Error estimates for band-limited spherical regularization wavelets in some inverse problems of satellite geodesy

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Abstract

In this paper we discuss a special class of regularization methods for solving the satellite gravity gradiometry problem in a spherical framework based on band-limited spherical regularization wavelets. Considering such wavelets as a result of a combination of some regularization methods with Galerkin discretization based on the spherical harmonic system we obtain the error estimates of regularized solutions as well as the estimates for regularization parameters and parameters of band-limitation.

1 Introduction

In a spherical framework the problems arising in satellite gradiometry can be formulated as an integral equation of the first kind

$$Af(x) := \frac{1}{4\pi R} \int\limits_{\Omega_R} \frac{d^2}{dr^2} \left(\frac{r^2 - R^2}{|x-y|^3} \right) f(y) dw_R(y) = g(x) \tag{1}$$

To be more specific, we assume a spherical surface of the earth $\Omega_R = \{x \in \mathbb{R}^3 ,$ $|x| = (x_1^2 + x_2^2 + x_3^2)^{1/2} = R \}$ as well as the orbit $\Omega_r = \{x \in \mathbb{R}^3, \ |x| = r\}$, $r > R$, (dw_R denotes the surface-element on Ω_R). From a physical point of view, $f(y), \ y \in \Omega_R$, represents the gravitational potential at the surface of

the earth Ω_R, $g(x)$, $x \in \Omega_r$, is the measured function at satellite altitude. For more details we refer the reader to [10,11].
In what follows Ω_ℓ denotes the surface of the earth Ω_R for $\ell = 1$ and Ω_r for $\ell = 2$. Furthermore, we use the Hilbert space $L^2(\Omega_\ell)$ containing all square-summable functions on the sphere Ω_ℓ equipped with the inner product

$$(f, g)_\ell = \int_{\Omega_\ell} f(y)g(y)d\omega_\ell(y)$$

and with the usual norm $\|f\|_{2,\ell} = (f, f)_\ell^{1/2}$.
Let $\{Y_{n,k},\ n = 0, 1, 2, ...;\ k = 1, 2, ..., 2n + 1\}$ be a set of spherical harmonics L^2-orthonormalized with respect to the unit sphere in Euclidean space R^3 (for more details see [2,9]). One of the central statements which relates a system of spherical harmonics $\{Y_{n,k},\ k = 0, 1, 2, ..., 2n + 1\}$ of order n to a Legendre polynomial of degree n

$$P_n(t) = (2^n n!)^{-1} \tfrac{d^n}{dt^n} \left[(t^2 - 1)^n\right]$$

is the addition theorem

$$\sum_{k=1}^{2n+1} Y_{n,k}(\xi)Y_{n,k}(h) = \tfrac{2n+1}{4\pi} P_n(\xi \cdot h)\ ,\ |\xi| = |h| = 1\ , \tag{2}$$

where $\xi \cdot h = \xi_1 h_1 + \xi_2 h_2 + \xi_3 h_3$. For later use we introduce the $L^2(\Omega_\ell)$-orthonormal system $\left\{Y_{n,k}^{(\ell)}\right\}$, $\ell = 1, 2$, given by

$$Y_{n,k}^{(\ell)}(y) = \tfrac{1}{r_\ell} Y_{n,k}\left(\tfrac{y}{|y|}\right),\ y \in \Omega_\ell,$$

where $r_1 = R$ and $r_2 = r$. The corresponding spherical harmonic spaces are denoted by

$$h_n(\Omega_\ell) = span\left\{Y_{n,k}^{(\ell)},\ k = 1, 2, ..., 2n + 1\right\}$$

and

$$H_m(\Omega_\ell) = \bigoplus_{n=0}^{m} h_n(\Omega_\ell).$$

Of particular interest for our considerations are spherical Sobolev spaces introduced in [3]. Starting with an unbounded self-adjoint strictly positive definite in $L^2(\Omega_\ell)$ operator

$$Lf(x) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}(n+\tfrac{1}{2})f_{n,j}^{(\ell)}\; Y_{n,j}^{(\ell)}(x)$$

where $f_{n,j}^{(\ell)} = (f, Y_{n,j}^{(\ell)})_\ell$, we introduce the space

$$E_s(\Omega_\ell) = \left\{ f : \|L^s f\|_{2,\ell}^2 = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}(n+\tfrac{1}{2})^{2s}\left(f_{n,j}^{(\ell)}\right)^2 < \infty \right\}.$$

