The Tangent Space at a Special Symplectic Instanton Bundle on $\mathbb{P}_{2n+1}$

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Introduction

Mathematical instanton bundles on \( \mathbb{P}_3 \) have their analogues in rank-2n instanton bundles on odd dimensional projective spaces \( \mathbb{P}_{2n+1} \). The families of special instanton bundles on these spaces, which generalize the special 't Hooft bundles on \( \mathbb{P}_3 \), were constructed and described in [OS] and [ST]. More general instanton bundles have recently been constructed in [AO2]. Let \( MI_{2n+1}(k) \) denote the moduli space of all instanton bundles on \( \mathbb{P}_{2n+1} \) with second Chern class \( c_2 = k \). In order to obtain a first impression of this space it is important to know its tangent dimension \( h^1 \text{End}(\mathcal{E}) \) at a stable bundle \( \mathcal{E} \) and the dimension \( h^2 \text{End}(\mathcal{E}) \) of the space of obstructions to smoothness.

In this paper we prove that for a special symplectic bundle \( \mathcal{E} \in MI_{2n+1}(k) \)

\[
h^2 \text{End}(\mathcal{E}) = (k - 2)^2 \binom{2n-1}{2}.
\]

Such bundles are stable by [AO1]. So for \( n \geq 2 \) the situation is quite different to that of \( \mathbb{P}_3 \), where this number becomes zero, which was shown in [HN]. Since \( H^i \text{End}(\mathcal{E}) = 0 \) for \( i \geq 3 \), our result and the Hirzebruch–Riemann–Roch formula, see Remark 2.4,

\[
h^1 \text{End}(\mathcal{E}) - h^2 \text{End}(\mathcal{E}) = -k^2 \binom{2n-1}{2} + k(8n^2) + 1 - 4n^2
\]

give

\[
h^1 \text{End}(\mathcal{E}) = 4(3n - 1)k + (2n - 5)(2n - 1).
\]

Therefore the dimension of \( MI_{2n+1}(k) \) grows linearly in \( k \), whereas the difference \( h^1 \text{End}(\mathcal{E}) - h^2 \text{End}(\mathcal{E}) \) becomes negative for \( n \geq 2 \) and grows quadratically in \( k \). A more important consequence, however, is that in general \( MI_{2n+1}(k) \) cannot be smooth at special symplectic bundles, see section 4 and [AO2].

In order to derive our result we fix a 2-dimensional vector space \( U \) and consider the natural action of \( SL(2) \) on \( \mathbb{P}_{2n+1} = \mathbb{P}(U \otimes S^n U) \) as in [ST]. The special instanton bundles are related to the \( SL(2) \)-homomorphisms \( \beta \), see 1.4, and are \( SL(2) \)-invariant. We prove that there is an isomorphism of \( SL(2) \)-representations

\[
H^2(\text{End} \mathcal{E}) \cong S^{k-3}(U) \otimes S^{k-3}(U) \otimes S^2(U \otimes S^{n-2}U).
\]

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Notation

- Throughout the paper \( K \) denotes an algebraically closed ground field of characteristic 0.
- \( U \) denotes a 2-dimensional \( K \)-vector space, \( S_n = S^n U \) its nth symmetric power and \( V_n = U \otimes S_n \).
- There is the natural exact sequence of \( GL(U) \)-equivariant maps for any \( k, n \geq 1 \)
  \[
  0 \to \Lambda^2 U \otimes S_{k-1} \otimes S_{n-1} \xrightarrow{\beta} S_k \otimes S_n \xrightarrow{\mu} S_{k+n} \to 0
  \]
  where \( \mu \) is the multiplication map and \( \beta \) is defined by \( (s \wedge t) \otimes f \otimes g \mapsto sf \otimes tg - tf \otimes sg \). This sequence splits and leads to the Clebsch–Gordan decomposition of \( S_k \otimes S_n \) by induction. When we tensorize the sequence with \( U \) we obtain the exact sequence
  \[
  0 \to \Lambda^2 U \otimes S_{k-1} \otimes V_{n-1} \xrightarrow{\beta} S_k \otimes V_n \xrightarrow{\mu} V_{k+n} \to 0.
  \]
- \( \mathbb{P} = \mathbb{P}_{2n+1} = \mathbb{P} V_n \) is the projective space of one dimensional subspaces of \( V_n \).
- The terms vector bundle and locally free sheaf are used synonymously.
- \( \mathcal{O}(d) \) denotes the invertible sheaf of degree \( d \) on \( \mathbb{P} \), \( \mathcal{O}^p \) the locally free sheaf of differential \( p \)-forms on \( \mathbb{P} \), such that \( \mathcal{O}^p(\mathbb{P}) = \Lambda^p \mathcal{Q}^\mathbb{P} \) where \( \mathcal{Q} = \mathcal{T}(-1) \) is the canonical quotient bundle on \( \mathbb{P} \).
- We use the abbreviations \( \mathcal{F}(d) = \mathcal{F} \otimes \mathcal{O}(d) \) for any sheaf \( \mathcal{F} \) of \( \mathcal{O} \)-modules on \( \mathbb{P} \), \( H^i \mathcal{F} = H^i(\mathcal{F}) = H^i(\mathbb{P} \mathcal{F}) \), \( h^i \mathcal{F} = \dim H^i \mathcal{F} \). If \( E \) is a finite dimensional \( K \)-vector space, \( E \otimes \mathcal{O} \) denotes the sheaf of sections of the trivial bundle \( \mathbb{P} \times E \), and \( E \otimes \mathcal{F} = (E \otimes \mathcal{O}) \otimes \mathcal{O} \mathcal{F} \). We also write \( m \mathcal{F} = K^m \otimes \mathcal{F} \).
- We use the Euler sequence \( 0 \to \Omega^1(\mathbb{P}) \to V_n^\vee \otimes \mathcal{O} \to \mathcal{O}(1) \to 0 \) and the derived sequences in its Koszul complex \( 0 \to \mathcal{O}^p(\mathbb{P}) \to \Lambda^p V_n^\vee \otimes \mathcal{O} \to \mathcal{O}^{p-1}(\mathbb{P}) \to 0 \) without extra mentioning.
- \( Ext^i(\mathcal{F}, \mathcal{G}) = Ext^i(\mathbb{P}, \mathcal{F} \mathcal{G}) \) for any two \( \mathcal{O} \)-modules \( \mathcal{F} \) and \( \mathcal{G} \).
1 Instanton bundles

