ON THE VANISHING DISPLACEMENT CURRENT LIMIT
FOR TIME-HARMONIC MAXWELL EQUATIONS

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On the "Vanishing Displacement Current Limit for Time-Harmonic Maxwell Equations"

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Abstract

This paper considers a transmission boundary-value problem for the time-harmonic Maxwell equations neglecting displacement currents which is frequently used for the numerical computation of eddy-currents. Across material boundaries the tangential components of the magnetic field $B$ and the normal component of the magnetization $\mu H$ are assumed to be continuous. This problem admits a hyperplane of solutions if the domains under consideration are multiply connected. Using integral equation methods and singular perturbation theory it is shown, that this hyperplane contains a unique point which is the limit of the classical electromagnetic transmission boundary-value problem for vanishing displacement currents. Considering the convergence proof, a simple constructive criterion how to select this solution is immediately derived.

Key words. time-harmonic Maxwell equations, transmission boundary-value problems, multiply connected domains, asymptotic analysis, singular perturbations, integral equation methods

AMS subject classification. 35Q60, 45F99, 78A22

1 Introduction

The investigations presented here were initiated by a common project with a company manufacturing generators for power plants. During the design phase of these machines, the minimization of the losses caused by induced eddy-currents is one of the primary goals. Since prototyping and measurements are extremely expensive, the use of numerical simulation software is gaining more and more importance.

Mathematically, the situation can be described by the setting displayed below.
A bounded domain of conducting material $D^- \subset \mathbb{R}^3$ is surrounded by some insulator in $D^+ = \mathbb{R}^3 \setminus D^-$, $D^+$ connected. In $D^+$ a current of density $J$ is given (usually a coil). $J$ produces an electromagnetic field which vice versa induces eddy-currents in $D^-$. 

Since generators usually rotate at a constant frequency, a classical electrodynamical treatment of the above setting delivers the following transmission boundary-value problem for the time-harmonic Maxwell equations.

\[
\begin{align*}
\text{curl } H^+ &= J - i\omega \varepsilon^+ E^+ \quad \text{in } D^+ \quad &\text{curl } H^- &= (\sigma^- - i\omega \varepsilon^-) E^- \quad \text{in } D^- , \\
\text{curl } E^+ &= i\omega \mu^+ H^+ \quad &\text{curl } E^- &= i\omega \mu^- H^- \quad \text{in } D^- . \\
\end{align*}
\]

\[
\begin{align*}
\n \cdot H^+ &= n \cdot H^- \\
\n \cdot E^+ &= n \cdot E^- & \text{on } \Gamma = \partial D^\pm , \\
\end{align*}
\]

where $n$ is the outer unit normal to the boundary of $D^-$, $w$ is the frequency $\mu^\pm, \varepsilon^\pm$ are the usual material parameters and $k^+ = \sqrt{\omega^2 \mu^\varepsilon^+}$ the wavenumber in $D^+$. It is well known, that under certain restrictions on $J$, the material behaviour and the domains $D^\pm$, (1) possesses a unique solution [7].

Considering the typical size of the frequency and the material constants of (1)

\[
w \approx 2\pi \cdot 50 \text{Hz}, \quad \varepsilon^\pm \approx 10^{-11} \frac{\text{As}}{\text{Vm}}, \quad \mu^\pm \approx 10^{-6} \frac{\text{Vs}}{\text{Am}}, \quad \sigma^- \approx 10^7 \frac{\text{V}}{\text{Am}}
\]

we see, that $\varepsilon^\pm$ are small parameters which introduce some kind of stiffness in the numerical treatment of the problem. Thus, neglecting the displacement currents $-i\omega \varepsilon^\pm E^\pm$ the corresponding asymptotic equations are considered. Since the equations in $D^+$ change to Pre-Maxwell type, also the radiation conditions are modified to $H^+(x), E^+(x) = o(1)$ for $|x| \to \infty$.

Moreover, it is common to replace the boundary condition $n \cdot E^+ = n \cdot E^-$ by $n \cdot (\mu^+ H^+) = n \cdot (\mu^- H^-)$ on $\Gamma$. Together we get the system of equations

\[
\begin{align*}
\text{curl } H^+ &= J \quad &\text{curl } H^- &= \sigma^- E^- \\
\text{curl } E^+ &= i\omega \mu^+ H^+ \quad &\text{curl } E^- &= i\omega \mu^- H^- \quad \text{in } D^-, \\
\end{align*}
\]

\[
\begin{align*}
\n \cdot H^+ &= n \cdot H^- \\
\n \cdot (\mu^+ H^+) &= n \cdot (\mu^- H^-) & \text{on } \Gamma ,
\end{align*}
\]
\[ H^+(x) = o(1), \quad E^+(x) = o(1), \quad |x| \to \infty, \]

which is frequently found in the engineering literature [3] as a model for eddy-current problems.

For the numerical treatment (2) is reformulated by eliminating \( J \) with the help of the law of Biot-Savart. The resulting set of equations in the unknowns \( H^\pm \) is

\[
\begin{align*}
\text{curl } H^+ &= 0 \quad \text{in } D^+ \\
\text{div } H^+ &= 0 \quad \text{in } D^+ \\
\frac{1}{\sigma^-} \text{curl } H^- &= i \omega \mu^- H^- \quad \text{in } D^- \\
\mathbf{n} \cdot \mathbf{A} H^+ - \mathbf{n} \cdot \mathbf{A} H^- &= c \\
\mathbf{n} \cdot (\mu^+ H^+) - \mathbf{n} \cdot (\mu^- H^-) &= f \quad \text{on } \Gamma, \\
H^+(x) &= o(1), \quad |x| \to \infty,
\end{align*}
\]

(3)

where \( c, f \) essentially depend on \( J \). To handle the differential equations in \( D^\pm \), the following argument was used.

\[
\text{curl } H^+ = 0 \Rightarrow H^+ = \nabla \varphi \quad \text{and} \quad \text{div } H^+ = 0 \Rightarrow \Delta \varphi = 0.
\]

But this is only valid if \( D^\pm \) are simply connected. For topological genus \( p > 0 \) the magnetic field is \( H^+ = \nabla \varphi + \sum_{i=1}^{p} h_i Z_i^+ \), where \( Z_i^+ \) are the \( p \) Neumann fields of \( D^+ \) (for the definition of \( Z_i^+ \) see Lemma 1 below) and \( h_i \in \mathbb{C} \). As was shown by one of the authors in [8], the \( h_i \) are free parameters in the sense that they are not determined neither by (2) nor by (3).

Following the above strategy (all \( h_i = 0 \)) does in general not lead to satisfactory results if \( D^\pm \) are multiply connected. Consider for example the massive aluminium cube of figure 2 which is surrounded by a concentric square loop.

\[ \text{Figure 2. Massive cube} \]

\[ \text{Figure 3. Computed eddy-currents} \]

\( D^\pm \) are simply connected and the computed eddy-currents in the plane of the loop show the behaviour which is expected by classical electrodynamics (figure 3).

Now we cut a small hole into the cube (figure 4) so that, the topological genus changes from 0 to 1. Let us take the value 0 for the free parameter (which is actually done automatically if you use the above reasoning). Due to continuity properties, the true eddy-currents in the plane of the loop should not change too much. But in fact, the numerical results show a very strange behaviour (figure 5).
So the central question now is, how to determine the free parameters correctly. In this paper, we show by using integral equation methods, that for $\partial D^\pm$ being $C^2$ and linear, homogeneous, isotropic materials in $D^\pm$ the solution of (1) converges for $\varepsilon \to 0$ in a certain Hölder norm to a specific solution of (2) resp. (3). This limiting procedure uniquely determines the free parameters of (2) resp. (3). From the proofs, a numerical algorithm for the computation of these parameters can be extracted. It is easily added to existing software, thus giving the correct eddy-current distributions for arbitrary topological genus $p$ (compare figure 5 with figure 6 for $p = 1$).

2 Assumptions and Notations

In this section we are going to specify all the technical prerequisites and notations we will use throughout the rest of the paper.

We consider an open, bounded domain $D^- \subset \mathbb{R}^3$ consisting of $m$ connected components $D^-_j$, $j \in \{1, \ldots, m\}$ of topological genus $p_j$ with boundaries $\Gamma_j$ of class $C^2$, $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$. The open complement $D^+ = \mathbb{R}^3 \setminus D^-$ is assumed to be connected. The resulting topological genus of $D^\pm$ is $p = \sum_{j=1}^m p_j$ and $\Gamma = \partial D^\pm = \bigcup_{j=1}^m \Gamma_j$. There exist $p$ surfaces $\Sigma^\pm_i \subset D^\pm$, $i \in \{1, \ldots, p\}$ so that $D^\pm \setminus \bigcup_{i=1}^p \Sigma^\pm_i$ are
simply connected. The boundary curves \( \gamma_i^\pm = \partial \Sigma_i^\pm \) are closed curves on \( \Gamma \). Moreover \( D^J \) denotes an open bounded domain with \( \partial D^J \) being \( C^2 \) and \( \bar{D}^J \subset D^+ \).

All the material coefficients occurring in the Maxwell equations are assumed to be real, positive constants in the corresponding domains \( D^+ \) resp. \( D^- \), unless stated otherwise.

Radiation conditions are always understood to hold uniformly for all directions \( \frac{x}{|x|} \). Concerning integrals, we suppress the arguments of the integrands wherever there is no danger of confusion. For constants occurring in estimates we will frequently use the same name in each step, even if the value changes. Moreover, \( A \) is the standard vector product in \( \mathbb{C}^3 \), the complex conjugate of some number \( z \in \mathbb{C} \) is denoted by \( \bar{z} \) and we write \( u \cdot v \) for \( u^T v, u, v \in \mathbb{C}^3 \).

For \( \alpha \in (0, 1] \), \( C^{0, \alpha}(D) \) equipped with

\[
\|f\|_{0, \alpha, D} = \sup_{x \in D} |f(x)| + \sup_{x, y \in D, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}
\]

denotes the Banach space of bounded, uniformly Hölder continuous functions on \( D \). Depending on the topology of \( D^\pm \), there exist two classes of special harmonic vector fields [6].

**Lemma 1 (Neumann fields)** Let \( p \) be the topological genus of \( D^\pm \). There exist exactly \( p \) linearly independent vector fields \( Z_i^\pm \in C^\infty(D^\pm) \cap C^{0, \alpha}(D^\pm), i \in \{1, \ldots, p\} \) with

\[
\text{curl } Z_i^\pm = 0, \quad \text{div } Z_i^\pm = 0 \quad \text{in } D^\pm, \\
\quad n \cdot Z_i^\pm = 0 \quad \text{on } \Gamma, \\
\quad Z_i^+(x) = O \left( \frac{1}{|x|^p} \right), \quad |x| \to \infty, \\
\quad \int_{\gamma_j^+} \tau \cdot Z_i^+ \, dl = \delta_{ij}, \\
\quad \int_{\gamma_j^-} \tau \cdot Z_i^- \, dl = 0, \quad \forall i, j \in \{1, \ldots, p\},
\]

\( n \) being the outer unit normal to \( D^- \) on \( \Gamma \).

**Lemma 2 (Dirichlet fields)** \( D^- \) consists of \( m \) connected components \( D_j^- \). There exist exactly \( m \) linearly independent vector fields \( Y_j^+ \in C^\infty(D^+) \cap C^{0, \alpha}(D^+), j \in \{1, \ldots, m\} \) with

\[
\text{curl } Y_j^+ = 0, \quad \text{div } Y_j^+ = 0 \quad \text{in } D^+, \\
\quad n \cdot Y_j^+ = 0 \quad \text{on } \Gamma, \\
\quad Y_j^+(x) = O \left( \frac{1}{|x|^p} \right), \quad |x| \to \infty.
\]

They are given by \( Y_j^+ = \text{grad } \varphi_j, j \in \{1, \ldots, m\}, \)

\[
\Delta \varphi_j = 0 \quad \text{in } D^+, \\
\varphi_j|_{\Gamma} = \delta_{ij}, \quad i, j \in \{1, \ldots, m\}, \\
\varphi_j(x) = o(1), \quad |x| \to \infty.
\]
In the sequel, we will frequently use the following Banach spaces.

**Definition 1**

- \( C^{0,a}(\Gamma) = \left\{ f \in C^{0,a}(\Gamma) \mid \int_{\Gamma} \mathcal{D} f = 0, \quad \forall j \in \{1, \ldots, m\} \right\} \), equipped with the norm \( \| \cdot \|_{0,a,\Gamma} \).
- \( T^{0,a}(\Gamma) = \left\{ a = (a_1, a_2, a_3)^T \mid a \in C^{0,a}(\Gamma), \quad n \cdot a = 0 \right\} \), equipped with the norm \( \| a \|_{T^{0,a},\Gamma} = \| a \|_{0,a,\Gamma} \), where \( n \) is the outer normal to \( D^- \) on \( \Gamma \).
- \( T^{0,a}_d(\Gamma) = \left\{ a \in T^{0,a}(\Gamma) \mid \text{Div} a \in C^{0,a}(\Gamma) \right\} \), equipped with the norm \( \| a \|_{T^{0,a}_d,\Gamma} = \max\{ \| a \|_{T^{0,a},\Gamma}, \| \text{Div} a \|_{0,a,\Gamma} \} \), where \( \text{Div} \) denotes the surface divergence on \( \Gamma \).
- \( T^{0,a}_s(\Gamma) = \left\{ a \in T^{0,a}(\Gamma) \mid \text{Div} a = 0, \quad \int_{\Gamma} a \cdot Z_i^+ \, ds = 0 \quad \forall i \in \{1, \ldots, p\} \right\} \), with the norm \( \| \cdot \|_{d_a,\Gamma} \).

Later on, we consider the dual systems

\begin{align}
(C^{0,a}(\Gamma), C^{0,a}(\Gamma), <\cdot, \cdot>) & \quad <u, v> = \int_{\Gamma} uv \, ds \quad (4) \\
(T^{0,a}_d(\Gamma), T^{0,a}(\Gamma), <\cdot, \cdot>) & \quad <u, v> = \int_{\Gamma} u \cdot v \, ds. \quad (5)
\end{align}

Moreover, we introduce for \( k \in \mathbb{C}, \ x, y \in \mathbb{R}^3, \ x \neq y \) the function

\[ \Phi_k(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \]

so that \( -\Phi_k(x, 0) \) is a fundamental solution of the Helmholtz operator \( \Delta + k^2 \) in \( \mathbb{R}^3 \) with wavenumber \( k \). Throughout the whole paper we assume that \( \text{Im}(k) \geq 0 \).

### 3 Sketch of the Paper

To establish the convergence proof mentioned in the introduction, we first consider a sequence of transmission boundary-value problems similar to (1),(2) where the current density \( J \) is removed at the expense of some additional inhomogeneities in the boundary conditions. All the results we are going to prove below are valid if \( \varepsilon^+, \varepsilon^- > 0 \) tend to 0 separately. For the sake of simplicity and to avoid cumbersome notations, we restrict ourselve to the case \( \varepsilon^+ = \varepsilon^- = \varepsilon > 0, \varepsilon \to 0 \). The system related to (1) is
Problem 1 Let $\varepsilon > 0$. For given $c, d \in T^\alpha_d(\Gamma)$ find $H^\pm, E^\pm \in C^1(D^\pm) \cap C^{0\alpha}(\overline{D}^\pm)$ so that

\[
\begin{align*}
\text{curl } H^+ &= -i\omega E^+ \quad \text{in } D^+ \\
\text{curl } E^+ - i\omega \mu^+ H^+ &= 0 \\
\text{div } E^+ &= 0 \\
\text{curl } H^- &= (\sigma^- - i\omega \varepsilon) E^- \quad \text{in } D^-, \\
\text{curl } E^- &= i\omega \mu^- H^- \\
n \wedge H^+ - n \wedge H^- &= c \\
n \wedge E^+ - n \wedge E^- &= d \quad \text{on } \Gamma, \\
\omega \mu^+ H^+(x) \wedge \frac{x}{|x|} - k^+ E^+(x) &= o\left(\frac{1}{|x|}\right), \quad |x| \to \infty,
\end{align*}
\]

where $k^+ = \sqrt{\omega^2 \mu^+ \varepsilon}$.

Problem 1 is well-posed [9]. We show, that the unique solution of Problem 1 converges in $C^{0\alpha}(\overline{D}^\pm)$ to the unique solution of

Problem 2 For $c, d \in T^\alpha_d(\Gamma)$ find $H^\pm, E^\pm \in C^1(D^\pm) \cap C^{0\alpha}(\overline{D}^\pm)$ with

\[
\begin{align*}
\text{curl } H^+ &= 0 \\
\text{curl } E^+ &= i\omega \mu^+ H^+ \quad \text{in } D^+ \\
\text{div } E^+ &= 0 \\
\text{curl } H^- &= (\sigma^- - i\omega \varepsilon) E^- \\
\text{curl } E^- &= i\omega \mu^- H^- \quad \text{in } D^-, \\
\int_{\Gamma_j} n \cdot E^+ ds &= 0 \quad \forall j \in \{1, \ldots, m\}, \\
n \wedge H^+ - n \wedge H^- &= c \\
n \wedge E^+ - n \wedge E^- &= d \quad \text{on } \Gamma, \\
H^+(z) = o(1), \quad E^+(x) = o(1), \quad |x| \to \text{cm}.
\end{align*}
\]

Moreover, every solution of the last problem also solves

Problem 3 For given $c \in T^\alpha_d(\Gamma), g \in C^{0\alpha}_d(\Gamma)$ and $h = (h_1, \ldots, h_p)^T \in \mathbb{C}^p$, find $H^\pm, E^* \in C^1(D^\pm) \cap C^{0\alpha}(\overline{D}^\pm)$ so that

\[
\begin{align*}
\text{curl } H^+ &= 0 \\
\text{curl } E^+ &= i\omega \mu^+ H^+ \quad \text{in } D^+ \\
\text{div } E^+ &= 0 \\
\text{curl } H^- &= (\sigma^- - i\omega \varepsilon) E^- \\
\text{curl } E^- &= i\omega \mu^- H^- \quad \text{in } D^-, \\
\int_{\Gamma_j} n \cdot E^+ ds &= 0 \quad \forall j \in \{1, \ldots, m\}, \\
\int_{\Gamma_i} \tau \cdot H^+ dl &= h_i \quad \forall i \in \{1, \ldots, p\}, \\
n \wedge (\mu^+ H^+) - n \cdot (\mu^- H^-) &= g \quad \text{on } \Gamma, \\
H^+(z) = o(1), \quad E^+(x) = o(1), \quad |x| \to \infty.
\end{align*}
\]

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for suitable $g$ and some special parameters $h_i$. Problem 3 corresponds to the set of equations (2) and the $h_i$ represent the parameters we want to determine.