On $E_s(\Omega_\ell)$ we are able to define an inner product

$$(f,g)_{s,\ell} = (L^s f, L^s g)_\ell = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}(n+\tfrac{1}{2})^{2s}\left(f_{n,j}^{(\ell)}\right)\left(g_{n,j}^{(\ell)}\right)$$

and the associated norm $\|f\|_{s,2,\ell} = (f,f)_{s,\ell}^{1/2}$. The spherical Sobolev space $\mathcal{H}_s(\Omega_\ell)$ is the completion of $E_s(\Omega_\ell)$ under the norm $\|\cdot\|_{s,2,\ell}$. In particular $\mathcal{H}_0(\Omega_\ell) = L^2(\Omega_\ell)$.

A straightforward calculation (see [11, p. 42], [4]) shows that if we consider (1) only at $x \in \Omega_r$ we can rewrite A in the form of a singular-value decomposition

$$\begin{aligned} Af(x) &= \int_{\Omega_R}\left(\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}\sigma_n Y_{n,j}^{(1)}(y)Y_{n,j}^{(2)}(x)\right) f(y)dw_R(y) \\ &= \sum_{n=0}^{\infty}\sigma_n \sum_{j=1}^{2n+1} Y_{n,j}^{(2)}(x) f_{n,j}^{(1)} \end{aligned} \tag{3}$$

where $\sigma_n = \left(\frac{R}{r}\right)^n (n+1)(n+2)r^{-2}$. Now we recognize that $\sigma_n \to 0$ for $n \to \infty$ and therefore A is compact. Remembering Hadamard's definition of a well-posed problem (existence, uniqueness, continuity of the inverse), we consequently see that the problem (1) is ill-posed as it violates the first and third condition. Due to the ill-posedness of the equation (1) a variety of regularization methods are being considered for an approximate solution, where particular emphasis must be put on balancing the data and the approximation error. In this paper we are concerned with a special class of regularization methods, as proposed in [4,11]. More precisely, the authors of these papers implement regularization methods based on filtered singular-value decomposition as a wavelet analysis. This enables us to pass over from one regularized solution to another by adding so-called detail information in terms of wavelets. It should be remarked that the idea to use wavelet based regularization techniques is realized already in [1,5]. The essential feature of the approach proposed in [4,11] is the ability to construct regularization wavelets on the sphere, for example. On the other hand, in the theory of spherical regularization wavelets the so-called band-limited case is among the most important. In this case the spherical regulariziaton wavelet packet as well as the spherical regularization scaling function and corresponding regularized solution of (1) belong to $H_m(\Omega_\ell)$. Therefore, using band-limited spherical regularization wavelets we reduce the number of wavelet coefficients. Moreover, the numerical realization can be performed by a fast pyramid scheme [13]. Lastly, the basis of $H_m(\Omega_\ell)$ is the system of spherical harmonics $\{Y_{n,k}^{(\ell)}\}$. On the other hand, it is not surprising that in our days geodesy is still dominated by spherical harmonics models for a global gravitational potential determination, i.e. up to now a table of spherical harmonic coefficients to a certain order is available.

A very important question when dealing with band-limited spherical regularization wavelets is the relation between the regularization level and the parameter of band-limitation m. Moreover, since in practice we are confronted with error affected right hand sides g_δ of (1) such that

$$\|g - g_\delta\|_{2,2} \leq \delta,$$

the relation of regularization level as well as the parameter of band-limitation m to the level of noise δ is another important question.

