Characterization of operators of positive scalar type

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Abstract

Let $X$ be a Banach lattice. Necessary and sufficient conditions for a linear operator $A : D(A) \to X$, $D(A) \subseteq X$, to be of positive $C^0$-scalar type are given. In addition, the question is discussed which conditions on the Banach lattice imply that every operator of positive $C^0$-scalar type is necessarily of positive scalar type.

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0 Introduction

A linear operator $A$ on a Banach space $X$ is a scalar-type-operator (or equivalently of scalar-type) on $[0, \infty)$ if $\sigma(A) \subseteq [0, \infty)$ and if there exists a spectral measure $E$ on the Borel subsets of $[0, \infty)$ such that

\[ D(A) = \{ x \in X : \lim_{n \to \infty} \int_0^n \lambda E(d\lambda)x \text{ exists} \} \]

and

\[ Ax = \lim_{n \to \infty} \int_0^n \lambda E(d\lambda)\quad x \in D(A). \]
0 INTRODUCTION

If $X$ is Banach lattice then we say that $A$ is of positive scalar type on $[0, \infty)$ if $A$ is of scalar type on $[0, \infty)$ with positive spectral measure $E$, i.e. $E(M)$ is a positive projection for all Borel measurable subsets $M \subseteq [0, \infty)$.

We refer the reader to [4] for a brief introduction into the history and the importance of scalar-type operators. Operators of positive scalar type where studied e.g. in [1]. In addition, it was shown in [2], if $A$ is an operator of scalar type on a cyclic Banach space $X$ then there exists an ordering and an equivalent norm on $X$, such that $X$ becomes a Banach lattice, and such that $X$ is of positive scalar type on $X$.

By $C_0[0, \infty)$ we denote the space of complex-valued functions on $[0, \infty)$ vanishing at infinity, and $\mathbf{L}(X)$ denotes the space of linear bounded operators on $X$. If $A$ is of scalar type on $[0, \infty)$ with spectral measure $E$ then there exists a bounded algebra homomorphism $T \Phi : C_0[0, \infty) \to \mathbf{L}(X)$ given by

$$\Phi(f)x = \int_0^\infty \lambda E(d\lambda)x.$$  

If we denote by $\rho_s$ the function $\rho_s(t) = 1/(s + t)$ then $\Phi(\rho_s) = (s + A)^{-1}$ for every $s > 0$. Operators for which such an algebra homomorphism exists are called $C^0$-scalar-type operators on $[0, \infty)$. If $X$ is a Banach lattice and if $\Phi$ is a positive algebra homomorphism then $A$ is said to be of positive $C^0$-scalar type.

We note that every operator of (positive) scalar type is of (positive) $C^0$-scalar type where the algebra homomorphism $\Phi$ is given by

$$\Phi(f) = \int_0^\infty f(t) E(dt), \quad f \in C_0[0, \infty).$$

Conversely, if a scalar-type-operator $A$ is of positive $C^0$-scalar type then $A$ automatically is of positive scalar type.

The notion of scalar type and $C^0$-scalar type operators lead to the following problems:

(1) Find conditions on $A$ (or on the resolvent of $A$ or the semigroup generated by $A$) which are necessary and sufficient for $A$ being of $C^0$-scalar type.
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(II) In which Banach spaces are all \( C^0 \)-scalar type operators of scalar type?

These two problems were discussed in detail in [6]. In this note we focus on the characterization of operators of positive scalar type.

1 Operators of positive scalar type

We study now operators acting on a Banach lattice \( X \). For elementary properties of positive operators in Banach lattices we refer the reader to [5].

Recall that the linear operator \( A \) on \( X \) is of positive scalar type on \([0, \infty)\) if \( A \) is of scalar type on \([0, \infty)\) with positive spectral measure \( E \), and that \( A \) is of positive \( C^0 \)-scalar type if \( A \) is of \( C^0 \)-scalar type with positive algebra homomorphism.

We pose the following two problems:

(I+) Find conditions on \( A \) (or on the resolvent of \( A \) or the semigroup generated by \( A \)) which are necessary and sufficient for \( A \) being of positive \( C^0 \)-scalar type.

(II+) In which Banach lattices are all operators of positive \( C^0 \)-scalar type operator of positive scalar type?

The answer to problem (I+) is the following

**Theorem 1.** Let \( A \) be a densely defined operator on \( X \) with \( \sigma(A) \subseteq [0, \infty) \).

Then the following assertions are equivalent:

(a) \( A \) is of positive \( C^0 \)-scalar type on \([0, \infty)\).

