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Karl-Heinz Küfer

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UNIVERSITÄT KAIERSLAUTERN
Fachbereich Mathematik
Erwin-Schrödinger-Straße
6750 Kaiserslautern

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A Simple Integral Representation for the Second Moments of Additive Random Variables on Stochastic Polyhedra

Karl-Heinz Küfer, University of Kaiserslautern, Erwin-Schrödinger-Straße, Post Box 3049, D-6750 Kaiserslautern

Summary:

Let $a_i$, $i := 1, \ldots, m$, be an i.i.d. sequence taking values in $\mathbb{R}^n$, whose convex hull is interpreted as a stochastic polyhedron $\mathcal{P}$. For a special class of random variables, which decompose additively relative to their boundary simplices, e.g., the volume of $\mathcal{P}$, simple integral representations of its first two moments are given in case of rotationally symmetric distributions in order to facilitate estimations of variances or to quantify large deviations from the mean.

1. Introduction

In their pioneering articles "Über die konvexe Hülle von $n$ zufällig gewählten Punkten I-II"[7,8], Renyi and Sulanke analyzed the expectation values of the number of edges, the area and the circumference of random convex polytopes generated by the convex hull of i.i.d. vectors on $\mathbb{R}^2$ for the first time. Their ideas and their techniques while treating random variables on stochastic polyhedra led to a variety of generalizing publications in different fields of research like the stochastic approximation theory of bounded convex sets, cf. Gruber[3] or Buchta[2] for a survey, or the stochastic complexity theory of algorithms, cf. Borgwardt[1]. Most of these articles exclusively deal with expectation values and don't give information about the concentration of the random variable's value around the mean. In many cases experimental evidence suggests that the distribution of the random variable is highly concentrated. For that reason it is of interest to obtain higher moments, especially the variance, of the variable investigated in order to quantify large deviations from the mean.

The aim of this paper is to give integral representations for the first two moments of a class of random variables on stochastic bounded polyhedra on $\mathbb{R}^n$, $n \geq 2$, in order to facilitate estimations of variances.

To explain our stochastic model precisely, let $a_1, \ldots, a_m$, $m \geq n$, be an i.i.d. sequence taking values in $\mathbb{R}^n$ distributed by an arbitrary rotationally symmetric distribution not concentrated in 0 and let $F(t) := P(\|x\|_2 \leq t)$ be the associated radial distribution function (RDF). If there is a function $f \in \mathcal{L}^1(\mathbb{R}^n)$ such that $F(r) = \int_{\|x\|_2 \leq r} f(x) \, dx$, $r > 0$, we call $f$ the density function of the underlying distribution. All theorems of the present paper are formulated for distributions with density functions in order to avoid a too difficult notation. (This is no restriction of generality because the set of distributions with density...
function is pointwise dense in the class of all rotationally symmetric distributions and so the general result is an easy gained limiting case, cf. Natanson[5].) In the above described respect

(1.1) \[ \mathcal{P} := \text{convhull}\{a_1, \ldots, a_m\} \]
is a randomly generated polyhedron on \( \mathbb{R}^n \).

For an axiomatic definition of the random variables we are dealing with we need some more notation. For any set of indices \( I := \{i_1, \ldots, i_n\} \subset \{1, \ldots, m\} \) with \( i_j \neq i_k \) for \( i \neq k \), let \( A_I := (a_{i_1}, \ldots, a_{i_n}) \) and \( S_I := \text{convhull}\{a_i \mid i \in I\} \). Furthermore, if the vectors \( a_i, i \in I \), are linearly independent, i.e. \( S_I \) is a simplex, \( H(A_I) := \text{affhull}\{S_I\} \) denotes the hyperplane containing \( S_I \) and if additionally \( 0 \notin H(A_I) \), \( H^{(1)}(A_I) \) denotes the closed halfspace generated by the hyperplane \( H(A_I) \) containing the origin whereas \( H^{(2)}(A_I) \) represents its closed complement. (The special case \( 0 \in H(A_I) \) doesn't matter because \( P(0 \in H(A_I)) = 0 \) for all distributions of our model.) \( h(A_I) \) denotes the distance of \( H(S_I) \) to the origin.

