TA 8l.

FORSCHUNG - AUSBILDUNG - WEITERBILDUNG

Bericht Nr. 67

CONTROLLABILITY INDICES FOR BEHAVIOUR SYSTEMS IN AR-REPRESENTATION

J. Hoffmann, D. Prätzel-Wolters

UNIVERSITÄT KAISERSLAUTERN
Fachbereich Mathematik
Arbeitsgruppe Technomathematik
Postfach 3049

6750 Kaiserslautern

Dezember 1991

1. Introduction

There is an extensive literature concerning feedback invariants of linear multivariable systems and their connection to control problems. Popov (1964) introduced feedback invariants in connection with a study of stability and linear optimal control.

The indices which are known as controllability indices in the literature were first identified as a complete set of invariants of the orbits of controllability pairs under state-coordinate, input-coordinate and feedback transformations in a paper by Brunovsky (1970). Popov (1972), Rosenbrock (1970) and Kalman (1971) published similar results nearly at the same time.

Rosenbrock (1970) and Kalman (1971) showed the connection between controllability indices and Kronecker indices of a singular matrix pencil. Wonham and Morse (1972) analysed controllability indices in the context of the geometric state space theory.

Rosenbrock (1970) and Rosenbrock and Hayton (1974) introduced dynamical indices of transfer functions and showed that they equal the controllability indices.

Wolovich (1974) identified the controllability indices with the help of coprime factorizations of transfer matrices, where the polynomial matrices are in "column proper form".

All these papers consider either the classical state-space setting or the transfer-function description.

A module theoretic approach to the definition of controllability indices and their controltheoretic properties was given in Forney (1975), Münzner and Prätzel-Wolters (1978) and Kailath (1980).

In the recent years singular linear systems and linear systems in autoregressive representation have become a major research topic in linear control theory. The investigation of the fine structure of controllability for these systems plays an important role, in particular, the above mentioned module theoretic approach has been extended to singular linear systems.

In Kučera and Zagalak (1988) "input controllability indices" for singular linear systems (E,A,B) are defined as minimal indices of the F[s]-module ker[sE-A,B] and an extension of Rosenbrock's pole-(invariant factor) assignment theorem is given. However, these controllability indices do not form a complete system of

invariants for the feedback-action on singular systems.
In a series of papers, Dai (1989), Shayman (1988), Karkanias and Heliopoulou (1989), Malabre et al. (1990) and recently Glüsing-Lüerßen (1991) refined the above concept of c.i.'s to obtain such sets of complete invariants for the feedback action.

For the more general class of linear systems in AR-representations there is no developed theory for the feedback equivalence, the pole-assignability-problem and the concept of controllability indices. However, recently Fagnani (1991) has introduced a geometric concept of controllability indices for general dynamical discrete time behaviour systems as defined in a series of pioneering papers of Willems ((1986a), (1986b), (1987), (1988), (1991)). In Willems' approach controllability is defined as an intrinsic system property which does neither depend on special dynamical properties like linearity, finite dimensionality etc. nor on the model representation. Consequently, the index list defined by Fagnani is given exclusively in terms of the behaviour.

In our paper we apply the module theoretic concepts introduced for behaviour systems in Hoffmann and Prätzel-Wolters (1991b) to construct a list of algebraic controllability indices for linear dynamical systems in AR-representation. Our approach is a straightforward extension of the characterization of controllability indices as minimal indices of the $F[s]$-modules $\ker[sE-A,B]$ in the state- space setting. It covers Fagnani's definition if the system class is restricted to linear, time-invariant complete behaviour systems with time axis $T=\mathbb{Z}$.

Section 2 contains some preliminary remarks concerning controllability of behaviour systems, in particular in AR-representation.

In Section 3 controllability indices are defined for linear systems in AR-representation. The obtained index list is shown to be equal to the Fagnani index list. A characterization of controllability via the controllability indices and an effective algorithm for their computation is given.

2. Preliminaries

In the recent years J.C. Willems developed in a series of papers a general theory of dynamical behaviour systems $\Sigma = (T,W,\mathcal{B})$ with time axis $T \subseteq \mathbb{R}$, signal alphabet W and behaviour $\mathcal{B} \subseteq W^T$ (see e.g. Willems (1986a, 1986b, 1987, 1988, 1991)).
Σ is called <u>time invariant</u> if T is an additive subgroup of $\mathbb{R}$ and $\mathcal{B}$ is invariant with respect to all t-shifts

$$\sigma^t: W^T \to W^T, \; w(\hat{t}) \longmapsto w(\hat{t}+t), \; t \in T .$$

A time invariant system Σ with time axis $T=\mathbb{Z}$ or $T=\mathbb{R}$ is called <u>controllable</u> if, for every w_1 and w_2 in $\mathcal{B}$, there exists $0 \leq t \in T$ and $w \in \mathcal{B}$ such that $w^- = w_1^-$ and $(\sigma^t w)^+ = w_2^+$, where

$$w^- := w|_{(-\infty,0)\cap T} \; , \; w^+ := w|_{[0,\infty)\cap T} \; .$$

Σ is said to be <u>complete</u> if

$$\left\{w \in \mathcal{B}\right\} \Longleftrightarrow \left\{w|_{[t_1,t_2]} \in \mathcal{B}|_{[t_1,t_2]}, \; \forall t_1,t_2 \in T, \; t_1 \leq t_2\right\}$$

