x=I_0+I_1, \tag{4}$$

say. For I_1 we obtain immediately

$$\lim_{\xi\to i} I_1 = \int_1^\infty \psi(i+x)(i+x)^{s-1}\mathrm{d}x = \int_1^\infty \psi(x)(i+x)^{s-1}\mathrm{d}x.$$

Concerning I_0 we use the functional equation II) of f. Then $f(\frac{-1}{z+1}) = (z+1)^k f(z)$, provided $\mathrm{Im}(z) > 0$. Passing to the function ψ yields $\psi(x) = (ix+1)^{-k}\psi(\frac{1}{x-i})$ if $\mathrm{Re}(x) > 0$. Thus we get

$$\begin{aligned} I_0 &= \int_0^1 \psi(\xi+x)(\xi+x)^{s-1}\mathrm{d}x = \int_0^1 \psi\left(\frac{1}{\xi-i+x}\right)(i\xi+1+ix)^{-k}(\xi+x)^{s-1}\mathrm{d}x \\ &= \int_{\xi-i}^{\xi-i+1} \psi\left(\frac{1}{x}\right)(ix)^{-k}(i+x)^{s-1}\mathrm{d}x = i^{-k}\int_{(\xi-i+1)^{-1}}^{(\xi-i)^{-1}} \psi(x)x^{k-2}(i+x^{-1})^{s-1}\mathrm{d}x. \end{aligned}$$

Here $\lim_{\xi\to i}(\xi-i+1)^{-1} = 1$ and $\lim_{\xi\to i}(\xi-i)^{-1} = \infty$. The integrand decreases exponentially at the upper limit of integration, so the integral converges for $\xi \to i$. Hence the limit $\lim_{\xi\to i} I(\xi) = \lim_{\xi\to i} I_0 + \lim_{\xi\to i} I_1$ exists and together with (4) we have

$$I(i) := \lim_{\xi\to i} I(\xi) = \int_i^{i+\infty} \psi(x)x^{s-1}\mathrm{d}x = \int_0^\infty \psi(x)(i+x)^{s-1}\mathrm{d}x.$$

This shows that we have proved the following result:

Theorem 2: *Let $f \in S_k$, $0 \le \delta < \frac{\pi}{2}$, $\xi = e^{i\delta}$, $s \in \mathbf{C}$, and define $T_0(s,\xi)$ as in Theorem 1. Then $T(s) := \lim_{\xi\to i} T_0(s,\xi)$ exists and equals*

$$T(s) = (2\pi)^s\Gamma(s)^{-1}\int_0^\infty \psi(x)(i+x)^{s-1}\mathrm{d}x.$$

Moreover,

$$\varphi(s) = T(s) + (-1)^{\frac{k}{2}}X(s)\overline{T(k-\overline{s})},$$

provided the Fourier coefficients of f are real numbers. Otherwise ψ has to be placed by $\overline{\psi}$ in the formula for $\overline{T(k-\overline{s})}$.

We are now ready to derive the analogue of the Riemann-Siegel integral formula. It then appears that a new type of functions enters the stage which has hitherto received much less attention than it deserves. We define it by

$$F(z) = 2\pi\int_0^\infty e^{2\pi xz}\psi(x)\mathrm{d}x, \quad f \in S_k,\ \psi(x) = f(ix),\ \mathrm{Re}(z) < 1. \tag{5}$$

Since $\psi(x) = O(e^{-2\pi x})$ for $x \to \infty$, it is easily seen that the integral converges absolutely and uniformly on compact subsets of the left half plane $\mathrm{Re}(z) < 1$. Hence F is holomorphic in $\mathrm{Re}(z) < 1$. Furthermore $F(z) = O(1)$ uniformly in $\mathrm{Re}(z) \le 1-\varepsilon$ for any $\varepsilon > 0$. Obviously, F is essentially the Laplace transform of the cusp form f. In our derivation it occurs in a natural way and it seems to play an important role in other investigations related to the Riemann-Siegel formula [2].

Consider the integral

$$I(s) = \frac{1}{2\pi i}\int_\Lambda e^{2\pi iz}F(z)z^{-s}\mathrm{d}z. \tag{6}$$

Here Λ denotes a loop around the positive imaginary axis, starting from $e^{-3\pi i/2}\infty$, running to $e^{-3\pi i/2}\varepsilon$ (where $0 < \varepsilon < 1$ is arbitrary), encircling the origin in the positive directon to $e^{\pi i/2}\varepsilon$, and finally returning to $e^{\pi i/2}\infty$. Thus the argument of z varies in the interval $-\frac{3\pi}{2} \le \arg(z) \le \frac{\pi}{2}$. Assume that s is confined to a compact subset of the complex plane. Then the integral $I(s)$ converges absolutely and uniformly. Hence I is an entire function. For the time being we shall assume $\mathrm{Re}(s) < 1$. Then ε above can be taken as small as we please. In particular we can let $\varepsilon \to 0$. Parametrizing the path of integration by $z = ue^{-3\pi i/2}$ and $z = ue^{\pi i/2}$, $u \geq 0$ real, we find

$$\begin{aligned} I(s) &= \frac{i}{2\pi i}\left(\int_{\infty}^{0} e^{-2\pi u}F(iu)(e^{-\frac{3\pi i}{2}}u)^{-s}\mathrm{d}u + \int_{0}^{\infty} e^{-2\pi u}F(iu)(e^{\frac{\pi i}{2}}u)^{-s}\mathrm{d}u\right) \\ &= -\frac{i}{2\pi i}e^{\frac{\pi i s}{2}}\left(e^{\pi i s} - e^{-\pi i s}\right)\int_{0}^{\infty} e^{-2\pi u}F(iu)u^{-s}\mathrm{d}u. \end{aligned} \tag{7}$$