We first show in the next section, that Problem 3 is well posed. Using this result we also obtain unique solvability and continuous dependence on the data for Problem 2. In the remaining sections, we establish an integral operator equation of the type $\mathcal{L}_\varepsilon e_\varepsilon = f_\varepsilon$, $e_\varepsilon = n \wedge E^+ |_\Gamma$, $E^+\varepsilon$ being the exterior electric field of the solution of Problem 1 for $\varepsilon > 0$ resp. Problem 2 and thus Problem 3 for $\varepsilon = 0$. Using the well-posedness of Problem 1 and Problem 2, we prove that $\mathcal{L}_\varepsilon e_\varepsilon \geq 0$ has a bounded inverse. Moreover $\mathcal{L}_\varepsilon$ and $f_\varepsilon$ converge in suitable norms to $\mathcal{L}_0$ and $f_0$, so that the solution $e_0$ of $\mathcal{L}_0 e_0 = f_0$ as $\varepsilon \to 0$. As a direct consequence, the tangential components on the boundary $\Gamma$ of the solutions of Problem 1 converge to those of Problem 2 and thus to the tangential components of a special solution of Problem 3. Afterwards we show, that this is already enough to get convergence of the fields in $C^{0,\alpha}(\overline{D^\pm})$.

In the multiply connected case, the definition of $\mathcal{L}_0$ resp. the convergence of $\mathcal{L}_\varepsilon$ to $\mathcal{L}_0$ causes some additional difficulties which are due to a singular perturbation of one of the operators involved. To outline the basic ideas, we first treat the case of simply connected domains and present the much more technical part for multiply connected $D^\pm$ separately.

Finally we return to our original problems (1), (2) and use the results obtained so far to prove convergence of the solutions of

Problem 4 Let $\varepsilon > 0$. For given $J \in C^1(\mathbb{R}^3)$, $\text{div} J = 0$, $\text{supp}(J) \subset D^J$ find $H^\pm, E^\pm \in C^1(D^\pm) \cap C^{0,\alpha}(\overline{D^\pm})$ so that

\[
\begin{align*}
\text{curl } H^+ &= J - i\omega \varepsilon E^+ & \text{in } D^+ \\
\text{curl } E^+ &= i\omega \mu^+ H^+ & \text{in } D^+ \\
\text{curl } H^- &= (a^- - i\omega \varepsilon)E^- & \text{in } D^-, \\
\text{curl } E^- &= i\omega \mu^- H^- & \text{in } D^-, \\
n \wedge H^+ &= n \wedge H^- & \text{on } \Gamma, \\
n \wedge E^+ &= n \wedge E^- & \text{on } \Gamma, \\
\omega \mu^+ H^+(x) A \frac{x}{|x|} - k^+ E^+(x) &= o\left(\frac{1}{|x|}\right), & |x| \to \infty,
\end{align*}
\]

where $k^+ = \sqrt{\omega^2 \mu^+ \varepsilon}$

to a solution of
For given $J \in C^1(\mathbb{R}^3)$, $\text{div} J = 0$, $\text{supp}(J) \subset D^j$ and $h = (h_1, \ldots, h_p)^T \in C\mathcal{P}$, find $H\pm, E\pm \in C^1(D\pm) \cap C^{0\alpha}(D\pm)$ so that

\[
\begin{align*}
curl H^+ &= J, \\
curl E^+ &= i\omega \mu^+ H^+ \quad \text{in } D^+, \\
div E^+ &= 0, \\
curl H^- &= \sigma^- E^-, \\
curl E^- &= i\omega \mu^- H^- \quad \text{in } D^-, \\
\int_{\Gamma_j} n \cdot E^+ ds &= 0 \quad \forall j \in \{1, \ldots, m\}, \\
\int_{\eta_i^+} \tau \cdot H^+ dl &= h_i \quad \forall i \in \{1, \ldots, p\}, \\
n \cdot H^+ &= n \cdot H^- \quad \text{on } \Gamma, \\
n \cdot (\mu^+ H^+) &= n \cdot (\rho^+ H^-) \quad \text{on } \Gamma, \\
H^+(x) &= 0(1), \quad E^+(x) = 0(1), \quad |x| \to \infty
\end{align*}
\]

for a specific choice of the circulations $h_i$.

4 Well-Posedness of Problem 3

A transmission boundary-value problem similar to Problem 3 was already investigated by one of the authors in [S]. For the following chapters we need some detailed information on the continuous dependence on the given data, which are not included there. These are finally obtained by some modifications of the proofs contained in [S]. It turns out, that things are even simplified in comparison to [S] and that some of the newly obtained intermediate results can also be used in later chapters. To keep the presentation short, we refer to [S] wherever possible.

A first uniqueness result for Problem 3 is found in [S].

Lemma 3 Two solutions of Problem 3 to the same data $c, g, h$ coincide in the fields $H^+, H^-$ and $E^-$. 

To prove existence and continuous dependence of the solution on the boundary data $c$ and $g$, we first, solve the case $h_i \equiv 0 \forall i \in \{1, \ldots, p\}$. We consider the following auxiliary problem, which is obtained from Problem 3 by discarding $E^+$.

Problem 6 For $c \in T_d^{0\alpha}(\Gamma)$, $g \in C_{\alpha}^\infty(\Gamma)$ given, find a solution $H\pm \in C^1(D\pm) \cap$
\[ C^{0,0}(D^-), \ E^- \in C^1(D^-) \cap C^{0,0}(D^-) \text{ of} \]

\[
\begin{align*}
\text{curl } H^+ &= 0 \quad \text{in } D^+ \\
\text{div } H^+ &= 0
\end{align*}
\]

\[
\begin{align*}
\text{curl } H^- &= \sigma^- E^- \\
\text{curl } E^- &= i \omega \mu^- H^- \quad \text{in } D^-.
\end{align*}
\]

\[
\int_{\Gamma_i} \tau \cdot H^+ dl = 0 \quad \forall i \in \{1, \ldots, p\},
\]

\[
- n \land H^+ - n \land H^- = c \quad \text{on } \Gamma,
\]

\[
\begin{align*}
n \cdot (\mu^+ H^+) - n \cdot (\mu^- H^-) &= 0 \\
H^+(x) &= o(1), \quad |x| \to \infty.
\end{align*}
\]

The following Lemma was shown in [8].

Lemma 4 Problem 6 has at most one solution.

To reduce Problem 6 to boundary integral equations, we will use the following operators.

Definition 2 For a \( a \in T_{\cup}^0(\Gamma) \), b \( b \in T_{\cup}^0(\Gamma) \), \( \lambda \in C^{0,0}(\Gamma) \), \( k \in \mathbb{C} \), \( \text{Im}(k) \geq 0 \) we define

\[
\begin{align*}
(M_k a)(x) &= 2n(x) \land \int_{\Gamma} \text{curl}_x (a(y)\Phi_k(x,y)) ds(y), \\
(M'_k a)(x) &= n(x) \land M_k n \land a(x), \\
(N_k a)(x) &= 2n(x) \cdot \int_{\Gamma} \text{curl}_x (a(y)\Phi_k(x,y)) ds(y), \\
(U_k a)(x) &= 2n(x) \land \int_{\Gamma} a(y)\Phi_k(x,y) ds(y), \\
(K_k \lambda)(x) &= 2 \int_{\Gamma} \lambda(y) \frac{\partial}{\partial n(y)} \Phi_k(x,y) ds(y), \\
(K'_k \lambda)(x) &= 2 \int_{\Gamma} \lambda(y) \frac{\partial}{\partial n(x)} \Phi_k(x,y) ds(y), \\
(S_k \lambda)(x) &= 2 \int_{\Gamma} \lambda(y)\Phi_k(x,y) ds(y), \\
(P_k \lambda)(x) &= 2n(x) \land \int_{\Gamma} n(y)\lambda(y)\Phi_k(x,y) ds(y), \\
(Q_k \lambda)(x) &= 2n(x) \cdot \int_{\Gamma} n(y)\lambda(y)\Phi_k(x,y) ds(y), \\
(R_k \lambda)(x) &= 2n(x) \land \int_{\Gamma} \text{grad}_x (\lambda(y)\Phi_k(x,y)) ds(y), \\
(T_k b)(x) &= (R_k (\text{div}_b))(x) + k^2 (U_k b)(x).
\end{align*}
\]

For \( \text{Im}(k) > 0 \) we set \( D_k \lambda = k^2 (I - K_k)^{-1} S_k \).

According to [1], we get the following mapping properties.
Lemma 5 For the linear operators defined above holds

\[
\begin{align*}
M_k & : T_0^{\alpha}(\Gamma) \to T_0^{\alpha}(\Gamma) \text{ compact}, \\
M'_k & : T^{\alpha}(\Gamma) \to T_0^{\alpha}(\Gamma) \text{ compact}, \\
N_k & : T_0^{\alpha}(\Gamma) \to C^{\alpha}(\Gamma) \text{ bounded}, \\
U_k, T_k & : T_0^{\alpha}(\Gamma) \to T_0^{\alpha}(\Gamma) \text{ bounded}, \\
K_k, K'_k, S_k, Q_k, D_k & : C^{\alpha}(\Gamma) \to C^{\alpha}(\Gamma) \text{ compact}, \\
P_k, R_k & : C^{\alpha}(\Gamma) \to T_0^{\alpha}(\Gamma) \text{ bounded}, \\
R_k - R_0 & : C^{\alpha}(\Gamma) \to T_0^{\alpha}(\Gamma) \text{ compact}.
\end{align*}
\]

\(M_k, M'_k\) are adjoint with respect to the dual system (5) and \(K_k, K'_k\) are adjoint with respect to (4). Moreover

\[
\langle T_k a, n \wedge b \rangle = \langle n \wedge a, T_k b \rangle \quad \forall a, b \in T_0^{\alpha}(\Gamma),
\]

where \(\langle \cdot, \cdot \rangle\) is the bilinear form of (5).

Now let us consider the vector fields

\[
\begin{align*}
H^+(x) &= \int_{\Gamma} \text{grad}_x (\lambda(y) \Phi_0(x, y)) \, ds(y), \\
H^-(x) &= \int_{\Gamma} \text{grad}_x (\lambda(y) \Phi_k(x, y)) \, ds(y) + \int_{\Gamma} n(y)(D_k \lambda(y) \Phi_k(x, y)) \, ds(y) \\
&\quad + \int_{\Gamma} \text{curl}_x (a(y) \Phi_k(x, y)) \, ds(y), \\
E^-(x) &= \frac{1}{\sigma^+} \left( \text{curl} H^- \right)(x),
\end{align*}
\]

where \(k = \sqrt{\omega \mu - \sigma^+}, \ \text{Im}(k) > 0\).

Lemma 6 If \(a \in T_0^{\alpha}(\Gamma), \lambda \in C^{\alpha}(\Gamma)\) the fields (6) fulfill the following conditions.

(i) \(H^\pm \in C^2(D^\pm) \cap C^{\alpha}(\bar{D}^\pm)\).

(ii) \(\text{curl} H^+ = 0, \ \text{div} H^+ = 0\) in \(D^+\), \(H^+(x) = o(1)\) and

\[
\int_{\Gamma_i} \tau \cdot H^+ \, dl = 0 \quad \forall i \in \{1, \ldots, p\}.
\]

(iii) \(\text{curl} H^-\) can be extended to \(C^{\alpha}(\bar{D}^-)\) and \(\text{div} H^- = 0\) in \(D^-\).

(iv) \(\text{curl} E^- = i\omega \mu H^-\).

(v) There exists a constant \(c_{\alpha}\) independent of \(a, \lambda, \delta\), so that

\[
\begin{align*}
\|H^-\|_{0, \alpha, D^-} & \leq c_{\alpha} \max \left\{ \|a\|_{2\alpha, \Gamma}, \|\lambda\|_{0, \alpha, \Gamma} \right\}, \\
\|\text{div} H^-\|_{0, \alpha, D} & \leq c_{\alpha} \|\lambda\|_{0, \alpha, \Gamma}, \\
\|\text{curl} H^-\|_{0, \alpha, D} & \leq c_{\alpha} \max \left\{ \|a\|_{2\alpha, \Gamma}, \|\lambda\|_{0, \alpha, \Gamma} \right\}, \\
\|H^+\|_{0, \alpha, D^+} & \leq c_{\alpha} \|\lambda\|_{0, \alpha, \Gamma}.
\end{align*}
\]
Proof: (i), (ii), the first part of (iii), (iv) and (v) follow directly from the properties of the layer potentials shown in [1]. It remains to be proven, that \( \text{div} \; H^{-} \) vanishes in \( D^{-} \). For \( x \in D^{-} \) we obtain

\[
(\text{div} \; H^{-})(x) = \Delta_{x} \int_{\Gamma} \lambda(y) \Phi_{k}(x, y) ds(y) + \int_{\Gamma} \text{div}_{x} n(y)(D_{k}\lambda)(y) \Phi_{k}(x, y) ds(y)
\]

\[
= -k^{2} \int_{\Gamma} \lambda(y) \Phi_{k}(x, y) ds(y) + \int_{\Gamma}(D_{k}\lambda)(y) n(y) \cdot \text{grad}_{x} \Phi_{k}(x, y) ds(y).
\]

Using the jump relations for the layer potentials [1], we get

\[
(\text{div} \; H^{-})(x) = -\frac{1}{2} k^{2}(S_{k}\lambda)(x) + \frac{1}{2} ((I - K_{k})D_{k}\lambda)(x) = 0
\]

since \( D_{k} = k^{2}(I - K_{k})^{-1}S_{k} \). Moreover, \( \text{div} \; H^{-} \) solves the scalar Helmholtz equation in \( D^{-} \) with wavenumber \( k \), \( \text{Im}(k) > 0 \), so that by the standard uniqueness result for this case the proof is completed.

Thus we observe, that for \( a \in T_{d}^{0\alpha}(\Gamma), \lambda \in C_{*}^{0\alpha}(\Gamma), H^{\pm}, E^{-} \) defined through (6) would be a solution of Problem 6, if the boundary conditions were fulfilled. Using again the jump conditions for the layer potentials, we can show the following lemma.

Lemma 7 Let \( a \in T_{d}^{0\alpha}(\Gamma), \lambda \in C_{*}^{0\alpha}(\Gamma) \). The corresponding fields \( H^{\pm}, E^{-} \) defined by (6) are a solution of Problem 6, if and only if \( a \) and \( \lambda \) solve

\[
A \begin{pmatrix} a \\ \lambda \end{pmatrix} = -2 \begin{pmatrix} c \\ g \end{pmatrix}, \quad A = \begin{pmatrix} M_{k} - I & R_{k} - R_{0} + P_{k}D_{k} \\ \mu^{-}N_{k} & \mu^{-}(-K_{k}' + I + Q_{k}D_{k}) - \mu^{+}(K_{0}' - I) \end{pmatrix}.
\]

The solvability of this operator equation is stated in the next theorem.

Theorem 1 \( A = A_{1} + A_{2}, A_{1}, A_{2} : T_{d}^{0\alpha}(\Gamma) \times C_{*}^{0\alpha}(\Gamma) \to T_{d}^{0\alpha}(\Gamma) \times C_{*}^{0\alpha}(\Gamma) \),

\[
A_{1} = \begin{pmatrix} -1 & 0 \\ \mu^{-}N_{k} & (\mu^{+} + \mu^{-})I \end{pmatrix}, \quad A_{2} = \begin{pmatrix} M_{k} & R_{k} - R_{0} + P_{k}D_{k} \\ 0 & \mu^{-}(-K_{k}' + Q_{k}D_{k}) - \mu^{+}K_{0}' \end{pmatrix}.
\]

\( A_{1} \) has a bounded inverse, \( A_{2} \) is compact. Moreover, \( A \) is injective and thus continuously invertible.

Proof: First let us show, that \( A_{1}, A_{2} \) really map \( T_{d}^{0\alpha}(\Gamma) \times C_{*}^{0\alpha}(\Gamma) \) into itself. Since for \( a \in T_{d}^{0\alpha}(\Gamma) \) we deduce

\[
\int_{\Gamma_{j}} (N_{k}a)(x) ds(x) = 2 \int_{\Gamma} \int_{\Gamma} n(x) \cdot \text{curl}_{x} (a(y)\Phi_{k}(x, y)) ds(y) ds(x) = 0
\]

by Stokes' Theorem, this is obvious for \( A_{1} \).

Now let us take a look at \( A_{2} \). For all \( j \in \{ 1, \ldots, m \} \) it is shown in [0], that

\[
1_{\Gamma_{j}}(x) = \begin{cases} 1 & x \in \Gamma_{j} \\ 0 & \text{else} \end{cases} \in C_{*}^{0\alpha}(\Gamma)
\]
is contained in the nullspace $N(I + K_0')$, so that
\[ \int_{\Gamma_j} K'_0 \lambda \, ds = < K'_0 \lambda, 1_{\Gamma_j} > = < \lambda, K_0(1_{\Gamma_j}) > = \lambda, -1_{\Gamma_j} > = -\int_{\Gamma_j} \lambda \, ds, \]

since $K_0$ and $K'_0$ are adjoint with respect to $< u, v > = \int_{\Gamma} uv \, ds$. But then $K'_0$ maps $C^{0,0}_0(\Gamma)$ into itself.

Moreover, $\text{div} \, H^-(x) = 0$ in $D^-$ so that
\[ 0 = \int_{D^-} \text{div} \, H^- \, dv = \int_{\Gamma_j} n \cdot H^- \, ds. \]

Using again the jump relations and the above defined operators we get
\[ 2 n \cdot H^- \big|_{\Gamma} = (I + K'_k + Q_k D_k)\lambda + N_k a. \]

But $N_k : T^{0,0}_d(\Gamma) \to C^{0,0}_0(\Gamma)$ and thus $K'_k + Q_k D_k$ maps $C^{0,0}_0(\Gamma)$ into itself.

$A_1$ is obviously continuously invertible and $A_2$ is compact due to the mapping properties (Lemma 5) of its single components (remember that $C^{0,0}_0(\Gamma)$ is a closed subspace of $C^{0,0}(\Gamma)$).

Now assume $(a, \lambda)^T$ to be a solution of $A(a, \lambda)^T = 0$. Corresponding to Lemma 7, the fields (6) defined via $a, \lambda$ are solutions of Problem 6 for $c = 0$ and $g = 0$, so that they have to vanish according to Lemma 4. Thus we conclude
\[ 0 = 2 n \cdot H^+ \big|_{\Gamma} = (K'_0 - I)\lambda. \]

But $N(I - K'_0) = \{0\} [1]$, i.e. $\lambda \equiv 0$. Moreover,
\[ 0 = 2 n \cdot H^- \big|_{\Gamma} = (M_k - I)a \]

and since $N(I - M_k) = \{0\}$ for $\text{Im}(k) > 0 [1]$, we also have $a = 0$ which completes the proof.