(b) \( t(t + A)^{-1} \) is uniformly bounded, \( (t + A)^{-1} \geq 0 \) and \( A^k(t + A)^{-2k} \geq 0 \) for \( k = 1, 2, \ldots \) and \( t > 0 \).

(c) \( A^k(1 + A)^{-(k+n)} \geq 0 \) for \( k, n = 0, 1, 2, \ldots \).
\( (d) \) \(-A \) generates a \( C_0 \)-semigroup \( (U(t))_{t \geq 0} \) such that \( U(t)X \subseteq D(A) \) for every \( t > 0 \), and \( A^k U(t) \geq 0 \) for \( k = 0, 1, 2, \ldots \) and \( t > 0 \).

The proof of Theorem 1 is an easy combination of the following ingredients:

(a) The proof of the characterization of \( C^0 \)-scalar type-operators given in [6], Theorem 6.

(b) A characterization of Stieltjes transforms of positive measures [7], Theorem VIII.17c., and its application to the resolvent \( (\lambda + A)^{-1}, \lambda > 0 \).

(c) A description of completely monotonic sequences [7], Theorem III.4a., and its application to the sequence \( ((1 + A)^{-n})_{n=0,1,2,\ldots} \).

(d) Bernstein’s theorem on the characterization of Laplace transforms of positive measures, and its application to the semigroup generated by \(-A\).

A partly answer to the second problem (II+) will follow from the solution of problem (II) given in [6], which we recall now. Assume \( A \) to be of scalar type on \([0, \infty)\) with spectral measure \( E \). Then \( A \) is of \( C^0 \)-scalar type on \([0, \infty)\) with corresponding algebra homomorphism \( \Phi \) given by

\[
\Phi f(x) = \int_0^\infty f(t) E(dt)x.
\]

If we denote by \( \Phi[x] : C_0[0, \infty) \to X \) the operator \( \Phi[x]f = \Phi f(x) \), and if we define \( E[x] \) to be the vector measure defined by \( E[x](E) = \mu(E)x \), then the" components" \( \Phi[x] \) of \( \Phi \) can be represented by the countably additive vector measure \( E[x] \), i.e.

\[
\Phi[x]f = \int_0^\infty f(t) E[x](dt).
\]

In [6] it is shown that the converse is also true, i.e. if \( A \) is a linear operator on \( X \) with \(( -\infty, 0) \subseteq \rho(A)\), then \( A \) is of scalar type on \([0, \infty)\) if and only if \( A \) is of \( C^0 \)-scalar type on \([0, \infty)\) and the components \( \Phi[x] \) of the corresponding
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algebra homomorphism $\Phi$ can be represented by a countably additive vector measure for all $x \in X$. The following theorem is an immediate consequence of the foregoing considerations.

**Theorem 2** The operator $A$ on $X$ is of positive scalar type on $[0, \infty)$ if and only if $A$ is of positive $C^0$-scalar type on $[0, \infty)$ with corresponding algebra homomorphism $\Phi$ and, for all $x \in X_+$, the, necessarily positive, components $\Phi[x]$ of $\Phi$ can be represented by a, necessarily positive, countably additive vector measure.

It is well known that every operator $T : C_0[0, \infty) \to X$ has a representation by a countably additive vector measure if and only if $X$ does not contain an isomorphic copy of the space of complex valued null sequences $c_0$ [6]. Hence, by the statement preceding Theorem 2, if $X$ does not contain an isomorphic copy of $c_0$ then every $C^0$-scalar type-operator on $X$ is of scalar type on $X$. Moreover, Doust [3] showed that if $c_0$ is contained in $X$ then there exists an operator of $C^0$-scalar type on $X$ which is not of scalar type.

Since we are interested in a characterization of those Banach lattices $X$ in which every operator of positive $C^0$-scalar type is of positive scalar type the following two open problems should be solved:

(III+) Give a characterization of those Banach lattices $X$ such that every positive operator $T : C_0[0, \infty) \to X$ can be represented by a, necessarily positive, countably additive vector measure.

(IV+) If $X$ is a Banach lattice with the property that not every positive operator $T : C_0[0, \infty) \to X$ can be represented by a countably additive vector measure is it then possible to construct an operator $A$ on $X$ which is of positive $C^0$-scalar type on $[0, \infty)$, but which is not of positive scalar type on $[0, \infty)$?
References


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