Using the definition (5) of $F(iu)$ gives

$$\begin{aligned} \int_0^\infty e^{-2\pi u}F(iu)u^{-s}\mathrm{d}u &= 2\pi\int_0^\infty e^{-2\pi u}u^{-s}\int_0^\infty e^{2\pi i x u}\psi(x)\mathrm{d}x\mathrm{d}u = 2\pi\int_0^\infty \psi(x)\int_0^\infty e^{-2\pi u(1-ix)}u^{-s}\mathrm{d}u\mathrm{d}x \\ &= (2\pi)^s\Gamma(1-s)\int_0^\infty \psi(x)(1-ix)^{s-1}\mathrm{d}x. \end{aligned}$$

The interchange of the order of integration is permitted by absolute convergence. Inserting this formula into (7), using $\pi^{-1}\Gamma(1-s)\sin\pi s = \Gamma(s)^{-1}$ and the formula from Theorem 2, we get *a fortiori*

$$I(s) = (2\pi)^s\Gamma(s)^{-1}\int_0^\infty \psi(x)(x+i)^{s-1}\mathrm{d}x = T(s).$$

The restriction $\mathrm{Re}(s) < 1$ can now be removed by analytic continuation, so that we have proved

Theorem 3: *Let $f \in S_k$ and $T(s)$ be defined as in Theorem 2. Then for all complex numbers s*

$$T(s) = \frac{1}{2\pi i}\int_\Lambda e^{2\pi i z}F(z)z^{-s}\mathrm{d}z,$$

where F is the Laplace transform of f as defined by (5).

Theorem 2 and 3 are the desired analogs of the Riemann-Siegel integral formula. In fact, the Laplce transform of the cusp form f occurs as a "kernel" in the integral for $T(s)$, exactly as the function $\csc(x) = 2i(e^{\pi i x} - e^{-\pi i x})^{-1}$ occurs in the formula for $f(s)$ of Section 0. This is no coincidence since the the cosecant function can be considered as the Laplace transform of the theta function. Also it can easily be shown [2], that F has simple poles at the positive integers, similar to the cosecant. Consequently we have a very close relationship between our formula and Riemann's, which can be proved along the same lines [1].

2. Cusps Forms for Congruence Subgroups. In this section we shall consider a much more general case using the same principle. In fact, we now allow cusp forms belonging to an arbitrary congruence subgroup of the modular group. Moreover, instead of the limit $\xi \to i$, which corresponds to the cusp -1, we can use any cusp $\frac{p}{q} \neq 0$ to play the same role as $\frac{p}{q} = -1$.

We use the standard notation. Thus let $f : \mathcal{H} \to \mathbf{C}$ be a function, $\gamma \in GL_2^+(\mathbf{Q})$, and k be a positive integer. Then define a new function $f|[\gamma]_k$ by $f|[\gamma]_k(z) = (\det\gamma)^{\frac{k}{2}}(cz+d)^{-k}f(\gamma z)$. If Γ is any congruence subgroup of the modular group, say $\Gamma(N) \subseteq \Gamma$, we let $S_k(\Gamma)$ denote the $\mathbf{C}$ vector space of cusp forms of weight k belonging to Γ. Consequently, $f \in S_k(\Gamma)$ if, and only if, the following conditions are satisfied:

I) $f : \mathcal{H} \to \mathbf{C}$ is holomorphic.
II) $f|[\gamma]_k = f$ for each $\gamma \in \Gamma$.
III) If $\gamma_0 \in SL_2(\mathbf{Z})$ and $g = f|[\gamma_0]_k$, then there exist complex numbers b_n such that

$$g(z) = \sum_{n=1}^\infty b_n e^{\frac{2\pi i n z}{N}},\ \mathrm{Im}(z) > 0.$$

If N is a positive integer set

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbf{Z}) | c \equiv 0(N),\right\},\ \Gamma_1(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbf{Z}) | a \equiv d \equiv 1(N), c \equiv 0(N),\right\}.$$

Then $\Gamma_0(N)$, $\Gamma_1(N)$ are congruence subgroups of level N and $\Gamma_1(N) \subseteq \Gamma_0(N)$. Let Γ denote any congruence subgroup. By conjugation, the study of $S_k(\Gamma)$ is reduced to that of $S_k(\Gamma_1(N))$ for suitable N. The space $S_k(\Gamma_1(N))$ of cusp forms of weight k can be further decomposed. Let χ be a Dirichlet charcter modulo N and

$$S_k(N,\chi) = \left\{ f \in S_k(\Gamma_1(N)) | f|[\gamma_0]_k = \chi(d) f,\ \gamma_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N) \right\}.$$

Then $S_k(\Gamma_1(N)) = \oplus_{\chi \bmod N} S_k(N,\chi)$. We shall therefore assume from now on that $f \in S_k(N,\chi)$. Each such f has a Fourier expansion $\sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$, $\mathrm{Im}(z) > 0$, since $f(z+1) = f(z)$. As before we shall also employ the function ψ_f defined by

$$\psi_f(x) = f\left(\frac{ix}{\sqrt{N}}\right) = \sum_{n=1}^{\infty} a(n) e^{-2\pi n x/\sqrt{N}}, \quad \mathrm{Re}(x) > 0. \tag{8}$$

We define the associated Dirichlet series and the Mellin transform of ψ_f through

$$L(s,f) = \sum_{n=1}^{\infty} a(n) n^{-s}, \quad R(s,f) = \int_0^{\infty} \psi_f(x) x^{s-1} \mathrm{d}x = \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s) L(s,f). \tag{9}$$

Now let $w_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ and define g by $g = f|[w_N]_k$, i.e. $g(z) = N^{-\frac{k}{2}} z^{-k} f(-\frac{1}{Nz})$. Then it follows easily $g \in S_k(N,\overline{\chi})$. Hence g has a Fourier expansion of the same type as f, and we write $g(z) = \sum_{n=1}^{\infty} b(n) e^{2\pi i n z}$, $\mathrm{Im}(z) > 0$. The relation $f|[w_N]_k = g$ then yields

$$\psi_f(x) = i^k x^{-k} \psi_g\left(\frac{1}{x}\right), \quad R(s,f) = i^k R(k-s,g). \tag{10}$$

We are now prepared to derive the analog of Theorem 2.