A direct consequence of Lemma 4, Lemma 6 and Lemma 7, Theorem 1 is

**Theorem 2** Problem 6 has a unique solution depending continuously on the given data $c$ and $g$.

Now we are coming back to our original task of determining solutions of Problem 3 with circulations $h_i = 0$, $i \in \{1, \ldots, p\}$. The only part which is still missing is a suitable electric field $E^+$. Let us therefore assume that we already have constructed a solution of Problem 6 as indicated above, that is we have solved $A \begin{pmatrix} a \\ \lambda \end{pmatrix} = -2 \begin{pmatrix} c \\ g \end{pmatrix}$ and computed $H^\pm, E^-$ via (6). This means
\[ H^+(x) = \int_{\Gamma} \text{grad}_x (\lambda(y)\phi_0(x, y)) \, ds(y) \]
with $\lambda \in C^0_{\alpha}(\Gamma)$, i.e. $\int_{\Gamma} \lambda \, ds = 0 \quad \forall j \in \{1, \ldots, m\}$. With the help of Lemma 29 from Appendix A, we get the existence of $b \in T^\alpha_d(\Gamma)$ with

$$\text{Div } b = \lambda \quad \text{on } \Gamma, \quad \|b\|_{\alpha, \Gamma} \leq c_\alpha \|\lambda\|_{\alpha, \Gamma}.$$  \hfill (7)

Now we define $E^+$ by

$$E^+(x) = i\omega \mu^+ \int_{\Gamma} \text{curl}_x (b(y) \Phi_0(x, y)) \, dy.$$  \hfill (8)

We immediately see, that $(\text{div } E^+)(x) = 0$ in $D^+$ and

$$\frac{1}{i\omega \mu^+} \left( \text{curl } E^+(x) \right) = \text{curl}_x \left( \int_{\Gamma} b(y) \Phi_0(x, y) \, ds(y) \right) = (\text{grad}_x \text{div } x - \Delta_x) \int_{\Gamma} b(y) \Phi_0(x, y) \, ds(y) = \text{grad}_x \left( \int_{\Gamma} \lambda(y) \Phi_0(x, y) \, ds(y) \right) = H^+(x).$$

Moreover

$$\int_{\Gamma} \mathbf{n}(y) E^+(y) \, ds(y) = i\omega \mu^+ \int_{\Gamma} \int_{\Gamma} \mathbf{n}(y) \text{curl}_y \left( b(z) \Phi_0(y, z) \right) \, ds(z) \, ds(y) = 0.$$  \hfill (9)

by Stokes' theorem. Due to the properties of the layer potential used in its definition, $E^+$ depends continuously on $b$ resp. $\lambda$ and thus on the data $c$ and $g$.

Collecting these results, we are able to extend the last theorem.

**Theorem 3** For any $c \in T^\alpha_d(\Gamma)$, $g \in C^\alpha_\infty(\Gamma)$ the fields $H^\pm$, $E^\pm$ defined through (6) and (8) by the unique solution $(a, \lambda)^T$ of $A \left( \begin{array}{c} a \\ \lambda \end{array} \right) = -2 \left( \begin{array}{c} c \\ g \end{array} \right)$ and $b$ from (7) solve Problem 3 with $h_i = 0, \ i \in \{1, \ldots, p\}$. Moreover

$$\max\{\|H^\pm\|_{\alpha, D^\pm}, \|E^\pm\|_{\alpha, D^\pm}\} \leq c_\alpha \max\{\|c\|_{\alpha, \Gamma}, \|g\|_{\alpha, \Gamma}\},$$

where $c_\alpha$ is independent of $g$ and $c$.

Now let us return to the general case $h_i \in \mathbb{C}, i \in \{1, \ldots, p\}$. Since Problem 3 is linear, it is enough to show solvability for $c = 0, g = 0$, and the $p$ different choices $h_i = \delta_{i\ell}, i, \ell \in \{1, \ldots, p\}$.

Let $i \in \{1, \ldots, p\}$ be fixed. By Theorem 3, a solution $H^+_i, E^+_i$ of Problem 3 for $g_i = 0, c_i = n \cdot \mathbf{A} \cdot Z^+_i \in T^\alpha_d(\Gamma), h^+_i = 0, i \in \{1, \ldots, p\}$, exists. Now define

$$H^+_i = \tilde{H}^+_i + Z^+_i \quad \text{in } D^+, \quad H^-_i = \tilde{H}^-_i \quad \text{in } D^-,$$

$$E^+_i = \tilde{E}^+_i + i\omega \mu^+ F^+_i \quad \text{in } D^+, \quad E^-_i = \tilde{E}^-_i \quad \text{in } D^-,$$
where \( F_i^+ \in C^1(D^+) \cap C^{0,\alpha}(\overline{D}^+) \) is a solution of
\[
\begin{align*}
\text{curl } F_i^+ &= Z_i^+, & \text{div } F_i^+ &= 0 \quad \text{in } D^+, \\
\int_{\Gamma_j} n \cdot F_i^+ ds &= 0 \quad \forall j \in \{1, \ldots, m\}, \\
F_i^+(x) &= o(1) \quad |x| \to \infty,
\end{align*}
\]
given by Lemma 27 in Appendix A. By direct calculations we verify, that \( H_i^\pm, E_i^\pm \) are a solution of Problem 3 with boundary conditions
\[
\begin{align*}
n \wedge H_i^+ - n \wedge H_i^- &= n \wedge \dot{H}_i^+ - n \wedge \dot{H}_i^- - n \wedge Z_i^+ = 0, \\
n \cdot (\mu^+ H_i^+) - n \cdot (\mu^- H_i^-) &= n \cdot (\mu^+ \dot{H}_i^+) - n \cdot (\mu^- \dot{H}_i^-) + n \cdot (\mu^+ Z_i^+) = 0
\end{align*}
\]
and circulations
\[
h_i = \int_{\gamma_i^+} \tau \cdot H_i^+ dl = \int_{\gamma_i^+} \tau \cdot \dot{H}_i^+ dl + \int_{\gamma_i^+} \tau \cdot Z_i^+ dl = \delta_{ii}.
\]
Together with Theorem 3, we obtain the final existence result of this section.

**Theorem 4** Problem 3 possesses a solution \( H^\pm, E^\pm \) with
\[
\max\{\|H^\pm\|_{0,\alpha,D^+}, \|E^\pm\|_{0,\alpha,D^+}\} \leq c_0 \max\{\|c\|_{d,1,\Gamma}, \|d\|_{d,1,\Gamma}, \|h\|_{\infty}\}.
\]

## 5 Well-Posedness of Problem 2

Now we consider Problem 2 and show how to construct solutions using the results from the last section.

**Lemma 8** Problem 2 has at most one solution.

**Proof:** Let us consider a solution \( H_i^\pm, E_i^\pm \) of Problem 2 for \( c = d = 0 \), \( H^+ \) is a harmonic vector field in \( D^+ \) with \( H^+(x) = o(1), |x| \to \infty \). Thus, \( H^+ \) already decays like \( O\left(\frac{1}{|x|}\right)\), so that we may apply the Gaussian theorem in the exterior domain \( D^+ \)
\[
i \omega \mu^+ \int_{D^+} H^+ \cdot \dot{H}^+ dx = - \int_{D^+} \text{curl } \dot{H}^+ \cdot E^+ - \dot{H}^+ \cdot \text{curl } E^+ dx
\]
\[
= \int_{\Gamma} n \cdot (\dot{H}^+ \wedge E^+) ds
\]
\[
= \int_{\Gamma} n \cdot (\dot{H}^- \wedge E^-) ds
\]
\[
= (\sigma^- + i \omega \varepsilon^-) \int_{D^-} E^- \cdot E^- dx - i \omega \mu^- \int_{D^-} H^- \cdot H^- dx.
\]
Considering the real part of the last equation, we get \( E^- = 0 \) in \( D^- \). Since \( i \omega \mu^- H^- = \text{curl } E^- \), \( H^- \) and thus also \( H^+ \) must vanish. Now \( E^+ \) is a solution
of
\[
\text{curl } E^+ = 0, \quad \text{div } E^+ = 0,
\]
\[
\int_{\Gamma_j} n \cdot E^+ ds = 0 \quad \forall j \in \{1, \ldots, m\},
\]
\[
n \wedge E^+ = 0 \quad \text{on } \Gamma,
\]
\[
E^+(x) = o(1), \quad |x| \to \infty
\]
which has to be identical to zero due to Lemma 26 from Appendix A.

To show existence, we start with the following remark.

**Lemma 9** A solution \( H^\pm, E^\pm \) of Problem 2 also solves Problem 3 with boundary data \( c \in T^\alpha_d(\Gamma) \) and \( g = -\frac{1}{i\omega} \text{Div } d \in C^\alpha_*(\Gamma) \) and some circulations \( h \in \mathbb{C}^p \).

**Proof:** By Lemma 22 from Appendix A, the surface divergence of \( n \wedge E^\pm \) on \( \Gamma \) is given by

\[
\text{Div } (n A E^\pm |_\Gamma) = -n \cdot \text{curl } E^\pm |_\Gamma = -i\omega \cdot n \cdot (\mu^H H^\pm)|_\Gamma.
\]

Taking the surface divergence of \( n A E^+ - n A E^- = d \) on \( \Gamma \) yields \( n \cdot (\mu^H H^+) - n \cdot (\mu^H H^-) = g \). Obviously, \( g \in C^\alpha(\Gamma) \) and since \( g = -\frac{1}{i\omega} \text{Div } d \) on any closed surface \( \Gamma_j, j \in \{1, \ldots, m\} \), we get \( \int_{\Gamma_j} g ds = 0 \) by Lemma 21 from Appendix A.

In the existence proof we reverse the order of argumentation. We consider the set of solutions of Problem 3 and determine suitable circulations \( h \in \mathbb{C}^p \) and a modified \( E^+ \), so that we get a solution of Problem 2.

**Theorem 5** Problem 2 has a unique solution \( H^\pm, E^\pm \) with

\[
\max \{ \| H^\pm \|_{0, D^\pm}, \| E^\pm \|_{0, D^\pm} \} \leq c \max \{ \| c \|_{d, \Gamma}, \| d \|_{d, \Gamma} \}.
\]

Moreover, the circulations \( h = (h_1, \ldots, h_p)^T \in \mathbb{C}^p, h_i = \int_{\Gamma} \mu \cdot H^+ dl \) solve a nonsingular linear \( p \times p \) system

\[
A h = b
\]

\[
a_{il} = \int_{\Gamma} (n A (E_l^+ - E_l^-)) \cdot Z_i^+ ds,
\]

\[
b_i = \int_{\Gamma} \left( d - n \wedge (E_i^+ - E_i^-) \right) \cdot Z_i^+ ds,
\]

\( i, l \in \{1, \ldots, p\} \), where \( H^\pm, E^\pm \) resp. \( H_i^\pm, E_i^\pm, i \in \{1, \ldots, p\} \) are arbitrary solutions of Problem 3 to the data \( c, g = -\frac{1}{i\omega} \text{Div } d, h_i = 0 \) resp. to \( c = 0, g = 0, h_i = \delta_{ii} \), \( i \in \{1, \ldots, p\} \).

**Proof:** Let \( c, d \in T^\alpha(\Gamma), h \in \mathbb{C}^p \) be given, define \( g = -\frac{1}{i\omega} \text{Div } d \in C^\alpha_*(\Gamma) \) and consider

\[
H_h^\pm = H_h^\pm + \sum_{i=1}^{p} h_i H_i^\pm, \quad E_h^\pm = E_h^\pm + \sum_{i=1}^{p} h_i E_i^\pm.
\]
$H^\pm_k, E^\pm_k$ fulfill all conditions of Problem 2 besides $n \wedge E^+_h = n \wedge E^-_h = d$ on $\Gamma$. If we could find a $F^+_h \in C^1(D^+) \cap C^{0,\alpha}(D^+)$ with

\[
\text{curl } F^+_h = 0, \quad \text{div } F^+_h = 0 \quad \text{in } D^+,
\]

\[
n \wedge F^+_h = e_h = d, \quad n \wedge (E^+_h - E^-_h) \quad \text{on } \Gamma,
\]

\[
\int_{\Gamma_j} F^+_h \cdot ds = 0, \quad j \in \{1, \ldots, m\},
\]

\[
F^+_h(x) = o(0), \quad |x| \to \infty,
\]

then

\[
H^\pm = H^\pm, \quad E^\pm = E^+_h + F^+_h \quad E^- = E^-_h
\]

would solve Problem 2. But, according to Lemma 26 from Appendix A, the above problem for $F^+_h$ is solvable if and only if

\[
\text{Div } e_h = 0, \quad \int_{\Gamma} e_h \cdot Z^+_i ds = 0, \quad i \in \{1, \ldots, p\}.
\]

For the surface divergence of $e_h$ we obtain

\[
\text{Div } e_h = \text{Div } d + i\omega \left( n \cdot (\mu^+ H^+_h) - n \cdot (\mu^- H^-_h) \right) = 0
\]

since $H^\pm_k, E^\pm_k$ are solutions of Problem 3 for $q = i \omega \text{Div } d$.

The second solvability condition of (10) is equivalent to the system of linear equations $Ah = b$ from the assumption. If this system is solvable, we can determine the above mentioned $F^+_h$ and thus obtain the existence of a solution of Problem 2.

Assume we have $\hat{h} \in \mathbb{C}^p$, with $Ah = 0$ and define

\[
\hat{H}^\pm = \sum_{i=1}^p \hat{h}_i H^\pm_i, \quad \hat{E}^\pm = \sum_{i=1}^p \hat{h}_i E^\pm_i.
\]

Since $H^\pm, E^\pm$ are not involved, $\hat{H}^\pm, \hat{E}^\pm$ are a solution of Problem 3 for homogeneous boundary data $c$ and $g$, with $\text{Div } \left( n \wedge (\hat{E}^+ - \hat{E}^-) \right) = -\frac{1}{i\omega} g = 0$. Moreover, $Ah = 0$ just means

\[
\int_{\Gamma} \left( n \wedge (\hat{E}^+ - \hat{E}^-) \right) \cdot Z^+_i ds = 0, \quad \forall i \in \{1, \ldots, p\},
\]

so that by Lemma 26 from Appendix A we can find a solution $\tilde{F}^+$ of (10) to the boundary value $\tilde{e} = -(n \wedge \hat{E}^+ - n \wedge \hat{E}^-)|_{\Gamma}$ and $\hat{H}^\pm, \hat{E}^\pm, \tilde{F}^+_h$ are a solution of Problem 2 for $c = d = 0$ on $\Gamma$. Due to the uniqueness result of Lemma 8, these fields have to vanish identically. But, on the other hand, the $\hat{h}_i$ are given by $\hat{h}_i = \frac{1}{\gamma_i} \int_{\gamma_i} \hat{H}^+ dl$, $i \in \{1, \ldots, p\}$, so that, $Ah = 0$ means $\hat{h} = 0$ and $A$ is nonsingular.

By this result we know that $Ah = b$ is always uniquely solvable. So we can determine the $\hat{h}_i$ and the field $F^+_h$ to obtain a solution $H^\pm, E^\pm$ of Problem 2 via (11). Since the solution of Problem 2 is unique, the second part of the assumption is shown.
To prove existence and continuous dependence of the solution on the data we repeat the above construction using the solutions $H^\pm_\varepsilon$, $E^\pm_\varepsilon$ resp. $H^\pm_I$, $H^\pm_I$ whose existence is guaranteed by Theorem 3. Since $H^\pm_\varepsilon$, $E^\pm_\varepsilon$ depend continuously on $c$ and $g = -\frac{1}{i\omega}\text{Div} d$, the right hand side $b$ of $Ah = b$ and thus the $h_i$ depend continuously on $c$ and $d$. Moreover, by Lemma 26 from Appendix A

$$
\|H^\pm_\varepsilon\|_{0,0,D^\pm} \leq c_\delta \|e_h\|_{0,0,\Gamma},
$$

$$
e_h = d - \left(n \wedge (E^+_\varepsilon - E^-_\varepsilon) + \sum_{i=1}^{p} h_1n \wedge (E^+_i - E^-_i)\right)
$$

so that $H^\pm$, $E^\pm$ depend continuously on $c$, $d$.

\[\blacksquare\]

6 The Integral Operator Equation

Now we are going to present the basic idea how to prove convergence of the solutions of Problem 1 to those of Problem 2 if $\varepsilon > 0$ tends to 0. As we will see, this is equivalent to the norm convergence of certain operators linked to interior and exterior boundary value problems.

We start our considerations by splitting the transmission boundary-value problems up into the following interior and exterior boundary-value problems.

**Problem 7** Let $\varepsilon > 0$ and $k^+ = \sqrt{\omega^2 \mu_+ - \varepsilon}$. For $e \in T^{0,0}(\Gamma)$ given, find a solution $H^+_\varepsilon$, $E^+_\varepsilon \in C^1(D^+) \cap C^{0,0}(D^+)$ of

$$
\begin{align*}
curl H^+_\varepsilon &= -i\omega e E^+_\varepsilon & \text{in } D^+, \\
curl E^+_\varepsilon &= i\omega \mu^+ H^+_\varepsilon & n \wedge E^+_\varepsilon = \varepsilon & \text{on } \Gamma,
\end{align*}
$$

$$
\omega \mu^+ H^+_\varepsilon(x) \frac{\varepsilon}{|x|} - k^+_\varepsilon E^+_\varepsilon(x) = o \left( \frac{1}{|x|} \right) \quad |x| \to \infty.
$$

**Problem 8** For $e \in T^{0,0}(\Gamma)$ given, find a solution $H^+_0$, $E^+_0 \in C^1(D^+) \cap C^{0,0}(D^+)$ of

$$
\begin{align*}
curl H^+_0 &= 0 & \text{in } D^+, \\
curl E^+_0 &= i\omega \mu^+ H^+_0 & n \wedge E^+_0 = \varepsilon & \text{on } \Gamma,
\end{align*}
$$

$$
\int_{\Gamma_j} n \cdot E^+_0 ds = 0 \quad \forall j \in \{1, \ldots, m\},
$$

$$
H^+_0(x) = o(1), \quad E^+_0(x) = o(1) \quad |x| \to \infty.
$$

**Problem 9** Let $\varepsilon \geq 0$ and $k^- = \sqrt{\omega^2 \mu^- - \varepsilon} + i\omega \sigma^- \mu^-$, $\text{Im}(k^-) > 0$. For $e \in T^{0,0}(\Gamma)$ given, find a solution $H^-\varepsilon$, $E^-\varepsilon \in C^1(D^-) \cap C^{0,0}(D^-)$ of

$$
\begin{align*}
curl H^-\varepsilon &= (\sigma - i\omega e) E^-\varepsilon & \text{in } D^-, \\
curl E^-\varepsilon &= i\omega \mu^- H^-\varepsilon & n \cdot E^-\varepsilon = e & \text{on } \Gamma.
\end{align*}
$$

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It is well known ([1],[4]), that all these boundary-value problems have unique solutions depending continuously on the data in the sense that for $\varepsilon \geq 0$

$$\|H^{\pm}_\varepsilon\|_{0,\alpha,\partial^+} + \|E^{\pm}_\varepsilon\|_{0,\alpha,\partial^+} \leq c_\alpha \|\varepsilon\|_{0,\alpha,\Gamma}.$$ 

Thus, the operators which map the tangential components of the electric field on I to those of the magnetic field are well defined.

**Definition 3** Consider a given boundary value $\varepsilon \in T^{0\alpha}_d(\Gamma')$ und the fields $H^{\pm}_\varepsilon$, $E^{\pm}_\varepsilon$ which are for $\varepsilon > 0$ the solution of Problem 7, for $\varepsilon = 0$ the solution of Problem 8 and the fields $H^{-}_\varepsilon$, $E^{-}_\varepsilon$ which uniquely solve Problem 9. Then we define for $\varepsilon \geq 0$ the linear operators $A^{\pm}_\varepsilon$ on $T^{0\alpha}_d(\Gamma')$ via

$$A^{\pm}_\varepsilon \varepsilon = n \wedge H^{\pm}_\varepsilon |_{\Gamma'}.$$ 

Using the continuous dependence of the solutions of Problem 7-9 on the boundary data together with Lemma 22 from Appendix A and the Lipschitz continuity of $n$ on $\Gamma$, we immediately get the following mapping properties for $A^{\pm}_\varepsilon$.