We call a simplex \( S_I \), whose associated hyperplane \( H(A_I) \) doesn't contain the origin, a boundary simplex of \( \mathcal{P} \) of the first kind, if \( a_j \in H^{(1)}(A_I) \) for \( j \in \bar{I} \), and we call it a boundary simplex of \( \mathcal{P} \) of the second kind, if \( a_j \in H^{(2)}(A_I) \) for \( j \in I \), \( I \) being the complementary set to \( I \) as a subset of \( \{1, \ldots, m\} \).

We relate the last definitions to the characteristic functions

(1.2) \[ \chi_i(A_I) := \begin{cases} 1 & \text{if } S_I \text{ is a boundary simplex of } \mathcal{P} \text{ of the } i\text{-th kind} \\ 0 & \text{else} \end{cases} \]

Now we are able to state the definition of the class of random variables investigated:

**Definition:**

A random variable \( \hat{Z} \) of a randomly generated bounded polyhedron \( \mathcal{P} \) is of additive type iff for almost all \( \mathcal{P} \):

(1.3) \[ \hat{Z}(\mathcal{P}) = \sum_{I \subset \{1, \ldots, m\} \mid |I| = n} [\chi_1(A_I) + \sigma \chi_2(A_I)]Z(A_I), \]

where \( \sigma \in \mathbb{R} \) is a fixed constant and \( Z : \mathbb{R}^{n \times n} \rightarrow \mathbb{R} \) is a continuous function invariant under orthogonal transformations and under exchanging columns of the argument.

Roughly speaking, a random variable on a polyhedron is of additive type if it can be additively decomposed resp. its boundary simplices.

The examples treated by Renyi and Sulanke in [5,6] are easily recognized as special cases, by taking \( \sigma = 1 \) and \( Z(A_I) = 1 \) for the polyhedron's number of boundary simplices, taking \( \sigma = -1 \) and \( Z(A_I) = |\text{convhull}(0, S_I)| \) for the volume and finally by taking \( \sigma = 1 \) and \( Z(A_I) = |S_I| \) for the surface area of the polyhedron, \( |\cdot| \) representing Lebesgue-measure of appropriate dimension.

Further examples of interest included are the weighted volume of \( \mathcal{P} \), to which the number of vertices can be reduced, the characteristic function \( \chi(\text{cone}(\mathcal{P}) = \mathbb{R}^n) \) or the average
number of \((u, v)\) shadow vertices of \(\mathcal{P}\), which is closely related to the analysis of average complexity of the simplex algorithm, cf. Borgwardt[1].

In the next section we give an integral representation for the first moment of a random variable of type (1.3), section 3 deals with the second moment. For concrete estimations of variances under slightly sharpened propositions we refer to a further article of the author being published.

2. Expectation values

The results in this short section are mainly due to Raynaud[6] and Borgwardt[1], who treated special cases of additive variables under rotationally symmetric distributions on \(\mathbb{R}^n\). Let

\begin{equation}
G_1(h) := P(x^{(n)} \leq h), \quad h \in \mathbb{R},
\end{equation}

\begin{equation}
G_2(h) := P(x^{(n)} > h) = 1 - G_1(h), \quad h \in \mathbb{R},
\end{equation}

\(x^{(n)}\) being the \(n\)-th coordinate of the vector \(x\), we state without proof, which can be given analogously to Borgwardt’s[1]:

Theorem 2.1:

For random variables \(Z\) of additive type on bounded stochastic polyhedra \(\mathcal{P}\) generated by a rotationally symmetric distribution with density function \(f\) holds while \(m \geq n\) and \(n \geq 2\):

\begin{equation}
E(\tilde{Z}) = \left(\frac{m}{n}\right) \int_{\mathbb{R}^{m-n}} [G_1^{m-n}(h) + \sigma G_2^{m-n}(h)] \Lambda_Z(h) dh
\end{equation}

with

\begin{equation}
\Lambda_Z(h) = |\omega_n| \int_{\mathbb{R}^{m-n-1}} Z^{(n)} \left(\begin{array}{c}
\bar{b}_1 \\
h \\
\bar{b}_n
\end{array}\right) |\det(B)| \prod_{i=1}^{n} f((\bar{b}_i, h)^T) d\bar{b}_i,
\end{equation}

where

\begin{equation}
B := \begin{pmatrix} \bar{b}_1 & \cdots & \bar{b}_n \\ 1 & \cdots & 1 \end{pmatrix}, \quad \bar{b}_i := (b_i^{(1)}, \ldots, b_i^{(n-1)})^T, \quad i = 1, \ldots, n
\end{equation}

and

\begin{equation}
|\omega_n| := \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}.
\end{equation}
Remarks on 2.1:
i) The functions $G_i(h)$ are easily evaluated by

$$G_2(h) = \int_{\mathbb{R}^{n-1}} g_0(t) dt, \quad g_0(t) := \int f((\tilde{b}, t)^T) d\tilde{b}$$

for distributions with density function $f$.

