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MULTIPARAMETER, POLYNOMIAL ADAPTIVE STABILIZERS

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1. Introduction

The problem of stabilizing a linear plant

$$\dot{y}(t) = ay(t) + bu(t), \quad x(t) \in R, \quad u(t) \in R \tag{1.1}$$

with a and b unknown numbers, $b \neq 0$, by a smooth controller of the form

$$u(t) = g(y(t), k(t), t) \tag{1.2}$$

$$\dot{k}(t) = f(y(t), k(t), t) \tag{1.3}$$

has been considered in a series of papers (Morse 1983, Nussbaum 1983, Willems and Byrnes 1984, Heymann et al. 1985). A variety of stabilizing control rules (1.2), (1.3) is known also for controllable and observable scalar systems (A,b,c) of higher order n which have (n–1) stable zeros and multivariable system (A,B,C) which are controllable, observable, minimum phase and satisfy det CB $\neq$ 0 (cf. Byrnes and Willems 1984, Mårtensson 1986). All these controllers are combinations of a *high gain adaptation law* $\dot{k} = f(y,k)$ with some switching devices inherent in $u = g(y,k,t)$.
More general the concept (1.2), (1.3) of universal adaptive stabilizers (UAS) has become a major research topic in adaptive control, and there are a number of results available for finite–dimensional linear systems (Byrnes and Willems (1984), Mårtensson (1986), Helmke and Prätzel–Wolters (1988), Owens et al (1989)), infinite–dimensional linear systems (Logemann and Zwart (1990), Logemann and Mårtensson (1990)), nonlinear systems (Byrnes and Isidori (1988), Mårtensson (1990), Ryan (1991), Nikitin and Schmid (1990)) and for the tracking problem (Helmke et al (1990)). The above list of

references is certainly not complete.

In most of these publications the proposed controllers are one parameter adaptive high gain feedback controllers. Very little work has been done on multiparameter–feedback laws in the context of UAS's (c.f. Schmid (1991) for a multiparameter version of the unbounded gain variation theorem).
In this paper we turn back to the initiating problem of Morse (1983):

– Do there exist smooth, rational or polynomial functions f and g in (1.2), (1.3) stabilizing all linear plants of the form (1.1)?

While Morse proved for scalar gains k(t) nonexistence of rational or polynomial UAS's for (1.1), Nussbaum (1983) gave the following example of an analytic UAS (f,g):

$$u = (k^2+1) \cos \frac{\pi k}{2} e^{k^2} \cdot y \tag{1.3}$$

$$\dot{k} = y(k+1) \tag{1.4}$$

To the best of our knowledge it has never been investigated, if there exists a polynomial UAS (f,g) for (1.1), if we allow for multiple gains $k = (k_1,\dots,k_p) \in \mathbb{R}^p$. In this paper we will show that indeed for even $p = 2l$, $l \in \mathbb{N}$, there exist such time–invariant polynomial pairs ($f:\mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^p$, $g:\mathbb{R} \times \mathbb{R}^p \to \mathbb{R}$) stabilizing all controllable systems of the form (1.1). Furthermore this result extends to scalar, higher order, minimum–phase, relative degree one systems. Usually the switching mechanism is in corporated in the feedback–function g, hence seperated from the gain adaptation. For our controller we combine gain adaptation and switching in one function f.

2. First order systems

In this section we analyse a certain family of multigain polynomial universal adaptive stabilizers for the class of first order linear systems:

$$L(1): \quad \dot{y} = ay + bu, \quad (a,b) \in \mathbb{R} \times \mathbb{R} \setminus \{0\}. \tag{2.1}$$

We need the following preparatory

2.1 Lemma:
Let $p(s) = s^{2n} + p_1 s^{2n-1} + \ldots + p_{2n-1} s + p_{2n} \in \mathbb{R}[s]$ be an unstable polynomial with real coefficients $p_j \in \mathbb{R}$, j=1,...,2n and complex zeroes

$$\begin{aligned} & s_j = \gamma_j + i\, w_j \quad \text{with} \\ & \gamma_j > 0,\ w_j \neq 0 \quad \text{for } j=1,\ldots,2n \end{aligned} \tag{2.2}$$

then

(i) Any nontrivial solution $z(\cdot) \not\equiv 0$ of the differential equation:

$$p(D)\, \bar{z} = 0 \tag{2.3}$$

has infinitely many zeros.