In Willems (1991) it is shown that every linear time-invariant complete system $\Sigma = (\mathbb{Z},\mathbb{R}^q,\mathcal{B})$ has an <u>autoregressive (AR)-representation</u>:

$$\mathcal{B} = \ker P(\sigma,\sigma^{-1}) \tag{2.1a}$$

$$P(s,s^{-1}) = P_L s^L + \ldots + P_\ell s^\ell \in \mathbb{R}^{p\times q}[s,s^{-1}] \tag{2.1b}$$

The operator

$$P(\sigma,\sigma^{-1}) : \begin{array}{ccc} (\mathbb{R}^q)^{\mathbb{Z}} & \longrightarrow & (\mathbb{R}^p)^{\mathbb{Z}} \\ w(t) & \longmapsto & P_L w(t+L)+\ldots+P_\ell w(t+\ell) \end{array} \quad , \; t \in \mathbb{Z}$$

is called a dipolynomial shift operator. If $\ell \geq 0$ then $P(\sigma,\sigma^{-1})$ is polynomial and denoted by $P(\sigma)$. q denotes the dimension of the signal alphabet space $W=\mathbb{R}^q$, whereas p, the number of equations representing $\mathcal{B}$, is flexible. However, among all dipolynomial matrices $P(s,s^{-1})$ satisfying (2.1a) there exist those with full row rank. They are unique up to multiplication from the left by unimodular matrices $U(s,s^{-1})$; there holds:

$$U(s,s^{-1}) \in \mathbb{R}^{p\times p}[s,s^{-1}] \text{ unimodular } \iff \det U = \alpha s^d, \quad \alpha \in \mathbb{R}\setminus\{0\},\ d \in \mathbb{Z} \qquad (2.2)$$

Introducing the dipolynomial degree function

$$\text{ddeg}: \mathbb{R}^{1\times q}[s,s^{-1}] \to \mathbf{N},\ \alpha_L s^L + \ldots + \alpha_\ell s^\ell \to L-\ell \qquad (2.3)$$

Willems (1991) calls a full row rank matrix P a <u>minimal lag description</u>, if among all full row rank AR-representations its total lag, i.e. the sum of the row degrees of P, is as small as possible.
For $T=\mathbb{Z}_+$, $\mathbb{R}$, $\mathbb{R}_+$ we consider analogous polynomial AR-representations with:

$$\mathcal{B} = \ker P(\sigma) \quad \text{resp.} \quad \mathcal{B} = \ker P(\tfrac{d}{dt}) \qquad (2.4)$$

where $P(s) \in \mathbb{R}^{p\times q}[s]$

Whether or not a behaviour system in AR-representation is controllable can be read off from the behavioural equations:

<u>2.1 Theorem [Willems (1991)]</u>

Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B})$ a dynamical system in AR-representation:

$$\mathcal{B} = \ker P(\sigma,\sigma^{-1}) ,$$

with $P(s,s^{-1}) \in \mathbb{R}^{p\times q}[s,s^{-1}]$ of full row rank. Then the following conditions are equivalent:

(i) Σ is controllable.

(ii) $\text{rank}_{\mathbb{C}}\ P(\lambda,\lambda^{-1}) = p \quad \text{for all } 0 \neq \lambda \in \mathbb{C}$ (2.5)

□

<u>2.2 Remark:</u>

For <u>$T = \mathbb{R}, \mathbb{R}_+$ resp. $\mathbb{Z}_+$</u> Theorem 2.1 remains true if we replace $P(s,s^{-1})$ by a <u>polynomial matrix</u> $P(s) \in \mathbb{R}^{p\times q}[s]$ and require (2.5) for all $\lambda \in \mathbb{C}$, i.e.

$$\text{rank}_{\mathbb{C}}\ P(\lambda) = p \ \underline{\forall \lambda \in \mathbb{C}} \qquad (2.5a)$$

□

A characterization of controllability in terms of the coefficient matrices of the representing dipolynomial resp. polynomial matrices which generalizes the classical controllability matrix in the state-space setting is derived in Hoffmann and Prätzel-Wolters (1991a). Furthermore, an effective numerical algorithm to test controllability is given in the above paper.

3. Controllability indices for AR-systems

In the literature there are several approaches for the investigation of controllability indices (c.i.) for different representations of linear systems (c.f. ex. Münzner and Prätzel-Wolters (1978)). Recently, Fagnani (1991) has introduced a general concept of c.i.'s for linear time-invariant dynamical systems $\Sigma=(T,\mathbb{R}^q,\mathcal{B})$ with time domain $T=\mathbb{Z}$ exclusively in terms of the behaviour $\mathcal{B}$, i.e. independent of a certain system representation. We suggest to call this approach the geometrical description of controllability indices. In the sequel we give a module theoretic definition of controllability indices for the special case of linear time-invariant complete systems and prove the equivalence of the two concepts. Furthermore, we also define c.i.'s for the time axis $T=\mathbb{Z}_+,\mathbb{R}_+,\mathbb{R}$.