ii) The function $\Lambda_Z(h)$ can be interpreted in the language of conditional expectation values by

$$\Lambda_Z(h) = E(Z(A_1) | h(A_1) = h) p(h)$$

where

$$p(h) := \omega_n \int_{\mathbb{R}^{n-1}} |\det(B)| \prod_{i=1}^n f(\tilde{b}_i, h)^T d\tilde{b}_i,$$

is the marginal density function of the distribution function $P(h(A_1) \leq h)$ for some set of indices $I$.

3. Second moments

Because of the nonlinearity of the functional $E(\tilde{Z}^2)$ it is much more complicated to give an integral formula for second moments of an additive type random variable $\tilde{Z}$ corresponding to (2.4). So, we deduce our result step by step and introduce further notation when necessary.

Let $\tilde{Z}$ be a random variable of additive type and $m \geq 2n \geq 4$, then by (1.3)

$$E(\tilde{Z}^2) = \sum_{i,j=1}^{2} \sigma^{i+j-2} \sum_{I,J \subset \{1, \ldots, m\}} \hat{c}_Z(I, J, i, j),$$

where $I = \{i_1, \ldots, i_n\}$ and $J = \{j_1, \ldots, j_n\}$ are arbitrary sets of indices $1, \ldots, m$ with cardinality $n$ and

$$\hat{c}_Z(I, J, i, j) := \int_{\mathbb{R}^m} \chi_i(A_I) \chi_j(A_J) Z(A_I) Z(A_J) \prod_{t=1}^m f(a_t) da_t,$$

$f$ being the density function of the underlying rotationally symmetric distribution. For reasons of symmetry $\hat{c}_Z(I, J, i, j)$ obviously depends only on the cardinality $k$ of the set $I \setminus J$ resp. the variables $i$ and $j$. Denoting $q_k$ for the number of pairs $(I, J)$, $I, J \subset \{1, \ldots, m\}$, with $|I \setminus J| = k$, we define

$$c_Z(k, i, j) := \hat{c}_Z(I, J, i, j)$$

if $|I \setminus J| = k$.
for \( i, j \in \{1, 2\} \) and \( k \in \{0, \ldots, n\} \). Now, we get from (3.1) by changing summations:

\[(3.4)\]
\[
E(\tilde{Z}^2) = \sum_{k=0}^{n} q_k \sum_{i,j=1}^{2} \sigma^{i+j-2} \epsilon_{Z}(k,i,j),
\]

where

\[(3.5)\]
\[
q_k = \binom{m}{n} \binom{n}{k} \binom{m-n}{k}, \quad k = 0, \ldots, n,
\]

which is obtained by simple combinatorial arguments.

In order to evaluate the terms \( \epsilon_{Z}(k,i,j) \) we define for \( i, j \in \{1, 2\} \) and \( k = 0, \ldots, n \):

\[(3.6)\]
\[
p_{i,j}(A_0, A_k) := \int_{\mathbb{R}^n}^{(m-n-k)} \chi_i(A_0)\chi_j(A_k) \prod_{\ell=n+k+1}^{m} f(a_{\ell})da_{\ell}
\]

with

\[(3.7)\]
\[
A_{\ell} := (a_{\ell+1}, \ldots, a_{\ell+n}), \quad \ell = 0, \ldots, n.
\]

\( p_{i,j}(A_0, A_k) \) equals the probability of \( A_0 \) being a boundary simplex of the \( i \)-th kind of \( P \) and \( A_k \) being a boundary simplex of the \( j \)-th kind of \( P \) at the same time.