(ii) For every $q(t) \in \mathbb{R}[t]$, $h \in \mathbb{N}$ and any solution z(.) of (2.3)

$$\hat{z}(t) := \int_0^t \int_0^{\tau_1} \ldots \int_0^{\tau_{h-1}} z(\tau_n)\, d\tau_n\, d\tau_{n-1} \ldots d\tau_1 + q(t)$$

has infinitely many zeros and

$$\liminf_{t\to\infty} \hat{z}(t) = -\infty \quad \text{and} \quad \limsup_{t\to\infty} \hat{z}(t) = +\infty \tag{2.4}$$

Proof:

(i) Every solution of (2.3) is of the form

$$z(t) = \sum_{j=1}^{M} e^{\gamma_j t} \left(r_j(t) \cos\, w_j t + m_j(t) \sin w_j t\right)$$

where M denotes the number of different roots of p(s) and $r_j(s)$, $m_j(s)$ are polynomials of degree v_j, where v_j denotes the multiplicity of s_j, j=1,...,2n. Let $\gamma_{max} := \max\{\gamma_1,\ldots,\gamma_\mu\}$. Then:

$$z(t) = e^{\gamma_{max} t}(r_{max}(t) \cos w_{max} t + m_{max}(t) \sin w_{max} t + f(t))$$

where

$$\lim_{t \to \infty} f(t) = \lim_{t \to \infty} \sum_{j \neq max}^{M} e^{-(\gamma_{max} - \gamma_j)t} (r_j(t) \cos w_j t + m_j(t) \sin w_j t) = 0$$

Furthermore, since

$$r_{max}(t) \cos w_{max} t + m_{max}(t) \cdot \sin w_{max} t$$
$$= (\sqrt{r_{max}(t)^2 + m_{max}(t)^2}) \cdot \cos (w_{max} t + \xi(t))$$

where $\lim_{t \to \infty} \xi(t) = \alpha < \infty$ exists, for $t \to \infty$ the zeroes of $z(t)$ converge to the zeros of $\cos (w_{max}\, t + \alpha)$.

(ii) An easy calculation shows that for $t \to \infty$, $\hat{z}(t)$ converges to:

$$e^{\gamma_{max} t} \sqrt{g_1(t)^2 + g_2(t)^2}) \cdot \cos (w_{max} t + \rho(t)) \tag{2.5}$$

where $g_1(\cdot)$ and $g_2(\cdot)$ are two real polynomials and $\lim_{t \to \infty} \rho(t) = \beta < \infty$ exists.
Hence $\hat{z}(t)$ has infinitely many zeros too.
Furthermore because $\gamma_{max} > 0$ and $g_1(t)$, $g_2(t)$ are non trivial polynomials the formulas in (2.4) are immediate consequences of (2.5).

□

2.2 Proposition:

The controller:

$$u(t) = (\sum_{j=1}^{2n} \lambda_i k_i(t))\, y(t) \tag{2.6a}$$

$$\dot{k}(t) = \begin{bmatrix} \dot{k}_1(t) \\ \vdots \\ \dot{k}_{2n}(t) \end{bmatrix} = y(t)^2 \cdot A \begin{bmatrix} k_1(t) \\ \vdots \\ k_{2n}(t) \end{bmatrix}, \; k(t_o) \neq 0$$

(2.6b)

where

(i) $(\lambda,A) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n \times 2n}$ is observable

(ii) $\sigma(A) \cap (\mathbb{C}\_ \cup \mathbb{R}) = \emptyset$ (2.6c)

is a UAS for $L(1)$.

Proof: The closed loop–system

$$\dot{y}(t) = (a+b < \lambda, k(t)>)\, y(t)$$
$$\dot{k}(t) = y(t)^2 \cdot A\, k(t)$$

is transformed by $z := y^2/_2$ and the rescaling of time $\tau := \int_0^t z(\sigma)\, d\sigma$ into the linear system:

$$\dot{z}(\tau) = \frac{dz}{d\tau} = a+b < \lambda, k(\tau) > \tag{2.7a}$$

$$\dot{k}(\tau) = \frac{dk}{d\tau} = Ak(\tau) \tag{2.7b}$$

From (2.7) we obtain by sucessive derivation:

$$z^{(2)}(\tau) = b < \lambda, A\, k(\tau) >$$
$$\vdots$$
$$z^{(2(n+1))}(\tau) = b < \lambda, A^{2n+1} k(\tau) >$$

