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UNIVERSAL ADAPTIVE STABILIZERS FOR ONE-DIMENSIONAL NONLINEAR SYSTEMS

Sergey Nikitin, Stephan Schmid
Fachbereich Mathematik
Universität Kaiserslautern
Postfach 3049
D - 6750 Kaiserslautern

UNIVERSITÄT KAISERSLAUTERN
Fachbereich Mathematik
Erwin-Schrödinger-Straße
D - 6750 Kaiserlautern

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Abstract

The purpose of this paper is twofold: first, to present universal adaptive stabilizers for large classes of one–dimensional nonlinear systems, and second, to derive results on the "decay rate" of the closed loop solutions, which for linear systems turns out to be exponential in the generic case.

1. Introduction

Universal adaptive stabilizers (UAS) are usually designed for the stabilization of large classes of linear systems [1]. There have also been investigations to allow for certain nonlinearities [2]. To this end the performance of UAS's for linear systems was examined when nonlinearities were present. But one typically had to assume that these nonlinearities were linearly bounded. The adaptive controllers introduced in [3] work for more general types of nonlinearities (and even for systems of relative degree greater than one), but are not universal in our sense, and require a more precise knowledge of the nonlinearity. Here we take a different approach. Known feedback and gain parameter adaptation laws are modified to design UAS's for large classes of nonlinear systems. As a result we came up with an UAS for all systems of the form $\dot{y} = p(y) + q(y)u$, $p(y)$, $q(y)$ being some (possibly unknown) polynomials with $p(0) = 0$, and $q(y) \neq 0$ for all $y \in \mathbb{R}$. We also investigate the "decay rate" of the closed loop solutions. There has been some effort to achieve asymptotic stability with exponential decay [4]. Here we show that for linear systems the "classical" UAS's generically have exponentially decaying closed loop solutions.

2. Universal Stabilizers for Nonlinear Systems

Let $g(\cdot) \in C^{\omega}(\mathbb{R})$ be a (given) analytic function satisfying

$$g(-y) = -g(y), \quad y \in \mathbb{R} \tag{1.a}$$

$$g(y) = 0 \Leftrightarrow y = 0 \tag{1.b}$$

$$g'(0) > 0. \tag{1.c}$$

We consider the class Σ_g of (nonlinear) systems having the form

$$\dot{y} = F(y) + bu \tag{2}$$

for some (unknown) real constant $b \neq 0$, and for some $F(\cdot) \in C^\omega(\mathbb{R})$ with

$h(\cdot) = \frac{F(\cdot)}{g(\cdot)} \in C^\omega(\mathbb{R})$ satisfying

$$|h(y)| \leq c,\ y \in \mathbb{R} \tag{3}$$

for some (unknown) constant $c > 0$.
To define an UAS for Σ_g let $V(\cdot) \in C(\mathbb{R})$ be any function with

$$V|_{\overline{\mathbb{R}}_+} \in C^\omega(\overline{\mathbb{R}}_+),\ V|_{\overline{\mathbb{R}}_-} \in C^\omega(\overline{\mathbb{R}}_-),$$

$$V(y) > 0,\ y \neq 0 \tag{4.a}$$

$$V(0) = 0 \tag{4.b}$$

$$\lim_{|y| \to \infty} V(y) = \infty \tag{4.c}$$

$$V'(y)y > 0,\ y \neq 0. \tag{4.d}$$

and let $N(\cdot) \in C^\omega(\mathbb{R})$ be a function of Nussbaum–type, i.e.