In this paper we consider band-limited spherical regularization wavelets as a result of a combination of various regularization methods with Galerkin discretization based on the spherical harmonic system and investigate both the above mentioned questions. From such point of view the second question is connected with error estimates of regularized solutions in terms of the level of noise δ. There are many papers devoted to the in-depth study of this problem. But as has been mentioned in [11, p.44] (see Section 2 of this paper too) we have to deal with the exponentially ill-posed integral equation (1). Tikhonov regularization for such ill-posed problems was investigated in [8]. In Section 2 we apply some elements of the technique from [8] to the case of general regularization methods.

The essence of the first above mentioned question consists in the relation between the level of Galerkin discretization and the regularization parameter. Probably such a question was considered for the first time in [12] and later in [6,7]. In Section 4 we extend this analysis to the geodetic exponentially ill-posed problems.

2 A general error estimate for the exponentially ill-posed problem (1)

In this section we consider the frequently used constraint of assuming that the exact solution f_* of (1) is in some fixed ball in a Sobolev space $\mathcal{H}_s(\Omega_R)$, $s > 0$. Recall that $\{\mathcal{H}_s(\Omega_\ell),\ s \in \mathbb{R}\}$ is a Hilbert scale generated by a selfadjoint, densely defined, unbounded operator L on $L^2(\Omega_\ell)$. We study a general class of regularization methods to reconstruct the solution f_* of (1) from noisy data g_δ, in which the approximate solutions f_γ^δ are defined by

$$f_\gamma^\delta = G_\gamma \left(L^{-2s} A^* A\right) L^{-2s} A^* g_\delta \tag{4}$$

As usual (see, for example, [14]), we suppose that $G_\gamma(\lambda)$ is a piecewise continuous function and satisfies the following assumption.

Assumption 1. *There exist constants $c_p < \infty$, $d_q < \infty$ such that with $c = \|L^{-s} A^* A L^{-s}\|_{L^2(\Omega_R)\to L^2(\Omega_R)}$ the following estimates are satisfied:*

$$\sup_{0\le\lambda\le c} |\lambda^p G_\gamma(\lambda)| \le c_p \gamma^{p-1},\ 0 \le p \le 1$$
$$\sup_{0\le\lambda\le c} |\lambda^q \left[1 - \lambda G_\gamma(\lambda)\right]| \le d_q \gamma^q,\ 0 \le q \le 1$$

Note that regularization methods defined by (4) have been studied in [14] in the case when A is a finitely smoothing operator. Now applying some ideas of [8] we obtain an estimate on the rate of convergence of f_γ^δ to f_* for the exponentially ill-posed problem (1), i.e. for an infinitely smoothing operator A.

Let

$$\nu(\varepsilon) := \left\{ \|f\|_{2,1} : f \in \mathcal{H}_s(\Omega_R),\ \|Af\|_{2,2} \le \varepsilon,\ \|L^s f\|_{2,1} \le 1 \right\}$$

Lemma 1. *There exists a constant* $c_{r,s}$ *depending only on* r *and* s, *such that*

$$\nu(\varepsilon) \leq c_{r,s} \log^{-s} \tfrac{1}{\varepsilon}.$$

Proof. Let ℓ_λ be the spectral measure of L, i.e.

$$\ell_\lambda f(x) = \sum_{n+\frac{1}{2}\leq\lambda} \sum_{j=1}^{2n+1} f_{n,j}^{(1)} Y_{n,j}^{(1)}(x).$$

From Theorem 2.10, Lemma 3.5 and Theorem 3.6 of [8] it follows that if for some constants $a > 0,\ b \geq 1,\ p > 0$

$$\|Af\|_{2,2}^2 \geq p^2 \int \exp(-b\lambda^a) d(\ell_\lambda f, f)_1$$

then there exists a constant c such that

$$\nu(\varepsilon) \leq c\left(\varphi^{-1}\left(\tfrac{\varepsilon^2}{p^2}\right)\right)^{1/2},$$

where

$$\varphi^{-1}(t) = \left(\frac{b}{\log\frac{1}{t}}\right)^{\frac{2s}{a}}.$$

Note that

$$\begin{aligned} p^2 \int \exp(-b\lambda^a) d(\ell_\lambda f, f)_1 &= p^2(\exp(-bL^a) f, f)_1 \\ &= p^2 \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} e^{-b(n+\frac{1}{2})^a} (f_{n,j}^{(1)})^2 \end{aligned}$$