Let $T=\mathbb{Z}$ and let supp(w) denote for every $w \in W^{\mathbb{Z}}$ the subset:

$$\operatorname{supp}(w) := \left\{t \in \mathbb{Z} \ , \ w(t) \neq 0\right\} \subset \mathbb{Z}$$

Let further $\mathcal{B}_t^+$, $t \in T$ denote the truncated behaviour spaces defined by:

$$\mathcal{B}_t^+ := \left\{w^+ \in \mathcal{B}^+ : \exists\ v \in \mathcal{B} \text{ with } v^+=w^+ \text{ and } \operatorname{supp}(v^-) \subseteq [-t,-1]\right\}$$

The $\mathcal{B}_t^+$ are linear subspaces satisfying:

$$\sigma^{-1}\mathcal{B}_o^+ \subseteq \mathcal{B}_o^+ \subseteq \mathcal{B}_1^+ \subseteq \dots \subseteq \mathcal{B}_t^+ \subseteq \dots \subseteq \mathcal{B}^+$$

The dimensions $m_t^+(\Sigma) := \dim C_t^+$, $t \in N_o$, of the quotient spaces

$$C_o^+ := \mathcal{B}_o^+ / \sigma^{-1}\mathcal{B}_o^+ \ , \quad C_t^+ := \mathcal{B}_t^+ / \mathcal{B}_{t-1}^+ \ , \quad t \geq 1$$

form a descending sequence $(m_t^+(\Sigma))_{t \in N_o}$.

Following Fagnani (1991) the numbers

$$c_i^+ := \#\left\{m_t^+(\Sigma) \geq i\right\} \ , \ 1 \leq i \leq m_o^+(\Sigma) \qquad (3.1)$$

are called the future controllability indices of Σ. An analogous construction with respect to the restrictions $\mathcal{B}_t^-$, $t \in \mathbb{Z}^-$ leads to the definition of past controllability indices.

3.1 Remark:

Observe that the sequence $(m_t^+(\Sigma))_{t\in N_0}$ will become constant after some $t_0 \in N_0$, but not necessary $m_t^+(\Sigma) = 0$ for $t \geq t_0$. Hence some the future controllability indices can be equal to ∞. If one only considers <u>finite memory systems</u>, all $c_i^+(\Sigma)$ are finite. Moreover, the past and future c.i.'s are equal in this case.

□

Furthermore, Fagnani showed that the c.i.'s are invariants with respect to a "controllability equivalence relation" on the set of all linear time-invariant behaviour systems defined as follows:

Two linear time-invariant systems $\Sigma_i = (\mathbb{Z}, \mathbb{R}^{q_i}, \mathcal{B}_i)$, i=1,2, are said to be <u>controllably equivalent</u> ($\Sigma_1 \simeq_c \Sigma_2$) if there exists a linear bijection ψ: $\mathcal{B}_1 \to \mathcal{B}_2$ such that:

(i) $\psi \circ \sigma^t = \sigma^t \circ \psi$ for all $t \in \mathbb{Z}$. (3.2)

(ii) For any $w_1, w_2 \in \mathcal{B}_1$ we have

$$w_1 \wedge w_2 \in \mathcal{B}_1 \Longleftrightarrow \psi(w_1) \wedge \psi(w_2) \in \mathcal{B}_2, \qquad (3.3)$$

and, if this is the case, then

$$\psi(w_1 \wedge w_2) = \psi(w_1) \wedge \psi(w_2) ,$$

where for $w_1, w_2 \in W^{\mathbb{Z}}$ we define

$$w_1 \wedge w_2(t) := \begin{cases} w_1(t) & \text{for } t<0 \\ w_2(t) & \text{for } t \geq 0 . \end{cases}$$

An equivalent condition for (3.3) is:

(iii) Let $w \in \mathcal{B}_1$; then

$$w^- = 0 \Longleftrightarrow (\psi(w))^- = 0$$
$$w^+ = 0 \Longleftrightarrow (\psi(w))^+ = 0 . \qquad (3.4)$$

Assume now that $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}(R))$ is a dynamical system in AR-representation:

$$\mathcal{B} = \ker R(\sigma,\sigma^{-1}) \tag{3.5a}$$

$$R(s,s^{-1}) = R_L s^L + \ldots + R_\ell s^\ell \in \mathbb{R}^{p\times q}[s,s^{-1}] \tag{3.5b}$$

$$\operatorname{rank}_{\mathbb{R}[s,s^{-1}]} R(s,s^{-1}) = p \tag{3.5c}$$

Here we implicitly assume that $p<q$; otherwise the following construction does not lead to a reasonable definition of c.i.'s; observe that $p=q$ corresponds to the autonomous case (compare Willems (1991)).

Interpreting $R(s,s^{-1})$ as the $\mathbb{R}[s,s^{-1}]$-linear mapping:

$$R(s,s^{-1}): \quad \begin{array}{ccc} \mathbb{R}^q[s,s^{-1}] & \longrightarrow & \mathbb{R}^p[s,s^{-1}] \\ x(s,s^{-1}) & \longmapsto & R(s,s^{-1})\cdot x(s,s^{-1}) \end{array}$$

we obtain that

$$M(R) := \ker R(s,s^{-1})$$

is a free $\mathbb{R}[s,s^{-1}]$-submodule of $\mathbb{R}^q[s,s^{-1}]$, satisfying:

$$M(R) = M(U\cdot R) \quad \text{for } U(s,s^{-1}) \in \mathbb{R}^{p\times p}[s,s^{-1}] \text{ unimodular}$$

Following the notation in Münzner and Prätzel-Wolters (1978) we call $M_\Sigma := M(R(s,s^{-1}))$ the "module of return to zero".

The list of polynomial indices (c.f. Münzner and Prätzel-Wolters (1978))

$$v(\Sigma) := (v_1(\Sigma),\ldots,v_m(\Sigma)),\ m := q-p$$

of the module

$$M_\Sigma \cap \mathbb{R}^q[s]$$

is called the list of (algebraic) controllability indices.