$$\sup_{k>0} \frac{1}{k} \int_0^k N(\sigma)d\sigma = +\infty \tag{5.a}$$

$$\inf_{k>0} \frac{1}{k} \int_0^k N(\sigma)d\sigma = -\infty \tag{5.b}$$

We introduce the following feedback and gain parameter adaptation law, resp.:

$$u = N(k)\, g(y) \tag{6.a}$$

$$\dot{k} = V'(y)\, g(y) \tag{6.b}$$

Theorem 1: Let $g(\cdot) \in C^{\omega}(\mathbb{R})$ satisfy the conditions (1). Then the compensator (6) satisfying (4) and (5) will stabilize any system (2) in the class Σ_g in the sense that for any initial value (y_0, k_0) a unique solution of the closed loop system (2), (6) exists for all $t \geq 0$ and satisfies

$$\lim_{t\to\infty} y(t) = 0 \tag{7.a}$$

$$\lim_{t\to\infty} k(t) = k_\infty \in \mathbb{R} \tag{7.b}$$

Remarks:

- For $g(y) = y$ Σ_g clearly contains all linear systems $\dot{y} = ay + bu$, $b \neq 0$. The general form of (6.b) includes the standard adaptation laws $\dot{y} = |y|^r$ for any $r \in \mathbb{N}$ (also $r = 1$), as well as for example $\dot{k} = \dfrac{|y|^r}{|y|^r+1}$ for any $r \in \mathbb{N}$.

- It can easily be checked that for $g(y) = y^{2k+1} + y$ Σ_g contains all polynomials with degree less or equal $2k+1$ and zero constant term. And for $g(y) = \sinh y$ Σ_g contains all polynomials with zero constant term.

Proof of Theorem 1: The derivative $\dot{V}$ w.r.t. the closed loop system (2), (6) is
$\dot{V}(y) = V'(y)g(y)(h(y) + bN(k))$.
Since $V'(y)g(y) \geq 0$ by (1), (4.d), this gives us because of (3)

$$\dot{V}(y) \leq V'(y)g(y)\,(c+bN(k))$$

implying by virtue of (6.b):

$$0 \leq V(y) \leq \int_{k_0}^{k} (c+bN(\sigma))d\sigma + V(y_0). \tag{8}$$

$k(t)$ is monotonically increasing since $\dot{k} \geq 0$ and $k(t)$ is bounded (where it exists) otherwise (5) would lead to a contradiction in (8). (4c) implies that $y(t)$ is also bounded (where it exists), hence the solution $(y(t), k(t))$ can be continued to all of $[0,\infty)$. We have shown (7.b).

To show (7.a), there is a sequence $t_i \uparrow \infty$ s.t. $\lim_{i\to\infty} \dot{k}(t_i) = 0$. By (6.b), (1) and (4) and since $y(t)$ is bounded we can conclude $\lim_{i\to\infty} y(t_i) = 0$. y can be written as a function of k and we have $\frac{dV}{dk} = h(y)+bN(k)$, hence

$$V(k) = \int_{k_0}^{k} (h(y)+bN(\sigma))d\sigma + V(k_0). \tag{9}$$

The integrand is bounded, so $V(k)$ is left continuous in k_∞. Thus, $V(y(t))$ converges as t goes to ∞ and, considering $V(y(t_i))$, the limit is zero. This proves (7.a). □

We have established asymptotic stability of the closed loop trajectories. To examine the "decay rate" the idea is to linearize (6.b) at $k = k_\infty$:

$$\frac{\partial(V'(y)g(y))}{\partial k} = \left(\frac{V''(y)g(y)}{V'(y)} + g'(y)\right) f'(k)$$

where $f(k) := \int_{k_0}^{k} (h(y)+bN(\sigma))d\sigma$. To evaluate this term at $k = k_\infty$ we note that clearly $y(t)$ cannot change signs, so $y(t)$ goes to zero either from left or right. But $V(\cdot)$ restricted to either side is analytic, hence for some $n \in \mathbb{N}$ $V^{(j)}(0) = 0$ for $0 \leq j < n$ and $V^{(n)}(0) \neq 0$, and we obtain

$$\left.\frac{\partial(V'(y)g(y))}{\partial k}\right|_{k=k_\infty} = ng'(0)\, f'(k_\infty).$$

Thus, for $z = k-k_\infty$, (6.b) reads

$$\dot{z} = ng'(0)\, f'(k_\infty)\, z + O(z^2).$$

Since z is monotonically increasing to zero this implies by (1.c) $f'(k_\infty) \leq 0$.
If $f'(k_\infty) < 0$, we can apply the comparison method [5] to the equation

$$|y|^{\cdot} = (f'(k_\infty) + O(k-k_\infty)\, g(|y|)$$

to obtain the result formulated in Theorem 2. What remains to show is that the case $f'(k_\infty) = 0$ "hardly ever" occurs. By (9) we have $f(k_\infty) = -V(y_0)$. For fixed k_0 we examine the set of y_0–values such that $f'(k_\infty) = 0$.