and set $a = 1,\ b = 2\log\frac{r}{R},\ p = \left(r\sqrt{Rr}\right)^{-1}$. Then from (3) and from the valid relation

$$\sigma_n^2 = \left(\tfrac{R}{r}\right)^{2n} \frac{(n+1)^2(n+2)^2}{r^4} \geq \frac{1}{r^4} e^{-2n\log\frac{r}{R}} \geq p^2 e^{-b(n+\frac{1}{2})^a}$$

we conclude that for any $f \in L^2(\Omega_R)$

$$\|Af\|_{2,2}^2 = \sum_{n=0}^{\infty} \sigma_n^2 \sum_{j=1}^{2n+1} \left(f_{n,j}^{(1)}\right)^2 \geq p^2 \int \exp(-b\lambda) d(\ell_\lambda f, f)_1.$$

Now the statement of the lemma follows immediately from the above mentioned results of [8]. □

Theorem 1. *Let the Assumption 1 be satisfied. Then for* $f_* \in \mathcal{H}_s(\Omega_R)$ *and* $\gamma = \delta^2$ *there holds the error estimate*

$$\left\|f_* - f_\gamma^\delta\right\|_{2,1} \leq c \log^{-s} \tfrac{1}{\delta} \tag{5}$$

where $c = c(r, s, \|f_*\|_{s,2,1})$ *is a constant independent of* δ.

Proof. We put $f_\gamma = G_\gamma(L^{-2s}A^*A)L^{-2s}A^*g$ and $T = AL^{-s}$. Then

$$f_\gamma - f_\gamma^\delta = G_\gamma(L^{-2s}A^*A)L^{-2s}A^*(g - g_\delta)$$

using the formula [14]

$$L^s G_\gamma(L^{-2s}A^*A) = G_\gamma(T^*T)L^s$$

and Assumption 1, we obtain the estimate

$$\begin{aligned}
\left\|L^s(f_\gamma - f_\gamma^\delta)\right\|_{2,1} &= \|L^s G_\gamma(L^{-2s}A^*A)L^{-2s}A^*(g - g_\delta)\|_{2,1} \\
&= \|G_\gamma(T^*T)T^*(g - g_\delta)\|_{2,1} \\
&\leq \|g - g_\delta\|_{2,2} \sup_{0 \leq \lambda \leq c} \left|\lambda^{1/2} G_\gamma(\lambda)\right| \\
&\leq c_{1/2}\gamma^{-1/2}\delta.
\end{aligned}$$

Moreover, it follows from Lemma 3.1 [15,p.34] that for any piecewise continuous function $G(\lambda)$

$$TG(T^*T) = G(TT^*)T, \; T^*G(TT^*) = G(T^*T)T^*.$$

Keeping in mind these formulas from the Assumption 1 we have

$$\begin{aligned}
\left\|A(f_\gamma - f_\gamma^\delta)\right\|_{2,2} &= \|AL^{-s}L^s G_\gamma(L^{-2s}A^*A)L^{-2s}A^*(g - g_\delta)\|_{2,2} \\
&= \|AL^{-s}G_\gamma(T^*T)L^{-s}A^*(g - g_\delta)\|_{2,2} \\
&= \|TG_\gamma(T^*T)T^*(g - g_\delta)\|_{2,2} \\
&= \|G_\gamma\left(T^*T\right)TT^*(g - g_\delta)\|_{2,2} \\
&\leq \|g - g_\delta\|_{2,2} \sup_{0 \leq \lambda \leq c} |\lambda G_\gamma(\lambda)| \\
&\leq c_1\delta
\end{aligned}$$

Let $c_0 = \max\{c_{1/2}, c_1\}$. As a result of the above mentioned estimates we obtain

$$\left\| L^s\left(\frac{f_\gamma - f_\gamma^\delta}{c_0\delta\gamma^{-1/2}}\right)\right\|_{2,1} \leq 1, \ \left\| A\left(\frac{f_\gamma - f_\gamma^\delta}{c_0\delta\gamma^{-1/2}}\right)\right\|_{2,2} \leq \sqrt{\gamma}.$$