Theorem 4: *Let N,k be positive integers, χ a Dirichlet character modulo N, $f \in S_k(N,\chi)$. For s,ξ complex with* $\mathrm{Re}(\xi) > 0$ *define*

$$T_0(s,f,\xi) = \left(\frac{2\pi}{\sqrt{N}}\right)^{s} \Gamma(s)^{-1} \int_{\xi}^{\xi+\infty} \psi_f(x) x^{s-1} \mathrm{d}x,$$

where $\psi_f(x) = f(\frac{ix}{\sqrt{N}})$. If p,q are integers with $pq \neq 0$ and $(p,q) = 1$ and if $\xi = i\frac{p}{q}\sqrt{N} e^{-i\delta}$ with $0 < \delta \leq \frac{\pi}{2}$ for $\frac{p}{q} > 0$ resp. $-\frac{\pi}{2} \leq \delta < 0$ for $\frac{p}{q} < 0$, then the limit $T(s,f,\frac{p}{q}) := \lim_{\xi \to ip\sqrt{N}/q} T_0(s,f,\xi)$ exists. Moreover, the associated Dirichlet series $L(s,f)$ has the representation

$$L(s,f) = T\left(s,f,\frac{p}{q}\right) + i^k X(s) T\left(k-s,g,-\frac{q}{pN}\right), \quad g = f|[w_N]_k, \quad X(s) = (2\pi/\sqrt{N})^{2s-k} \Gamma(k-s)/\Gamma(s)$$

.

Proof: The method is exactly parallel to that of Section 1, but the behaviour at the cusps has to be analyzed more carefully. We first have for any complex ξ with $\mathrm{Re}(\xi) > 0$

$$R(s,f) = \int_0^{\xi\infty} \psi_f(x) x^{s-1} \mathrm{d}x = \left(\int_0^{\xi} + \int_{\xi}^{\xi\infty}\right) \psi_f(x) x^{s-1} \mathrm{d}x. \tag{11}$$

Applying the transformation formula in (10) and turning the line of integration we get

$$R(s,f) = \int_{\xi}^{\xi+\infty} \psi_f(x) x^{s-1} \mathrm{d}x + i^k \int_{\xi^{-1}}^{\xi^{-1}+\infty} \psi_g(x) x^{k-s-1} \mathrm{d}x. \tag{12}$$

Hence using the definition of $T_0(s,f,\xi)$

$$L(s,f) = T_0(s,f,\xi) + i^k X(s) T_0(k-s,g,\xi^{-1}), \quad X(s) = \left(\frac{2\pi}{\sqrt{N}}\right)^{2s-k} \frac{\Gamma(k-s)}{\Gamma(s)}, \quad g = f|[w_N]_k.$$

We now consider special values of the parameter ξ, namely $\xi = i\frac{p}{q}\sqrt{N}e^{-i\delta}$, with $0 < \delta \le \frac{\pi}{2}$ for $p/q > 0$ and $-\frac{\pi}{2} \le \delta < 0$ for $p/q < 0$. In order to justify the limit operation $\delta \to 0$ we first observe $\psi_f(\xi) = f\left(\frac{i\xi}{\sqrt{N}}\right) = f\left(-\frac{p}{q}e^{-i\delta}\right)$. Here $-\frac{p}{q}e^{-i\delta} = -\frac{p}{q}+w$ where $w = \frac{p}{q}(1-e^{-i\delta})$. Thus $\mathrm{Im}(w) = \frac{p}{q}\sin\delta > 0$ and $w \to 0$ is equivalent to $\delta \to 0$. Let b, d be integers satisfying $pd + bq = 1$. Consequently, $A := \left(\begin{smallmatrix} p & b \\ -q & d \end{smallmatrix}\right) \in SL_2(\mathbf{Z})$. Moreover, let $f_1 := f|[A]_k$. Then f_1 has a Fourier expansion of the shape $f_1(z) = \sum_{n=1}^{\infty} c_n e^{2\pi i n z/N}$, $\mathrm{Im}(z) > 0$. This follows from property III) above. By definiton $f(Az) = (-qz + d)^k f_1(z)$. In particular $Az = w - \frac{p}{q}$ where $z = -\frac{1}{q^2 w} + \frac{d}{q}$. This implies

$$\psi_f(\xi) = f\left(w - \frac{p}{q}\right) = f(Az) = (-qz + d)^k f_1(z) = (qw)^{-k} f_1\left(-\frac{1}{q^2 w} + \frac{d}{q}\right).$$

The Fourier series for f_1 shows that $f_1(-\frac{1}{q^2 w} + \frac{d}{q})$ decays exponentially provided w tends to zero. As this is equivalent to $\delta \to 0$ and $\xi \to i\frac{p}{q}\sqrt{N}$, we see that $\psi_f(\xi)$ decays exponentially as $\xi \to i\frac{p}{q}\sqrt{N}$. Hence the limit operation $\xi \to i\frac{p}{q}\sqrt{N}$ in (12) is permitted, which concludes the proof of the theorem.

Having derived this basic representation for $T_0(s, f, \xi)$ it is now easy to get the general version of the Riemann-Siegel integral formula. We state the final result, whose proof is practically the same as that of Theorem 3, as follows.

Theorem 5: *Let N, k be positive integers, χ a Dirichlet character modulo N, $f \in S_k(N, \chi)$. Let s be a complex number and p, q be positive integers with $(p, q) = 1$. Define $T(s, f, \frac{p}{q})$ as in Theorem 4. Then*

$$T\left(s, f, \frac{p}{q}\right) = \frac{1}{2\pi i}\int_{\Lambda} e^{2\pi i \frac{p}{q} z} z^{-s} \mathcal{L}\{f, \frac{p}{q}\}(z)\mathrm{d}z,$$

where Λ denotes a loop around the positive imaginary axis. If $g = f|[w_N]_k$, then the associated Dirichlet series satisfies

$$L(s, f) = T\left(s, f, \frac{p}{q}\right) + i^k X(s)\overline{T\left(k - \overline{s}, \overline{g}, \frac{q}{pN}\right)}.$$

In this formula

$$\mathcal{L}\{f, \frac{p}{q}\}(z) = 2\pi\int_0^{\infty} e^{2\pi x z}\psi_f\left(i\frac{p}{q}\sqrt{N} + x\sqrt{N}\right)\mathrm{d}x$$

is the Laplace transform with parameter $\frac{p}{q}$ of ψ_f. It is holomorphic in the left half plane $\mathrm{Re}(z) < 1$.