1.1 An instanton bundle on $\mathbb{P} = \mathbb{P}_{2n+1}$ with instanton number $k$ or a $k$-instanton is an algebraic vector bundle $E$ on $\mathbb{P}$ satisfying:

(i) $E$ has rank $2n$ and Chern polynomial $c(E) = (1 - h^2)^{-k} = 1 + kh^2 + \ldots$.

(ii) $E$ has natural cohomology in the range $-2n - 1 \leq d \leq 0$, that is for any $d$ in that range $h^d E(d) \neq 0$ for at most one $i$.

A $k$-instanton bundle $E$ is called symplectic if there is an isomorphism $E \cong E^\vee$ satisfying $\varphi^\vee = -\varphi$. In this case the spaces $A$ and $B$ below are Serre-duals of each other, since $H^{2n}(E(-2n - 1))^\vee \cong H^1 E^\vee(-1) \cong H^1 E(-1)$.

**Remark:** In the original definition in [OS] the additional conditions

(iii) $E$ is simple, that is $\text{Hom}(E, E) = K$,

(iv) the restriction of $E$ to a general line is trivial

are imposed. It was shown in [AO1] that (iii) is already a consequence of (i) and (ii). Condition (iv) seems to be independent but we do not need it in this paper. By [ST] special instantons satisfy (iv).

1.2 Let now $A, B, C$ be vector spaces of dimensions $k, k, 2n(k - 1)$ respectively. A pair of linear maps

$$A \xrightarrow{a} B \otimes \Lambda^2 V_n^\vee, \quad B \otimes V_n^\vee \xrightarrow{b} C$$

corresponds to a pair of sheaf homomorphisms

$$A \otimes \mathcal{O}(-1) \xrightarrow{\tilde{a}} B \otimes \Omega^1(1), \quad B \otimes \Omega^1(1) \xrightarrow{\tilde{b}} C \otimes \mathcal{O}.$$  

Here $\tilde{a}$ is the composition of the induced homomorphisms $A \otimes \mathcal{O}(-1) \rightarrow B \otimes \Lambda^2 V_n^\vee \otimes \mathcal{O}(-1) \rightarrow B \otimes \Omega^1(1)$ and $\tilde{b}$ is the composition of the induced homomorphismus $B \otimes \Omega^1(1) \rightarrow B \otimes V_n^\vee \otimes \mathcal{O} \rightarrow C \otimes \mathcal{O}$. Conversely, $a$ and $b$ are determined by $\tilde{a}$ and $\tilde{b}$ respectively as $H^0(\tilde{a}(1))$ and $H^0(\tilde{b}^\vee)^\vee$. Moreover, the sequence

$$A \otimes \mathcal{O}(-1) \xrightarrow{\tilde{a}} B \otimes \Omega^1(1) \xrightarrow{\tilde{b}} C \otimes \mathcal{O} \quad (1)$$

is a complex if and only if the induced sequence

$$A \rightarrow B \otimes \Lambda^2 V_n^\vee \rightarrow C \otimes V_n^\vee$$

is a complex. We say that (1) is a **monad** if it is a complex and if in addition $\tilde{a}$ is a subbundle and $\tilde{b}$ is surjective.
Proposition 1.3 The cohomology sheaf $\mathcal{E} = \text{Ker } \hat{b}/\text{Im } \hat{a}$ of a monad (1) is a $k$-instanton and conversely any $k$-instanton is the cohomology of a monad (1). Moreover, the spaces $A, B, C$ of such a monad can be identified with $H^{2n}\mathcal{E}(-2n-1)$, $H^1\mathcal{E}(-1)$, $H^1\mathcal{E}$ respectively.