Cayley–Hamilton implies that there exist $p_1,\ldots,p_{2n} \in \mathbb{R}$ such that

$$< \lambda, A^{2n}\, k > = -p_1 < \lambda, A^{2n-1}\, k > - \ldots - p_{2n-1} < \lambda, Ak > - p_{2n} < \lambda, k >$$

Thus

$$z^{(2n+2)} + p_1 z^{(2n+1)} + \ldots + p_{2n} z^{(2)} = 0.$$

Moreover observability of (λ,A) together with the assumption $k(t_0) \neq 0$ implies $< \lambda, Ak(t) > \not\equiv 0$ and by lemma 2.1. $z^{(2)}(\tau)$ has infinitely many zeros. Finally from part (ii) of lemma 2.1 we conclude that

$$z(\tau) = \int_0^\tau \int_0^t z^{(2)}(\sigma)\, d\sigma\, d\tau + q_1 \tau + q_2$$

has infinitely many zeros too. But $z(\tau) = {y^2(\tau)}/_2$ and therefore $\tau = \int_0^t z(\sigma)\, d\sigma$ remains bounded. This implies $y(t) \in L_2\ [0,\infty)$ and $\lim_{t\to\infty} k(t) = \lim_{t\to\infty} e^{\alpha \int_0^t y(\sigma)^2\, d\sigma} k(0) = k_\infty < \infty$. Finally because $y(\cdot)$ is L_2 and $k(\cdot)$ is bounded, $\dot{y}(\cdot)$ is L_2 too and thus $\lim_{t\to\infty} y(t) = 0$.

□

2.3 Remarks:

(i) Proposition 2.2 shows that, contrary to the one parameter case, there exist 2n–parameter, n = 1,2..., universal adaptive stabilizes with polynomial control and adaptation laws.

(ii) Fig. 1 illustrates the behaviour of $y(\cdot)$ and $k(\cdot)$ for the system

$$\dot{y} = 4.9y - 0{,}5u, \quad y(0) = 1,\ k(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$u = [0\ \ 1]\, ky, \quad \dot{k} = y^2 \cdot \begin{bmatrix} 0 & -26 \\ 1 & 2 \end{bmatrix} k$$