3. The function $\zeta(s)\zeta(s+1)$. We have already remarked that other types of Dirichlet series also admit integral representations similar to those given above. In this section we derive several formulas for $\zeta(s)\zeta(s+1)$. It corresponds essentially to the Dedekind eta function. In this case, however, $\eta(z)$ is not a cusp form, but has a simple pole at each cusp. This different behaviour causes some additional difficulties. They can be overcome by first applying a suitable integration by parts and then proceeding as before.

If we let $R(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$, the function R satisfies $R(s) = R(1 - s)$, which is nothing but the functional equation for $\zeta(s)$. Moreover, R is meromorphic, having only simple poles at $s = 0$, $s = 1$, with residues being equal to -1 and 1, respectively. We further define

$$f(s) = \tfrac{1}{2}R(s)R(s + 1). \tag{13}$$

From $R(s) = R(1 - s)$ we get the functional equation $f(s) = f(-s)$. Also, f is holomorphic in the complex plane, except for simple poles at $s = -1, 1$ and a double pole at $s = 0$. Using the definition of R and the duplication formula for the gamma function, we obtain

$$f(s) = (2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s + 1). \tag{14}$$

Inserting the Dirichlet series

$$\zeta(s)\zeta(s+1) = \sum_{n=1}^{\infty} \sigma_{-1}(n) n^{-s}, \quad \sigma_{-1}(n) = \sum_{d|n} d^{-1}, \tag{15}$$

which converges absolutely for $\sigma = \mathrm{Re}(s) > 1$, and using the gamma integral, yields the basic formula

$$f(s) = \int_0^\infty \psi(x) x^{s-1} \mathrm{d}x, \quad \sigma > 1, \quad \psi(x) = \sum_{n=1}^{\infty} \sigma_{-1}(n) e^{-2\pi n x}, \quad \mathrm{Re}(x) > 0. \tag{16}$$

Obviously, ψ is holomorphic in the half plane $\mathrm{Re}(x) > 0$. This function is well known, since it essentially equals the logarithm of the Dedekind eta function, viz. $\log \eta(ix) = -\frac{\pi x}{12} - \psi(x)$. Some important properties can be deduced by inverting the Mellin transform (16). This gives

$$\psi(x) = \frac{1}{2\pi i} \int_{(c)} f(s) x^{-s} \mathrm{d}s, \quad c > 1.$$

For example, using the functional equation $f(s) = f(-s)$, substituting $-s = w$ in the integral, shifting the contour to the right and evaluating the residues, leads to the formula

$$\psi(x) = \frac{\pi}{12} x^{-1} - \frac{\pi}{12} x + \frac{1}{2} \log x + \psi\left(\frac{1}{x}\right). \tag{17}$$

Here, as usual, $\mathrm{Re}(x) > 0$ and $\log x$ takes its principal value, i.e. $|\arg(x)| < \frac{\pi}{2}$.

Besides this well known function ψ, we need a new one, ψ_1, which we define through

$$\psi_1(x) = \sum_{n=1}^{\infty} \sigma_{-1}(n) n^{-1} e^{-2\pi n x}, \quad \mathrm{Re}(x) > 0. \tag{18}$$

Its chief properties are summarized in

Theorem 6: *For complex x with positive real part let $\psi_1(x)$ be defined by (18). Then the function ψ_1 is holomorphic in the right half plane $\mathrm{Re}(x) > 0$ and $\psi_1' = -2\pi\psi$. Moreover, $\psi_1(x) = 2\pi \int_x^\infty \psi(u)\mathrm{d}u$, where the path of integration may be any rectifiable curve extending to infinity and lying in a sector $|\arg(u)| \leq \frac{\pi}{2} - \delta < \frac{\pi}{2}$ ($\delta > 0$ fixed). Finally, ψ_1 satisfies the functional equation*

$$\psi_1(x) = A \log x + B + Cx \log x + Dx + Ex^2 - 2\pi x \sum_{n=1}^{\infty} \sigma_{-1}(n) E_2\left(\frac{2\pi n}{x}\right). \tag{19}$$

Here $A, \ldots, E$ are suitable complex numbers not depending on x, and the function E_2 (generalized exponential integral) is given by $E_2(z) = \int_1^\infty e^{-zt} t^{-2} \mathrm{d}t$, $\mathrm{Re}(z) \geq 0$.

Proof: All statements are easy deductions from the definitions (16) and (18) of ψ and ψ_1, except the last two concerning the functional equation. To show these, one may proceed as in the case of $\psi(x)$, using the Mellin transform

$$\int_0^\infty \psi_1(x) x^{s-1} \mathrm{d}x = (2\pi)^{-s} \Gamma(s) \zeta(s+1) \zeta(s+2) = \frac{2\pi}{s} f(s+1),$$

and its reciprocal

$$\frac{1}{2\pi} \psi_1(x) = \frac{1}{2\pi i} \int_{(c)} f(s+1) \frac{x^{-s}}{s} \mathrm{d}s, \quad c > 0.$$

Applying the functional equation for f, shifting the contour appropriately, taking into account the double poles at $s = -1, 0$, as well as the simple one at $s = 1$, leads to

$$\psi_1(x) = A \log x + B + Cx \log x + Dx + Ex^2 - 2\pi x \frac{1}{2\pi i} \int_{(c)} f(s) \frac{x^s}{s+1} \mathrm{d}s, \ c > 1.$$