3.2 Remarks:

a) Another possible way to introduce c.i.'s is to define them as the dipolynomial indices of M_Σ (cf. Hoffmann and Prätzel-Wolters (1991b)). However, these two sets of integers coincide. Since the definition via the polynomial module is also valid for the case $T = \mathbb{Z}_+, \mathbb{R}_+, \mathbb{R}$, we have chosen it.

b) If Σ is in state space form, i.e.

$$\Sigma = (\mathbb{Z}, \mathbb{R}^{n+m}, \mathcal{B}((sI_n - A, B))), \quad (A,B) \in \mathbb{R}^{n\times(n+m)},$$

then the list $c(\Sigma)$ coincides with the list of the ordinary c.i.'s for state space systems. This is a consequence of the special form; the subsets $\mathcal{B}_t^+$ admit in this case:

$$\mathcal{B}_o^+ = \left\{ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in (\mathbb{R}^{n+m})^{\mathbb{N}_o} : w_1(k) = -A^{k-1}Bw_2(0) - \ldots - Bw_2(k-1), \right.$$

$$\left. k \geq 1,\ w_1(0) = 0 \text{ and } w_2(k) \in \mathbb{R}^m \text{ for } k \geq 0 \right\}$$

$$\sigma^{-1}\mathcal{B}_o^+ = \left\{ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathcal{B}_o^+ : w_2(0) \in \ker B \right\}$$

$$\mathcal{B}_k^+ = (\sigma\ \mathcal{B}_{k-1}^+)^+ \quad \text{for } k \geq 1$$

and hence:

$$\dim C_o^+ = \dim \left(\mathcal{B}_o^+ / \sigma^{-1}\mathcal{B}_o^+ \right) = n - \dim \ker B = \dim \operatorname{im} B,$$

$$\dim C_k^+ = \dim \left(\mathcal{B}_k^+ / \mathcal{B}_{k-1}^+ \right)$$

$$= \dim \operatorname{Im}(B, AB, \ldots, A^{k-1}B) / \operatorname{Im}(B, AB, \ldots, A^{k-2}B)$$

Note that the form of $\sigma^{-1}\mathcal{B}_o^+$ as calculated above contradicts the characterization

$$\sigma^{-1}\mathcal{B}_o^+ = \left\{ w \in \mathcal{B}_o^+ : w(0) = 0 \right\}$$

given in Fagnani (1991). □

The geometric and algebraic controllability indices coincide:

3.3 Theorem:

Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}(R))$ where R satisfies (3.5). Then $c(\Sigma) = v(\Sigma)$.

□

For the proof of Theorem 3.3 we need the following lemmata:

3.4 Lemma:

Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}(R))$ where R satisfies (3.5). Furthermore, let $U \in \mathbb{R}^{p\times p}[s,s^{-1}]$ unimodular and $Q \in \mathbb{R}^{q\times q}$ nonsingular such that

$$P := URQ = \sum_{k=0}^{deg} P_k s^k =: \sum_{k=0}^{deg} (\bar{P}_k, \tilde{P}_k) s^k$$

with $P_o \neq 0$, rank $\bar{P}_{deg} = p$ and $\tilde{P}_{deg} = 0_{p\times(q-p)}$.

Then URQ is strict system equivalent to the state space form $(sI_{deg\cdot p} - A_\Sigma, B_\Sigma)$ where:

$$A_\Sigma := \begin{pmatrix} 0 & \cdots & 0 & -\bar{P}_o \cdot \bar{P}_{deg}^{-1} \\ I_p & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & \vdots \\ 0 & \cdots\ 0 & I_p & -\bar{P}_{deg-1} \cdot \bar{P}_{deg}^{-1} \end{pmatrix}, \quad B_\Sigma = \begin{pmatrix} \tilde{P}_o \\ \vdots \\ \vdots \\ \tilde{P}_{deg-1} \end{pmatrix}$$

Proof:

We will show that there exist matrices $M_{1e}, M_{2e} \in \mathbb{R}^{deg\cdot p\times deg\cdot p}[s]$ unimodular and $K \in \mathbb{R}^{deg\cdot p\times(q-p)}[s]$ such that

$$M_{1e}(sI_{deg\cdot p} - A_\Sigma, B_\Sigma) \begin{pmatrix} M_{2e} & K \\ 0 & I_{q-p} \end{pmatrix} = \begin{pmatrix} I_{(deg-1)\cdot p} & 0 \\ 0 & P \end{pmatrix}. \qquad (3.6)$$

Now

$$(sI_{deg\cdot p} - A_\Sigma, B_\Sigma)$$

$$= \begin{pmatrix} sI_p & 0 & \cdots & \cdots & 0 & \bar{P}_o \cdot \bar{P}_{deg}^{-1} & \tilde{P}_o \\ -I_p & sI_p & 0 & & \vdots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \vdots \\ \vdots & & \ddots & \ddots & sI_p & \bar{P}_{deg-2} \cdot \bar{P}_{deg}^{-1} & \tilde{P}_{deg-2} \\ 0 & \cdots & \cdots & 0 & -I_p & sI_p + \bar{P}_{deg-1} \cdot \bar{P}_{deg}^{-1} & \tilde{P}_{deg-1} \end{pmatrix}.$$