Then from Lemma 1 there follows

$$\left\|f_\gamma - f_\gamma^\delta\right\|_{2,1} \leq c_0 c_{r,s}\frac{\delta}{\sqrt{\gamma}}\log^{-s}\frac{1}{\sqrt{\gamma}} \tag{6}$$

Now we note that

$$f_* - f_\gamma = (I - G_\gamma(L^{-2s}A^*A)L^{-2s}A^*A)f_*.$$

Applying the same tricks as above we have

$$\begin{aligned}
\left\|L^s(f_* - f_\gamma)\right\|_{2,1} &= \|(L^s - G_\gamma(T^*T)\,L^{-s}A^*AL^{-s}L^s)f_*\|_{2,1} \\
&= \|(I - G_\gamma(T^*T)T^*T)L^s f_*\|_{2,1} \\
&\leq \|f_*\|_{s,2,1}\sup_{0\leq\lambda\leq c}|1 - \lambda G_\gamma(\lambda)| \\
&\leq d_0\,\|f_*\|_{s,2,1}\,.
\end{aligned}$$

$$\begin{aligned}
\left\|A(f_* - f_\gamma)\right\|_{2,2} &= \|AL^{-s}L^s(f_* - f_\gamma)\|_{2,2} \\
&= \|T(I - G_\gamma(T^*T)T^*T)L^s f_*\|_{2,2} \\
&= \|(I - G_\gamma(TT^*)TT^*)TL^s f_*\|_{2,2} \\
&\leq \|f_*\|_{s,2,1}\sup_{0\leq\lambda\leq c}\left|(1 - \lambda G_\gamma(\lambda))\lambda^{1/2}\right| \\
&\leq d_{1/2}\gamma^{1/2}\,\|f_*\|_{s,2,1}\,.
\end{aligned}$$

From these inequalities and the notation $d = \max\{d_0, d_{1/2}\}$ one can conclude that

$$\left\| L^s\left(\frac{f_* - f_\gamma}{d\|f_*\|_{s,2,1}}\right)\right\|_{2,1} \leq 1\,, \ \left\| A\left(\frac{f_* - f_\gamma}{d\|f_*\|_{s,2,1}}\right)\right\|_{2,1} \leq \sqrt{\gamma}.$$

Then by virtue of Lemma 1

$$\|f_* - f_\gamma\|_{2,1} \leq d c_{r,s}\log^{-s}\frac{1}{\sqrt{\gamma}} \tag{7}$$

For $\gamma = \delta^2$ this estimate together with (6) yields the desired estimate (5)

$$\begin{aligned} \left\| f_* - f_\gamma^\delta \right\|_{2,1} &\leq \left\| f_\gamma - f_\gamma^\delta \right\|_{2,1} + \| f_* - f_\gamma \|_{2,1} \\ &\leq c_0 c_{r,s} \frac{\delta}{\sqrt{\gamma}} \log^{-s} \frac{1}{\sqrt{\gamma}} + d c_{r,s} \log^{-s} \frac{1}{\sqrt{\gamma}} \; . \\ &\leq c \log^{-s} \frac{1}{\delta} \end{aligned}$$

The theorem is proved. □

Note that in the case of the Tikhonov regularization, i.e. $G_\gamma(\lambda) = (\gamma + \lambda)^{-1}$, the assertion of the Theorem 1 follows from [8].

3 Spherical regularization wavelet packets

In this section we follow the approach [4,11] and our aim is to show that the regularized solution f_γ^δ is obtainable by decomposition and reconstruction of the noisy right hand side g_δ with respect to the wavelet basis. We start with the representation

$$f_\gamma^\delta(x) = G_\gamma(L^{-2s}A^*A)L^{-2s}A^*g_\delta = \int\limits_{\Omega_R} k_{G_\gamma}(x,y) g_\delta(y) dw_r(y),$$

where

$$k_{G_\gamma}(x,y) = \sum_{n=0}^{\infty} \sum_{i=1}^{2n+1} G_\gamma \left(\frac{\sigma_n^2}{\left(n+\frac{1}{2}\right)^{2s}} \right) \frac{\sigma_n}{\left(n+\frac{1}{2}\right)^{2s}} Y_{n,i}^{(1)}(x) Y_{n,i}^{(2)}(y).$$