<u>Theorem 2:</u> Let k_0 be fixed. Assume there exist $A > 0$, $\epsilon > 0$, such that

$$|h'(y)| \leq A|V'(y)| \quad \text{for } |y| \leq \epsilon \tag{10}$$

There is a discrete set $S \subset \mathbb{R}$ without finite limit points such that the following holds for all $y_0 \in \mathbb{R} \setminus S$:
The closed loop solution y(t) satisfies

$$|y(t)| \leq \psi(t)$$

where $\psi(\cdot)$ is given by
$$\begin{aligned} \dot{\psi} &= -\beta\, g(\psi) \\ \psi(0) &= \phi(|y_0|); \end{aligned}$$

hereby $\phi(\cdot)$ is some strictly monotonically increasing function with $\phi(0) = 0$ and $\beta > 0$ is some constant.

<u>Proof:</u> It only remains to show that the set of y_0–values such that $f'(k_\infty) = 0$ is discrete and has not finite limit points. But $f'(k_\infty) = h(0) + bN(k_\infty)$, and clearly the set of zeroes of $\varphi(k) = h(0) + bN(k) \in C^\omega$ has these properties.
Consider now the differential equation (for $z^\pm = V(y)|_{y\in\mathbb{R}^\pm}$)

$$\frac{dz^{\pm}}{dk} = h(V^{\pm}(z^{\pm})) + bN(k) \tag{11}$$

where $V^{\pm}(z) = \begin{cases} V^{-1}(z) & z > 0 \quad (\text{where } V = V|_{\mathbb{R}^{\pm}}) \\ 0 & z \leq 0 \end{cases}$.

Since $h(\cdot) \in C^{\omega}$ and because of (10) equation (11) satisfies the conditions of existence and uniqueness of solutions in any compact set. Therefore, if $y_0\hat{y}_0 > 0$, $V(y_0) \neq V(\hat{y}_0)$ implies $k_\infty(V(y_0)) \neq k_\infty(V(\hat{y}_0))$, where $k_\infty(V(y_0))$ is the first zero of the solution of (11) with initial condition $z^s(k_0) = V(y_0)$; $s = \text{sign } y_0$. Consequently, the corresponding set of initial conditions y_0 is also discrete and has no finite limit points. □

At this point we specialize to the case $g(y) = y$. All linear systems

$$\dot{y} = ay + bu, \; b \neq 0, \tag{2.a}$$

are in Σ_g, and condition (10) is always met, so the closed loop solution of (2.a), (6) will tend to zero exponentially for almost all initial conditions (the set where this doesn't hold depends of course on the system parameters) with decay rate $a + b\,N(k_\infty)$. $k(t)$ is monotonically increasing to some finite limit $k_\infty = k$, which is the smallest $k \geq k_0$, such that $f(k) = -V(y_0)$ (see Fig. 1).