$$\sigma(A) = \{1 \pm 5i\}$$

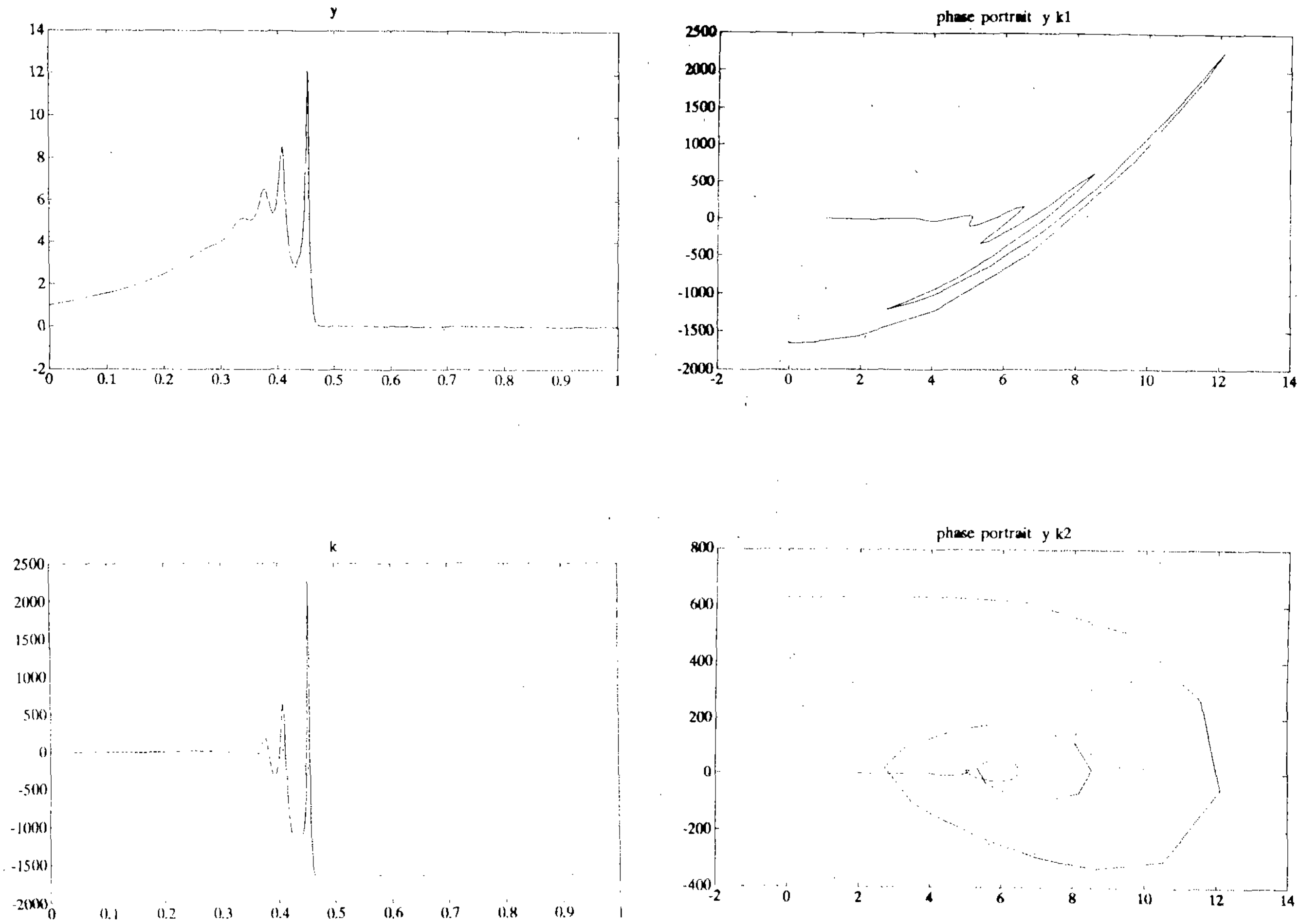

Figure 1

3. Relative degree one minimum phase systems

In this section L denotes the set of all scalar, controllable and observable linear systems (A,b,c) satisfying:

$$\det\ cb \neq 0 \tag{3.1a}$$

$$\det \begin{bmatrix} sI-A & b \\ c & 0 \end{bmatrix} \neq 0 \text{ for } s \in \mathbb{C}_+ \tag{3.1b}$$

i.e. the systems $(A,b,c) \in L$ are minimum phase and relative degree one systems. We show that the UAS's described in section 2 stabilize L too.

For T finite instead of $T = \infty$ the following lemma is frequently applied in adaptive control texts. The essential point here is, that the constant C in formula (3.3) can be selected T–independent.

3.1. Lemma: Let the scalar system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t) \\ y(t) &= c\,x(t), \quad x(0) = x_0 \end{aligned} \tag{3.2}$$

be asymptotically stable with decay rate $\alpha < 0$, i.e. $\|e^{At}\| \leq Me^{\alpha t}$. Then for every $u(\cdot) \in L_2(\mathbb{R}_+)$ we have:

$$\left|\int_0^\infty u(\sigma)\, y(\sigma)\, d\sigma\right| \leq C \cdot \int_0^\infty u(\sigma)^2\, d\sigma \tag{3.3}$$

Proof: The solution $y(\cdot)$ of (3.2) is given by:

$$y(t) = c\, e^{At} x_0 + c \int_0^t e^{A(t-\tau)}\, bu(\tau)\, d\tau,$$

hence

$$\begin{aligned} \left|\int_0^\infty y(\tau)\, u(\tau)\, d\tau\right| &\leq k_1 \int_0^\infty u(\tau)^2\, d\tau \\ &+ k_2 \cdot \int_0^\infty e^{\alpha\tau} |u(\tau)| \int_0^\tau e^{-\alpha\theta}\, |bu(\theta)|\, d\theta\, d\tau \end{aligned} \tag{3.4}$$

Consider the bounded linear operator

$$H : L_{2,-\alpha}(0,+\infty)_t \longrightarrow L_{2,-\alpha}(0,+\infty)$$

with $H(u)(t) := \int_0^t u(\tau)\, d\tau$ where $L_{2,-\alpha}(0,+\infty)$ denotes the space of all continuous functions $u(\cdot)$ for which $\|u\|_\alpha^2 := \int_0^\infty e^{-2\alpha\tau}\, u^2(\tau)\, d\tau$ is finite. It is easily verfied that:

$H(t^n) = \dfrac{t^{n+1}}{n+1}$ and

$$\|H(t^n)\|_\alpha^2 = \frac{[2(n+1)]!}{(2\alpha)^{2n+3}(n+1)^2}$$

From this we conclude that for any $p(t) \in \mathbb{R}[t]$:

$$\frac{\|H(p)\|_\alpha^2}{\|\ p\ \|_\alpha^2} \le \frac{1}{\alpha^2}$$

However $\mathbb{R}[t]$ is dense in $L_{2,-\alpha}(0,\infty)$, thus $\|H\|_\alpha = \frac{1}{2}$ and the second summand on the right hand side of (3.4) is bounded by $k_3 \int_0^\infty u(\tau)^2 d\tau$ where k_3 is a suitable constant.