**Lemma 10** For $\varepsilon \geq 0$ the operators $A^{\pm}_\varepsilon : T^{0\alpha}_d(\Gamma') \rightarrow T^{0\alpha}_d(\Gamma)$ are bounded.

With the help of the last definition, we are now able to derive an operator equation depending on $\varepsilon$, whose solvability is equivalent to the solvability of Problem 1 for $\varepsilon > 0$ resp. Problem 2 for $\varepsilon = 0$.

**Theorem 6** Let $c, d, e_\varepsilon \in T^{0\alpha}_d(\Gamma')$ and $H^{\pm}_\varepsilon$, $E^{\pm}_\varepsilon$ be the solution of Problem 7 for $\varepsilon > 0$ resp. Problem 8 for $\varepsilon = 0$ to the boundary value $e_\varepsilon$. Moreover, let $H^{-}_\varepsilon$, $E^{-}_\varepsilon$ be the solution of Problem 9 to the boundary data $e_\varepsilon = d$. Then $H^{\pm}_\varepsilon$, $E^{\pm}_\varepsilon$ solve Problem 1 for $\varepsilon > 0$ resp. Problem 2 for $\varepsilon = 0$. If and only if

$$L_\varepsilon e_\varepsilon = f_\varepsilon, \quad L_\varepsilon = A^{\pm}_\varepsilon - A_\varepsilon, \quad f_\varepsilon = c - A^{-}_\varepsilon d.$$ (1.2)

**Proof:** $H^{\pm}_\varepsilon$, $E^{\pm}_\varepsilon$ fulfill the differential equations and radiation conditions of Problem 1 resp. Problem 2. It remains to be shown, that the correct boundary conditions are taken on.

According to the assumptions $n \wedge E^{\pm}_\varepsilon - n \wedge E^{-}_\varepsilon = d$ on $\Gamma$. Moreover

$$n \wedge H^{\pm}_\varepsilon |_{\Gamma} = A^{\pm}_\varepsilon (n \wedge E^{\pm}_\varepsilon |_{\Gamma}) = A^{\pm}_\varepsilon e_\varepsilon$$

$$n \wedge H^{-}_\varepsilon |_{\Gamma} = A^{-}_\varepsilon (n \wedge E^{-}_\varepsilon |_{\Gamma}) = A^{-}_\varepsilon (e_\varepsilon - d)$$

so that the boundary condition $n \wedge H^{\pm}_\varepsilon - n \wedge H^{-}_\varepsilon = c$ holds if and only if $e_\varepsilon$ solves (1.2).

Using the results about the transmission boundary value problems obtained in section 4 and 5, we can show unique solvability of (1.2) for all $\varepsilon \geq 0$.

**Theorem 7** $L_\varepsilon : T^{0\alpha}_d(\Gamma) \rightarrow T^{0\alpha}_d(\Gamma)$ has a bounded inverse for all $\varepsilon \geq 0$. 

Proof: Let \( f \in T^0_\alpha(\Gamma) \) and \( H_e^\pm, E_e^\pm \) be the unique solution of Problem 1 for \( \varepsilon > 0 \) resp. Problem 2 for \( \varepsilon = 0 \) with boundary data \( c = f \) and \( d = 0 \). According to the last theorem we get \( \mathcal{L}_e c_e = f \) where \( e_e = n \wedge E_e^\pm|_\Gamma \), so that \( \mathcal{L}_e \) is surjective. Moreover by the continuous dependence of \( H_e^\pm, E_e^\pm \) on the boundary data we immediately obtain \( \| e_e \|_{\alpha, \Gamma} \leq c_\alpha \| f \|_{\alpha, \Gamma} \).

Now consider \( e_e \in T^0_d(\Gamma) \) with \( \mathcal{L}_e c_e = 0 \) and the solutions \( H_e^\pm, E_e^\pm, \varepsilon \geq 0 \) of Problem 7 resp. Problem 8 and Problem 9 to the boundary values \( e_e \). Then \( n \wedge E_e^\pm = n \wedge E_0^\pm \) on \( \Gamma \) and since \( \mathcal{L}_e = A_e^+ - A_e^- \), \( A_e^\pm(n \wedge E_e^\pm) = n \wedge H_e^\pm \), we also have \( n \wedge H_e^\pm = n \wedge H_0^\pm \). So \( H_e^\pm, E_e^\pm \) are the unique solution of Problem 1 for \( \varepsilon > 0 \) resp. Problem 2 for \( \varepsilon = 0 \) to the boundary values \( c = 0 \) and \( d = 0 \) and thus have to vanish. But this means \( e_e = n \wedge E_e^\pm|_\Gamma = 0 \) and \( \mathcal{L}_e \) is injective.

7 Representation of \( \mathcal{L}_e \) and Convergence in the Simply Connected Case

The strategy of the convergence proof is now obvious. Equation (12) is uniquely solvable for all \( \varepsilon \geq 0 \) and equivalent to the transmission boundary value problems in the sense defined above. If we can show, that for \( \varepsilon \) tending to 0 the operators \( A_e^\pm \) converge to \( A_0^\pm \) in the norm induced by \( T^0_\alpha(\Gamma) \), we can conclude that the solution \( e_e \) of (12) tends to the solution \( e_0 \) of \( \mathcal{L}_0 c_0 = f_0 \). Thus, the tangential components \( n \wedge E_e^\pm|_\Gamma \) of the solution \( H_e^\pm, E_e^\pm \) of Problem 1 converge to the tangential components \( n \wedge E_0^\pm|_\Gamma \) of the solution \( H_0^\pm, E_0^\pm \) of Problem 2 for \( \varepsilon = 0 \). As we will see below, this is already enough to ensure the convergence of \( H_e^\pm, E_e^\pm \) to \( H_0^\pm, E_0^\pm \) in \( C^0(\overline{\Omega}) \).

Because the whole problem is reduced to the convergence of \( A_e^\pm \) to \( A_0^\pm \) for \( \varepsilon \) tending to 0, it is necessary to derive a more explicit form of these operators than the one given in Definition 3. This form is obtained via the following representation results for the solutions of the interior and exterior boundary value problems given above.

Lemma 11 For \( \mathbf{c} \in T^0_d(\Gamma), \mathbf{x} \in D^\pm \) define

\[
(M_e^\pm \mathbf{c})(\mathbf{x}) = \int_\Gamma \text{curl}_x (c(y) \Phi_k(x, y)) ds(y),
\]
\[
(T_e^\pm \mathbf{c})(\mathbf{x}) = \int_\Gamma \text{curl}_x \text{curl}_x (c(y) \Phi_k(x, y)) ds(y)
= \int_\Gamma \text{Div} c(y) \text{grad}_x \Phi_k(x, y) ds(y) + k^2 \int_\Gamma c(y) \Phi_k(x, y) ds(y).
\]

Then \( M_e^\pm, T_e^\pm : T^0_d(\Gamma) \to C^0(\overline{\Omega}) \) and

\[
\| M_e^\pm \mathbf{c} \|_{\alpha, D^\pm}, \| T_e^\pm \mathbf{c} \|_{\alpha, D^\pm} \leq c_\alpha \| \mathbf{c} \|_{\alpha, \Gamma},
\]

where \( c_\alpha \) is independent of \( \mathbf{c} \in T^0_d(\Gamma) \).
Lemma 12 Let $E^+_{\varepsilon}, H^+_{\varepsilon}$ be a solution of Problem 7 for $\varepsilon > 0$ resp. Problem 8 for $\varepsilon = 0$ and $E^-_{\varepsilon}, H^-_{\varepsilon}$ be a solution of Problem 9. Then

$$
M^+_{k_+}(n \wedge H^+_{\varepsilon}|r) + \frac{1}{i\omega \mu^+} T^+_{k_+}(n \wedge E^+_{\varepsilon}|r) = H^+_{\varepsilon} \text{ in } D^+,
$$

$$
M^-_{k^-}(n \wedge H^-_{\varepsilon}|r) + \frac{1}{i\omega \mu^-} T^-_{k^-}(n \wedge E^-_{\varepsilon}|r) = -H^-_{\varepsilon} \text{ in } D^-,
$$

$$
M^-_{k_+}(n \wedge E^-_{\varepsilon}|r) + \frac{1}{\sigma^2 - i\omega \varepsilon} T^-_{k_+}(n \wedge H^-_{\varepsilon}|r) = -E^-_{\varepsilon} \text{ in } D^-.
$$

The proof of Lemma 11 and Lemma 12 is exactly the same as for the standard representation theorems which can be found for example in [1].

Approaching the boundary $\Gamma$ with $x$ from the exterior resp. the interior domain and using the jump relations for the layer potentials involved as well as the operators of Definition 2, we get

Lemma 13 Let $E^+_{\varepsilon}, H^+_{\varepsilon}$ be a solution of Problem 7 for $\varepsilon > 0$ resp. of Problem 8 for $\varepsilon = 0$ and $E^-_{\varepsilon}, H^-_{\varepsilon}$ be a solution of Problem 9. Then

$$
(I - M_{k_+})(n \wedge H^+_{\varepsilon}|r) = \frac{1}{i\omega \mu^+} T_{k_+}(n \wedge E^+_{\varepsilon}|r),
$$

$$
(I + M_{k^-})(n \wedge H^-_{\varepsilon}|r) = -\frac{1}{i\omega \mu^-} T_{k^-}(n \wedge E^-_{\varepsilon}|r).
$$

If the operators $I - M_{k_+}$ resp. $I + M_{k^-}$ were continuously invertible for all $\varepsilon \geq 0$, we would directly obtain explicit representations for $A^\pm_{\varepsilon}$. According to Lemma 23 from Appendix A, this is true for $I + M_{k^-}$ since $\text{Im}(k^-) > 0$. For topological genus $p \geq 1$ we do not get the desired representation for $I - M_{k_+}$ since $I - M_k$ is singularly perturbed for $k \to 0$.

Theorem 8 For the operator $A^-_{\varepsilon}$ holds

$$
A^-_{\varepsilon} = -\frac{1}{i\omega \mu^-}(I + M_{k^-})^{-1} T_{k^-}, \quad 0 \leq \varepsilon.
$$

Depending on the domains $D^\pm$ there exists a constant $\delta > 0$, so that

$$
A^+_{\varepsilon} = \frac{1}{i\omega \mu^+}(I - M_{k_+})^{-1} T_{k_+}, \quad 0 < \varepsilon \leq \delta.
$$

Moreover

$$
\lim_{\varepsilon \to 0} \|A^-_{\varepsilon} - A^-_0\| = 0
$$

in the operator norm induced by $T^0_{d}(\Gamma)$.

If $D^\pm$ are simply connected, the representation (14) also holds for $\varepsilon = 0$ and

$$
\lim_{\varepsilon \to 0} \|A^+_{\varepsilon} - A^+_0\| = 0
$$

in the operator norm induced by $T^0_{d}(\Gamma)$.
Proof: The results on the representations (13),(14) are obtained as described above. The convergence results are then a consequence of Lemma 34 from Appendix B.

Corollary 1 Let $D$ be simply connected. Then

$$\lim_{\varepsilon \to 0} \|\mathcal{L}_\varepsilon - \mathcal{L}_0\| = 0$$

in the operator norm induced by $T_\varepsilon^{0,\alpha}(\Gamma)$.

8 Singular Perturbations and the Multiply Connected Case

Now we have to overcome the difficulty that (14) is not valid for $\varepsilon = 0$ if $D^*$ are multiply connected. We start our considerations of this case by modifying some general results on singular perturbation problems from [1] so that they apply to our special situation.

Theorem 9 Let $X, Y$ be Banach-spaces $<\cdot, \cdot>: X \times Y \to \mathbb{C}$ a nondegenerate bilinear form and $K \subset \mathbb{C}$ be a subset of complex numbers with accumulation point $0 \in K$. Consider a family of compact linear operators $\{A_k : X \to X \mid k \in K\}$ and their adjoints $A_k' : Y \to X$ with respect to $<\cdot, \cdot>$. Assume the operators $L_k = I - A_k$ to be injective for $k \neq 0$, the Riesz-number of $L_0$ to be one and

$$A_k = A_0 + CK^2 + D(k) \quad (15)$$

where $C : X \to X$ is a bounded linear operator independent of $k$ and $D(k)$ is linear with $\|D(k)\| = o(k^2)$ for $k \to 0$.

Moreover let $\{H_k : X \to X \mid k \in K\}$ be a family of bounded linear operators with $\lim_{k \to 0} \|H_k - H_0\| = 0$, $\{b_1, \ldots, b_p\}$ be a basis of the nullspace $N(L_0')$, $L_0' = 1 - A_0'$, and assume

$$<H_k q, b_i> = g_i(q)k^2 + h_i(q, k), \quad (16)$$

$$|g_i(q)| \leq c_i \|q\|, \quad |h_i(q, k)| \leq d_i(k)\|q\|, \quad d_i(k) = o(k^2), \quad k \to 0$$

for all $i \in \{1, \ldots, p\}$, where the constants $c_i$ are independent of $k$ and $q \in X$ and the functions $d_i(k)$ are independent of $q \in X$.

For $q \in X$ fixed, there exists a $\phi_0 \in X$ with

$$L_0\phi_0 = H_0q \quad (17)$$

so that for the unique solution $\phi_k$, $k \neq 0$ of

$$L_k\phi_k = H_kq, \quad (18)$$

holds

$$\|\phi_k - \phi_0\| \leq d(k)\|q\|, \quad d(k) = o(1), \quad k \to 0,$$

where $d(k)$ is independent of $q$. 

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Proof: In a first step we reduce the solution of (18) (which is in general posed in an infinite dimensional space X) to an equation posed in the finite dimensional nullspace \( N(L_0) \).

Since the Riesz-number of \( L_0 \) is 1, we have \( X = N(L_0) \oplus L_0(X) \) and define the projection operator

\[
P : X = N(L_0) \oplus L'(X) \to N(L_0).
\]

According to [1] the linear operator

\[
L_0^+ = (L_0 P)^{-1}(I - P),
\]

is bounded. Furthermore we define

\[
Q_k = L_0^+(L_0 - L_k), \quad F_k^+ = (I - Q_k)^{-1}L_0^+.
\]

The existence and continuity of \((I - Q_k)^{-1}\) follows from (15) and a Neumann series expansion for sufficiently small \(|k|\). From now on we assume without loss of generality that \( I - Q_k \) is boundedly invertible for all \( k \in K \). A straightforward calculation using the above introduced operators shows that

\[
\phi_k = \psi_k + (I - Q_k)^{-1} \chi_k, \quad \psi_k = \begin{cases} F_k^+ H_k q & k \neq 0 \\ L_0^+ H_0 q & k = 0 \end{cases}, \quad \chi_k \in N(L_0).
\]

For \( k \neq 0 \), \( \phi_k \) is a solution of (18) if and only if \( \chi_k \in N(L_0) \) is a solution of

\[
PL_k(I - Q_k)^{-1} \chi_k = PH_k q - PL_k \psi_k
\]

(for details see [1], Chapter 1). By (20)

\[
\phi_k - \phi_0 = \psi_k - \psi_0 + (I - Q_k)^{-1} \chi_k - \chi_0.
\]

To show convergence of \( \phi_k \) to \( \phi_0 \) we treat both parts of the right hand side of (22) separately.

Using (30) we find

\[
\psi_k - \psi_0 = (F_k^+ - L_0^+) H_k q + L_0^+ (H_k - H_0) q.
\]

Therefore

\[
\|\psi_k - \psi_0\| \leq \|F_k^+ - L_0^+\| \|H_k\| \|q\| + \|L_0^+\| \|H_k - H_0\| \|q\|.
\]

But \( \lim_{k \to 0} \|F_k^+ - L_0^+\| = 0 \) and the assumption \( \lim_{k \to 0} \|H_k - H_0\| = 0 \) imply

\[
\|\psi_k - \psi_0\| \leq e_1(k) \|q\|, \quad e_1(k) = o(1) \quad k \to 0,
\]

where \( e_1(k) \) is independent of \( q \).

In the next step we prove a similar estimate for \( \|\chi_k - \chi_0\| \). Considering the left hand side of (21) we obtain by a straightforward calculation using (15) and \( L_k = I - A_k \)

\[
PL_k(I - Q_k)^{-1}|_{N(L_0)} = -\left(k^2 PC|_{N(L_0)} + H(k)|_{N(L_0)}\right),
\]

(24)
Because of \( \lim_{k \to 0} \|Q_k\| = 0 \) and \( \|D(k)\| = o(k^2) \), \( k \to 0 \), we conclude that \( H(k) \) satisfies
\[
\|H(k)\| = o(k^2), \quad k \to 0.
\]

Let \( \chi_k \in N(L_0) \), \( k \neq 0 \) be a solution of \( PL_k(I - Q_k)^{-1} \chi_k = 0 \), i.e. \( \chi_k \) solves (21) with \( q = 0 \). With the equivalence of (18) and (21) for \( k \neq 0 \), we conclude that \( \phi_k \) is a solution of \( L_k \phi_k = 0 \), \( k \neq 0 \), i.e. by uniqueness \( \phi_k = 0 \). Since \( q = 0 \) we get using (20)
\[
\chi_k = (I - Q_k)(\phi_k - F_k^+ H_k q) = 0.
\]

So we know that the mapping \( PL_k(I - Q_k)^{-1} |_{N(L_0)} \) is invertible and by (24)
\[
\left( PL_k(I - Q_k)^{-1} |_{N(L_0)} \right)^{-1} = -k^{-2} \left( PC|_{N(L_0)} + \tilde{H}(k)|_{N(L_0)} \right)^{-1}
\]
where
\[
k(k) = k^{-2} H(k), \quad \|\tilde{H}(k)\| = o(1), \quad k \to 0.
\]

Now we consider the right hand side of (21). Given the basis \( \{b_1, \ldots, b_p\} \) of \( N(L_0) \) we find a basis \( \{a_1, \ldots, a_p\} \) of \( N(L_0) \) such that
\[
\langle a_i, b_j \rangle = \delta_{ij} \quad \forall i, j \in \{1, \ldots, p\}.
\]

The projector (19) is given as \( Pq = \sum_{i=1}^p <q, b_i> a_i \), so that by (16)
\[
P H_k q = \sum_{i=1}^p <H_k q, b_i> a_i = \sum_{i=1}^p g(q) k^2 a_i + \sum_{i=1}^p h_i(q, k) a_i = g(q) k^2 + h(q, k)
\]
with \( g(q), h(q, k) \in N(L_0) \), \( g(q) \) being independent of \( k \). Obviously
\[
\|g(q)\| \leq c \|q\|, \quad \|h(q, k)\| \leq e_2(k) \|q\|, \quad e_2(k) = o(k^2), \quad k \to 0
\]
with \( c \) being independent of \( q \) and \( k \), \( e_2(k) \) being independent of \( q \).

