TOYING WITH JORDAN MATRICES

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It is shown that an important resolvent estimate is instable under small perturbations.

In many applications operators $T$ are of interest which fulfill the condition: the resolvent set $\rho(T)$ contains the negative reals, and there is a constant $\gamma > 0$, such that for all positive reals $s$ the "resolvent estimate" holds

(res) \[ \| (sI + T)^{-1} \| \leq \frac{\gamma}{s}. \]

In the theory of semigroups (see e.g. J.A. GOLDSTEIN [1, p. 20]) or in the theory of evolutionary integral equations (J. PRUSS [4, p. 69]) this condition is well known. In the investigation of regularization methods for ill-posed equations in Banach spaces the condition (res) is necessary for the convergence of the so-called Lavrentiev regularization (see R. PLATO [3], E. SCHOCK/V. PHONG [6], SPECKERT [7]).

The following simple example will show that the condition (res) is unstable under arbitrary small perturbations. We will construct operators $J$ built up with Jordan matrices which act on block subspaces of $\ell_p$.

By $J_n(\lambda)$ we denote an $(n \times n)$-Jordan matrix with the eigenvalue $\lambda$ and with 1’s in the upper diagonal. The following Lemma is well known.

**Lemma 1.** Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Then

\[
J_n(\lambda)^{-1} = \begin{pmatrix}
\lambda^{-1} & -\lambda^{-2} & \cdots & (-1)^{n-1} & \lambda^{-n} \\
\lambda^{-1} & \cdots & (-1)^{n-2} & \lambda^{-n+1} \\
\vdots & \ddots & \ddots & \ddots \\
0 & & \ddots & \ddots \\
& & & & \lambda^{-1}
\end{pmatrix}
\]

**Lemma 2.** In $X = \ell_p^n = \left( \mathfrak{F}^n, \| \cdot \|_p \right)$ for $0 < \lambda \leq 1$ holds

\[ \| J_n(\lambda)^{-1} \| \geq \lambda^{-n}. \]
**Proof.** Let $e_n$ be the $n^{th}$ unit vector, then
\[ \| J_n(\lambda)^{-1} e_n \|^p_p = \sum_{k=1}^{n} \lambda^{-pk} \geq \lambda^{-pn}. \]

Let $D_n(\lambda)$ be the $(n \times n)$-diagonal matrix with the eigenvalue $\lambda$, then for $\varepsilon > 0$
\[ D_n(\lambda) + \varepsilon J_n(0) = \varepsilon J_n \left( \frac{\lambda}{\varepsilon} \right). \]

**Proposition 3.** Let $(\lambda_k)$ be a sequence of non-negative reals converging to zero, $\varepsilon > 0$, $n \in \mathbb{N}$ and
\[ J_\varepsilon = \bigoplus_{k=1}^{\infty} \varepsilon J_n \left( \frac{\lambda_k}{\varepsilon} \right). \]
Then for all $s$ with $0 < s \leq \varepsilon$
\[ \left\| (s I + J_\varepsilon)^{-1} \right\| \geq \frac{\varepsilon^{n-1}}{s^n}. \]

**Proof.** We have
\[ sI + J_\varepsilon = \bigoplus_{k=1}^{\infty} \varepsilon J_n \left( \frac{\lambda_k + s}{\varepsilon} \right). \]
By Lemma 2 for $x = e_{nm}$
\[ \left\| (sI + J_\varepsilon)^{-1} x \right\|^p_p = \sum_{k=1}^{n} \varepsilon^{-p} \frac{\varepsilon^{pk}}{(\lambda_{m+k})^p} \geq \left( \frac{\varepsilon^{n-1}}{(\lambda_{m+n})^p} \right)^p, \]
thus
\[ \left\| (sI + J_\varepsilon)^{-1} \right\| \geq \sup_m \frac{\varepsilon^{n-1}}{(\lambda_{m+n})^p} = \frac{\varepsilon^{n-1}}{s^n}. \]

Let $D = \bigoplus_{k=1}^{\infty} D_n(\lambda_k)$, then
\[ \| D - J_\varepsilon \| = \varepsilon \]
because $D - J_\varepsilon = \bigoplus_{k=1}^{\infty} \varepsilon J_n(0)$, and $J_n(0)$ acts as a shift operator.

**Corollary 4.** The operator $D$ fulfills (res), but for every $\varepsilon > 0$ the operator $J_\varepsilon$, which is an $\varepsilon$-perturbation of $D$, violates this condition.
In [4] it is shown, that the Lavrentiev regularization diverges, if (res) is violated. On the other hand, in the treatment of ill-posed problems the discussion of perturbations is indispensable, hence the Lavrentiev method is useless for ill-posed equations in Banach spaces.

The smallest positive integer \( r \), such that for all positive integers \( k \)

\[ \text{Ker} \left( \lambda I - T \right)^r = \text{Ker} \left( \lambda I - T \right)^{r+k} \]

and

\[ \text{Range} \left( \lambda I - T \right)^r = \text{Range} \left( \lambda I - T \right)^{r+k} \]

is called the Riesz number of the compact operator \( T \). (see e.g. R. KRESS [2, p. 27]).

For compact selfadjoint operators in Hilbert spaces the Riesz number of each eigenvalue \( \lambda \neq 0 \) is equal to unity. An operator \( T \) in a Hilbert space \( X \) is said to be highly non-selfadjoint, if for any positive integer \( n \) there is a finite dimensional \( T \)-invariant subspace \( E \subset X \), such that the restriction of \( T \) onto \( E \) has an eigenvalue \( \lambda \) with Riesz number \( > n \).

The following construction of a highly non-selfadjoint operator is a refinement of the construction above.
Let \( (\lambda_k) \) be a sequence of non-negative reals, tending to zero. Then

\[ T = \bigoplus_{k=1}^{\infty} J_k(\lambda_k) \]

is highly non-selfadjoint, since \( \lambda_k \) is an eigenvalue of \( T \) with Riesz-number \( k \). The spectrum of \( T \) is the unit disk \( \Delta \) and the sequence \( (\lambda_k) \).

\[ sI - T = \bigoplus_{k=1}^{\infty} J_k(s - \lambda_k) . \]

The entry with number \((1,k)\) of the matrix \((sI - T)^{-1}\) has the modulus \( |s - \lambda_k|^{-k} \) hence

\[ \sup_{k} |s - \lambda_k|^{-k} < \infty, \text{ if } s \notin \{\lambda_k, k \in \mathbb{N}\} \text{ or } |s| > 1 \text{ (since } \lim_{k} \lambda_k = 0) \].

Of course, \( T \) is not compact.

Although each matrix \( J_k(0) \) is nilpotent, \( N = \bigoplus_{k=1}^{\infty} J_k(0) \) is not quasinilpotent, since the spectrum of \( N \) is the unit disk \( \Delta \).

REFERENCES


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