Sketch of a proof: if a monad (1) is given it is easy to derive the properties of the definition. Conversely using Beilinson's spectral sequence, Riemann–Roch and in particular (ii), one obtains a monad with the identification of the vector spaces as in [OS]. The map $b$ is then nothing but the natural map $H^1\mathcal{E}(-1) \otimes V_n^\vee \to H^1\mathcal{E}$ and the map $a$ is given as the composition of the cup product

$$H^{2n}\mathcal{E}(-2n-1) \otimes \Lambda^2 V_n \to H^{2n}\mathcal{E} \otimes \Omega^{2n-1}(-1)$$

and the natural isomorphisms

$$H^{2n}\mathcal{E} \otimes \Omega^{2n-1}(-1) \cong H^{2n-1}\mathcal{E} \otimes \Omega^{2n-2}(-1) \cong \ldots \cong H^1\mathcal{E}(-1)$$

arising from the Koszul sequences and condition (ii), see [V] in case of $\mathbb{P}_3$.

1.4 Existence and special instanton bundles: Using the special structure $V_n = U \otimes S^n U$ and the Clebsch–Gordan type exact sequence

$$0 \to \Lambda^2 U \otimes S_{k-2} \otimes V_{n-1} \xrightarrow{\beta} S_{k-1} \otimes V_n \xrightarrow{\alpha} V_{k+n-1} \to 0,$$

see notation, we define the special homomorphism

$$S_{k-1}^\vee \otimes \Omega^1(1) \xrightarrow{b} \Lambda^2 U^\vee \otimes S_{k-2}^\vee \otimes V_{n-1}^\vee \otimes \mathcal{O}$$

by $b = \beta^\vee$. We denote $\mathcal{N} = \text{Ker } (\hat{b})$. It was shown in [ST] that $\hat{b}$ is surjective and that

$$H^0\mathcal{N}(1) \subset S_{k-1}^\vee \otimes H^0\Omega^1(2)$$

can be identified with a canonical injective $GL(U)$–homomorphism

$$S_{2n+k-1}^\vee \otimes \Lambda^2 U^\vee \cong S_{k-1}^\vee \otimes \Lambda^2 V_n^\vee,$$

dual to the map

$$S_{k-1} \otimes \Lambda^2 V_n \to S_{2n+k-1} \otimes \Lambda^2 U$$

which is defined by $f \otimes (s \otimes g) \wedge (t \otimes h) \mapsto (fgh) \otimes (s \wedge t)$.

In order to construct instanton bundles we have to find $k$-dimensional subspaces $A \subset S_{2n+k-1}^\vee \otimes \Lambda^2 U^\vee \subset S_{k-1}^\vee \otimes \Lambda^2 V_n^\vee$

such that the induced homomorphism $\hat{a}$ is a subbundle. By [ST], Lemma 3.7.1, this is the case exactly when $\mathbb{P}A \subset \mathbb{P}(S_{2n+k-1}^\vee)$ does not meet the secant variety $\text{Sec}_n(C_{2n+k-1})$ of $(n-1)$-dimensional secant planes of the canonical rational curve
$C_{2n+k-1}$ of $\mathbb{P}S_{2n+k-1}^\vee$, given by $u \mapsto u^{2n+k-1}$. By dimension reasons such subspaces exist, [ST], 3.7, and hence instanton bundles exist.

A $k$–instanton bundle $\mathcal{E}$ is called special if the map $b$ of its monad is isomorphic to the $GL(U)$–homomorphism $\beta^\vee$, that is if there are isomorphisms $\varphi$ and $\psi$ and $g \in GL(V_n)$ with the commutative diagram

$$
\begin{array}{ccc}
H^1\mathcal{E}(-1) \otimes V_n^\vee & \xrightarrow{b} & H^1\mathcal{E} \\
\varphi \otimes g^\vee \downarrow & & \downarrow \psi \\
S_{k-1}^\vee \otimes V_n^\vee & \xrightarrow{\beta^\vee} & \Lambda^2 V_n^\vee \otimes S_{k-2}^\vee \otimes V_n^\vee.
\end{array}
$$

Whereas in [ST] the family of all special $k$–instanton bundles was described, examples of different types of general instanton bundles were found in [AO2].