Using (27) and (20) a straightforward calculation implies
\[
P H_k q - PL_k \psi_k = (g(q) + PC \psi_0 + G(q, k)) k^2,
\]

\[
G(q, k) = PC(F_k^+ H_k - L_0^+ H_0) q + k^{-2} PD(k)(F_k^+ H_k q) + k^{-2} h(q, k).
\]

By \( \lim_{k \to 0} \|F_k^+ - L_0^+\| = 0 \), \( \lim_{k \to 0} \|H_k - H_0\| = 0 \), the assumption \( \|D(k)\| = o(k^2) \), \( k \to 0 \), and (28) we obtain
\[
\|G(q, k)\| \leq e_3(k) \|q\|, \quad e_3(k) = o(1), \quad k \to 0,
\]
and \( e_3(k) \) being independent of \( q \).

With these results we are able to consider \( \chi_k \to \chi_0 \). According to (21),(25),(29)
\[
\chi_k = \left( PL_k(I - Q_k)^{-1} |_{N(L_0)} \right)^{-1} (PH_k q - PL_k \psi_k)
\]
\[
= - \left( PC|_{N(L_0)} + \tilde{H}(k)|_{N(L_0)} \right)^{-1} (g(q) + PC \psi_0 + G(q, k)).
\]
Following [1] the mapping $PC|_{N(L_0)}$ is continuously invertible. Let
\[ \chi_0 = -(PC|_{N(L_0)})^{-1}(g(q) + PC\psi_0) \]
i.e.
\[ \chi_k - \chi_0 = -(PC|_{N(L_0)} + \tilde{H}(k)|_{N(L_0)})^{-1} - (PC|_{N(L_0)})^{-1} (g(q) + PC\psi_0) \]
\[ = (PC|_{N(L_0)} + \tilde{H}(k)|_{N(L_0)})^{-1} G(q, k). \]
But (28) ensures $\|g(q)\| \leq c\|q\|$ and therefore
\[ \|g(q) + PC\psi_0\| = \|g(q) + PCL_0^+H_0q\| \leq c\|q\|, \]
with $c$ being independent of $q$. Using the relations (26) and (30) we obtain
\[ \|\chi_k - \chi_0\| \leq \varepsilon_4(k)\|q\|, \quad \varepsilon_4(k) = o(0), \quad k \to 0, \quad (31) \]
\varepsilon_4(k) being independent of $q$.
Collecting all results we see, that (22) provides
\[ \|\phi_k - \phi_0\| \leq \|\phi_k - \psi_0\| + \|I - Q_k\|^{-1} \|\chi_k - \chi_0\| + \|I - Q_k\|^{-1} \|\chi_0\|. \]
Using $\lim_{k \to 0} \|Q_k\| = 0$, (23) and (31) we draw the conclusion
\[ \|\phi_k - \phi_0\| = d(k)\|q\|, \]
where $d(k)$ is independent of $q$, $d(k) = o(1)$ as $k \to 0$. Finally $\phi_0$ solves (17) since
\[ \|L_0\phi_0 - H_0q\| = \|L_0(\phi_0 - \phi_k) + (L_0 - L_k)\phi_k + L_k\phi_k - H_0q\| \leq \|L_0\|\|\phi_k - \phi_0\| + \|L_k - L_0\|\|\phi_k\| + \|H_k - H_0\|\|q\|. \]

In order to prove uniform Hölder convergence of the operators $A_k^+$ we apply the last theorem to
\[ (I - M_{k^+})(n \land H_{\varepsilon}^+) = \frac{1}{i\omega \mu^+ T_{k^+}} (n \land E_{\varepsilon}^+), \quad \varepsilon \to 0 \]
(see Lemma 13 and Theorem 8). It remains to be checked that the operators $M_{k^+}$ and $T_{k^+}$ fulfill all the assumptions of Theorem 9.
Since $k^+ = \sqrt{\omega^2 \mu^+ \varepsilon}$, the limit $\varepsilon \to 0$ is equivalent to $k^+ \to 0$. To simplify the notation, we use $k$ instead of $k^+$ if there is no danger of confusion.

**Lemma 14** Let $K = \{t \mid t \in \mathbb{C}, \operatorname{Im}(t) \geq 0, |t| < \delta\}$, $\delta$ small enough. Then $M_k : T_0^o(\Gamma) \to T_0^o(\Gamma)$, $k \in K$, is a family of compact operators with adjoint $M_k^* : T_0^o(\Gamma) \to T_0^o(\Gamma)$ with respect to the dual system (5). $L_k = I - M_k$ is
continuously invertible on $T_{d}^{0,\alpha}(\Gamma)$ for all $k \in K$, $k \neq 0$. Moreover the Riesz-number of $L_0$ is one and

$$M_k = M_0 + Ck^2 + D(k),$$

where $C : T_{d}^{0,\alpha}(\Gamma) \to T_{d}^{0,\alpha}(\Gamma)$ is a bounded linear mapping independent of $k$ and $D(k) : T_{d}^{0,\alpha}(\Gamma) \to T_{d}^{0,\alpha}(\Gamma)$ is linear such that

$$\|D(k)\| = o(k^2), \quad k \to 0.$$

**Proof:** Using Lemma 5, Lemma 23 from Appendix A, the definition of the set $K$ and Fredholm’s alternative, the first part is obvious. The result concerning the Riesz-number of $L_0$ is found in [1].

To check the asymptotic behaviour of $M_k$ we consider an arbitrary $q \in T_{d}^{0,\alpha}(\Gamma)$ and decompose

$$(M_kq)(x) - (M_0q)(x) = 2n(x) \wedge \int_{\Gamma} \text{curl}_x (q(y) (\Phi_k(x,y) - \Phi_0(x,y))) \, ds(y)$$

$$= k^2 (Cq)(x) + (D(k)q)(x),$$

where

$$(Cq)(x) = -\frac{1}{4\pi} n(x) \wedge \int_{\Gamma} \text{curl}_x (q(y) |x - y|) \, ds(y),$$

$$(D(k)q)(x) = 2n(x) \wedge \int_{\Gamma} \text{curl}_x (q(y) \left( \Phi_k(x,y) - \Phi_0(x,y) + \frac{k^2}{8\pi} |x - y| \right)) \, ds(y).$$

To prove the assertions concerning the operators $C$ and $D$ we use the same techniques as [1] to examine surface potentials. Therefore we concentrate on the essential steps.

The operator $C$ can be written as

$$(Cq)(x) = \frac{1}{4\pi} n(x) \wedge \int_{\Gamma} q(y) \wedge \frac{x - y}{|x - y|} \, ds(y).$$

Due to the boundedness of $I'$ we obtain for the surface potential $C$ with kernel $K(x,y) = \frac{x - y}{|x - y|}$ the mapping property

$$C : T_{d}^{0,\alpha}(\Gamma) \to T_{d}^{0,\alpha}(\Gamma), \quad \|Cq\|_{0,\alpha} \leq c_1 \|q\|_{\infty},$$

where $c_1$ is independent of $q$. To check the surface divergence of $Cq$, we define

$$E(c) = \frac{1}{4\pi} \int_{\Gamma} \text{curl}_x \left( \text{curl}_x q(y) |x - y| \right) ds(y), \quad 2 \in D^-.$$

Using the identity $\text{curl}_x \text{curl}_x = \text{grad}_x \text{div}_x - \Delta_x$ and Lemma 21 from Appendix A one immediately arrives at

$$E(x) = \frac{1}{4\pi} \int_{\Gamma} (\text{Div}_x q)(y) \frac{x - y}{|x - y|} ds(y) - \frac{1}{4\pi} \int_{\Gamma} q(y) \frac{2}{|x - y|} ds(y)$$

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for all $x \in D^-$. In complete analogy with [1] we can extend $E(x)$ in a Hölder continuous fashion up to the boundary $\Gamma$ such that $\|E\|_{0, \Gamma} \leq c_2 \|q\|_{d, \Gamma}$ with $c_2$ being independent of $q$. Lemma 22 provides $\text{Div } Cq = -n \cdot E|_\Gamma$ and the Lipschitz continuity of the unit outward normal $n$ on $\Gamma$ leads to

$$\|\text{Div } Cq\|_{0, \Gamma} \leq c_3 \|q\|_{d, \Gamma},$$

c_3 independent of $q$. Therefore

$$C : T^{\alpha}_d(\Gamma) \to T^{\alpha}_d(\Gamma), \quad \|Cq\|_{d, \Gamma} \leq c_4 \|q\|_{d, \Gamma},$$

where $c_4$ is independent of $q$.

It remains to consider the operator $D(k)$. For $q \in T^{\alpha}_d(\Gamma)$ we can write

$$(D(k)q)(x) = -2n(x) \wedge \int_\Gamma q(y) \wedge G(x, y) ds(y),$$

with

$$G(x, y) = \text{grad } x \left( \Phi_k(x, y) - \Phi_0(x, y) + \frac{k^2}{8\pi} |x - y| \right). \quad (32)$$

Using a power series expansion of $\Phi_k(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$ we obtain

$$G(x, y) = \frac{x-y}{4\pi} (ik)^3 \sum_{n=0}^{\infty} \frac{(ik|x-y|)^n}{n! (n+1)! (n+3)}.$$

Due to the boundedness of $\Gamma$ we find a ball $B_d$ such that $\Gamma \subset B_d$. With the power series representation of the kernel $G(x, y)$ we get for arbitrary $x_1, x_2 \in B_d, y \in \Gamma$ with $|x_1 - x_2| \leq \frac{1}{2}|x_1 - y|$ the estimate

$$|G(x_1, y) - G(x_2, y)| \leq 2 \frac{|x_1 - x_2|}{4\pi} |k^3 e^{2ds} \leq c_5 |k^3 | |x_1 - x_2|,$$

where $s = \omega \sqrt{\delta \mu^*}$ is an upper bound for $|k|$, $k \in K$, and $c_5$ is independent of $k, x_1, x_2$ and $y$. According to [1] we get

$$\frac{1}{k^3} D(k) : T^{\alpha}_d(\Gamma) \to T^{\alpha}_d(\Gamma), \quad \frac{1}{k^3} D(k)q \|_{0, \Gamma} \leq c_6 \|q\|_{\infty},$$

c_6 independent of $k$ and $q$. In order to examine the surface divergence we consider for $x \in D^-$

$$F(x) = 2 \text{curl } x \int_\Gamma \text{curl } x \left( q(y) \left( \Phi_k(x, y) - \Phi_0(x, y) + \frac{k^2}{8\pi} |x - y| \right) \right) ds(y)$$

$$= 2 \int_\Gamma (\text{Div } q)(y) G(x, y) ds(y) + 2 \int_\Gamma q(y) H(x, y) ds(y),$$

where $G(x, y)$ was defined in (32) and $H(x, y) = k^2 (\Phi_k(x, y) - \Phi_0(x, y))$. Because of $|H(x, y)| \leq Mk^3$ the potential $F$ can be extended in a Hölder continuous fashion up
to the boundary \( \Gamma \). Lemma 22 from Appendix A provides \( \text{Div} \ D(k)q = -n \cdot F |_{\Gamma} \) and from the continuous dependence of the layer potentials on their densities we obtain
\[
\| \text{Div} \ D(k)q \|_{\alpha, \Gamma} \leq c_{8} |k^{3}|(\|q\|_{\alpha, \Gamma} + \|\text{Div} q\|_{\alpha, \Gamma}).
\]
Thus
\[
\frac{1}{k^{3}} D(k) : T_{d}^{0} (\Gamma) \rightarrow T_{d}^{0} (\Gamma), \quad \\| \frac{1}{k^{3}} D(k)q \|_{\alpha, \Gamma} \leq c_{8} \|q\|_{\alpha, \Gamma},
\]
where \( c_{8} \) is independent of \( k \) and \( q \).

**Lemma 15** The linear operators \( T_{k} : T_{d}^{0} (\Gamma) \rightarrow T_{d}^{0} (\Gamma) \) from **Definition 2** are bounded with \( \lim_{k \rightarrow 0} \| T_{k} - T_{0} \| = 0 \). \( Z_{i}^{-} |_{\Gamma}, \ i \in \{ 1, \ldots, p \} \) are a basis of \( N(I - M_{0}) \) and for arbitrary \( q \in T_{d}^{0} (\Gamma) \) holds
\[
<T_{k}q, Z_{i}^{-} > = g_{i}(q)k^{2} + h_{i}(q, k) \quad i \in \{ 1, \ldots, p \},
\]
where \( g_{i}(q) \) is independent of \( k \) for all \( i \in \{ 1, \ldots, p \} \). Moreover
\[
|g_{i}(q)| \leq c_{i} \|q\|_{\alpha, \Gamma}, \quad |h_{i}(q, k)| \leq d_{i}(k) \|q\|_{\alpha, \Gamma}, \quad d_{i}(k) = o(k^{2}), \quad k \rightarrow 0,
\]
\( \forall i \in \{ 1, \ldots, p \} \) with \( c_{i} \) being independent of \( k \) and \( q \) resp. \( d_{i}(k) \) is independent of \( q \).

**Proof:** The results about the convergence of the \( T_{k} \) resp. the basis of \( N(I - M_{0}) \) are shown in Lemma 34 in Appendix B resp. Lemma 23 in Appendix A.

Consider \( T_{k} \) and \( n \cdot A \ Z_{i}^{-} \) on \( \Gamma \), which lies in \( T_{d}^{0} (\Gamma) \) since \( \text{Div} \ (n \cdot A \ Z_{i}^{-}) = -n \cdot \text{curl} \ Z_{i}^{-} = 0 \) on \( \Gamma \) by Lemma 22 from Appendix A. For arbitrary \( i \in \{ 1, \ldots, p \} \) and \( q \in T_{d}^{0} (\Gamma) \) holds due to Lemma 5
\[
<T_{k}q, Z_{i}^{-} > = - < T_{k}q, (n \cdot (n \cdot A \ Z_{i}^{-})) > \\
= - < n \cdot q, T_{k}(n \cdot A \ Z_{i}^{-}) > \\
= < q, n \cdot A T_{k}(n \cdot A \ Z_{i}^{-}) >, \tag{33}
\]
\( < \cdot, \cdot > \) being the bilinear form of the dual system (5). But
\[
\left( T_{k}(n \cdot A \ Z_{i}^{-}) \right)(x) = 2n(x) \cdot \int_{\Gamma} \text{grad} \ x \left( \text{Div} \ (n(y) \cdot A \ Z_{i}^{-}(y))\Phi_{k}(x, y) \right) ds(y) \\
+ 2k^{2}n(x) \cdot \int_{\Gamma} n(y) \cdot A \ Z_{i}^{-}(y)\Phi_{k}(x, y) ds(y).
\]
Using again \( \text{Div} \ (n \cdot A \ Z_{i}^{-}) = 0 \) on \( \Gamma \) we get
\[
\left( T_{k}(n \cdot A \ Z_{i}^{-}) \right)(x) = 2k^{2}n(x) \cdot \int_{\Gamma} n(y) \cdot A \ Z_{i}^{-}(y)\Phi_{0}(x, y) ds(y) \\
+ 2k^{2}n(x) \cdot \int_{\Gamma} n(y) \cdot A \ Z_{i}^{-}(y)(\Phi_{k}(x, y) - \Phi_{0}(x, y)) ds(y).
\]
According to (33) we find a decomposition
\[
<T_{k}q, Z_{i}^{-} > = g_{i}(q)k^{2} + h_{i}(q, k),
\]
with
\[ g(q) = \langle q, 2n A \left( n A \int n A \Phi_0 ds \right) \rangle > \]
\[ h_i(q, k) = \langle q, 2k^2 n A \left( n A \int \Phi_k - \Phi_0 ds \right) \rangle > \]

The continuity of the bilinear form \( \langle \cdot, \cdot \rangle \) provides
\[ |g_i(q)| \leq C_i \|q\|_{d, a, \Gamma}, \quad |h_i(q, k)| \leq d_i(k) \|q\|_{d, a, \Gamma}, \quad \sigma(\tau) = o(k^2), \quad k \to 0 \]
with \( c_i \) being independent of \( k \) and \( q \) resp. \( d_i(k) \) being independent of \( q \).

Now we come to the final result of this section.

**Theorem 10** Let \( A_\varepsilon^+ : T_\varepsilon^{0, \alpha}(\Gamma) \to T_\varepsilon^{0, \alpha}(\Gamma) \) be the operators of Definition 3. Then
\[ \lim_{\varepsilon \to 0} \| A_\varepsilon^+ - A_0^+ \| = 0 \]
in the induced operator norm.

**Proof:** In Lemma 14 and Lemma 15 we have shown that, for arbitrary \( q \in T_\varepsilon^{0, \alpha}(\Gamma) \)
\[ (I - M_0)q = \frac{1}{i \omega t^{+}} T_0 q, \quad k_\varepsilon^+ = \sqrt{\omega^2 t^{+} \varepsilon} \in K, \quad \varepsilon \to 0. \tag{34} \]
fits into the frame of the singular perturbation problems considered in Theorem 9.

For \( k_\varepsilon^+ \in K, k_\varepsilon^+ \neq 0 \) we obtain from Theorem 8 and (34)
\[ \phi_{k_\varepsilon^+} = A_\varepsilon^+ q. \]

Given an arbitrary \( q \in T_\varepsilon^{0, \alpha}(\Gamma) \), Theorem 9 provides a unique \( \phi_0 \) which fulfills
\[ (I - M_0)q = \frac{1}{i \omega t^{+}} T_0 q, \quad \| \phi_{k_\varepsilon^+} - \phi_0 \|_{d, a, \Gamma} \leq d(k_\varepsilon^+) \|q\|_{d, a, \Gamma}, \]
where \( d(k_\varepsilon^+) \) is independent of \( q \), \( d(k_\varepsilon^+) = o(1) \) as \( k_\varepsilon^+ \to 0 \). Defining \( A_0^+ \) by \( A_0^+ q = \phi_0 \) we obtain
\[ \| A_\varepsilon^+ - A_0^+ \| = \sup_{q \in T_\varepsilon^{0, \alpha}(\Gamma), \phi \neq 0} \frac{\| A_\varepsilon^+ q - A_0^+ q \|_{d, a, \Gamma}}{\|q\|_{d, a, \Gamma}} \]
\[ = \sup_{q \in T_\varepsilon^{0, \alpha}(\Gamma), \phi \neq 0} \frac{\| \phi_{k_\varepsilon^+} - \phi_0 \|_{d, a, \Gamma}}{\|q\|_{d, a, \Gamma}} = d(k_\varepsilon^+). \tag{35} \]

This shows that the bounded linear operators \( A_\varepsilon^+ \) converge uniformly for \( \varepsilon \to 0 \) to the bounded linear operator \( A_0^+ \).