Using definitions (3.3) and (3.6) we obtain

\[(3.8)\]
\[
\epsilon_{Z}(k,i,j) = \int_{\mathbb{R}^n}^{(m)} \chi_i(A_0)\chi_j(A_k)Z(A_0)Z(A_k) \prod_{\ell=1}^{m} f(a_{\ell})da_{\ell}
\]
\[
= \int_{\mathbb{R}^n}^{(n+k)} p_{i,j}(A_0, A_k)Z(A_0)Z(A_k) \prod_{\ell=1}^{n+k} f(a_{\ell})da_{\ell}
\]

proving the

**Proposition 3.1:**

For random variables \( \tilde{Z} \) of additive type, rotationally symmetric distributions with density function \( f \) and \( m \geq 2n \geq 4 \) holds:

\[(3.9)\]
\[
E(\tilde{Z}) = \sum_{k=0}^{n} q_k \sum_{i,j=1}^{2} \sigma^{i+j-2} \epsilon_{Z}(k,i,j)
\]

with

\[(3.10)\]
\[
q_k = \binom{m}{n} \binom{n}{k} \binom{m-n}{k}
\]
and

\( (3.11) \quad e_Z(k, i, j) = \int_{\mathbb{R}^n} p_{i,j}(A_0, A_k) Z(A_0) Z(A_k) \prod_{\ell=1}^{n+k} f(a_\ell) da_\ell, \)

where

\( (3.12) \quad p_{i,j}(A_0, A_k) = P(A_0 \text{ is boundary simplex of } \mathcal{P} \text{ of the } i\text{-th kind} \land A_k \text{ is boundary simplex of } \mathcal{P} \text{ of the } j\text{-th kind}). \)

After this first step of evaluation we look at \( p_{i,j} \) more closely. By use of the obvious equations

\( (3.13) \quad \chi_i(A_\ell) = \prod_{j=0}^{k} \chi(a_j \in H^{(i)}(A_\ell)) \prod_{j=n+k+1}^{m} \chi(a_j \in H^{(i)}(A_\ell)) \)

for \( i = 1, 2 \) and \( \ell = 0, \ldots, n \) we gain

\( (3.14) \quad p_{i,j}(A_0, A_k) = (\tilde{G}_{i,j}(A_0, A_k))^{m-n-k} \chi(S_k \subset H^{(i)}(A_0)) \chi(S_0 \subset H^{(j)}(A_k)) \)

with

\( (3.15) \quad \tilde{G}_{i,j}(A_0, A_k) := P(a \in (H^{(i)}(A_0) \cap H^{(j)}(A_k))). \)

and \( S_\ell \) defined analogously to (3.7).

The quantities \( \tilde{G}_{i,j} \) can be illustrated geometrically as weighted areas dividing the measure space \((\mathbb{R}^n, \mathcal{P})\) in four parts, cf. figure 1.

**Figure 1:**

![Figure 1](image)

More formally holds:

\( (3.16) \quad \sum_{i,j=1}^{2} \tilde{G}_{i,j}(A_0, A_k) = 1. \)
Elementary geometric insight makes clear that $G_{i,j}(A_0, A_k)$ for \( k = 1, \ldots, n \) depends only on the distance $h_1 = h(A_0)$ of the hyperplane $H(A_k)$ to the origin, the distance $h_2 = h(A_k)$ of the hyperplane $H(A_k)$ to the origin and on the angle $\varphi(A_0, A_k)$ enclosed by the normal vectors $n_1$ of $H(A_0)$ and $n_2$ of $H(A_k)$, cf. figure 1. Clearly, $\|n_1\|_2 = h_1$ and $\|n_2\|_2 = h_2$.

The natural objective now is to represent $e_Z(k, i, j)$ for $k = 1, \ldots, n$ by an integral of the form

$$
(3.17) \quad e_Z(k, i, j) = \int_0^\infty \int_0^\pi G_{i,j}^{m-n-k}(h_1, h_2, \varphi) \Lambda_{i,j,k,z}(h_1, h_2, \varphi) \, d\varphi \, dh_1 \, dh_2
$$

with appropriately chosen functions $\Lambda_{i,j,k,z}$ and

$$
(3.18) \quad G_{i,j}(h_1, h_2, \varphi) := \tilde{G}_{i,j}(A_0, A_k).
$$

For $k = 0$ we obtain for $i, j \in \{1, 2\}$:

$$
(3.19) \quad \tilde{G}_{i,j}(A_0, A_0) = \begin{cases} G_i(h(A_0)) & i = j \\ 0 & i \neq j \end{cases}
$$

$G_i$ being defined by (2.1) resp. (2.2), which leads to

$$
(3.20) \quad e_Z(0, i, j) = \begin{cases} \int_0^\infty G_i^{m-n}(h) \Lambda_{Z^2}(h) \, dh & i = j \quad , i, j \in \{1, 2\} \\ 0 & i \neq j \end{cases}
$$

with $\Lambda_{Z^2}$ as in (2.4).