Successive multiplication from the left by the unimodular matrices

$$\begin{bmatrix} I_p & & & \\ & \ddots & & \\ & & I_p & sI_p \\ & & & I_p \end{bmatrix}, \begin{bmatrix} I_p & & & & \\ & \ddots & & & \\ & & I_p & sI_p & \\ & & & I_p & \\ & & & & I_p \end{bmatrix}, \ldots, \begin{bmatrix} I_p & sI_p & & \\ & I_p & & \\ & & \ddots & \\ & & & I_p \end{bmatrix}$$

yields the matrix

$$\begin{bmatrix} 0 & \cdots\cdots & 0 & X_1 & Y_1 \\ -I_p & \ddots & \vdots & \vdots & \vdots \\ 0 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots \quad 0 & X_{deg-1} & Y_{deg-1} \\ 0 & \cdots \quad \cdots \quad 0 & -I_p & X_{deg} & Y_{deg} \end{bmatrix}$$

where

$$X_i := s^{deg-i+1}I_p + s^{deg-i}\bar{P}_{deg-1}\cdot\bar{P}_{deg}^{-1} + \ldots + \bar{P}_{i-1}\cdot\bar{P}_{deg}^{-1} \in \mathbb{R}^{p\times p}[s]$$

and

$$Y_i := s^{deg-i}\tilde{P}'_{deg-1} + \ldots + \tilde{P}_{i-1} \in \mathbb{R}^{p\times(q-p)}[s]$$

for $i=1,\ldots,deg$.

Multiplying successively from the right by the unimodular matrices

$$\begin{bmatrix} I_p & & & & \\ & \ddots & & & \\ & & I_p & X_{deg} & Y_{deg} \\ & & & I_p & 0 \\ & & & & I_{q-p} \end{bmatrix}, \ldots, \begin{bmatrix} I_p & & X_2 & Y_2 \\ & \ddots & & \\ & & I_p & \\ & & & I_p & \\ & & & & I_{q-p} \end{bmatrix}$$

and

$$\begin{bmatrix} I_p & & & & \\ & \ddots & & & \\ & & I_p & & \\ & & & \bar{P}_{deg} & \\ & & & & I_{q-p} \end{bmatrix}.$$

one obtains

$$\begin{bmatrix} 0 & \cdots\cdots & 0 & P \\ -I_p & \ddots & \vdots & 0 \\ 0 & \ddots \ddots & \vdots & \vdots \\ \vdots & \ddots \ddots \ddots & \vdots & \vdots \\ \vdots & \ddots \ddots & 0 & \vdots \\ 0 & \cdots \;\cdots\; 0 & -I_p & 0 \end{bmatrix},$$

which gets transformed by elementary row transformations into

$$\begin{bmatrix} I_{(\deg-1)\cdot p} & 0 \\ 0 & P \end{bmatrix}.$$

In total all the transformations are of the form (3.6). □

3.5 Lemma:

(i) Let $\Sigma_1 = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}(R))$ where R satisfies (3.5). Let $\Sigma_2 = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}(RT))$ where $T \in \mathbb{R}^{q\times q}$ is nonsingular. Then:

(a) $c(\Sigma_1) = c(\Sigma_2)$

(b) $v(\Sigma_1) = v(\Sigma_2)$.

(ii) Let $\Sigma_i = (\mathbb{Z}, \mathbb{R}^{q_i}, \mathcal{B}(R_i))$, $R_i = (T_i, U_i)$, $q_i = \ell_i + m$, $T_i \in \mathbb{R}^{\ell_i \times \ell_i}[s]$, $\det T_i \neq 0$, $U_i \in \mathbb{R}^{\ell_i \times m}[s]$, i=1,2. Furthermore, assume that $T_i^{-1}U_i$ is strictly proper rational for i=1,2 and that Σ_1 and Σ_2 are strictly system equivalent. Then:

(a) $v(\Sigma_1) = v(\Sigma_2)$

(b) $c(\Sigma_1) = c(\Sigma_2)$

Proof:

(i) (a) Define ψ: $\mathcal{B}(R) \longrightarrow \mathcal{B}(RT)$, $w \to T^{-1}w$. Then ψ is an isomorphism and clearly satisfies conditions (3.2) and (3.4). Hence $\Sigma_1 \simeq_c \Sigma_2$.

(b) The mapping $M_{\Sigma_1} \cap \mathbb{R}^q[s] \longrightarrow M_{\Sigma_2} \cap \mathbb{R}^q[s]$,

$x(s) \to T^{-1}\cdot x(s)$ is a (polynomial) degree-preserving $\mathbb{R}[s]$-isomorphism, which implies $v(\Sigma_1) = v(\Sigma_2)$.

(ii) (a) (Compare Theorem 3.4 in Münzner and Prätzel-Wolters (1978).)

(b) By the definition of strict system equivalence there exists $q \geq \max(\ell_1, \ell_2)$ and polynomial matrices M_{1e}, M_{2e} and Y with M_{1e}, M_{2e} unimodular such that

$$M_{1e}\begin{pmatrix} I_{q-\ell_1} & 0 & 0 \\ 0 & T_1 & U_1 \end{pmatrix} = \begin{pmatrix} I_{q-\ell_2} & 0 & 0 \\ 0 & T_2 & U_2 \end{pmatrix}\begin{pmatrix} M_{2e} & -Y \\ 0 & I_m \end{pmatrix} \tag{3.7}$$

Let $\tilde{\Sigma}_i = (\mathbb{Z}, \mathbb{R}^{q+m}, \mathcal{B}(\tilde{R}_i))$ where $\tilde{R}_i = \begin{pmatrix} I_{q-\ell_i} & 0 & 0 \\ 0 & T_i & U_i \end{pmatrix} \in \mathbb{R}^{q\times(q+m)}[s]$,

$i=1,2$. Then $c(\Sigma_i) = c(\tilde{\Sigma}_i)$ for $i=1,2$. Define:

$$\psi:\ \mathcal{B}(\tilde{R}_1) \longrightarrow \mathcal{B}(\tilde{R}_2)\ ,\ w \longmapsto \begin{pmatrix} M_{2e}(\sigma) & -Y(\sigma) \\ 0 & I_m \end{pmatrix} w \tag{3.8}$$

Then ψ is an isomorphism (c.f. (3.7)) which commutes with the shift σ. It remains to show that (3.4) is satisfied.