By the definition of \( A_\varepsilon^+ \), \( \varepsilon > 0 \) we know that \( A_\varepsilon^+ q = n \wedge H_\varepsilon^+ |_{\Gamma}, H_\varepsilon^+, E_\varepsilon^+ \) being the unique solution of Problem 7 to the boundary value \( q \in T_\varepsilon^{0, \alpha}(\Gamma) \). But \( E_\varepsilon^+ \) also solves
\[ \Delta E_\varepsilon^+ + (k_\varepsilon^+)^2 E_\varepsilon^+ = 0 \quad \text{in } D^+ \]
\[ \text{div } E_\varepsilon^+ = 0, \quad n \wedge E_\varepsilon^+ = q \quad \text{on } \Gamma, \]

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with radiation condition
\[ \text{curl } E_e^+(x) \wedge \frac{x}{|x|} - ik_e^+ E_e^+(x) = o \left( \frac{1}{|x|} \right), \quad |x| \to \infty. \]

Let \( H_0^+, E_0^+ \) be the unique solution of Problem 8 to the boundary value \( q \in T^{\alpha}_d(\Gamma) \). Theorem 5.9 from [1] and \( \text{curl } E_0^+ = i\omega \mu^+ H_0^+ \) lead to
\[
\lim_{\varepsilon \to 0} \| n \wedge H_\varepsilon^+ - n \wedge H_0^+ \|_{\text{do},\Gamma} = 0.
\]

Since \( n \wedge H_\varepsilon^+ |_{\Gamma} = A_\varepsilon^+ q \) for \( \varepsilon \geq 0 \) we finally get
\[
\| (A_0^+ - \hat{A}_0^+) q \|_{\text{do},\Gamma} \leq \| n \wedge H_\varepsilon^+ - n \wedge H_0^+ \|_{\text{do},\Gamma} + \| A_\varepsilon^+ - \hat{A}_0^+ \| \| q \|_{\text{do},\Gamma},
\]
and by (35),(36) \( A_0^+ \) and \( \hat{A}_0^+ \) coincide so that \( A_\varepsilon^+ \) converges in norm to \( A_0^+ \).

**Corollary 2** Let \( D^\pm \) be of topological genus \( p \geq 0 \) and \( \mathcal{L}_\varepsilon, \varepsilon \geq 0 \) the operators introduced in Theorem 6. Then
\[
\lim_{\varepsilon \to 0} \| \mathcal{L}_\varepsilon - \mathcal{L}_0 \| = 0
\]
in the operator norm induced by \( T^{\alpha}_d(\Gamma) \).

## 9 Convergence of the Electromagnetic Fields

Before showing convergence of the corresponding fields in the domains \( D^\pm \) we focus on their tangential components on the boundary \( \Gamma \).

**Lemma 16** Consider sequences \( c_\varepsilon, d_\varepsilon \in T^{\alpha}_d(\Gamma), \varepsilon > 0 \) such that
\[
\lim_{\varepsilon \to 0} \| c_\varepsilon - c_0 \|_{\text{do},\Gamma} = 0, \quad \lim_{\varepsilon \to 0} \| d_\varepsilon - d_0 \|_{\text{do},\Gamma} = 0.
\]

Let \( H_\varepsilon^+, E_\varepsilon^+ \) be the solution of Problem 1 for \( \varepsilon > 0 \) to the boundary inhomogeneities \( c_\varepsilon, d_\varepsilon \in T^{\alpha}_d(\Gamma) \). Moreover, let \( H_0^+, E_0^+ \) be the solution of Problem 2 with the boundary inhomogeneities \( c_0, d_0 \in T^{\alpha}_d(\Gamma) \). Then
\[
\lim_{\varepsilon \to 0} \| n \wedge H_\varepsilon^+ - n \wedge H_0^+ \|_{\text{do},\Gamma} = 0, \quad \lim_{\varepsilon \to 0} \| n \wedge E_\varepsilon^+ - n \wedge E_0^+ \|_{\text{do},\Gamma} = 0.
\]

**Proof:** Assume \( 0 \leq \varepsilon < \delta \) and define \( e_\varepsilon = n \wedge E_\varepsilon^+ |_{\Gamma} \). According to Theorem 6 we get \( \mathcal{L}_\varepsilon e_\varepsilon = f_\varepsilon, \mathcal{L}_\varepsilon = \hat{A}_\varepsilon^+ - A_\varepsilon^-, f_\varepsilon = c_\varepsilon - A_\varepsilon^- d_\varepsilon \). Using \( \lim_{\varepsilon \to 0} \| A_\varepsilon^- - A_0^- \| \) from Theorem 8 and the assumptions on \( c_\varepsilon, d_\varepsilon \) we obtain
\[
\lim_{\varepsilon \to 0} \| f_\varepsilon - f_0 \|_{\text{do},\Gamma} = 0.
\]
Due to Theorem 7 the operator $L_\varepsilon, \varepsilon \geq 0$ has a bounded inverse. Moreover, Corollary 2 shows that

$$\lim_{\varepsilon \to 0} \|L_\varepsilon - L_0\| = 0$$

in the operator norm induced by $T^{\varepsilon_0}_2(\Gamma)$. Thus

$$\lim_{\varepsilon \to 0} \|L_\varepsilon^{-1} - L_0^{-1}\| \leq \lim_{\varepsilon \to 0} \left( \|L_\varepsilon^{-1}\| \|L_\varepsilon - L_0\| \|L_0^{-1}\| \right) = 0$$

and since $e_\varepsilon = L_\varepsilon^{-1}f_\varepsilon, \varepsilon \geq 0$, we conclude

$$\lim_{\varepsilon \to 0} \|e_\varepsilon - e_0\|_{d\alpha, \Gamma} = \lim_{\varepsilon \to 0} \|L_\varepsilon^{-1}e_\varepsilon - L_0^{-1}e_0\|$$

$$\leq \lim_{\varepsilon \to 0} \left( \|L_\varepsilon^{-1} - L_0^{-1}\| \|f_\varepsilon\|_{d\alpha, \Gamma} + \|L_0^{-1}\| \|f_\varepsilon - f_0\|_{d\alpha, \Gamma} \right)$$

$$= 0.$$}

So

$$\lim_{\varepsilon \to 0} \|n A E^+_\varepsilon - n A E^+_0\|_{d\alpha, \Gamma} = 0.$$

Now we turn our attention to the magnetic fields. By the definition of the operators $A^+_\varepsilon$ we get,

$$n \wedge H^+_\varepsilon |_{\Gamma} - n \wedge H^+_0 |_{\Gamma} = A^+_\varepsilon e_\varepsilon - A^+_0 e_0.$$

Due to the continuity and the convergence properties of $A^+_\varepsilon$ we conclude

$$\lim_{\varepsilon \to 0} \|n \wedge H^+_\varepsilon - n \wedge H^+_0\|_{d\alpha, \Gamma} = 0.$$

The rest of the assertion is shown with the help of the boundary conditions on $\Gamma$

$$n \wedge E^+_\varepsilon - n \wedge E^+_0 = n \wedge E^-_\varepsilon - n \wedge E^-_0 + d_\varepsilon - d_0,$$

$$n \wedge H^+_\varepsilon - n \wedge H^+_0 = n \wedge H^-_\varepsilon - n \wedge H^-_0 + c_\varepsilon - c_0.$$

**Theorem 11**: Under the assumptions of Lemma 16 holds

$$\lim_{\varepsilon \to 0} \|H^+_\varepsilon - H^+_0\|_{0, \alpha, D^\pm} = 0, \quad \lim_{\varepsilon \to 0} \|E^+_\varepsilon - E^+_0\|_{0, \alpha, D^\pm} = 0.$$

**Proof**: $E^+_\varepsilon$ satisfies

$$\triangle E^+_\varepsilon + (k^\pm_\varepsilon)^2 E^+_\varepsilon = 0 \quad \text{in} \ D^+, \quad \text{div} \ E^+_\varepsilon = 0 \quad \text{on} \ \Gamma, \quad \text{curl} \ E^+_\varepsilon(x) \wedge \frac{x}{|x|} - i k^\pm_\varepsilon E^+_\varepsilon(x) = o \left( \frac{1}{|x|} \right), \quad |x| \to \infty$$

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with \( k^+ = \sqrt{\omega^2 \mu^+ \varepsilon} \) and \( E^+_0 \) solves
\[
\begin{align*}
\Delta E^+_0 &= 0 \quad \text{in } D^+, \\
\text{div } E^+_0 &= 0 \quad \text{on } \Gamma', \\
E^+_0(x) &= o(1), \quad |x| \to \infty.
\end{align*}
\]
Moreover
\[
\int_{\Gamma_j} n(y) \cdot E^+_0(y) ds(y) = 0 \quad \forall j \in \{1, \ldots, m\}.
\]
Using Theorem 5.9 from [1] and the convergence of the tangential fields provided by the previous lemma, we arrive at
\[
\lim_{\varepsilon \to 0} \| E^+_\varepsilon - E^+_0 \|_{0,0,D^+} = 0, \quad \lim_{\varepsilon \to 0} \| \text{curl } E^+_\varepsilon - \text{curl } E^+_0 \|_{0,0,D^+} = 0
\]
and because of \( \text{curl } E^+_\varepsilon = i \omega \mu^+ H^+_\varepsilon \) it is clear that \( \lim_{\varepsilon \to 0} \| H^+_\varepsilon - H^+_0 \|_{0,0,D^+} = 0 \).

In order to show convergence of the fields in the domain \( D^- \) we use the representations of the fields given in Lemma 12,
\[
H^-_\varepsilon = -M^-_{k^-}(n \wedge H^-_\varepsilon) - \frac{1}{i \omega \mu^-} T^-_{k^-}(n \wedge E^-_\varepsilon), \quad k^- = \sqrt{\omega^2 \mu^- \varepsilon + i \omega \sigma^- \mu^-}.
\]
Thus
\[
H^-_\varepsilon - H^-_0 = -M^-_{k^-}(n \wedge H^-_\varepsilon - n \wedge H^-_0) - (M^-_{k^-} - M^-_{k^0})(n \wedge H^-_0) - \frac{1}{i \omega \mu^-} T^-_{k^-}(n \wedge E^-_\varepsilon - n \wedge E^-_0) - \frac{1}{i \omega \mu^-} (T^-_{k^-} - T^-_{k^0})(n \wedge E^-_0).
\]
Lemma 16 together with Lemma 32 and Lemma 33 from Appendix I3 provides
\[
\lim_{\varepsilon \to 0} \| H^-_\varepsilon - H^-_0 \|_{0,0,\bar{D}^-} = 0.
\]
In completely the same way \( \lim_{\varepsilon \to 0} \| E^-_\varepsilon - E^-_0 \|_{0,0,\bar{D}^-} = 0 \) is shown.

**Corollary 3** Let \( H^\pm_\varepsilon, E^\pm_\varepsilon \) be the solution of Problem 1 to the boundary values \( c_\varepsilon, d_\varepsilon \in T^0_{\varepsilon}(\Gamma) \),
\[
\lim_{\varepsilon \to 0} \| c_\varepsilon - c_0 \|_{1,0,\bar{D}^+} = 0, \quad \lim_{\varepsilon \to 0} \| d_\varepsilon - d_0 \|_{1,0,\bar{D}^-} = 0.
\]
Then \( H^\pm_\varepsilon, E^\pm_\varepsilon \) converge in \( C^0(\bar{D}) \) to a solution \( H^\pm_0, E^\pm_0 \) of Problem 3 to the boundary values \( c_0, g_0 = -\frac{1}{i \omega} \text{Div } d_0 \) with circulations \( h = (h_1, \ldots, h_p)^T \) given by the linear system \( (9) \) of Theorem 5.

**Proof:** The above result follows directly from Theorem 11 and Theorem 5.

Now we are coming back to Problem 4 resp. Problem 5, where we prescribe a current density \( J \) in the outer domain \( D^+ \) and homogeneous transmission conditions on \( \Gamma \).
In order to apply the previous convergence results to this kind of transmission problem we use volume potentials to transform the inhomogeneity \( J \) of the differential equations into inhomogeneities \( c_\varepsilon, d_\varepsilon \) of the boundary conditions.
Lemma 17 Let \( J \) be given as in Problem 4 resp. Problem 5. For \( \varepsilon \geq 0 \) there exist \( H_\varepsilon^J, E_\varepsilon^J \in C^1(\mathbb{R}^3) \cap C^{0\alpha}(\mathbb{R}^3) \) such that

\[
\text{curl } H_\varepsilon^J = J - i\omega \varepsilon E_\varepsilon^J, \quad \text{curl } E_\varepsilon^J = i\omega \mu^+ H_\varepsilon^J, \quad \text{div } E_\varepsilon^J = 0 \quad \text{in } D^+.
\]

\[
\int_{\Gamma_j} n \cdot E_\varepsilon^J \, ds = 0 \quad \forall j \in \{1, \ldots, m\}
\]

(37)

with

\[
\omega \mu^+ H_\varepsilon^J(x) \wedge \frac{x}{|x|} - k_z^+ E_\varepsilon^J(x) = O \left( \frac{1}{|x|^2} \right), \quad |x| \to \infty
\]

for \( \varepsilon > 0 \) respectively

\[
H_0^J(x) = o(1), \quad E_0^J(x) = o(1), \quad |x| \to \infty
\]

for \( \varepsilon = 0 \). Moreover

\[
\int_{\Gamma_i^p} \tau \cdot H_0^J \, dl = 0 \quad \forall i \in \{1, \ldots, p\}
\]

(38)

and

\[
\lim_{\varepsilon \to 0} \| H_\varepsilon^J - H_0^J \|_{0, n} = 0, \quad \lim_{\varepsilon \to 0} \| E_\varepsilon^J - E_0^J \|_{0, n, D^+} = 0.
\]

Proof: Throughout the whole proof we use \( k \) instead of \( k^+ \). For \( \varepsilon \geq 0 \) we define

\[
E_\varepsilon^J(x) = i\omega \mu^+ \int_{\Omega^J} J(y) \Phi_k(x, y) \, dv(y),
\]

(39)

\[
H_\varepsilon^J(x) = \text{curl } x \int_{\Omega^J} J(y) \Phi_k(x, y) \, dv(y).
\]

(40)

Lemma 35 from Appendix B provides \( H_\varepsilon^J, E_\varepsilon^J \in C^1(\mathbb{R}^3) \cap C^{0\alpha}(\mathbb{R}^3) \). and

\[
\lim_{\varepsilon \to 0} \| H_\varepsilon^J - H_0^J \|_{0, D^+} = 0, \quad \lim_{\varepsilon \to 0} \| E_\varepsilon^J - E_0^J \|_{0, D^+} = 0.
\]

By definition \( \text{curl } E_\varepsilon^J(x) = i\omega \mu^+ H_\varepsilon^J(x) \) and because of \( \text{div } J = 0, J|_{\partial D^+} = 0 \) we obtain using Lemma 28 from Appendix A

\[
\left( \text{div } E_\varepsilon^J \right)(x) = -i\omega \mu^+ \int_{\partial D^+} n(y) \cdot J(y) \Phi_k(x, y) \, dv(y) = 0,
\]

i.e.

\[
\left( \text{curl } H_\varepsilon^J \right)(x) = \left( \text{grad } x \text{div } x - \Delta x \right) \int_{\Omega^J} J(y) \Phi_k(x, y) \, dv(y)
\]

\[
= \frac{1}{i\omega \mu^+} \text{grad div } E_\varepsilon^J + J(x) + \frac{k^2}{i\omega \mu^+} E_\varepsilon^J(x)
\]

\[
= J(x) - i\omega \varepsilon E_\varepsilon^J(x).
\]

In the case \( \varepsilon = 0 \) the boundedness of \( D^+ \) provides \( H_0^J(x), E_0^J(x) = o(1) \) as \( |x| \to \infty \). For \( \varepsilon > 0 \) it is shown in [7] that the fields \( E_\varepsilon^J, H_\varepsilon^J \) meet the corresponding radiation condition.
An application of the Gaussian theorem yields
\[
\int_{D_j} n(y) \cdot E'_\varepsilon(y) ds(y) = \int_{D_j} \left( \text{div} \ E'_\varepsilon \right) (y) dv(y) = 0
\]
for arbitrary \( j \in \{1, \ldots, m\} \), where the \( D_j^- \) are the connected components of the interior domain \( D^- \).

Moreover \( \eta_i^+ = \partial \Sigma_i^- \), \( \Sigma_i^- \) \( \subset D^- \) and by Stokes' Theorem
\[
\int_{\eta_i^+} \tau \cdot H'_0 dl = \int_{\Sigma_i^-} n \cdot \text{curl} \ H'_0 ds = \int_{\Sigma_i^-} n \cdot J ds = 0
\]
since \( \text{supp}(J) \subset D^- \) \( \subset D^+ \).

By direct calculations the following result is easily verified.

**Lemma 18** Let \( \hat{H}^\pm, \hat{E}^\pm \) be a solution of Problem 1 to \( c = -n \wedge H^J, d = -n \wedge E^J \).
Then
\[
H^+ = \hat{H}^+ + H^J, \quad E^+ = \hat{E}^+ + E^J, \quad H^- = \hat{H}^-, \quad E^- = \hat{E}^-
\]
are a solution of Problem 4.

**Lemma 19** Let \( \hat{H}^\pm, \hat{E}^\pm \) be a solution of Problem 3 to \( c = -n \wedge H^J_0, g = \frac{1}{i\omega} \text{Div} (n \wedge E^J_0) \) and circulations \( h = (h_1, \ldots, h_p)^T \in \mathbb{CP} \). Then
\[
H^+ = \hat{H}^+ + H^J_0, \quad E^+ = \hat{E}^+ + E^J_0, \quad H^- = \hat{H}^-, \quad E^- = \hat{E}^-
\]
solve Problem 5 with the same circulations \( h \).

**Proof:** \( H^\pm, E^\pm \) fulfill the differential equations and radiation conditions of Problem 5. Due (37), (38) we obtain
\[
\int_{\eta_i^+} n \cdot E^+ ds = 0 \quad \forall j \in \{1, \ldots, m\},
\]
\[
\int_{\eta_i^+} \tau \cdot H^+ dl = h_i \quad \forall i \in \{1, \ldots, p\},
\]
where \( h_i \) are the circulations of \( \hat{H}^+ \). Moreover on \( \Gamma \) we have
\[
n \wedge H^+ - n \wedge H^- = n \wedge \hat{H}^+ - n \wedge \hat{H}^- + n \wedge H^J_0 = c + n \wedge H^J_0 = 0
\]
and since \( g = \frac{1}{i\omega} \text{Div} (n \wedge E^J_0) = -\frac{1}{i\omega} n \cdot \text{curl} E^J_0 = -n \cdot (\mu^+ H^J_0) \) we deduce
\[
n \cdot (\mu^+ H^+) - n \cdot (\mu^- H^-) = n \cdot (\mu^+ \hat{H}^+) - n \cdot (\mu^- \hat{H}^-) + n \cdot (\mu^+ H^J_0) = g + n \cdot (\mu^+ H^J_0) = 0.
\]

As a direct consequence we obtain
Lemma 20 Problem $4$ is uniquely solvable. Problem $5$ is solvable and $H_0^\pm, E^\pm_0$ are uniquely determined.

Proof: Existence is obtained by using the existence results of Problem $1$ resp. Problem $3$ together with the last two Lemmata. The uniqueness results of these problems directly carry over to Problem $4$ resp. Problem $5$.

Now we prove the final convergence theorem.

Theorem 12 The unique solution $H_0^\pm, E^\pm_0$ of Problem $4$ converges in $C^{0\alpha}(\hat{D})$ to $H^\pm, E^\pm$, $\hat{D}$ being a solution of Problem $5$ with circulations $h_\iota$ given by the nonsingular linear system $Ah = b$ with coefficients

$$a_{i\ell} = \int_\Gamma (n_\iota \cdot (E^+_\iota - E^-_\iota)) \cdot Z^+_\ell ds,$$

$$b_\iota = -\int_\Gamma (n_\iota \cdot (E^+_\iota - E^-_\iota)) \cdot Z^+ds,$$

$i, \ell \in \{1, \ldots, p\}$, where $H^\pm, E^\pm$ resp. $H^+_\iota, E^+_\iota, l \in \{1, \ldots, p\}$ are arbitrary solutions of Problem $5$ for $h_\iota = 0$ resp. for $f = 0, h_\iota = \delta_\iota, i \in \{1, \ldots, p\}$.