**Remark 1.5** If $\mathcal{E}$ is special and symplectic then, in addition to the special $GL(U)$–homomorphism $b = \beta^\vee$ of its monad, the map $a$ is given by an element $\alpha \in S_{2n+2k-2}^\vee$ as $a = \kappa \circ \tilde{\alpha}$ where $S_{k-1} \xrightarrow{\tilde{\alpha}} S_{2n+k-1}^\vee$ is defined by $\tilde{\alpha}(f)(g) = \alpha(fg)$ and $S_{2n+k-1}^\vee \xrightarrow{\alpha} S_{k-1}^\vee \otimes \Lambda^2 V_n^\vee$ is as above, [ST], 4.3 and 5.8. In particular $a$ is a $GL(U)$–homomorphism, too, and can be represented by a persymmetric matrix.

**Remark 1.6** It is shown in [AO1] that special symplectic instanton bundles are stable in the sense of Mumford–Takemoto.
2 Representing $\text{Ext}^2(\mathcal{E}, \mathcal{E})$

**Proposition 2.1** Let $\mathcal{E}$ be a symplectic $k$-instanton and let $\mathcal{N}$ be the kernel of the monad (1). Then $\text{Ext}^2(\mathcal{E}, \mathcal{E}) \cong H^2(\mathcal{N} \otimes \mathcal{N})$.

Proof: The monad (1) gives rise to the exact sequences

$$0 \rightarrow \mathcal{N} \rightarrow B \otimes \Omega^1(1) \overset{\hat{b}}{\rightarrow} C \otimes \mathcal{O} \rightarrow 0$$

(2)

and

$$0 \rightarrow A \otimes \mathcal{O}(-1) \rightarrow \mathcal{N} \rightarrow \mathcal{E} \rightarrow 0.$$  

(3)

After tensoring we have the exact sequences

$$0 \rightarrow A \otimes \mathcal{N}(-1) \rightarrow \mathcal{N} \otimes \mathcal{N} \rightarrow \mathcal{E} \otimes \mathcal{N} \rightarrow 0$$  

(4)

and

$$0 \rightarrow A \otimes \mathcal{E}(-1) \rightarrow \mathcal{N} \otimes \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{E} \rightarrow 0.$$  

(5)

Since $\mathcal{E} \cong \mathcal{E}^\vee$ we obtain $\text{Ext}^2(\mathcal{E}, \mathcal{E}) \cong H^2(\mathcal{E} \otimes \mathcal{E})$. Sequence (2) implies $h^2 \mathcal{N}(-1) = h^2 \mathcal{N}(-1) = 0$ and from this and (3) also $h^2 \mathcal{E}(-1) = h^2 \mathcal{E}(-1) = 0$. Now sequences (4) and (5) yield isomorphisms $H^2(\mathcal{E} \otimes \mathcal{E}) \cong H^2(\mathcal{N} \otimes \mathcal{E}) \cong H^2(\mathcal{N} \otimes \mathcal{N})$. \hfill $\Box$

2.2 In order to represent $H^2(\mathcal{N} \otimes \mathcal{N})$ we note that the sequence (2) is part of the exact diagram

$$\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{N} \\
\downarrow & & \downarrow \\
0 & \rightarrow & B \otimes \Omega^1(1) \overset{\hat{b}}{\rightarrow} C \otimes \mathcal{O} \rightarrow 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & H \otimes \mathcal{O} \\
\downarrow & & \downarrow \\
B \otimes \mathcal{O}(1) & = & B \otimes \mathcal{O}(1) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$  

(6)

where $H$ is the kernel of the operator $\hat{b}$, which is surjective because $\hat{b}$ is surjective.

The left–hand column of (6) gives us after tensoring by $\Omega^1(1)$

$$B \otimes H^0 \Omega^1(2) \cong H^1(\mathcal{N} \otimes \Omega^1(1)) \oplus H^2(\mathcal{N} \otimes \Omega^1(1)) = 0.$$  

(7)

Since $\hat{b}$ is the Beilinson representation of $\mathcal{N}$, we have the commutative diagram

$$\begin{array}{ccc}
H^1 \mathcal{N}(-1) \otimes H^0 \mathcal{O}(1) & \xrightarrow{\text{can}} & H^1 \mathcal{N} \\
\parallel & & \parallel \\
B \otimes V^\vee & \xrightarrow{\hat{b}} & C
\end{array}$$  

(8)
Moreover, \( \delta \) in (7) coincides also with cup:

\[
\begin{align*}
B \otimes H^0\Omega^1(2) & \xrightarrow{\delta} H^1(\mathcal{N} \otimes \Omega^1(1)) \\
\cong & H^1(\mathcal{N}(\mathcal{N} \otimes \Omega^1(1)) \\
H^1(\mathcal{N}(-1) \otimes H^0\Omega^1(2)
\end{align*}
\]

(9)

Tensoring the top row of (6) with \( \mathcal{N} \) and using (7) we obtain the following diagram with exact row:

\[
\begin{array}{c}
0 \rightarrow H^1(\mathcal{N} \otimes \mathcal{N}) \rightarrow B \otimes H^1(\mathcal{N} \otimes \Omega^1(1)) \rightarrow C \otimes H^1(\mathcal{N}) \rightarrow H^2(\mathcal{N} \otimes \mathcal{N}) \rightarrow 0 \\
\| \| \\
B \otimes B \otimes \Lambda^2 V^\vee_n \overset{\Phi}{\rightarrow} C \otimes C.
\end{array}
\]