Let $\Pi:\ (\mathbb{R}^{q+m})^{\mathbb{Z}} \longrightarrow (\mathbb{R}^m)^{\mathbb{Z}}$ denote the projection $\binom{x}{y} \longmapsto y$ and let $w \in \mathcal{B}(\tilde{R}_1)$. Assume $w^- = 0$. Then by (3.8) $\Pi(\psi(w))^- = \Pi w^- = 0$; now

$$(1-\Pi)(\psi(w))^- = \begin{pmatrix} 0_{q-\ell_2} \\ T_2^{-1}U_2(\sigma)\Pi(\psi(w))^- \end{pmatrix}$$ since $T_2^{-1}U_2$ is strictly

proper rational, and hence $(1-\Pi)(\psi(w))^- = 0$. The converse implication $(\psi(w))^- = 0 \Rightarrow w^- = 0$ is proven analogously because ψ^{-1} is of the form

$$\psi^{-1}:\ \mathcal{B}(\tilde{R}_2) \longrightarrow \mathcal{B}(\tilde{R}_1)\ ,\ w \longmapsto \begin{pmatrix} M_{2e}^{-1}(\sigma) & M_{2e}^{-1}Y(\sigma) \\ 0 & I_m \end{pmatrix} w \tag{3.9}$$

with M_{2e}^{-1} polynomial.

Assume $w^+ = 0$. Since M_{2e} and Y are polynomial there holds: $(\psi(w))^+ = \psi(w^+)$, which gives $(\psi(w))^+ = 0$. Furthermore, the implication $(\psi(w))^+ = 0 \Rightarrow w^+ = 0$ is an immediate consequence of the unimodularity of M_{2e} and (3.9). Summarizing, there holds $\Sigma_1 \approx_c \Sigma_2$.

□

Proof of Theorem 3.3:

Let P := URQ, A_Σ and B_Σ as defined in Lemma 3.4 and let $\Sigma_1 := (\mathbb{Z},\mathbb{R}^q,\mathcal{B}(P))$ and $\Sigma_2 := (\mathbb{Z},\mathbb{R}^{(deg-1)\cdot p+q},\mathcal{B}(sI_{deg\cdot p}-A_\Sigma,B_\Sigma))$. Since left multiplication of R by a unimodular U does not change the behaviour we obtain $c(\Sigma) = c(\Sigma_1)$ and $v(\Sigma) = v(\Sigma_1)$ by Lemma 3.5 (i). By Lemma 3.4 Σ_1 and Σ_2 are strict system equivalent and satisfy the assumptions of Lemma 3.5 (ii), hence $c(\Sigma_1) = c(\Sigma_2)$ and $v(\Sigma_1) = v(\Sigma_2)$. By Remark 3.2 b) the list $c(\Sigma_2)$ coincides with the list of ordinary c.i.'s for state space systems, which is identical to $v(\Sigma_2)$ (c.f. Theorem 3.3 in Münzner and Prätzel-Wolters (1978)).

□

3.6 Remark:

For arbitrary Rosenbrock-type polynomial system matrices:

$$R(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} \in \mathbb{R}^{(\ell+p)\times(\ell+m)}[s] \qquad (3.10a)$$

$$\det T(s) \not\equiv 0, \ (VT^{-1}U+W) \text{ strict proper rational} \qquad (3.10b)$$

as well as for singular state-space systems

$$E\dot{x} = Ax+Bu \qquad (3.11a)$$

$$E,A \in \mathbb{R}^{n\times n}, \ B \in \mathbb{R}^{n\times m}, \ \det[sE-A] \not\equiv 0 \qquad (3.11b)$$

the lists of controllability indices defined in the literature (c.f. Münzner and Prätzel-Wolters (1978) and Glüsing-Lüerßen (1991)) coincide with the list $v(\Sigma)$ of $\Sigma = (\mathbb{Z},\mathbb{R}^{\ell+m},\mathcal{B}(T(s),U(s)))$ with T,U as in (3.10) respectively the list $v(\tilde{\Sigma})$ with $\tilde{\Sigma} = (\mathbb{Z},\mathbb{R}^{n+m},\mathcal{B}(sE-A,B))$ and E,A,B from (3.11).

□

Let $\Sigma = (\mathbb{Z},\mathbb{R}^q,\mathcal{B}(R))$ be again a dynamical system in AR-representation with R satisfying (3.5). Let further

$$f(s,s^{-1}) = \left[f_1,\ldots,f_{\binom{q}{p}}\right]$$

be the vector of all p×p-minors f_i of R. Willems (1991) defines the <u>Mc Millan degree</u> of Σ, $Mm(\Sigma)$, as:

$$Mm(\Sigma) = Mm(R) = \text{ddeg } f(s,s^{-1}) \tag{3.12}$$

$Mm(\Sigma)$ is well defined because $Mm(R) = Mm(UR)$ for any unimodular U. Even $Mm(RQ) = Mm(R)$ is true for nonsingular constant matrices Q.