Proof: From Lemma $18$ we know that the unique solution $H^\pm, E^\pm$ of Problem $4$ can be written as

$$H^\pm = \hat{H}^\pm + H^\pm_0,$$

$$E^\pm = \hat{E}^\pm + E^\pm_0,$$

where $\hat{H}^\pm, \hat{E}^\pm$ is the unique solution of Problem $1$ for $c_\iota = -n \wedge H^\pm_0, d_\iota = -n \wedge E^\pm_0$.

Due to Lemma $17$ $H^\pm, E^\pm$ converge in $C^{0\alpha}(D^\pm)$ to $H^\pm_0, E^\pm_0$ and thus $c_\iota, d_\iota$ converge in $T^{0\alpha}_{d}(\Gamma, \hat{D})$ to $c_\iota = -n \wedge H^\pm_0, d_\iota = -n \wedge E^\pm_0$. According to Corollary $3$, $\hat{H}^\pm, \hat{E}^\pm$ converge in $C^{0\alpha}(\hat{D}^\pm)$ to $\hat{H}^\pm_0, \hat{E}^\pm_0$, which solve Problem $3$ for the boundary data $c_0, g_0 = -\frac{1}{\nu} \text{Div} d_0$. Therefore $H^\pm_\iota, E^\pm_\iota$ converge in $C^{0\alpha}(\hat{D}^\pm)$ to $H^\pm_0, E^\pm_0$.

$$H^\pm_0 = \hat{H}^\pm_0 + H^\pm_0, \quad E^\pm_0 = \hat{E}^\pm_0 + E^\pm_0, \quad H^- = \hat{H}_0, \quad E^- = \hat{E}_0^-,$$

which is due to Lemma $19$ a solution of Problem $5$ with circulations $h_\iota = \hat{h}_\iota = \int_{\Omega^\pm} h \cdot \hat{h}_0^\iota dl$.

Let $H^\pm, E^\pm$ resp. $H^+_\iota, E^+_\iota, l \in \{1, \ldots, p\}$ be given as in the assumption and define

$$\hat{H}^\iota = H^\iota - H^\iota_0, \quad \hat{E}^\iota = E^\iota - E^\iota_0, \quad H^\iota = H^- + \hat{H}^\iota, \quad E^\iota = E^- + \hat{E}^\iota.$$

Then $\hat{H}^\iota, \hat{E}^\iota$ resp. $H^\iota, E^\iota_\iota, l \in \{1, \ldots, p\}$ are solutions of Problem $3$ to the data $c_0, g_0, h_\iota = 0$ resp. to $c = 0, g = 0, h_\iota = \delta_\iota, i \in \{1, \ldots, p\}$. According to Corollary 3, the circulations $h = (h_1, \ldots, h_p)^T$ of $H^\iota_0$ are given by the nonsingular system $Ah = b$ with coefficients

$$a_{i\ell} = \int_{\Gamma} (n_\iota \cdot (E^+_\iota - E^-_\iota)) \cdot Z^+_\ell ds,$$

$$b_\iota = \int_{\Gamma} (n_\iota \cdot (E^+_\iota - E^-_\iota)) \cdot Z^+ds,$$

$i, \ell \in \{1, \ldots, p\}$. Since $d_0 = -n \wedge E^+_0$ we finally get

$$b_\iota = -\int_{\Gamma} (n_\iota \cdot (E^+_\iota + \hat{E}^+_\iota - E^-_\iota)) \cdot Z^+ds = -\int_{\Gamma} (n_\iota \cdot (E^+_\iota - E^-_\iota)) \cdot Z^+ds$$

which completes the proof.
10 Appendix A

The following four Lemmata can be found in [1].

**Lemma 21** Let \(a \in T_d^0(\Gamma)\). Then \(\int_\Gamma \text{Div}a\,ds = 0\) and for \(x \in R^3 \setminus \Gamma\) holds

\[
\int_\Gamma \text{div}_x(a(y)\Phi_k(x,y))\,ds(y) = \int_\Gamma (\text{Div}a)(y)\Phi_k(x,y)\,ds(y).
\]

**Lemma 22** Consider \(D_{ho}^+ = \{x = z + hn(z)\mid z \in \Gamma, \quad 0 \leq h \leq h_0\}\) and \(E\) being \(C^1\) in the interior of \(D_{ho}^+\) or \(D_{ho}^-\), so that curl \(E\) can be continuously extended to \(D_{ho}^+\) or \(D_{ho}^-\). Then

\[
\text{Div} \left( n \times E \right) = -n \cdot \text{curl} E |_{\Gamma}.
\]

**Lemma 23** For \(\text{Im}(k) > 0\), \(N(I + M_k) = \{0\}\). There exists \(\delta > 0\), so that for all \(k \neq 0\), \(\text{Im}(k) \geq 0, |k| < \delta, N(I + M_k) = \{0\}\). Moreover, for \(k = 0\) we obtain

\[
N(I + M_0') = \text{span}\{Z_i^\pm, \ldots, Z_p^\pm\},
\]

\[
N(I + M_0) = \text{span}\{n \wedge Z_i^\pm, \ldots, n \wedge Z_p^\pm\}.
\]

**Lemma 24** The matrices \(Z^\pm = (z_{ij}^\pm), z_{ij}^\pm = \int_\Gamma Z_i^\pm \cdot (n \wedge Z_j^\pm)\,ds, i, j \in \{1, \ldots, p\}\), \(Z_i^\pm\) being the Neumann yields of \(D^\pm\), are nonsingular.

**Lemma 25** The operator \(M_0\) maps \(T_d^0(\Gamma)\) into itself and \(I + M_0\) has a bounded inverse in \(T_d^0(\Gamma)\).

**Proof:** Let \(a \in T_d^0(\Gamma)\) and consider \(F(x) = \int_\Gamma \text{curl}_x(a(y)\Phi_0(x,y))\,ds(y)\). For \(x \in D^+\)

\[
\text{curl} F(x) = \text{grad}_x \int_\Gamma (\text{Div}a)(y)\Phi_0(x,y)\,ds(y) = 0
\]

since \(\text{Div}a = 0\). By the jump conditions for the above layer potential we get

\[
M_0a = 2n \times F |_{\Gamma} - a,
\]

so that

\[
\text{Div} (M_0a) = -2n \cdot \text{curl} F |_{\Gamma} = 0
\]

by Lemma 22.

On the other hand, \(N(I + M_0') = \text{span}\{Z_1^+, \ldots, Z_p^+\}\) and \(M_0, M_0'\) are adjoint with respect to (5). Therefore

\[
\int_\Gamma M_0a \cdot Z_i^+\,ds = <M_0a, Z_i^+> = <a, M_0'Z_i^+> = -<a, Z_i^+> = -\int a \cdot Z_i^+\,ds = 0
\]

and the first part of the assertion is shown.

To show that \(I + M_0\) has a bounded inverse in \(T_d^0(\Gamma)\), we will use the Riesz theory. \(M_0\) is compact in \(T_d^0(\Gamma)\) and therefore in \(T_d^0(\Gamma)\). Now assume, that \(a \in T_d^0(\Gamma)\),
Using Lemma 23, we know that \( a \) is given by \( a = \sum_{i=1}^{p} x_i n \wedge Z_i^- \) and since \( a \in T_{\alpha}^{0,\alpha}(\Gamma) \),

\[
0 = \int_{\Gamma} a \cdot Z_i^+ ds = \sum_{i=1}^{p} x_i \int_{\Gamma} (n \wedge Z_i^-) \cdot Z_i^+ ds = \sum_{i=1}^{p} z_i^+ x_i.
\]

But following Lemma 24, the \( p \times p \) matrix with coefficients \( z_i^+ \) is nonsingular. Thus \( a = 0 \), \( N(I + M_0) = \{0\} \) and \( I + M_0 \) has a bounded inverse in \( T_{\alpha}^{0,\alpha}(\Gamma) \).

**Lemma 26** The boundary value problem \( F \in C^2(\mathcal{D}^+) \cap C^0(\mathcal{D}^+) \)

\[
\begin{align*}
\text{curl} \ F &= 0, \\
\int_{\Gamma_j} n \cdot F ds &= 0, \\
\text{div} \ F &= 0 & \forall j \in \{1, \ldots, m\}, \\
\text{on} \ \Gamma, \\
F(z) &= o(1), & |x| \to \infty
\end{align*}
\]

is uniquely solvable if and only if \( c \in T_{\alpha}^{0,\alpha}(\Gamma) \). In this case there exists a constant \( c_{\alpha} \) such that

\[
\|F\|_{0,0,\alpha} \leq c_{\alpha}\|c\|_{\alpha,\Gamma}.
\]

**Proof:** Let \( F \) be a solution of the problem. By Lemma 22, we get \( \text{Div} \ c = 0 \), so that \( c \in T_{\alpha}^{0,\alpha}(\Gamma) \). Since \( Z_i^+(x) = O \left( \frac{1}{|x|^m} \right) \) for \( |x| \to \infty \) we can use Green's formula

\[
\int_{\Gamma} c \cdot Z_i^+ ds = \int_{\Gamma} n \cdot (F \wedge Z_i^+) ds = -\int_{\mathcal{D}^+} Z_i^+ \cdot \text{curl} \ F - F \cdot \text{curl} Z_i^+ dv = 0,
\]

and \( c \) is also contained in \( T_{\alpha}^{0,\alpha}(\Gamma) \).

Let us start the proof of the second direction by showing uniqueness. Assume \( F \) is a solution to \( c = 0 \). But then, \( F \) has to be a linear combination of Dirichlet fields \( F = \sum_{j=1}^{m} f_j Y_j^+ \). Since \( Y_j^+ = \text{grad} \ \varphi_j \), \( \varphi_j \big|_{\Gamma_1} = \delta_{ji} \), we obtain

\[
\int_{\Gamma} n \cdot Y_j^+ ds = \int_{\Gamma} n \cdot Y_j^+ \varphi ds = -\int_{\mathcal{D}^+} \text{div} \ (Y_j^+ \varphi) dv = -\int_{\mathcal{D}^+} Y_j^+ \cdot Y_i^+ dv
\]

and

\[
0 = \sum_{i=1}^{m} f_i \int_{\Gamma_i} n \cdot F ds = \sum_{j=1}^{m} \sum_{i=1}^{m} f_j \bar{f}_i \int_{\Gamma_i} n \cdot Y_j^+ ds = -\sum_{j=1}^{m} \sum_{i=1}^{m} f_j \bar{f}_i \int_{\mathcal{D}^+} Y_j^+ Y_i^+ dv = \int_{\mathcal{D}^+} F \cdot F ds_{\alpha}.
\]

\[37\]
Thus $F = O$.

To show existence, we use the ansatz $F(x) = \int_\Gamma \text{curl } z \left(a(y) \Phi_0(x, y)\right) ds(y)$ for $a \in T^{0\alpha}_*(\Gamma)$. Due to the properties of the above layer potential [1], all conditions of the boundary value problem besides $n \cdot F = c$ on $\Gamma$ are fulfilled. Taking the limit for $x \in D^+$ to $\Gamma$ leads to $(I + M_0)a = 2c, a, c \in T^{0\alpha}_*(\Gamma)$ which is according to Lemma 25 uniquely solvable with $\|a\|_{\alpha, \Gamma} \leq c_\alpha\|c\|_{\alpha, \Gamma}$. Since

$$\|F\|_{0\alpha, D^+} \leq c_\alpha\|a\|_{\alpha, \Gamma},$$

the proof is complete. □

**Lemma 27** For all $i \in \{1, \ldots, p\}$ there exists a field $E \in C^1(D^+) \cap C^{0\alpha}(D^+)$ with

$$\text{curl } E = Z_i^+, \quad \text{div } E = 0 \quad \text{in } D^+, \quad \int_\Gamma n \cdot E ds = 0 \quad \forall j \in \{1, \ldots, m\},$$

$$E(x) = o(1) \quad |x| \to \infty.$$ 

**Proof:** Let $e = n \cdot Z_i^+ \in T^{0\alpha}_*(\Gamma)$. Then $\text{Div } e = -n \cdot \text{curl } Z_i^+ |_\Gamma = 0$ according to Lemma 22 and

$$\int_\Gamma e \cdot Z_i^+ ds = \int_\Gamma (n \cdot Z_i^+) \cdot Z_i^+ ds$$

$$= \int_{D^+} \text{div } (Z_i^+ \wedge Z_i^+) dv$$

$$= -\int_{D^+} \text{curl } Z_i^+ \cdot \text{curl } Z_i^+ dv$$

$$= 0,$$

so that $e \in T^{0\alpha}_*(\Gamma)$. Thus we can solve the boundary value problem of Lemma 26 with $c = e$ and the solution $F$ is of the form

$$F = \text{curl } \tilde{E}, \quad \tilde{E}(x) = \int_\Gamma a(y) \Phi_0(x, y) ds(y)$$

with $a \in T^{0\alpha}_*(\Gamma)$. But for $H = F - Z_i^+$ holds

$$\text{curl } H = 0, \quad \text{div } H = 0 \quad \text{in } D^+, \quad \int_\Gamma n \cdot H ds = 0, \quad \forall j \in \{1, \ldots, m\},$$

$$n \wedge H = 0 \quad \text{on } \Gamma,$$

$$H(x) = o(1) \quad |x| \to \infty,$$

so that $H$ vanishes due to the uniqueness result of Lemma 26 and $F = Z_i^+$ resp. $\text{curl } \tilde{E} = Z_i^+$. In general $\int_\Gamma n \cdot \tilde{E} ds \neq 0$. But this is easily corrected by subtracting
the gradient of the solution of the boundary value problem
\[ \Delta u = 0 \quad \text{in } D^+, \]
\[ \partial_n u = u_j \quad \text{on } \Gamma_j, \quad j \in \{1, \ldots, m\}, \]
\[ u(x) = o(1), \quad |x| \to \infty \]
with \( u_j = \int_{\Gamma_j} n \cdot \tilde{E} ds \left( \int_{\Gamma_j} ds \right)^{-1}. \) This problem has a unique solution in \( C^2(D^+) \cap C^{1\alpha}(\tilde{D}^+) \) \[ [1]. \]

**Lemma 28** Let \( D \subset \mathbb{R}^3 \) be open, bounded with \( \partial D \) being \( C^2 \). If \( a \in C^1(D) \cap C(\tilde{D}) \) the volume potential
\[ V(x) = \int_D a(y) \Phi_k(x, y) dv(y) \]
lies in \( C^1(\mathbb{R}^3) \) and
\[ (\text{curl } V)(x) = \int_D (\text{curl } a)(y) \Phi_k(x, y) dv(y) - \int_{\partial D} n(y) \wedge a(y) \Phi_k(x, y) ds(y), \]
\[ (\text{div } V)(x) = \int_D (\text{div } a)(y) \Phi_k(x, y) dv(y) - \int_{\partial D} n(y) \cdot a(y) \Phi_k(x, y) ds(y). \]
If \( a \in C^1(D) \cap C^{0\alpha}(\tilde{D}) \) then \( V \in C^2(D) \).

This lemma, which is for example found in [6], is used to show the following useful result.

**Lemma 29** To every \( \lambda \in C^{0\alpha} (\Gamma) \) exists an \( \lambda \in T^{0\alpha}_d (\Gamma) \) with
\[ \text{Div } a = \lambda, \quad ||a||_{0\alpha, \Gamma} \leq c_\alpha ||\lambda||_{0\alpha, \Gamma}. \]

**Proof:** We first consider the boundary value problem
\[ \Delta u = 0 \quad \text{in } D^+, \quad \partial_n u - \lambda \quad \text{on } \Gamma. \]
Let us try to find a solution of (41) in form of a single layer potential
\[ u(x) = \int_{\Gamma} \varphi(y) \Phi_0(x, y) ds(y) \]
with density \( \varphi \in C^{0\alpha} (\Gamma) \). From the jump relations for (42) we obtain the integral equation
\[ (I + K'_0) \varphi = 2\lambda. \]
From Lemma 5 we know, that \( K'_0 \) is compact in \( C^{0\alpha}(\Gamma) \). Moreover, in the proof of Theorem 1 we have shown, that \( K'_0 \) maps \( C^{0\alpha}(\Gamma) \) into itself and thus is also compact in \( C^{0\alpha}(\Gamma) \). According to [1], there exists a basis \( \psi_l \) of \( N(I + K'_0) \) with
\[ \int_{\Gamma_j} \psi_l ds = \delta_{jl}, \quad j, l \in \{1, \ldots, m\}, \]

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so that \( N(I + K_0') = \{0\} \) in \( C_\omega^0(\Gamma) \). Thus \( I + K_0' \) has a bounded inverse in \( C_\omega^0(\Gamma) \) and (43) has a unique solution \( \varphi \in C_\omega^+(\Gamma) \) with

\[
\|\varphi\|_{\omega, \Gamma} \leq c_1 \|\lambda\|_{\omega, \Gamma}.
\]

Using the properties of the single layer potential, we obtain

\[
u, \nabla u \in C_\omega^0(D^-), \quad \max\{\|u\|_{\omega, D^-}, \|\nabla u\|_{\omega, D^-}\} \leq c_2 \|\lambda\|_{\omega, \Gamma}.
\]

Now we consider the corresponding exterior boundary value problem

\[
\Delta v = 0 \text{ in } D^+, \quad \partial v = \lambda \text{ on } \Gamma, \quad v(x) = o(1) \quad |x| \to \infty.
\]

Using again a single layer potential ansatz, it is shown in [1], that a unique strong solution exists with

\[
v, \nabla v \in C_\omega^0(D^+), \quad \max\{\|v\|_{\omega, D^+}, \|\nabla v\|_{\omega, D^+}\} \leq c_3 \|\lambda\|_{\omega, \Gamma}.
\]

Let \( w = \nabla u \), where \( u \) is the solution of (41). Then \( w \in C^1(D^-) \cap C_\omega^0(D^-) \) and \( \nabla w = 0 \), \( \text{div } w = 0 \) in \( D^- \). For the volume potential

\[
F(x) = \int_{D^-} w(y) \Phi_0(x, y) ds(y)
\]

Lemma 28 now provides

\[
\left(\text{curl } F\right)(x) = -\int_{\Gamma} n(y) \wedge w(y) \Phi_0(x, y) ds(y),
\]

\[
\left(\text{div } F\right)(x) = -\int_{\Gamma} n(y) \cdot w(y) \Phi_0(x, y) ds(y).
\]

Since \( w |_{\Gamma} \in C_\omega^0(\Gamma) \) we get [1]

\[
\text{curl } F, \text{ div } F \in C^1(D^-) \cap C_\omega^0(D^-)
\]

with

\[
\|\text{curl } F\|_{\omega, D^-} \leq c_4 \|w\|_{\omega, D^-}, \quad \|\text{div } F\|_{\omega, D^-} \leq c_5 \|w\|_{\omega, D^-}.
\]

Using the continuous dependence of \( w \) on \( \lambda \), we conclude

\[
\|\text{curl } F\|_{\omega, D^-} \leq c_6 \|\lambda\|_{\omega, \Gamma}, \quad \|\text{div } F\|_{\omega, D^-} \leq c_7 \|\lambda\|_{\omega, \Gamma}.
\]

Consider now the surface potential

\[
G(x) = \int_{\Gamma} n(y)v(y)\Phi_0(x, y) ds(y).
\]

Then \( G \in C^2(D^-) \cap C_\omega^0(D^-) \), \( \text{curl } G, \text{ div } G \in C^1(D^-) \cap C_\omega^0(D^-) \) and

\[
\|\text{curl } G\|_{\omega, D^-} \leq c_8 \|v\|_{\omega, \Gamma} \leq c_9 \|\lambda\|_{\omega, \Gamma}.
\]
In the final step, we consider \( \mathbf{A} = G - F \in C^2(D^-) \). Then

\[
\begin{align*}
\left( \text{div } \mathbf{A} \right)(x) &= \left( \text{div } G \right)(x) - \left( \text{div } F \right)(x) \\
&= \int_{\Gamma} v(y) n(y) \cdot \nabla_x \Phi_0(x, y) ds(y) + \int_{\Gamma} n(y) \cdot w(y) \Phi_0(x, y) ds(y) \\
&= \int_{\Gamma} \left( \partial_n v \right)(y) \Phi_0(x, y) - v(y) \partial_n \Phi_0(x, y) ds(y) \\
&= 0
\end{align*}
\]

in \( D^- \) and for \( \mathbf{B} = \text{curl } \mathbf{A} = \text{curl } G - \text{curl } F \) holds

\[
\| \mathbf{B} \|_{0, \alpha, D^-} \leq c_{10} \| \lambda \|_{0, \alpha, \Gamma}
\]

and

\[
\text{curl } \mathbf{B} = -\mathbf{w} \quad \text{in } D^-.
\]

Taking \( \mathbf{a} = \mathbf{z} \wedge \mathbf{B} \mid_{\Gamma} \) we see, that \( \mathbf{a} \in T^{\alpha \gamma}_d(\Gamma) \) with

\[
\text{Div } \mathbf{a} = -n \cdot \text{curl } \mathbf{B} = n \cdot \mathbf{w} = n \cdot \nabla u = \lambda
\]

\( \mathbf{d} \)

\[
\| \mathbf{a} \|_{\alpha, \gamma, \Gamma} \leq c_{\alpha} \| \lambda \|_{0, \alpha, \Gamma}.
\]

\section{Appendix B}

In this section we consider some asymptotic properties of the single- and double layer potentials used in the convergence theorems. The first lemma is a slight modification of a result from [1].