(10)

It follows that

\[
H^2(\mathcal{N} \otimes \mathcal{N}) = \text{Coker}(\Phi) = \text{Ker}(\Phi^\vee).
\]

(11)

**Lemma 2.3** The induced operator \( \Phi \) is the composition \( B \otimes B \otimes \Lambda^2 V^\vee_n \overset{\text{id} \otimes \sigma}{\rightarrow} B \otimes B \otimes V^\vee_n \otimes V^\vee_n \overset{\Phi}{\rightarrow} C \otimes C \), where \( \sigma \) denotes the canonical desymmetrization.

**Proof:** The computation of \( \Phi \) is achieved by the diagram

\[
\begin{array}{c}
B \otimes B \otimes \Lambda^2 V^\vee_n \xrightarrow{id_{B \otimes B} \otimes \sigma} B \otimes B \otimes V^\vee_n \otimes V^\vee_n \\
\| \| \\
B \otimes H^1(\mathcal{N} \otimes \Omega^1(1)) \xrightarrow{\text{id} \otimes H^0(\iota(1))} B \otimes H^1(\mathcal{N} \otimes \Omega^1(1)) \\
\| \| \\
B \otimes H^1(\mathcal{N}(\mathcal{N} \otimes \Omega^1(1)) \xrightarrow{id_{B \otimes V^\vee_n} \otimes \text{cup}} B \otimes V^\vee_n \otimes H^1(\mathcal{N}) \\
\| \| \\
C \otimes H^1(\mathcal{N}) \xrightarrow{\text{id} \otimes H^1(\mathcal{N}) \otimes \iota} B \otimes V^\vee_n \otimes H^1(\mathcal{N}) \\
\| \| \\
C \otimes C \xrightarrow{b \otimes \text{id}_C} B \otimes V^\vee_n \otimes C
\end{array}
\]

In this diagram \( \iota \) denotes the canonical inclusion \( \Omega^1(1) \hookrightarrow V^\vee_n \otimes \mathcal{O} \), and up to \( \Lambda^2 V^\vee_n \cong H^0\Omega^1(2) \) and \( V^\vee_n \cong H^0\mathcal{O}(1) \) the map \( \sigma \) can be identified with \( H^0(\iota(1)) \). Therefore, the square I is commutative. Square II is a canonically induced diagram of cup–operations and commutative using \( B \cong H^1(\mathcal{N}(-1)) \). The triangle III is induced by the
commutative triangle

\[
\begin{array}{c}
B \otimes \mathcal{N} \otimes \Omega^1(1) \xrightarrow{id \otimes b} B \otimes V^\vee \otimes \mathcal{N} \\
\downarrow b \otimes id \quad \check{\otimes} \quad b \otimes id \\
C \otimes \mathcal{N}
\end{array}
\]

and hence commutative, and the commutativity of IV results just from the identification \( H^1 \mathcal{N} \cong C \). Now by definition the composition of the left-hand column is \( \Phi \) and the composition of the right-hand column is \( id_B \otimes id_{V^\vee \otimes b} \) since \( b \) is defined by (8). It follows that

\[
\Phi = (b \otimes id_C) \circ (id_B \otimes id_{V^\vee \otimes b}) \circ (id_{B \otimes \mathcal{N}} \otimes \sigma) = (b \otimes b) \circ (id \otimes \sigma).
\]

**Remark 2.4** If \( \mathcal{E} \) is a \( k \)-instanton bundle it is easily checked that \( h^i \mathcal{E}(d) = h^i \mathcal{E}^\vee(d) = 0 \) for \( i \geq 2 \) and \( d \geq -1 \). Using \( \mathcal{E}^\vee \otimes \mathcal{N} \) again it follows that \( Ext^i(\mathcal{E}, \mathcal{E}) = H^i(\mathcal{E}^\vee \otimes \mathcal{N}) = 0 \) for \( i \geq 3 \). This and the Riemann–Roch formula, which can also ad hoc be derived from the monad representation, give

\[
h^1(\mathcal{E}^\vee \otimes \mathcal{E}) - h^2(\mathcal{E}^\vee \otimes \mathcal{E}) = -k^2 \binom{2n-1}{2} + 8kn^2 - 4n^2 + 1.
\]
3 Determination of Ext²(ℰ, ℰ)