Inserting (14), (15) and noting

$$\frac{1}{2\pi i}\int_{(c)}\Gamma(s)\frac{z^s}{s+1}\mathrm{d}s = E_2\left(\frac{1}{z}\right),$$

proves the assertion. Another, and possibly more direct, way to obtain the modular relation (19) consists in using the corresponding equation (17) for ψ and integrating. For,

$$\psi_1(x) = 2\pi\int_x^\infty \psi(u)\mathrm{d}u = 2\pi\int_x^1 \psi(u)\mathrm{d}u + \psi_1(1).$$

If $\mathrm{Re}(x) > 0$, we get from (17)

$$\begin{aligned}\int_x^1 \psi(u)\mathrm{d}u &= \int_x^1\left[\frac{\pi}{12}u^{-1} - \frac{\pi}{12}u + \frac{1}{2}\log u + \psi\left(\frac{1}{u}\right)\right]\mathrm{d}u\\ &= -\frac{\pi}{12}\log x - \frac{\pi}{24}(1-x^2) + \frac{1}{2}(-1 - x\log x + x) + \int_1^{x^{-1}}\psi(u)u^{-2}\mathrm{d}u.\end{aligned}$$

Write the last integral as

$$\int_1^\infty \psi(u)u^{-2}\mathrm{d}u - \int_{x^{-1}}^\infty \psi(u)u^{-2}\mathrm{d}u = \int_1^\infty \psi(u)u^{-2}\mathrm{d}u - x\int_1^\infty \psi\left(\frac{u}{x}\right)u^{-2}\mathrm{d}u,$$

where

$$\int_1^\infty \psi\left(\frac{u}{x}\right)u^{-2}\mathrm{d}u = \sum_{n=1}^\infty \sigma_{-1}(n)E_2\left(\frac{2\pi n}{x}\right).$$

This concludes the derivation of (19) and gives the values

$$A = -\frac{\pi^2}{6},\ C = -\pi,\ D = \pi,\ E = \frac{\pi^2}{12},\ B = \psi_1(1) + 2\pi\int_1^\infty \psi(u)u^{-2}\mathrm{d}u - \frac{\pi^2}{12} - \pi.$$

Hence, Theorem 6 is completely proved.

Note that the infinite series involving E_2 is absolutely convergent. This follows from $E_2(z) = O(e^{-z}),\quad z \to \infty,\ |\arg(z)| < \pi$.

On the basis of the preceding investigations, we are now going to derive a new formula for $f(s)$, as given by the following result.

Theorem 7: *For $s \in \mathbf{C}$, except $s = \pm 1$ and $s = 0$*

$$f(s) = \frac{s-1}{2\pi}\int_0^\infty \psi_1(x)(x+i)^{s-2}\mathrm{d}x - \frac{s+1}{2\pi}\int_0^\infty \psi_1(x)(x-i)^{-s-2}\mathrm{d}x + H(s),$$

where H is defined by

$$H(s) = -\frac{\pi i}{12}e^{\frac{\pi i s}{2}}\left(\frac{1}{s-1} + \frac{1}{s+1}\right) + \frac{e^{\frac{\pi i s}{2}}}{2s}\left(\frac{\pi i}{2} - \frac{1}{s}\right).$$

Proof: Let w be a complex number having positive real part and assume temporarily $\sigma > 1$. If $\arg(w) = \varphi$ with $|\varphi| < \frac{\pi}{2}$, we may turn the line of integration in (16) around the origin. Splitting the integral at $x = w$ and applying the functional equation (17) to the finite part (i.e. from 0 to w), we thus obtain

$$f(s) = \int_w^{\infty e^{i\varphi}} \psi(x)x^{s-1}\mathrm{d}x + \int_{w^{-1}}^{\infty e^{-i\varphi}} \psi(x)x^{-s-1}\mathrm{d}x + H(s,w),$$

with

$$H(s,w) = \frac{\pi}{12}\left(\frac{w^{s-1}}{s-1} - \frac{w^{s+1}}{s+1}\right) + \frac{w^s}{2s}\left(\log w - \frac{1}{s}\right). \tag{20}$$

As usual, $\log w$ denotes the principal branch of the logarithm, where $|\operatorname{Im}(\log w)| < \frac{\pi}{2}$. Integrating by parts yields using Theorem 6

$$f(s) = \frac{1}{2\pi}\psi_1(w)w^{s-1} + \frac{s-1}{2\pi}\int_w^{\infty e^{i\varphi}} \psi_1(x)x^{s-2}\mathrm{d}x + $$
$$+ \frac{1}{2\pi}\psi_1(w^{-1})w^{s+1} - \frac{s+1}{2\pi}\int_{w^{-1}}^{\infty e^{-i\varphi}} \psi_1(x)x^{-s-2}\mathrm{d}x + H(s,w).$$

Since $\psi_1(x)$ decays exponentially as $\operatorname{Re}(x) \to \infty$, the paths of integration can again be rotated, so as to run in a direction parallel to the positive real axis. Thus

$$f(s) = \frac{w^s}{2\pi}\left[\psi_1(w)w^{-1} + \psi_1(w^{-1})w\right] + \frac{s-1}{2\pi}\int_w^{w+\infty} \psi_1(x)x^{s-2}\mathrm{d}x -$$
$$- \frac{s+1}{2\pi}\int_{w^{-1}}^{w^{-1}+\infty} \psi_1(x)x^{-s-2}\mathrm{d}x + H(s,w). \tag{21}$$