If we apply the addition theorem (2), k_{G_γ} reduces to a so-called radial basis function which depends only on the inner product

$$\frac{x \cdot y}{Rr} = \frac{x_1 y_1 + x_2 y_2 + x_3 y_3}{Rr}$$

$$k_{G_\gamma}(x,y) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi Rr} \frac{\sigma_n}{\left(n+\frac{1}{2}\right)^{2s}} G_\gamma \left(\frac{\sigma_n^2}{\left(n+\frac{1}{2}\right)^{2s}} \right) P_n \left(\frac{x \cdot y}{Rr} \right).$$

Let $\{\gamma_j\}$ be a strictly decreasing sequence of real numbers satisfying

$$\lim_{j \to \infty} \gamma_j = 0 \; , \; \lim_{j \to -\infty} \gamma_j = \infty.$$

The function

$$\phi_j(t) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \hat{\phi}_j(n) P_n(t)$$

with

$$\hat{\phi}_j(n) = \frac{\sqrt{\sigma_n}}{\left(n+\frac{1}{2}\right)^s} \sqrt{G_{\gamma_j}\left(\frac{\sigma_n^2}{\left(n+\frac{1}{2}\right)^{2s}}\right)} \tag{8}$$

is called the scale discrete spherical regularization scaling function corresponding to the regularization method (4). Moreover, we call $\{\Psi_j,\ j \in Z\}$, defined by

$$\Psi_j(t) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \hat{\Psi}_j(n) P_n(t)$$

with

$$\hat{\Psi}_j(n) = [\left(\hat{\phi}_{j+1}(n)\right)^2 - \left(\hat{\phi}_j(n)\right)^2]^{1/2} \tag{9}$$

the scale discrete spherical regularization wavelet packet with respect to (4). As in [4,11] we define the dilation operators $D_p,\ p \in Z$, acting on the families $\{\phi_j\},\ \{\Psi_j\}$ in the following way: $D_p\phi_j = \phi_{j+p},\ D_p\Psi_j = \Psi_{j+p}$.
In particular, we obtain $\Psi_j = D_j\Psi_0$. Thus, we refer to Ψ_0 and ϕ_0 as "mother wavelet packet" and "mother scaling function" respectively. Moreover, we define a rotation (shifting) operator $R_z^{(\ell)}$ on $\Omega_\ell,\ \ell = 1, 2$, by

$$R_z^{(\ell)}\phi_j(x) = R_z^{(\ell)} D_j \phi_0(x) = \sum_{n=0}^{\infty} \sum_{i=1}^{2n+1} \hat{\phi}_j(n) Y_{n,i}^{(1)}(z) Y_{n,i}^{(\ell)}(x),\ x \in \Omega_\ell \tag{10}$$

In the same manner as (10) we define $R_z^{(\ell)} D_j \Psi_0(x),\ x \in \Omega_\ell$ which can be interpreted as a dilated and rotated copy of the mother wavelet Ψ_0. From Theorem 4.6 [4] it follows that

$$f^{\delta}_{\gamma_k}(x) = \int\limits_{\Omega_R} \left(\int\limits_{\Omega_r} R^{(2)}_z \phi_0(y) g_\delta(y) dw_r(y) \right) R^{(1)}_z \phi_0(x) dw_R(z)$$
$$+ \sum_{j=0}^{\infty} \int\limits_{\Omega_R} \left(\int\limits_{\Omega_r} R^{(2)}_z D_j \Psi_0(y) g_\delta(y) dw_r(y) \right) R^{(1)}_z D_j \Psi_0(x) dw_R(z)$$

This formula shows the essential characteristic of regularization wavelets. By adding the so-called detail information of level k as the difference of two smoothings of two consecutive scales $k+1$ and k (see (9)), we change the regularized solution from $f^{\delta}_{\gamma_k}$ to $f^{\delta}_{\gamma_{k+1}}$ thereby satisfying $f^{\delta}_{\gamma_k} \longrightarrow f_*$ in the case that $\delta \longrightarrow 0$, $\gamma_k \sim \delta^2$ (see Theorem 1).