□

3.2 Theorem:
The controller (2.6) is a UAS for L.

Proof: Let $\Sigma = (A,b,c) \in L(n)$ and select a basis in $\mathbb{R}^n$ such that Σ admits the following decomposition:

$$\dot{x}_1 = A_1 x_1 + A_2 y \tag{3.5a}$$

$$\dot{y} = ay + bu + A_4 x_1 \tag{3.5b}$$

where $(a,b) \in \mathbb{R} \times \mathbb{R}\backslash\{0\}$ and $\sigma(A_1) \subset \mathbb{C}^-$ (consequence of (3.1)). Applying (2.6), multiplying (3.5b) by y and integration gives:

$$\begin{aligned}\tfrac{1}{2}y(t)^2 - \tfrac{1}{2}y(0)^2 &= \int_0^t (a+b < \lambda,k(\sigma) >)\, y(\sigma)^2 \, d\sigma \\ &\quad + A_4 \int_0^t x_1(\sigma)\, y(\sigma)\, d\sigma\end{aligned} \tag{3.6}$$

and by lemma 3.1 applied to (3.6) we obtain:

$$\begin{aligned}\tfrac{1}{2}y(t)^2 - \tfrac{1}{2}y(0)^2 &\le \int_0^t \hat{a} y(\sigma)^2 \, d\sigma + b < \lambda, \int_0^t k(\sigma)\, y(\sigma)^2 \, d\sigma > \\ &= \hat{a} \int_0^t y(\sigma)^2 d\sigma + b < \lambda,\ A^{-1}\ [k(t) - k(0)] >\end{aligned} \tag{3.7}$$

Substituting $\tau := \int_0^t y(\sigma)^2 \, d\sigma$ and inserting $k(t) = e^{A\tau}\, k(0)$ the right hand side of (3.7) is transformed into:

$$\hat{a}\tau + b <\lambda, e^{A\tau}(A^{-1}k(0))> + b\ \lambda\ A^{-1}k(0) \tag{3.8}$$

Now $z(\tau) := b <\lambda, e^{A\tau}(A^{-1}k(0))> \not\equiv 0$ and satisfies:

$$p(D)\ z(\cdot) \equiv 0 \quad \text{where } p(s) = \det\ [sI-A].$$

Hence if $\tau \to \infty$ then by lemma 2.1 (iii) the right hand side of (3.7) admits arbitrary large negative values, contradicting the inequalitiy (3.7). Therefore $\tau \in L_\infty$, $y \in L_2$, $\dot{y} \in L_2$ and

$$\lim_{t \to \infty} y(t) = 0, \quad \lim_{t \to \infty} k(t) = e^{A\tau_\infty} k(0)$$

where

$$\tau_\infty = \|y(\cdot)\|_2.$$

$\square$

3.3 Remarks:

(i) It is easily proved that theorem 3.2 remains true if the system states in 3.2 are disturbed by L_2–functions $d(\cdot)$.

(ii) The multi–gain adaptation laws (2.6) give rise to further investigations of combined gain–adaptations and switchings in higher dimensional parameter spaces which may stabilize a wider of class of linear systems than L. At least the UAS's of the form

$$\dot{k} = y^2, \; u = N(k)y \tag{3.9}$$

with $N(k)$ a switching function are restrictive in the following sense:

- If $N(k)$ is assumed to be a Nussbaum function, i.e.

$$\limsup_{k \to \infty} \frac{1}{k} \int_0^k N(\sigma)\, d\sigma = -\infty, \quad \lim_{k \to \infty} \frac{1}{k} \int_0^k N(\sigma)\, d\sigma = -\infty \tag{3.10}$$

  then controllers of the form (3.9) cannot universally stabilize systems of relative degree higher than 2, because for $n \to \infty$ any polynomial of the form $z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ with $a_0 \to \pm \infty$ has unstable roots.