Up to this point w was aribitrary, subject only to the condition $\operatorname{Re}(w) > 0$. Now we specialize to $w = e^{i\varphi}$, where $\varphi = \frac{\pi}{2} - \delta$ and $0 < \delta < \frac{\pi}{2}$. We are going to show that we may let δ tend to 0, i.e. $w \to i$ in (21). This is, of course, obvious for $H(s,w)$. To discuss the integrals, observe that

$$w = ie^{-i\delta} = \sin\delta + i\cos\delta = i + \delta + O(\delta^2), \quad w^{-1} = -ie^{i\delta} = \sin\delta - i\cos\delta = -i + \delta + O(\delta^2),$$

for $\delta \to 0$. Hence, setting $w = i + u$ and $w^{-1} = -i + u'$, we find $u = \delta + O(\delta^2)$, $u' = \delta + O(\delta^2)$. Moreover, $\operatorname{Re}(u) > 0$, $\operatorname{Re}(u') > 0$ and $|\arg(u)|, |\arg(u')| \le \frac{\pi}{4}$ (say), provided δ is small enough. Applying this to the first integral in (21) yields

$$\int_w^{w+\infty} \psi_1(x)x^{s-2}\mathrm{d}x = \int_u^{u+\infty} \psi_1(x+i)(x+i)^{s-2}\mathrm{d}x = \int_u^{u+\infty} \psi_1(x)(x+i)^{s-2}\mathrm{d}x.$$

But Theorem 6 shows that $\psi_1(x) = O(\log x)$ for x tending to 0 in a sector $|\arg(x)| \le \frac{\pi}{4}$. Consequently, the last integral converges as $u \to 0$, i.e. $\delta \to 0$, $w \to i$. The same argument applies to

$$\int_{w^{-1}}^{w^{-1}+\infty} \psi_1(x)x^{-s-2}\mathrm{d}x = \int_{u'}^{u'+\infty} \psi_1(x)(x-i)^{-s-2}\mathrm{d}x.$$

It remains to discuss the bracketed term in (21). With the same notation as before, namely $w = i + u$, $w^{-1} = -i + u'$, we have using Theorem 6

$$\begin{aligned}\psi_1(w)w^{-1} + \psi_1(w^{-1})w &= \psi_1(u)w^{-1} + \psi_1(u')w \\ &= \left[A\log u + B + O(u\log u)\right]w^{-1} + \left[A\log u' + B + O(u'\log u')\right]w \\ &= (A\log\delta + B)(w^{-1} + w) + O(\delta\log\delta) = O(\delta\log\delta),\end{aligned}$$

since $w^{-1} + w = u + u' = O(\delta)$. Thus, as $\delta \to 0$ (equivalently $w \to i$), the bracketed term in (21) vanishes, giving

$$f(s) = \frac{s-1}{2\pi}\int_i^{i+\infty} \psi_1(x)x^{s-2}\mathrm{d}x - \frac{s+1}{2\pi}\int_{-i}^{-i+\infty} \psi_1(x)x^{-s-2}\mathrm{d}x + H(s,i).$$

The restriction $\sigma > 1$ made at the beginning of the proof can now be dropped by analytic continuation, excluding only the poles of $H(s,i)$. Substitution of the variable of integration and defining $H(s) = H(s,i)$ finally concludes the proof of Theorem 7.

The formula given by this result seems to be new. Its validity depends essentially on the properties of ψ_1, which permitted the limiting value $w = i$ in (21). We were thus able to perform the same limit operation as in Sections 1 and 2, despite the fact that $\psi(x)$ grows like $(x-i)^{-1}$ for $x \to i$. Now the Riemann-Siegel formula can be deduced in the usual manner. As a preliminary step, we divide both sides of the formula of Theorem 7 by $(2\pi)^{-s}\Gamma(s)$ to get

Theorem 8: *For any $s \in \mathbf{C}$ except $s = 0, 1, 2, \ldots$*

$$\zeta(s)\zeta(s+1) = T(s) + X(s)\overline{T(-\bar{s})} + (2\pi)^s\Gamma(s)^{-1}H(s),$$

where

$$T(s) = (2\pi)^{s-1}\Gamma(s-1)^{-1}\int_0^\infty \psi_1(x)(x+i)^{s-2}\mathrm{d}x, \quad X(s) = (2\pi)^{2s}\frac{\Gamma(-s)}{\Gamma(s)},$$

and $H(s)$ is defined as in Theorem 7.

Proof: This is obvious from Theorem 7 and formula (14). The points $s = 0, 1, 2, \ldots$ excluded are poles of $X(s)$ and $H(s)$.

To proceed further, we are going to transform $T(s)$ into a loop integral around the positive imaginary axis. We require some preliminary considerations.

With ψ_1 as in (18) we define a new function F by

$$F(z) = 2\pi\int_0^\infty e^{2\pi xz}\psi_1(x)\mathrm{d}x, \quad \mathrm{Re}(z) < 1. \tag{22}$$

It follows from (18) that $\psi_1(x) = O(e^{-2\pi x})$ as $x \to \infty$. Moreover $\psi_1(x) = O(\log x)$ as x tends to 0 by Theorem 6. Consequently, the above integral converges absolutely and uniformly provided $\mathrm{Re}(z) \le 1 - \varepsilon < 1$. This shows that F is a holomorphic function in the left half plane $\mathrm{Re}(z) < 1$. Furthermore we have

$$|F(z)| \le 2\pi\int_0^\infty e^{2\pi x\,\mathrm{Re}(z)}\psi_1(x)\mathrm{d}x = O\big((1-\mathrm{Re}(z))^{-1}\big)$$

if $\mathrm{Re}(z) < 1$ and this implies that F is uniformly bounded in any half plane $\mathrm{Re}(z) \le 1 - \varepsilon$ if $\varepsilon > 0$ is fixed. Now assume $\sigma < 2$ and consider

$$I(s) = \int_0^\infty e^{-2\pi u}F(iu)u^{1-s}\mathrm{d}u,$$

which is absolutely and uniformly convergent in any strip $\sigma_0 \le \sigma \le \sigma_1 < 2$. Note that $F(iu) = O(1)$ as stated above. Using the definition of F, we get

$$I(s) = 2\pi\int_0^\infty e^{-2\pi u}u^{1-s}\int_0^\infty e^{2\pi xiu}\psi_1(x)\mathrm{d}x\mathrm{d}u = 2\pi\int_0^\infty \psi_1(x)\int_0^\infty e^{-u(2\pi-2\pi ix)}u^{1-s}\mathrm{d}u\mathrm{d}x,$$

where the interchange of the order of integration is permitted by absolute convergence. The inner integral takes the value $(2\pi)^{s-2}(1-ix)^{s-2}\Gamma(2-s)$, hence

$$e^{\frac{\pi i}{2}(s-2)}I(s) = (2\pi)^{s-1}\Gamma(2-s)\int_0^\infty \psi_1(x)(x+i)^{s-2}\mathrm{d}x. \tag{23}$$