4 Galerkin discretization and band-limited spherical regularization wavelets

In this section we want to draw the attention to band-limited spherical regularization wavelets. As in [11, p.113] we assume that there exists an increasing sequence of non-negative integers $\{m_j\}$ such that for $j \in Z$

$$R^{(\ell)}_z D_j \phi_0,\ R^{(\ell)}_z D_j \Psi_0 \in H_{m_j}(\Omega_\ell),\ \ell = 1,2 \qquad (11)$$

Let Q_m be the orthogonal projector on $H_m(\Omega_R)$, that is

$$Q_m f(x) = \sum_{n=0}^{m} \sum_{i=1}^{2n+1} Y^{(1)}_{n,i}(x) f^{(1)}_{n,i}.$$

It is not hard to verify that

$$f^{\delta}_{\gamma j}(x) = \int\limits_{\Omega_R} \left(\int\limits_{\Omega_r} R^{(2)}_z D_j \phi_0(y) g_\delta(y) dw_r(y) \right) R^{(1)}_z D_j \phi_0(x) dw_R(z)$$

Keeping in mind this formula from (11) we conclude that in the band-limited case $f^{\delta}_{\gamma_j} \in H_{m_j}(\Omega_R)$. Therefore

$$f^{\delta}_{\gamma_j} = Q_m f^{\delta}_{\gamma_j} = Q_{m_j} G_{\gamma_j}(L^{-2s} A^* A) L^{-2s} A^* g_\delta$$

On the other hand, a straightforward calculation shows that

$$\begin{aligned} Q_m G_\gamma(L^{-2s}A^*A)L^{-2s} &= G_\gamma(L^{-2s}Q_mA^*AQ_m)L^{-2s} \\ &= G_\gamma(L^{-2s}A^*A)L^{-2s}Q_m \end{aligned} \qquad (12)$$

Thus, the only possibility to obtain the band-limited spherical regularization wavelets (11) is to apply regularization methods defined by functions $G_{\gamma_j}(\lambda)$ to the discretized equation $AQ_mf(x) = g_\delta(x)$ instead of $Af(x) = g_\delta(x)$. Then by virtue of (12) the function

$$f_{\gamma,m}^\delta = G_\gamma(L^{-2s}A^*A)L^{-2s}Q_mA^*g_\delta$$

gives the approximate solution of (1).

Theorem 2. *Let the Assumption 1 be satisfied and* $f_* \in \mathcal{H}_s(\Omega_R)$. *Then*

$$\left\|f_* - f_{\gamma,m}^\delta\right\|_{2,1} \leq c\log^{-s}\tfrac{1}{\sqrt{\gamma}}\left(1 + \tfrac{\delta}{\sqrt{\gamma}} + \tfrac{1}{\gamma m^{s-2}}\left(\tfrac{R}{r}\right)^{m+1}\right),$$

where c *is the constant independent of* δ, γ, m.

Proof. By definition

$$\begin{aligned} \left\|L^{-s}\left(I - Q_m\right)f\right\|_{2,1}^2 &= \left\|\left(I - Q_m\right)L^{-s}f\right\|_{2,1}^2 \\ &= \sum_{n=m+1}^{\infty}\sum_{i=1}^{2n+1}\left(n + \tfrac{1}{2}\right)^{-2s}\left(f_{n,i}^{(1)}\right)^2 \leq \left(m + \tfrac{3}{2}\right)^{-2s}\left\|f\right\|_{2,1}^2 \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \left\|\left(I - Q_m\right)A^*f\right\|_{2,1}^2 &= \sum_{n=m+1}^{\infty}\sigma_n^2\sum_{i=1}^{2n+1}\left(f_{n,i}^{(2)}\right)^2 \leq \sigma_{m+1}^2\left\|f\right\|_{2,2}^2 \\ &= \left(\tfrac{R}{r}\right)^{2(m+1)}\tfrac{(m+2)^2(m+3)^2}{r^4}\left\|f\right\|_{2,2}^2 \leq cm^4\left(\tfrac{R}{r}\right)^{2m+2}\left\|f\right\|_{2,2}^2 \end{aligned}.$$