In the following we will establish (3.17) by three successive simultaneous rotations of the vectors $a_\ell$, $\ell = 1, \ldots, n + k$.

**First Rotation:**

The normal vector $n_1$ of the hyperplane $H(A_0)$ have the polar representation

$$
(3.21) \quad (h_1, \psi_1, \ldots, \psi_{n-1})^T,
$$

where $h_1 \in \mathbb{R}_0^+$, $\psi_1 \in [0, 2\pi)$, $\psi_\ell \in [0, \pi)$ for $\ell = 2, \ldots, n - 1$.

Furthermore, let $\tilde{d}_1, \ldots, \tilde{d}_n$ be an orthonormal basis of $\mathbb{R}^n$, with $\tilde{d}_n = n_1^0$. This basis can be chosen as

$$
(3.22) \quad \tilde{d}_n(\ell) = \begin{cases} \sin \psi_1 \sin \psi_2 \ldots \sin \psi_{n-1} & \text{if } \ell = 1 \\ \cos \psi_{\ell-1} \sin \psi_\ell \ldots \sin \psi_{n-1} & \text{if } \ell = 2, \ldots, n - 1 \\ \cos \psi_{n-1} & \text{if } \ell = n \end{cases}
$$

and

$$
\tilde{d}_\ell = \frac{\sin \psi_1 \ldots \sin \psi_\ell}{\sin \psi_1 \ldots \sin \psi_{\ell-1}} \partial \tilde{d}_n, \quad \ell = 1, \ldots, n - 1.
$$
Now define the orthogonal matrix $R_1 := (d_1, \ldots, d_n)$ and the new variables $b_\ell := R_1^{-1} a_\ell$ for $\ell = 1, \ldots, n + k$ with $b_\ell := (\bar{b}_\ell, h_1)^T$.

The Jacobian $\Phi_1$ of this transformation satisfies

\begin{equation}
|\det(\Phi_1)| = \sin \psi_2 (\sin \psi_3)^2 \ldots (\sin \psi_{n-1})^{n-2} |\det(B)|,
\end{equation}

as one easily proves like Raynaud[6] with $B$ defined in (2.5).

Using the formula

\begin{equation}
\int_0^{2\pi} \int_0^\pi \int_0^\pi |\det(\Phi_1)| \sin \psi_2 \sin \psi_3 \ldots \sin \psi_{n-1} d\psi_{n-1} \ldots d\psi_2 d\psi_1 = |\omega_n| |\det(B)|,
\end{equation}

we obtain for $e_Z(k, i, j)$ the representation:

\begin{equation}
e_Z(k, i, j) := \int_0^{2\pi} \int_0^\pi \int_0^\pi p_{i, j}(\tilde{A}_0, \tilde{A}_k) \tilde{\lambda}_k, Z(\tilde{A}_0, \tilde{A}_k) \prod_{i=n+1}^{n+k} f((\bar{b}_i, h_1)^T) db_i \, dh_1,
\end{equation}

where now defining $\tilde{A}_0 := R_1^{-1} A_0$ and $\tilde{A}_k := R_1^{-1} A_k$ holds:

\begin{equation}
p_{i, j}(\tilde{A}_0, \tilde{A}_k) = G_{i, j}^{m-n-k}(h_1, h_2(\tilde{A}_k), \varphi(\tilde{A}_0, \tilde{A}_k)) \chi(\tilde{A}_k \subset H^{(i)}(\tilde{A}_0) \wedge \tilde{1}_0 \subset H^{(j)}(\tilde{A}_k))
\end{equation}

and where

\begin{equation}
\tilde{\lambda}_k, Z(\tilde{A}_0, \tilde{A}_k) = |\omega_n| \int_{\mathbb{R}^{n-1}} Z(\tilde{A}_0) Z(\tilde{A}_k) |\det(B)| \prod_{\ell=1}^{n} f((\bar{b}_\ell, h_1)^T) db_\ell.
\end{equation}

Geometrically, by this first rotation the hyperplane $H(A_0)$ containing the boundary simplex $A_0$ was rotated into a hyperplane normal to the unit vector $e_n$, cf. figure 2.