Next we consider

$$J(s) = \frac{1}{2\pi i}\int_\Lambda e^{2\pi iz}z^{1-s}F(z)\mathrm{d}z, \tag{24}$$

where Λ is the loop around the positive imaginary axis as described right after the analogous expression (6) for cusp forms. Using the properties of F, we conclude that J is an entire function of s. It is now easy to relate $J(s)$ to $T(s)$ as before. Let us assume that $\sigma < 2$. Then we may let tend ε to 0 in our parametrization of Λ and thus we obtain

$$J(s) = \frac{1}{2\pi i}\left(e^{-\frac{3\pi i}{2}(2-s)}\int_\infty^0 e^{-2\pi u}u^{1-s}F(iu)\mathrm{d}u + e^{\frac{\pi i}{2}(2-s)}\int_0^\infty e^{-2\pi u}u^{1-s}F(iu)\mathrm{d}u\right)$$
$$= \frac{1}{2\pi i}e^{\frac{\pi i}{2}(s-2)}I(s)(e^{-\pi i s} - e^{\pi i s}) = -\frac{1}{\pi}(2\pi)^{s-1}\sin\pi s\Gamma(2-s)\int_0^\infty \psi_1(x)(x+i)^{s-2}\mathrm{d}x$$

by (23). Since $\Gamma(s-1)\Gamma(2-s) = -\pi/\sin\pi s$, we get $J(s) = T(s)$. The restriction of σ to values less than 2 can now be removed by analytic continuation. We have thus proved

Theorem 9: *If Λ denotes a loop around the positive imaginary axis as described above, then*

$$T(s) = \frac{1}{2\pi i}\int_\Lambda e^{2\pi i z}z^{1-s}F(z)\mathrm{d}z$$

for any complex s. The function F is defined by (22).

Again we have found an analogue of the Riemann-Siegel formula. As in the case of cusp forms, one may ask whether a more general version of Theorem 9 is possible. For this purpose it is certainly necessary to investigate the behaviour of ψ_1 in the vicinity of an arbitrary cusp, which we will do next.

If z is complex and $|z| < 1$ let

$$g(z) = -\log\prod_{m=1}^\infty(1-z^m) = \sum_{n=1}^\infty\sum_{m=1}^\infty\frac{1}{n}z^{mn}.$$

This function is related to $\psi(x)$ and to the logarithm of the Dedekind eta function and has been studied in a classical work by Rademacher [6]. In our notation $g(e^{-2\pi x}) = \psi(x)$ if $\mathrm{Re}(x) > 0$. Let p, q be integers such that $q > 0$, $(p,q) = 1$, and denote by p' any solution of the congruence $pp' \equiv -1 \bmod q$. Then, as Rademacher has shown ([6], p. 317, (1.45))

$$g\left(e^{-\frac{2\pi x}{q}+\frac{2\pi i p}{q}}\right) = g\left(e^{-\frac{2\pi}{xq}+\frac{2\pi i p'}{q}}\right) - \frac{\pi}{12q}(x - x^{-1}) + \frac{1}{2}\log x - \pi i S(p',q). \tag{25}$$

In this formula S denotes the well-studied Dedekind sum

$$S(p',q) = \sum_{\mu=1}^{q-1}\frac{\mu}{q}\left(\frac{p'\mu}{q} - \left[\frac{p'\mu}{q}\right] - \frac{1}{2}\right),$$

with $[t]$ being the greatest integer not exceeding t. To employ our function ψ, we replace x by $\overline{x}$ in (25) and take complex conjugates on both sides. The result is

$$\psi\left(\frac{x}{q} + i\frac{p}{q}\right) = \psi\left(\frac{1}{xq} + i\frac{p'}{q}\right) - \frac{\pi}{12q}(x - x^{-1}) + \frac{1}{2}\log x + \pi i S(p',q). \tag{26}$$

The next two results correspond to Theorems 6 and 7, respectively.

Theorem 10: *Let p, q be integers, $q > 0$, $(p,q) = 1$ and let p' be a solution of $pp' \equiv -1 \bmod q$. Then, if* $\mathrm{Re}(x) > 0$

$$\psi_1\left(\frac{x}{q} + i\frac{p}{q}\right) = A_q\log x + B_{pq} + C_q x\log x + D_{pq}x + E_q x^2 - \frac{2\pi x}{q}\sum_{n=1}^\infty \sigma_{-1}(n)e^{-\frac{2\pi i n p'}{q}}E_2\left(\frac{2\pi n}{xq}\right).$$

Here A_q, B_{pq}, C_q, D_{pq}, E_q are complex numbers not depending on x. In fact

$$A_q = -\frac{\pi^2}{6q^2},\ C_q = -\frac{\pi}{q},\ D_{pq} = \frac{\pi}{q} - \frac{2\pi^2 i}{q} S(p',q),\ E_q = \frac{\pi^2}{12q^2},$$

and

$$B_{pq} = -\frac{\pi}{q} - \frac{\pi^2}{12q^2} + \frac{2\pi^2 i}{q} S(p',q) + \frac{2\pi}{q}\int_1^\infty \psi\left(\frac{u}{q} + i\frac{p'}{q}\right) u^{-2}\mathrm{d}u + \psi_1\left(\frac{1}{q} + i\frac{p}{q}\right).$$

Proof: Using the definition of ψ_1 we have

$$\psi_1\left(\frac{x}{q} + i\frac{p}{q}\right) = 2\pi \int_{\frac{x}{q}+i\frac{p}{q}}^\infty \psi(u)\mathrm{d}u.$$

The integral equals

$$\psi_1\left(\frac{x}{q} + i\frac{p}{q}\right) = 2\pi \int_{\frac{x}{q}+i\frac{p}{q}}^{\frac{1}{q}+i\frac{p}{q}} \psi(u)\mathrm{d}u + \psi_1\left(\frac{1}{q} + i\frac{p}{q}\right) = \frac{2\pi}{q}\int_x^1 \psi\left(\frac{u}{q} + i\frac{p}{q}\right)\mathrm{d}u + \psi_1\left(\frac{1}{q} + i\frac{p}{q}\right).$$

As in the proof of Theorem 6 the assertion follows from (26) by integration, q.e.d.