\textbf{Lemma 30} Let \( K(x, y) \) be \( \text{continuous for all } x \in \mathbb{R}^3, y \in \Gamma \) with \( x \neq y \). Assume

\[
|K(x, y)| \leq C_1|x - y|^{-1} + C_2
\]

and

\[
|K(x_1, y) - K(x_2, y)| \leq C_3|x_1 - x_2| + C_4\frac{|x_1 - x_2|}{|x_1 - y|} + C_5\frac{|x_1 - x_2|}{|x_1 - y|^2}
\]

for all \( x, x_1, x_2 \in \mathbb{R}^3, y \in \Gamma \) such that \( 2|x_1 - x_2| \leq |x_1 - y| \), where \( C_1, \ldots, C_5 \) are independent of \( x_1, x_2, y \). Then the generalized potential

\[
u(x) = \int_{\Gamma} K(x, y)c(y) ds(y)
\]

with \( c \in C(\Gamma) \) is well defined for all \( x \in \mathbb{R}^3 \). Moreover \( u \in C^{0,\beta}(\mathbb{R}^3) \) for all \( \beta \in (0, 1) \) and

\[
\| u \|_{0, \beta} \leq C_\beta \| c \|_{\infty, \Gamma}, \quad C_\beta \leq M \sum_{i=1}^5 C_i,
\]

where \( M \) only depends on \( \Gamma \) and \( \beta \).
Proof: The first estimate ensures existence of the integral \( u(x) = \int_{\Gamma} K(x, y)c(y)ds(y) \) with a weakly singular kernel. Using (44) the Hölder continuity is shown in complete analogy with [1].

**Lemma 31** For \( c \in T^{0,\alpha}_d(\Gamma) \), \( x \in D^\pm \) and \( \text{Im} (k) \geq 0 \) define

\[
(S^\pm_k)(x) = \int_{\Gamma} c(y)\Phi_k(x, y)ds(y).
\]

Then \( S^\pm_k \) can be extended to \( \bar{D}^\pm \) in a Hölder continuous fashion such that

\[
\|S^\pm_k c\|_{0,\alpha, D^\pm} \leq c_\alpha \|c\|_{0,\alpha, \Gamma}
\]

with \( c_\alpha \) being independent of \( c \in T^{0,\alpha}_d(\Gamma) \). Moreover

\[
\lim_{k \to k_0} \|S^\pm_k - S^\pm_{k_0}\| = 0
\]

holds in the induced operator norm.

**Proof:** Existence and extension properties are shown in [1]. For \( x, y \in \mathbb{R}^3, x \neq y \) and \( r = |x - y| \) define

\[
K(x, y) = \frac{\delta_{kr} - e^{ikr}}{4\pi r}.
\]

Using a meanvalue argument one immediately arrives at

\[
|e^{ikr_1} - e^{ikr_2}| \leq 3|k||r_1 - r_2|
\]

for arbitrary \( r_1, r_2 \geq 0 \), \( \text{Im} (k) \geq 0 \), i.e.

\[
|K(x, y)| \leq M_1|k - k_0|.
\]

\( M_1 \) is independent of \( k, k_0, x \) and \( y \). Considering the decomposition

\[
K(x_1, y) - K(x_2, y) = \frac{e^{ik_0r_1} - e^{ik_0r_2}}{4\pi r_1}(e^{i(k-k_0)r_1} - 1)
\]

\[
+ \frac{e^{ik_0r_2}}{4\pi r_1}(e^{i(k-k_0)r_1} - e^{i(k-k_0)r_2})
\]

\[
+ \frac{e^{ik_0r_2}}{4\pi}(e^{i(k-k_0)r_2} - 1) \left( \frac{r_2 - r_1}{r_1r_2} \right),
\]

\( r_1 = |x_1 - y|, r_2 = |x_2 - y| \) and applying (47) we get

\[
|K(x_1, y) - K(x_2, y)| \leq |k - k_0||r_2 - r_1|(M_2|k_0| + \frac{M_3}{r_1}),
\]

(49)
where $M_2, M_3$ are independent of $k, k_0, x_1, x_2, y$. By Lemma 30 and the special choice
\[ C_1 = 0, \quad C_2 = M_1 |k - k_0|, \quad C_4 = M_2 |k - k_0| |k_0|, \]
\[ C_5 = 0, \]

we obtain for arbitrary $c \in T_{d}\Gamma (\Gamma ) \|S_k^+ c - S^- k_0 c\|_{0, 0} \leq C_\alpha \|c\|_{\alpha \Gamma}$ and $|k - k_0|$ enters each $C_i, i \in \{1, \ldots, 5\}$ in a linear fashion. Therefore we finally conclude
\[ \lim_{k \to k_0} \|S_k^+ - S^- k_0\| = 0 \]
in the induced operator norm. The corresponding result for $S_k^-, S_0^-$ is shown in a similar manner.

**Lemma 32** For $M_k^\pm$ from Lemma 11 holds
\[ \lim_{k \to k_0} \|M_k^\pm - M_k^0\| = 0 \]
in the corresponding induced operator norm.

**Proof:** Because of
\[ \text{curl}_x (c(y) \Phi_k (x, y) - c(y) \Phi_{k_0} (x, y)) = \text{grad}_x (\Phi_k (x, y) - \Phi_{k_0} (x, y)) \wedge c(y) \]
we consider the integral kernel
\[ G(x, y) = \frac{x - y}{r} - i k_0 K(x, y) + \frac{x - y}{4 \pi r^2} i (k - k_0) e^{ikr} - \frac{x - y}{r^2} K(x, y), \]
where $K(x, y)$ was introduced in (46). This leads to
\[ |G(x, y)| \leq M_2 |k - k_0||k_0| + M_1 |k - k_0||x - y|^{-1} \]
for all $x, y \in \mathbb{R}^3, x \neq y$. $M_1, M_2$ are independent of $k, k_0, x$ and $y$. Given $x_1, x_2, y \in \mathbb{R}^3$ and $r_1 = |x_1 - y| > 0, r_2 = |x_2 - y| > 0$, we can decompose
\[ G(x_1, y) - G(x_2, y) = \]
\[ = \left( \frac{x_1 - y}{r_1} - \frac{x_2 - y}{r_2} \right) i k_0 K(x_1, y) + i k_0 \frac{x_2 - y}{r_2} (K(x_1, y) - K(x_2, y)) + i (k - k_0) \left( \frac{x_1 - y}{4 \pi r_1^2} - \frac{x_2 - y}{4 \pi r_2^2} \right) e^{ikr_1} + \frac{x_2 - y}{4 \pi r_2^2} (e^{ikr_1} - e^{ikr_2}) + \frac{x_2 - y}{r_2} \left( \frac{x_1 - y}{r_1^2} - \frac{x_2 - y}{r_2^2} \right) K(x_1, y) - \frac{x_2 - y}{r_2^2} (K(x_1, y) - K(x_2, y)). \]

Using (47) and (49) we see that
\[ |K(x_1, y) - K(x_2, y)| \leq M_4 |k_0| + \frac{M_5}{r_1} |k - k_0| |x_1 - x_2|, \]

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\[ |G(x_1, y) - G(x_2, y)| \leq \left( D_1 |k_0|^2 + D_2 \frac{|k_0|}{r_1} + D_3 \frac{|k| + |k_0|}{r_2} \right) |k - k_0| |x_1 - x_2| \\
+ \left( \frac{D_4}{r_1 r_2} + \frac{D_5}{r_1^2} \right) |k - k_0| |x_1 - x_2|, \]

with some \( D_1, \ldots, D_5 \) independent of \( k, k_0, x_1, x_2 \) and \( y \). Given \( x_1, x_2 \in \mathbb{R}^3, y \in \Gamma, \) \( r_1 = |x_1 - y| > 0, r_2 = |x_2 - y| > 0 \) and \( 2|x_1 - x_2| \leq |x_1 - y| \), we obtain by applying Lemma 30 with the special choice

\[
C_1 = M_1 |k - k_0|, \quad C_2 = M_2 |k - k_0| |k_0|, \\
C_3 = D_1 |k - k_0| |k_0|^2, \quad C_4 = (D_2 |k_0| + 2D_3(|k| + |k_0|)) |k - k_0|, \\
C_5 = (2D_4 + D_5)|k - k_0|,
\]

the estimate

\[
\|M^+_k c - M^+_k \|_{0, \alpha, D^+} \leq C_0 \|c\|_{0, \alpha, \Gamma}.
\]

\( k - k_0 \) enters each \( C_i, i \in \{1, \ldots, 5\} \) in a linear fashion. Therefore (45) provides \( \lim_{k \to k_0} \|M^-_k - M^-_{k_0}\| = 0 \) in the induced operator norm. The convergence \( \lim_{k \to k_0} \|M^-_k - M^-_{k_0}\| = 0 \) is shown in a similar manner.  

**Lemma 33** For \( T^\pm_k \) from Lemma 11 holds

\[
\lim_{k \to k_0} \|T^\pm_k - T^\pm_{k_0}\| = 0
\]
in the corresponding induced operator norm.

**Proof:** For \( c \in T^0_\alpha(\Gamma) \) and \( x \in D^\pm \) define

\[
(F^\pm_k c)(x) = \int_\Gamma (\text{Div} c)(y) \text{grad}_x \Phi_k(x, y) ds(y).
\]

Because of \( c \in T^0_\alpha(\Gamma) \), i.e. \( \text{Div} c \in C^\alpha_\alpha(\Gamma) \), \( F^\pm_k \) can be extended in a Hölder continuous fashion up to \( \bar{D}^\pm \). Using the properties of \( G(x, y) = \text{grad}_x \Phi_k(x, y) \to \Phi_k(x, y) \) mentioned in the proof of Theorem 32 we obtain \( \lim_{k \to k_0} \|F^\pm_k - F^\pm_{k_0}\| = 0 \). Applying Lemma 31 we see that \( \lim_{k \to k_0} \|k^2 S^\pm_k - k_0^2 S^\pm_{k_0}\| = 0 \), i.e. \( \lim_{k \to k_0} \|T^\pm_k - T^\pm_{k_0}\| = 0 \).

Now we are in the position to consider the integral operators for the potentials on the boundary \( \Gamma \).

**Lemma 34** For the operators \( M_k, T_k : T^0\alpha(\Gamma) \to T^0\alpha(\Gamma) \) from Definition 2 holds

\[
\lim_{k \to k_0} \|M_k - M_{k_0}\| = 0, \quad \lim_{k \to k_0} \|T_k - T_{k_0}\| = 0
\]
in the induced operator norm.
Proof: Lemma 32 and Lemma 33 provide

\[ \| M_k c - M_{k_0} c \|_{0, \Gamma} \leq d_1(k) \| c \|_{d_0, \Gamma} \quad \| T_k c - T_{k_0} c \|_{0, \Gamma} \leq d_2(k) \| c \|_{d_0, \Gamma}, \]

where \( d_1(k) \) and \( d_2(k) \) are independent of \( c \in T^{\omega}_\infty(\Gamma) \) and \( d_1(k) \), \( d_2(k) = o(1) \) as \( k \to k_0 \).

For \( x \in D^- \) define

\[
F(x) = \text{curl} \int_\Gamma (c(y) \Phi_k(x, y)) \, ds(y)
\]

\[
= \text{grad} \int_\Gamma (\text{Div } c)(y) \Phi_k(x, y) \, ds(y) + k^2 \int_\Gamma c(y) \Phi_k(x, y) \, ds(y).
\]

In connection with Lemma 22 from Appendix A we see

\[
(\text{Div } M_k c)(x) = -n(x) \cdot \int_\Gamma (\text{Div } c)(y) \text{grad } \Phi_k(x, y) \, ds(y) - (\text{Div } a)(x)
\]

\[
- k^2 n(x) \cdot \int_\Gamma c(y) \Phi_k(x, y) \, ds(y).
\]

Therefore Lemma 33 leads to \( \| \text{Div } M_k c - \text{Div } M_{k_0} c \|_{0, \Gamma} \leq d_3(k) \| c \|_{d_0, \Gamma} \) where \( d_3(k) \) is independent of \( c \in T^{\omega}_\infty(\Gamma) \) and \( d_3(k) = o(1) \) as \( k \to k_0 \). Thus \( \lim_{k \to k_0} \| M_k - M_{k_0} \| = 0 \) in the induced operator norm in \( T^{\omega}_\infty(\Gamma) \).

In order to show convergence for \( T_k \) we define for \( x \in D^- \)

\[
G(x) = \text{curl} \int_\Gamma (\text{Div } c)(y) \text{grad } \Phi_k(x, y) \, ds(y) + k^2 \text{curl} \int_\Gamma c(y) \Phi_k(x, y) \, ds(y).
\]

Using again Lemma 22 from Appendix A we conclude for \( x \in \Gamma \)

\[
(\text{Div } T_k c)(x) = -n(x) \cdot G(x) = -k^2 n(x) \cdot \int_\Gamma \text{curl } x (c(y) \Phi_k(x, y)) \, ds(y).
\]

Lemma 32 states that \( \| \text{Div } T_k c - \text{Div } T_{k_0} c \|_{0, \Gamma} \leq d_4(k) \| c \|_{d_0, \Gamma} \) with \( d_4(k) \) being independent of \( c \in T^{\omega}_\infty(\Gamma) \), \( d_4(k) = o(1) \) as \( k \to k_0 \). Finally we arrive at \( \lim_{k \to k_0} \| T_k - T_{k_0} \| = 0 \) in the induced operator norm in \( T^{\omega}_\infty(\Gamma) \).

Lemma 35 Let \( J \subset C^1(\mathbb{R}^3) \) and \( D^J \) open, bounded, \( D^J \subset D^+ \), \( \partial D^J \) being \( C^2 \), such that \( \text{supp}(J) \subset D^J \). The volume potential

\[
V_k(x) = \int_{D^J} J(y) \Phi_k(x, y) \, dv(y)
\]

satisfies \( V_k, \text{curl } V_k \in C^1(\mathbb{R}^3) \cap C^{0, \alpha}(\mathbb{R}^3) \) and

\[
\lim_{k \to 0} \| V_k - V_0 \|_{0, D^+} = 0, \quad \lim_{k \to 0} \| \text{curl } V_k - \text{curl } V_0 \|_{0, D^+} = 0.
\]
Proof: Due to the assumptions we have $J \in C^1(\mathbb{R}^3) \cap C^0(\mathbb{R}^3)$ and the regularity of $V_k$ can be proven using Theorem 8.1 from [2].

Analysing the convergence behaviour we can write

$$V_k(x) - V_0(x) = \int_{D^J} J(y)(\Phi_k(x, y) - \Phi_0(x, y))dv(y) = \int_{D^J} J(y)K(x, y)dv(y),$$

where $K(x, y)$ is given by (46) and fulfills due to (48), (49) for $x, x_1, x_2 \in \mathbb{R}^3, y \in D^J$

$$|K(x, y)| \leq M_1|k|, \quad (50)$$

$$|K(x_2, y) - K(x_1, y)| \leq M_2|k|\frac{|x_1 - x_2|}{|x_1 - y|}, \quad (51)$$

where $M_1$ is independent of $k, x, y$ and $M_2$ is independent of $x_1, x_2, y, k$. From (50) we immediately get

$$\|V_k - V_0\|_{\infty, D^+} \leq M_3|k|,$$

$M_3$ being independent of $k$. Considering $B = \{x \in \mathbb{R}^3 | x_1 - y < 1\}$ we conclude

$$\int_{D^J} \frac{1}{|x_1 - y|}dv(y) \leq \int_B \frac{1}{|x_1 - y|}dv(y) + \int_{D^J \setminus B} \frac{1}{|x_1 - y|}dv(y) \leq 2\pi + \text{vol}(D^J),$$

and by (52)

$$|(V_k(x_2) - V_0(x_2)) - (V_k(x_1) - V_0(x_1))| \leq \|J\|_\infty \int_{D^J}|K(x_2, y) - K(x_1, y)|dv(y) \leq M_4|k| |x_1 - x_2|$$

where $M_4$ is independent of $x_1, x_2$ and $k$. Therefore we have

$$\lim_{k \to 0} \|V_k - V_0\|_{0^*, D^+} = 0.$$

By Lemma 28 from Appendix A we know that

$$(\text{curl } V_k)(x) = \int_{D^J} (\text{curl } J)(y)\Phi_k(x, y)(x, y)dv(y).$$

Applying the same arguments as above provides

$$\lim_{k \to 0} \|\text{curl } V_k - \text{curl } V_0\|_{0^*, D^+} = 0.$$
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