**Figure 2:**

Next, by a second simultaneous rotation of the vectors $b_\ell$ we will establish $h_2$ und $\varphi$ as free quantities. This transformation is only required for $n \geq 3$, for that reason let $n \geq 3$ in the following.
Second Rotation:

The normal vector $\vec{n}_2$ of the hyperplane $H(\hat{A}_k)$ may be written in the form (cf. figure 2):

$$\vec{n}_2 = (\vec{n}_2, \vec{n}_2^{(n)})^T.$$  

The truncated vector $\vec{n}_2 \in \mathbb{R}^{n-1}$ have the polar coordinates $(\|\vec{n}_2\|_2, \hat{\psi}_1, \ldots, \hat{\psi}_{n-2})^T$ with $\hat{\psi}_1 \in [0, 2\pi)$, $\hat{\psi}_\ell \in [0, \pi)$ for $\ell = 2, \ldots, n - 2$. Additionally $\hat{\vec{d}}_1, \ldots, \hat{\vec{d}}_{n-1}$ be an orthonormal basis of $\mathbb{R}^{n-1}$, where $\hat{\vec{d}}_{n-1} = \vec{n}_2^0$. Like (3.22) we define

$$\varphi_\ell := \begin{cases} \sin \hat{\psi}_1 \sin \hat{\psi}_2 \ldots \sin \hat{\psi}_{n-2} & \text{if } \ell = 1 \\ \cos \hat{\psi}_{\ell-1} \sin \hat{\psi}_\ell \ldots \sin \hat{\psi}_{n-2} & \text{if } \ell = 2, \ldots, n - 2 \\ \cos \hat{\psi}_{n-2} & \text{if } \ell = n - 1 \end{cases}$$

$$\hat{\vec{d}}_{\ell} = \frac{\sin \hat{\psi}_1 \ldots \sin \hat{\psi}_{\ell} \hat{\psi}_{\ell-1}}{\sin \hat{\psi}_1 \ldots \sin \hat{\psi}_{\ell-1}} \frac{\partial \hat{\vec{d}}_{n-1}}{\partial \hat{\psi}_{\ell}}, \quad \ell = 1, \ldots, n - 2.$$  

We define the matrix $R_2 := (\hat{\vec{d}}_1, \ldots, \hat{\vec{d}}_{n-1})$ and the new variables $\hat{c}_\ell$ by $\hat{c}_\ell := (\vec{c}_\ell, c_{\ell}^{(n)})^T$ with $\vec{c}_\ell := \gamma_{\ell}^{-1} \hat{\vec{d}}_{\ell}$ and by $c_{\ell}^{(n)} := \ell^{(n)}_\ell$ for $\ell = 1, \ldots, n + k$. In the following $\hat{c}_\ell$ often will be notated by $\hat{c}_\ell = (\vec{c}_\ell, c_{\ell}^{(n-1)}, c_{\ell}^{(n)})^T$.

For $\ell = k + 1, \ldots, n + k$ the vector $c_{\ell}^{(n-1)}$ satisfies the linear equation

$$\varphi_\ell h_{\ell} - 2 \sin \frac{\varphi_\ell}{2} c_{\ell}^{(n-1)} = 0$$

for $\varphi \in [0, \pi]$. Especially for $\varphi \in (0, \pi)$ holds, cf. figure 3:

$$c_{\ell}^{(n-1)} = \frac{h_{\ell} - \cos \varphi c_{\ell}^{(n)}}{\sin \varphi}.$$  