- In Schmid (1991) it is shown that in any adaptive controller of the form (3.9), which universally stabilizes a system class $\overline{L}$ with $L(1) \subset \overline{L}$ necessarily the switching function $N(k)$ satisfies the Nussbaum–conditions (3.10).

□

Fig. 2 illustrates the behaviour of $y(\cdot)$ and $k(\cdot)$ for the minimum phase relative degree one system:

$$x = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \; x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} x,$$

controlled by the same controller as applied to the first order system (c. Fig. 1).

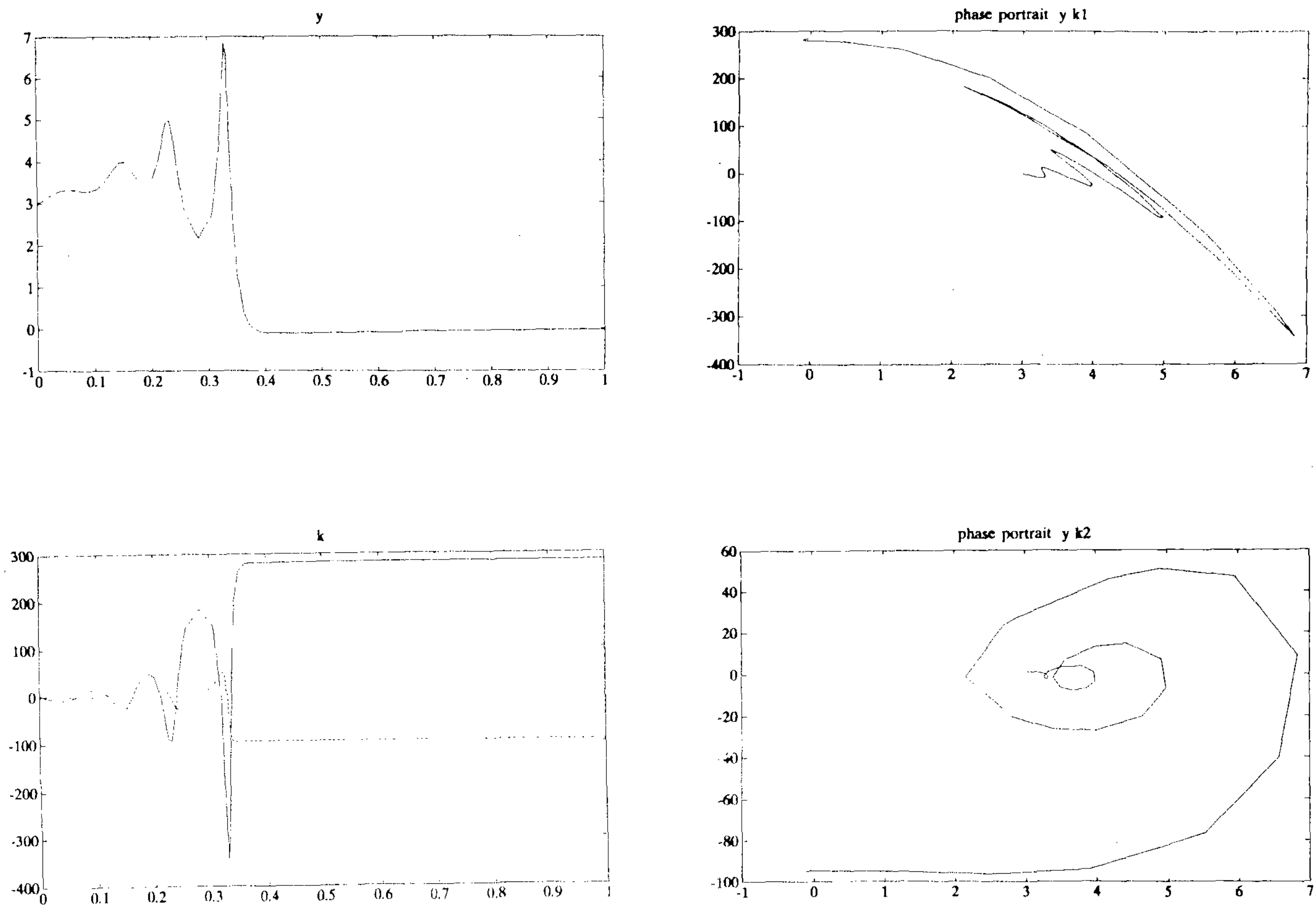

Figure 2

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