3.7 Theorem:

Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}(R))$ where R satisfies (3.5). Let further $v(\Sigma) = (v_1,\ldots,v_m)$ be the list of controllability indices of Σ. Then:

$$\Sigma \text{ controllable}$$

$$\Leftrightarrow \quad Mm(\Sigma) = \sum_{i=1}^{m} v_i$$

Proof:

Transform R to $P = URQ = (\bar{P},\tilde{P})$ as in Lemma 3.4 with rank $P_o=p$. These transformations leave controllability invariant, i.e. $\mathcal{B}(R)$ controllable $\Leftrightarrow$ $\mathcal{B}(P)$ controllable, and $Mm(R) = Mm(P)$.
Now by Theorem 3.4 (ii) <=> (iii) in Hoffmann and Prätzel-Wolters (1991b) we have

$$Mm(P) = \sum_{i=1}^{p} \mu_i \tag{3.13}$$

where $\mu := (\mu_1,\ldots,\mu_p)$ is the index list associated with the module $\mathbb{R}^{1\times p}[s,s^{-1}]\cdot P$, i.e. the μ_i's are the lags of a minimal lag description of $\mathcal{B}(P)$ (see Hoffmann and Prätzel-Wolters (1991b)).
However, controllability of Σ is equivalent to controllability of $\mathcal{B}(P)$ where P is a polynomial Rosenbrock-type system matrix. For these matrices we have:

$$\mathcal{B}(P) \text{ controllable} \Leftrightarrow \deg(\det \bar{P}) = \sum_{i=1}^{m} v_i$$

This together with $\deg(\det \bar{P}) = \sum_{i=1}^{p} \mu_i$ and (3.13) proves the result. □

Finally, based on Lemma 3.4 we obtain an effective <u>algorithm</u> for the calculation of the controllability indices.
Starting with a system $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}(R))$ satisfying (3.5) we first construct a strictly system equivalent state-space system $(A_\Sigma, B_\Sigma) \in \mathbb{R}^{\deg\cdot p\times(\deg-1)p+q}$ according to Lemma 3.4. For an explicit construction of the transformation matrices (Q,U) compare Hoffmann (1991). Note that (A_Σ, B_Σ) is not uniquely determined; however, all possible state-space systems generate the same index list $\nu(\Sigma)$. Having obtained (A_Σ, B_Σ) we determine $\nu(\Sigma)$ by the Kalman-Rosenbrock deleting procedure.

3.8 Example:
Consider the nonsingular system of difference equations:

$$\begin{aligned} &w_1(t+2)+3w_3(t+2)+6w_4(t+2)+3w_5(t+2)+\\ &+2w_1(t+1)+w_2(t+1)-w_3(t+1)+w_5(t+1)+\\ &\qquad +w_1(t)+2w_2(t)+2w_4(t)+3w_5(t) = 0\\ &\qquad\qquad 2w_1(t+2)+w_4(t) = 0\ ,\ t\in\mathbb{Z} \end{aligned}$$

with the associated dynamical system $\Sigma = (\mathbb{Z}, \mathbb{R}^5, \mathcal{B}(R))$, where:

$$R(s,s^{-1}) := \begin{pmatrix} s^2+2s+1 & s+2 & 3s^2-s & 6s^2+2 & 3s^2+s+3 \\ 2s^2 & 0 & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 5}[s,s^{-1}]$$

Hence p=2, q=5 and deg=2. The 2×2-minors of R are

$$-2s^2(s+2),\ -2s^2(3s^2-s),\ s^2+2s+1-2s^2(6s^2+2),\ -2s^2(3s^2+s+3),$$
$$0,\ s+2,\ 0,\ 3s^2-s,\ 0,\ -(3s^2+s+3).$$

Since there is one minor not equal to zero,

$$\operatorname{rank}_{\mathbb{R}[s,s^{-1}]} R(s,s^{-1}) = 2\ .$$

Furthermore, simple calculations show that the gcd of the above minors is a dipolynomial unit, which yields the controllability of Σ (c.f. Willems (1991)). Now write

$$R(s,s^{-1}) = \begin{pmatrix} 1 & 0 & 3 & 6 & 3 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix} s^2 + \begin{pmatrix} 2 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} s + \begin{pmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}\ .$$

Observe that R is polynomial; moreover, R is a (dipolynomial) minimal lag description with $Mm(\Sigma) = 4$. Define $Q \in \mathbb{R}^{5\times 5}$ by

$$Q := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} .$$

Then

$$P(s) := R(s)\cdot Q = \begin{pmatrix} 1 & 3 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix} s^2 + \begin{pmatrix} 2 & -1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} s +$$

$$+ \begin{pmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} .$$

Obviously, P is in the form as in Lemma 3.4. For the matrices A_Σ and B_Σ we obtain

$$A_\Sigma = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{3} & -\frac{7}{6} \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad B_\Sigma = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Moreover,

$$(B_\Sigma, A_\Sigma B_\Sigma, A_\Sigma^2 B_\Sigma, A_\Sigma^3 B_\Sigma) =$$

$$= \begin{pmatrix} 2 & 2 & 3 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & \frac{7}{3} & \frac{8}{3} & \frac{11}{3} & \frac{7}{9} & -\frac{5}{18} & \frac{11}{9} & \frac{7}{27} & -\frac{16}{27} & \frac{11}{27} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and the controllability indices of Σ are $v_1=2 > v_2=v_3=1$.
$\sum_{i=1}^{3} v_i = 4 = \deg \cdot p = Mm(\Sigma)$.