We are now able to determine Ext²(ℰ, ℰ) as a GL(2)-representation space in case of a special instanton bundle. In that case b is the dual of the operator β : Λ²U ⊗ Sₖ₋₂ ⊗ Vₙ₋₁ → Sₖ₋₁ ⊗ Vₙ, see notation or 1.4. Then Φᵥ is the composition of β ⊗ β and the multiplication map Vₙ ⊗ Vₙ → Λ²Vₙ. In order to simplify we choose a fixed basis s, t ∈ U and the isomorphism Λ²U ≅ k given by s ∧ t. Then

\[ S_{k-2} ⊗ S_{k-2} ⊗ V_{n-1} ⊗ V_{n-1} \xrightarrow{\Phi'_v} S_{k-1} ⊗ S_{k-1} ⊗ Λ²V_n \]

is explicitly given by

\[ \Phi'_v(g ⊗ g' ⊗ v ⊗ v') = sg ⊗ sg' ⊗ (tv ∧ tv') - sg ⊗ tg' ⊗ (tv ∧ sv') - tg ⊗ sg' ⊗ (sv ∧ tv') + tg ⊗ tg' ⊗ (sv ∧ sv'). \]

In order to determine the kernel of Φᵥ we consider the GL(U)-homomorphism

\[ S_{k-3} ⊗ S_{k-3} ⊗ V_{n-2} ⊗ V_{n-2} \xrightarrow{\epsilon'} S_{k-2} ⊗ S_{k-2} ⊗ V_{n-1} ⊗ V_{n-1} \]

defined similarly by

\[ \epsilon'(f ⊗ f' ⊗ u ⊗ u') = sf ⊗ sf' ⊗ tu ⊗ tu' - sf ⊗ tf' ⊗ su ⊗ tu' - tf ⊗ sf' ⊗ tu ⊗ su' + tf ⊗ tf' ⊗ su ⊗ su'. \]

Up to the order of factors the map ε' is the tensor product β' ⊗ β' where β' : Sₖ₋₃ ⊗ Vₙ₋₂ → Sₖ₋₂ ⊗ Vₙ₋₁ is defined as β. Hence, ε' is injective. Finally, we define ε as the composition

\[ S_{k-3} ⊗ S_{k-3} ⊗ S²V_{n-2} \xrightarrow{id ⊗ \epsilon'} S_{k-3} ⊗ S_{k-3} ⊗ V_{n-2} ⊗ V_{n-2} \xrightarrow{\epsilon'} S_{k-2} ⊗ S_{k-2} ⊗ V_{n-1} ⊗ V_{n-1} \]

where ε is the canonical desymmetrization. Then also ε is injective.

**Proposition 3.1** \((S_{k-3} ⊗ S_{k-3} ⊗ S²V_{n-2}, ε)\) is the kernel of \(Φ'_v\).

**Proof:** A straightforward computation shows that Im(ε) ⊂ Ker(Φᵥ). In order to show equality we reduce Ker(Φᵥ) modulo Im(ε) using canonical bases of the vector spaces. A more elegant proof using Clebsch–Gordan decompositions seems much harder to achieve. Let us denote the bases as follows:

- basis of \(S_{k-3}\): \(e_α = s^{k-3-α}t^α\) \(0 ≤ α ≤ k - 3\)
- basis of \(S_{k-2}\): \(f_α = s^{k-2-α}r^α\) \(0 ≤ α ≤ k - 2\)
- basis of \(S_{k-1}\): \(g_α = s^{k-1-α}z^α\) \(0 ≤ α ≤ k - 1\)
- basis of \(V_{n-2}\): \(u_µ = s ⊗ s^{n-2-µ}t^µ\) \(0 ≤ µ ≤ n - 2\)
- basis of \(V_{n-1}\): \(x_µ = s ⊗ s^{n-1-µ}t^µ\) \(0 ≤ µ ≤ n - 1\)
- basis of \(V_n\): \(y_µ = s ⊗ s^{n-µ}t^µ\) \(0 ≤ µ ≤ n\)
- basis of \(V_n\): \(z_µ = t ⊗ s^{n-µ}t^µ\) \(0 ≤ µ ≤ n\)
For the basis \( f_\alpha \otimes f_\beta \otimes x_\mu \otimes x_\nu, f_\alpha \otimes f_\beta \otimes x_\mu \otimes x_\nu, f_\alpha \otimes f_\beta \otimes x_\mu \otimes x_\nu, f_\alpha \otimes f_\beta \otimes x_\mu \otimes x_\nu \) we use the index tuplets \((\alpha, \beta, \mu, \nu), (\alpha, \beta, \mu, \nu), (\alpha, \beta, \mu, \nu), (\alpha, \beta, \mu, \nu)\) respectively. The set of these indices will be ordered lexicographically with the additional assumption that always \( \mu < \nu \). Then, for example, \((\alpha, \beta, \mu, \nu) < (\alpha, \beta, \mu, \nu)\).

Accordingly, the coefficients of an element \( \xi \in S_{k-2} \otimes S_{k-2} \otimes V_{n-1} \otimes V_{n-1} \) will be denoted by \( c(\alpha, \beta, \mu, \nu), c(\alpha, \beta, \mu, \nu), c(\alpha, \beta, \mu, \nu), c(\alpha, \beta, \mu, \nu) \).