Now the same steps as in the proof of Theorem 1 lead to the estimates

$$\begin{aligned} \left\|L^s\left(f_\gamma^\delta - f_{\gamma,m}^\delta\right)\right\|_{2,1} &= \left\|L^sG_\gamma(L^{-2s}A^*A)L^{-2s}\left(I - Q_m\right)A^*g_\delta\right\|_{2,1} \\ &= \left\|G_\gamma(T^*T)L^{-s}(I - Q_m)A^*g_\delta\right\|_{2,1} \\ &\leq \sup_{0\leq\lambda\leq c}\left|G_\gamma(\lambda)\right|\left\|L^{-s}(I - Q_m)A^*g_\delta\right\|_{2,1} \\ &\leq \tfrac{c_0}{\gamma}\left\|L^{-s}(I - Q_m)(I - Q_m)A^*g_\delta\right\|_{2,1} \\ &\leq \tfrac{c_0}{\gamma}\left(m + \tfrac{3}{2}\right)^{-s}\left\|\left(I - Q_m\right)A^*g_\delta\right\|_{2,1} \\ &\leq \tfrac{c}{\gamma m^{s-2}}\left(\tfrac{R}{r}\right)^{m+1}\left\|g_\delta\right\|_{2,2} \end{aligned}$$

Moreover,

$$\begin{aligned}
\left\|A(f_\gamma^\delta - f_{\gamma,m}^\delta)\right\|_{2,2} &= \left\|AL^{-s}L^sG_\gamma\left(L^{-2s}A^*A\right)^{-2s}(I-Q_m)\,A^*g_\delta\right\|_{2,2} \\
&= \left\|TG_\gamma\left(T^*T\right)L^{-s}(I-Q_m)A^*g_\delta\right\|_{2,2} \\
&= \left\|G_\gamma(TT^*)TL^{-s}(I-Q_m)A^*g_\delta\right\|_{2,2} \\
&\le \sup_{0\le\lambda\le c}\left|G_\gamma(\lambda)\lambda^{1/2}\right| \left\|L^{-s}\left(I-Q_m\right)A^*g_\delta\right\|_{2,2} \\
&\le \tfrac{c}{\sqrt{\gamma}m^{s-2}}\left(\tfrac{R}{r}\right)^{m+1}\|g_\delta\|_{2,2}
\end{aligned}$$

Finally, we apply Lemma 1 in a way analogous to that used before. Then as a result of the above mentioned estimates we obtain

$$\left\|f_\gamma^\delta - f_{\gamma,m}^\delta\right\|_{2,1} \le \tfrac{c}{\gamma m^{s-2}}\left(\tfrac{R}{r}\right)^{m+1}\|g_\delta\|_{2,2}\log^{-s}\tfrac{1}{\sqrt{\gamma}}.$$

The assertion of the theorem follows from the last inequality and (6),(7). By inspecting the result of Theorem 2 for an error free right hand side ($\delta = 0$) we can see that it has a sense to take the parameter of band-limitation m_j for band-limited spherical regularization wavelets as $m_j \sim \log\frac{1}{\gamma_j}$. From Theorem 2 and (7) it follows that in this case

$$\left\|f_* - f_{\gamma_j,m_j}\right\|_{2,1} \le c\log^{-s}\tfrac{1}{\sqrt{\gamma_j}}.$$

On the other hand, using error affected data the exponentially ill-posed character of the original integral equation (1) leads to a much larger error level in the solution in comparison to the data error. Namely, for $m_j \sim \log\frac{1}{\gamma_j}$, $\gamma_j \sim \delta^2$

$$\left\|f_* - f_{\gamma_j,m_j}^\delta\right\|_{2,1} \le c\log^{-s}\tfrac{1}{\delta}.$$

Nevertheless, as appears from the numerical test example [4], the results are still applicable and agree with the above mentioned estimates.

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