□

For the time axis $T = \mathbb{Z}_+, \mathbb{R}_+, \mathbb{R}$ there does not exist a geometrical description of the controllability indices of $\Sigma = (T, \mathbb{R}^q, \mathcal{B})$. However, if Σ is a linear time-invariant system in AR-representation, i.e. $\mathcal{B} = \mathcal{B}(R)$ where $R(s)$ is a polynomial $p\times q$-matrix satisfying $\operatorname{rank}_{\mathbb{R}[s]} R(s) = p$, then the developed algebraic construction carries over completely to the $\mathbb{R}[s]$-linear mapping:

$$R(s) : \begin{array}{l} \mathbb{R}^q[s] \longrightarrow \mathbb{R}^p[s] \\ x(s) \longmapsto R(s)x(s) \end{array}$$

and the associated module

$$M(R) = \ker R(s) \subset \mathbb{R}^q[s] .$$

4. Conclusion

The purpose of this paper was the construction of controllability indices for dynamical AR-systems in a module theoretic framework. The obtained list of controllability indices coincides with the index list introduced by Fagnani (1991) in a geometric framework.
Moreover, several existing concepts of controllability indices for different representations of linear systems are shown to be special cases of the new definition.
Finally, an effective algorithm for the calculation of the controllability indices was derived.

References

Brunovsky, P. (1970): A classification of linear controllable systems; Kybernetica, 3, 173-187

Dai, L. (1989): State feedback and transfer matrix in singular systems; Int. J. Control, 49, 905-920

Fagnani, F. (1991): Controllability and observability indices for linear time-invariant systems: A new approach; System and Control Letters, 17, 243-250

Forney, G.D. (1975): Minimal bases of rational vector spaces, with applications to multivariable linear systems; SIAM Journal on Control, 13, 493-520

Glüsing-Lüerßen, H. (1991): Gruppenaktionen in der Theorie singulärer Systeme; Dissertation, Fachbereich Mathematik/ Informatik, Universität Bremen

Hoffmann, J. (1991): Kontrollierbarkeit und Beobachtbarkeit dynamischer Systeme - ein algebraischer Zugang; Diplomarbeit, Fachbereich Mathematik, Universität Kaiserslautern

Hoffmann, J. and Prätzel-Wolters, D. (1991a): Controllability tests for behaviour systems in AR-representation; Universität Kaiserslautern, Berichte der Arbeitsgruppe Technomathematik, Bericht Nr. 59

Hoffmann, J. and Prätzel-Wolters, D. (1991b): Dipolynomial minimal bases and linear systems in AR-representation; Universität Kaiserslautern, Berichte der Arbeitsgruppe Technomathematik, Bericht Nr. 64

Kailath, T. (1980): Linear systems, Prentice-Hall

Kalman, R.E. (1971): Kronecker invariants and feedback; Proc. Conf. on ordinary differential equations, Math. Research Center, Naval Research Lab., Washington D.C.

Karcanias, N. and Heliopoulou, H. (1989): A classification of minimal bases for singular systems; in: International Symposium on the Mathematical Theory of Networks and Systems, Amsterdam

Kučera, V. and Zagalak, P. (1988): Fundamental theorem of state feedback for singular systems; Automatica, 24, 653-658

Malabre, M., Kučera, V. and Zagalak, P. (1990): Reachability and controllability indices for linear descriptor systems; Syst. Contr. Let., 15, 119-123

Münzner, H.F. and Prätzel-Wolters, D. (1978): Minimalbasen polynomialer Moduln, Strukturindizes und BRUNOVSKY-Transformationen; Universität Bremen, Veröffentlichung des Forschungsschwerpunkts Dynamische Systeme, Report Nr. 1

Popov, V.M. (1964): Hyperstability and optimality of automatic systems with several control functions; Rev. Roumaine Sci. Techn. Elektrotechn. et Engrg., 9, 629-690

Popov, V.M. (1972): Invariant description of linear time-invariant controllable systems; SIAM Journal on Control, 10, 252-264

Rosenbrock, H.H. (1970): State space and multivariable theory; Wiley-Interscience, New York

Rosenbrock, H.H. and Hayton, G.E. (1974): Dynamical indices of a transfer function matrix; Int. J. Control, 20, 177-189

Shayman, M.A. (1988): Homogeneous indices, feedback invariants and control structure theorem for generalized linear systems; SIAM J. Contr. and Opt., 26, 387-400

Willems, J.C. (1986a): From time series to linear system - Part I: Finite-dimensional linear time invariant systems; Automatica, 22, 561-580

Willems, J.C. (1986b): From time series to linear system - Part II: Exact modelling; Automatica, 22, 675-694

Willems, J.C. (1987): From time series to linear system - Part III: Approximate modelling; Automatica, 23, 87-115

Willems, J.C. (1988): Models for dynamics; Dynamics reported, 2, 171-269

Willems, J.C. (1991): Paradigms and puzzles in the theory of dynamical systems; IEEE Transactions on Automatic Control, 36, 259-294

Wolovich, W.A. (1974): Linear multivariable systems, Springer

Wonham, W.M. and Morse, A.S. (1972): Feedback invariants of linear multivariable systems; Automatica, 8, 93-100