By the formula for \( c_I \) we obtain the

**Lemma 3.2** Let \( \xi \in S_{k-2} \otimes S_{k-2} \otimes V_{n-1} \otimes V_{n-1} \).

(i) The coefficient of \( \Phi^\vee(\xi) \) at the basis element \( g_\alpha \otimes g_\beta \otimes y_\mu \wedge y_\nu \) in \( S_{k-1} \otimes S_{k-1} \otimes \Lambda^2 V_n \) is

\[
- c(\alpha, \beta, \mu - 1, \nu - 1) - c(\alpha, \beta, \nu - 1, \mu - 1) - c(\alpha - 1, \beta, \mu, \nu - 1) + c(\alpha - 1, \beta, \nu, \mu) + c(\alpha - 1, \beta - 1, \mu, \nu) - c(\alpha - 1, \beta - 1, \nu, \mu).
\]

Here we agree that each of these coefficients is 0 if one of \( \alpha, \beta, \mu - 1, \nu - 1 \) \( \notin [0, k - 2] \) or if one of \( \mu, \nu, \mu - 1, \nu - 1 \) \( \notin [0, n - 1] \).

(ii) Analogous statements hold for the coefficient of \( \Phi^\vee(\xi) \) at \( g_\alpha \otimes g_\beta \otimes y_\mu \wedge y_\nu \) for \( \mu < \nu \) (without bars) and at \( g_\alpha \otimes g_\beta \otimes \bar{y}_\mu \wedge \bar{y}_\nu \) for \( \mu < \nu \) (with two bars).

**Lemma 3.3** Let the notation be as above. If \( \Phi^\vee(\xi) = 0 \) then:

(i) If \( c(\alpha, \beta, \mu, \nu) \) is the first non-zero coefficient of \( \xi \) (in the lexicographical order), then \( 0 < \mu \leq \nu \).

(ii) If \( c(\alpha, \beta, \mu, \nu) \) is the first non-zero coefficient of \( \xi \), then \( \mu \neq 0, \nu \neq 0 \).

(iii) \( c(\alpha, \beta, \mu, \nu) \) is never a first non-zero coefficient of \( \xi \).

(iv) If \( c(\alpha, \beta, \mu, \nu) \) is the first non-zero coefficient of \( \xi \), then \( 0 < \mu \leq \nu \).

Proof: (i) Let \( c(\alpha, \beta, \mu, \nu) \) be the first coefficient of \( \xi \). Then, by Lemma 3.2 the coefficient of \( 0 = \Phi^\vee(\xi) \) at \( g_\alpha \otimes g_\beta \otimes y_{\mu + 1} \wedge y_{\nu + 1} \) is

\[
0 = c(\alpha, \beta, \mu, \nu) - c(\alpha, \beta, \nu, \mu) - c(\alpha, \beta - 1, \mu, \nu + 1) + c(\alpha, \beta - 1, \nu, \mu + 1) - c(\alpha - 1, \beta, \mu + 1, \nu) + c(\alpha - 1, \beta, \nu + 1, \mu) - \ldots
\]

Since \( c(\alpha, \beta, \mu, \nu) \) is the first coefficient, only the first two in this formula could be non-zero because the others have smaller index in the lexicographical order. Hence

\[
c(\alpha, \beta, \mu, \nu) = c(\alpha, \beta, \nu, \mu).
\]
If $\mu > \nu$ then $c(\alpha, \beta, \nu, \mu)$ would be earlier and non-zero. Hence, $\mu \leq \nu$. Assume now that $\mu = 0$. The coefficient of $\Phi^v(\xi)$ of $g_\alpha \otimes g_{\beta+1} \otimes y_0 \wedge y_{\nu+1}$ is

$$0 = c(\alpha, \beta + 1, -1, \nu) - c(\alpha, \beta + 1, \nu, -1) - c(\alpha, \beta, -1, \nu + 1) + c(\alpha, \beta, \nu, 0) + \ldots$$

In this sum all but $c(\alpha, \beta, \nu, 0)$ are automatically zero because $(\alpha - 1, \beta, \ldots) \leq (\alpha, \beta, 0, \nu)$ and $-1$ occurs. Hence, $c(\alpha, \beta, 0, \nu) = c(\alpha, \beta, \nu, 0) = 0$, contradiction.