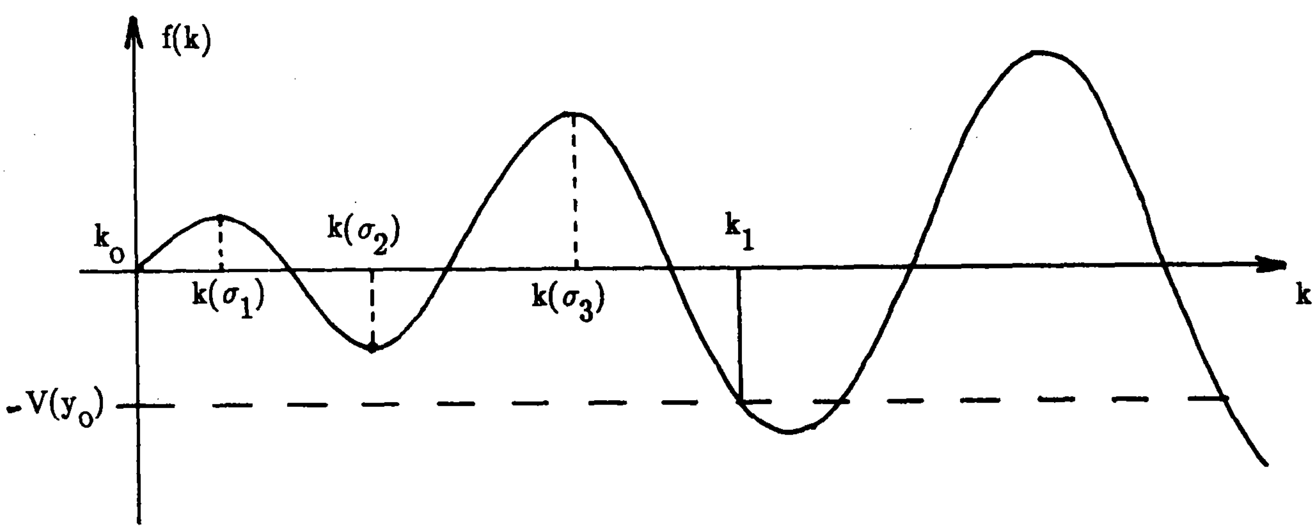

Figure 1

So, the number of switchings between the "right" and the "wrong" sign equals the number of extremal points of f(k) between k_o and k_1. In the example that is shown in Fig. 1 we have three switching times σ_1, σ_2, σ_3.

4. Concluding Remarks

In a forthcoming paper we will extend these results to higher dimensional nonlinear affine systems. Here we only want to mention the following simple extension of our results:
Let $g(\cdot), r(\cdot) \in C^{\omega}(\mathbb{R})$ be (given) analytic functions with $g(\cdot)$ satisfying (1) as before and $r(\cdot)$ satisfying

$$r(\cdot) \geq \alpha > 0. \tag{1.d}$$

We consider the larger class $\Sigma_{g,r} \supset \Sigma_g$ of systems having the form

$$\dot{y} = F(y) + Q(y)u \tag{2*}$$

with $F(\cdot) \in C^{\omega}(\mathbb{R})$, $h(\cdot) = \frac{F(\cdot)}{g(\cdot)} \in C^{\omega}(\mathbb{R})$ satisfying (3) as before and $Q(\cdot) \in C^{\omega}(\mathbb{R})$ satisfying

$$0 < d_1 \leq |Q(y)| \leq d_2 r(y), \; y \in \mathbb{R} \tag{3*}$$

for some (unknown) constants $d_1, d_2 > 0$.
And we take the following feedback and gain parameter adaptation laws

$$u = N(k) \frac{g(y)}{r(y)} \tag{6.a*}$$

$$\dot{k} = V'(y)\, g(y) \tag{6.b}$$

where $V(\cdot)$, $N(\cdot)$ are as previously introduced ((4) and (5), resp.). Then the corresponding versions of Theorems 1 and 2 are true.

The proofs remain practically the same if one uses the new time variable
$\tau = \int_0^t r(y(\sigma))\, d\sigma$ and the following additional condition for Theorem 2:
there exist $\bar{A}$, $\delta > 0$ such that $|(Q(y)/r(y))'| < \bar{A}|V'(y)|$ for $|y| \le \delta$.
As an example, if $g(y) = \sinh y$, $r(y) = \cosh y$, then $\Sigma_{g,r}$ contains all systems having the form

$$\dot{y} = p(y) + q(y)\, u$$

where $p(y)$, $q(y)$ are some polynomials with $p(0) = 0$ and $g(y) \neq 0$ for all $y \in \mathbb{R}$.

5. References

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