Theorem 11: *Let p,q be positive integers, $(p,q)=1$. Then*

$$f(s) = \frac{s-1}{2\pi}\int_0^\infty \psi_1\left(x + i\frac{p}{q}\right)\left(x + i\frac{p}{q}\right)^{s-2}\mathrm{d}x - \frac{s+1}{2\pi}\int_0^\infty \psi_1\left(x - i\frac{p}{q}\right)\left(x - i\frac{p}{q}\right)^{-s-2}\mathrm{d}x$$
$$+ e^{\frac{\pi i s}{2}}\left(\frac{p}{q}\right)^s \frac{i}{2\pi} D(p,q) + H\left(s, i\frac{p}{q}\right),$$

where $D(p,q) = \frac{\pi^2}{6pq}\log\frac{p}{q} - \frac{q}{p}B_{pq} + \frac{p}{q}\overline{B_{qp}}$, and B_{pq} is defined as in Theorem 10. Moreover, $H(s,w)$ is given by (20).

Proof: We start from equation (21) which was proved for any complex w with $\mathrm{Re}(w) > 0$ and for $\sigma > 1$. The last condition can be dropped, avoiding only the singularities of $H(s,w)$. Let δ be real, $0 < \delta \le \frac{\pi}{2}$ and set $w = i\frac{p}{q}e^{-i\delta}$. We shall show that $\delta \to 0$, i.e. $w \to i\frac{p}{q}$ is permissible in (21). We write $w = \frac{p}{q}(i+u)$, $w^{-1} = \frac{q}{p}(-i+u')$. Hence $\mathrm{Re}(u) > 0$, $\mathrm{Re}(u') > 0$ and $u = \delta + O(\delta^2)$, $u' = \delta + O(\delta^2)$ as $\delta \to 0$. The first integral in (21) is equal to

$$\int_{\frac{p}{q}u}^{\frac{p}{q}u+\infty} \psi_1\left(x + i\frac{p}{q}\right)\left(x + i\frac{p}{q}\right)^{s-2}\mathrm{d}x,$$

which converges absolutely for $u \to 0$ since $\psi_1(x+i\frac{p}{q})$ grows only logarithmically at $x=0$ by the previous theorem. The other integral is treated similarly. Thus it remains to show that $\lim_{\delta\to 0}\left[\psi_1(w)w^{-1} + \psi_1(w^{-1})w\right]$ exists. From the definition of w,u and Theorem 10 we obtain

$$\psi_1(w) = \psi_1\left(\frac{up}{q} + i\frac{p}{q}\right) = A_q\log(up) + B_{pq} + O(\delta\log\delta)$$
$$= A_q\log\delta + A_q\log p + B_{pq} + O(\delta\log\delta).$$

Similarly,

$$\psi_1(w^{-1}) = \overline{\psi_1\left(\frac{\overline{u'}q}{p} + i\frac{q}{p}\right)} = A_p\log\delta + A_p\log q + \overline{B_{qp}} + O(\delta\log\delta).$$

Combining these formulas we get

$$\psi_1(w)w^{-1} + \psi_1(w^{-1})w = -i\frac{q}{p}A_q\log p + i\frac{p}{q}A_p\log q - i\frac{q}{p}B_{pq} + i\frac{p}{q}\overline{B_{qp}} + O(\delta\log\delta).$$

Taking the limit $\delta \to 0$, we obtain the assertion of the theorem, q.e.d.

The next two results give generalizations of Theorems 8 and 9. As the proofs do not involve any new idea, we omit them.

Theorem 12: *Let p, q be positive integers, $(p,q) = 1$. Then*

$$\zeta(s)\zeta(s+1) = T\left(s, \frac{p}{q}\right) + \overline{T\left(-\overline{s}, \frac{q}{p}\right)} X(s) + i(2\pi)^{s-1} \frac{e^{\frac{\pi i s}{2}}}{\Gamma(s)} \left(\frac{p}{q}\right)^s D(p,q) + (2\pi)^s \Gamma(s)^{-1} H\left(s, i\frac{p}{q}\right),$$

with the function T defined by

$$T\left(s, \frac{p}{q}\right) = (2\pi)^{s-1}\Gamma(s-1)^{-1} \int_0^\infty \psi_1\left(x + i\frac{p}{q}\right)\left(x + i\frac{p}{q}\right)^{s-2} \mathrm{d}x,$$

$X(s)$, $D(p,q)$ as in Theorems 8 and 11, respectively, and finally

$$H\left(s, i\frac{p}{q}\right) = -\frac{\pi i}{12} e^{\frac{\pi i s}{2}} \left(\frac{p}{q}\right)^{s-1} \left(\frac{1}{s-1} + \frac{p^2 q^{-2}}{s+1}\right) + \frac{e^{\frac{\pi i s}{2}}}{2s}\left(\frac{p}{q}\right)^s \left(\frac{\pi i}{2} + \log\frac{p}{q} - \frac{1}{s}\right).$$

Theorem 13: *The function T from Theorem 12 admits the integral representation*

$$T\left(s, \frac{p}{q}\right) = \frac{1}{2\pi i}\int_\Lambda e^{2\pi i \frac{p}{q} z} z^{1-s} F\left(z, \frac{p}{q}\right) \mathrm{d}z, \quad F\left(z, \frac{p}{q}\right) = 2\pi \int_0^\infty e^{2\pi x z} \psi_1\left(x + i\frac{p}{q}\right) \mathrm{d}x.$$

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