The statements (ii), (iii), (iv) are proved analogously. \hfill $\square$

Now we continue the proof of Proposition 3.1. We reduce an element $\xi \in Ker(\Phi^v)$ to $0 \ mod \ Im(\epsilon)$ using Lemma 3.3.

a) Assume that the first non-zero coefficient of $\xi$ is

$$c(\alpha, \beta, \mu, \nu).$$

Then by Lemma 3.3 $0 < \mu \leq \nu$. Then the element

$$\xi' = \xi - c(\alpha, \beta, \mu, \nu)c(e_\alpha \otimes e_\beta \otimes u_{\mu-1} \cdot u_{\nu-1})$$

belongs to $Ker(\Phi^v)$. We have

$$c(e_\alpha \otimes e_\beta \otimes u_{\mu-1} \cdot u_{\nu-1})$$
$$= f_\alpha \otimes f_\beta \otimes (x_\mu \otimes x_\nu + x_\nu \otimes x_\mu)$$
$$- f_\alpha \otimes f_{\beta+1} \otimes (x_{\mu-1} \otimes x_\nu + x_\nu \otimes x_{\mu-1})$$
$$- f_{\alpha+1} \otimes f_\beta \otimes (x_\mu \otimes x_{\nu-1} + x_\nu \otimes x_{\mu-1})$$
$$+ f_{\alpha+1} \otimes f_{\beta+1} \otimes (x_{\mu-1} \otimes x_{\nu-1} + x_{\nu-1} \otimes x_{\mu-1})$$

and therefore $\xi'$ is a sum of monomials of index $> (\alpha, \beta, \mu, \nu)$. Hence, we can assume that $\xi \ mod \ Im(\epsilon)$ has no coefficient with index $(\alpha, \beta, \mu, \nu)$.

b) By Lemma 3.3 we can assume that the first non-zero coefficient of $\xi$ has index $(\alpha, \beta, \mu, \nu)$ or $(\alpha, \beta, \bar{\mu}, \bar{\nu})$. In the first case we know by Lemma 3.3 that $0 < \mu, \nu$. When we consider again

$$\xi' = \xi - c(\alpha, \beta, \bar{\mu}, \bar{\nu})c(e_\alpha \otimes e_\beta \otimes u_{\mu-1} \cdot \bar{u}_{\nu-1})$$

we have $\Phi^v(\xi') = 0$ and $\xi'$ is a sum of monomials of index $> (\alpha, \beta, \mu, \nu)$. Hence, we may assume that $\xi \ mod \ Im(\epsilon)$ has $c(\alpha, \beta, \bar{\mu}, \bar{\nu})$ as first non-zero coefficient. Again by Lemma 3.3 $0 < \mu, \nu$ and

$$\xi' = \xi - c(\alpha, \beta, \bar{\mu}, \bar{\nu})c(e_\alpha \otimes e_\beta \otimes \bar{u}_{\mu-1} \cdot \bar{u}_{\nu-1})$$

is a sum of monomials of index $> (\alpha, \beta, \bar{\mu}, \bar{\nu})$.

This finally shows that $\xi = 0 \ mod \ Im(\epsilon)$.

This completes the proof of Proposition 3.1.
4 Conclusions

By Proposition 2.1, Proposition 3.1, (11) and Lemma 2.3 we have determined the space $Ext^2(\mathcal{E}, \mathcal{E})$. Together with Remark 2.4 we obtain

**Theorem 4.1** For any special symplectic $k$-instanton bundle $\mathcal{E}$ on $\mathbb{P}_{2n+1}$,

1. $Ext^2(\mathcal{E}, \mathcal{E}) \cong S_{k-3}^\vee \otimes S_{k-3}^\vee \otimes S^2V_{n-2}^\vee$

2. $\dim Ext^2(\mathcal{E}, \mathcal{E}) = (k - 2)^2(2n-1)$

3. $\dim Ext^1(\mathcal{E}, \mathcal{E}) = 4k(3n - 1) + (2n - 5)(2n - 1)$.

Let $MI_{2n+1}(k)$ denote the open part of the Maruyama scheme of semi-stable coherent sheaves on $\mathbb{P}_{2n+1}$ with Chern polynomial $(1 - h^2)^{-k}$ consisting of instanton bundles. By [AO1] any special symplectic instanton bundle $\mathcal{E}$ is stable. Therefore, $Ext^1(\mathcal{E}, \mathcal{E})$ can be identified with the tangent space of $MI_{2n+1}(k)$ at $\mathcal{E}$. In [AO2] deformations $\mathcal{E}'$ of special symplectic instanton bundles in $MI_{2n+1}(k)$ have been found for $n = 2$ and $k = 3, 4$ which satisfy $Ext^2(\mathcal{E}', \mathcal{E}') = 0$. This shows that in these cases there are components $MI'_{2n+1}(k)$ of $MI_{2n+1}(k)$ of the expected dimension $4(3n - 1)k + (2n - 5)(2n - 1)$ containing the set of special instanton bundles. In particular, see [AO2]:

for $k = 3, 4$ the moduli space $MI_5(k)$ is singular at least in special symplectic bundles.

However, in case $2n + 1 = 3$ we obtain the vanishing result of [HN]:

any special $k$-instanton bundle $\mathcal{E}$ on $\mathbb{P}_3$ satisfies $Ext^2(\mathcal{E}, \mathcal{E}) = 0$ and is a smooth point of $MI_3(k)$,

since any rank-2 instanton bundle is symplectic.
References