The evaluation of the second rotation’s Jacobian $\Phi_2$ requires lengthy calculations omitted here. $\Phi_2$ satisfies the relation

$$|\text{det}(\Phi_2)| = \frac{1}{\sin^2 \varphi} \prod_{i=2}^{n-2} \sin \psi_i |\text{det}(C)|,$$

which leads to

$$\int_0^{2\pi} \int_0^\pi \ldots \int_0^{\pi} |\text{det}(\Phi_2)| \hat{\psi}_1 \ldots \hat{\psi}_{n-2} = \frac{|\omega_{n-1}|}{\sin^2 \varphi} |\text{det}(\hat{C})|$$

corresponding to (3.24), where

$$\hat{C} := \begin{pmatrix} \vec{c}_{k+1} & \ldots & \vec{c}_{n+k} \\ \frac{c_{k+1}^{(n)} - \cos \varphi h_2}{\sin \varphi} & \ldots & \frac{c_{n+k}^{(n)} - \cos \varphi h_2}{\sin \varphi} \end{pmatrix}.$$
For detailed proofs of (3.32) and (3.33) the interested reader is referred to Küfer[4]. The objective of the second rotation was to move both normal vectors $n_1$ and $n_2$ of the hyper-planes $H(\hat{A}_0)$ and $H(\hat{A}_k)$ into the $(e_{n-1}, e_n)$-plane, cf. figure 3.

![Figure 3:](image)

Having performed the second rotation we come to the desired form (3.17) for the quantities $e_Z(k, i, j)$. It holds for $k = 1, \ldots, n$ and $i, j \in \{1, 2\}:

\[\Lambda_{i,j,k,Z}(h_1, h_2, \varphi) = \frac{\omega_n |\omega_{n-1}|}{\sin^2 \varphi} \int\int \int K_{1,i,j} K_{2,i,j} \lambda_{k,Z}(\hat{c}_1, \ldots, \hat{c}_{n+k}) \ d\hat{\mu}_k \]

where

\[K_{1,i,j} := \begin{cases} (-\infty, h_1) & \text{if } i = 1 \\ [h_1, \infty) & \text{if } i = 2 \end{cases}, \quad K_{2,i,j} := \begin{cases} (-\infty, d_1) & \text{if } j = 1 \\ [d_1, \infty) & \text{if } j = 2 \end{cases},\]

\[\lambda_{k,Z}(\hat{c}_1, \ldots, \hat{c}_{n+k}) := |\det(\hat{C}_k)|Z(\hat{A}_k)|\det(\hat{C}_0)|Z(\hat{A}_0),\]

and

\[d\hat{\mu}_k := \prod_{t=1}^{k+n} f(\hat{c}_t) \prod_{t=1}^{k} d\hat{c}_t^{(n-1)} \prod_{t=n+1}^{n+k} d\hat{c}_t^{(n)} \prod_{t=1}^{n+k} d\hat{c}_t,\]

the matrices $\hat{C}_0$ and $\hat{C}_k$ taking forms

\[\hat{C}_0 = \begin{pmatrix} \vec{c}_1 & \cdots & \vec{c}_k & \vec{c}_{k+1} & \cdots & \vec{c}_n \\ \hat{c}_1^{(n-1)} & \cdots & \hat{c}_k^{(n-1)} & d_1 & \cdots & d_1 \end{pmatrix} \]

\[d\hat{\mu}_k := \prod_{t=1}^{k+n} f(\hat{c}_t) \prod_{t=1}^{k} d\hat{c}_t^{(n-1)} \prod_{t=n+1}^{n+k} d\hat{c}_t^{(n)} \prod_{t=1}^{n+k} d\hat{c}_t,\]

the matrices $\hat{C}_0$ and $\hat{C}_k$ taking forms

\[\hat{C}_0 = \begin{pmatrix} \vec{c}_1 & \cdots & \vec{c}_k & \vec{c}_{k+1} & \cdots & \vec{c}_n \\ \hat{c}_1^{(n-1)} & \cdots & \hat{c}_k^{(n-1)} & d_1 & \cdots & d_1 \end{pmatrix} \]
and

\[
\hat{C}_k = \begin{pmatrix}
\hat{c}_{k+1} & \ldots & \hat{c}_n & \hat{c}_{n+1} & \ldots & \hat{c}_{n+k} \\
d_2 & \ldots & d_2 & \frac{\hat{c}_{n+1} - \cos \varphi h_2}{\sin \varphi} & \ldots & \frac{\hat{c}_{n+k} - \cos \varphi h_2}{\sin \varphi}
\end{pmatrix}
\]

Furthermore, \( \hat{A}_0 := R_{2}^{-1} A_0 \) and \( \hat{A}_k := R_{2}^{-1} A_k \). The quantities \( d_\ell, \ell = 1, 2 \), defined by

\[
d_1 := \frac{h_2 - \cos \varphi h_1}{\sin \varphi}; \quad d_2 := \frac{h_1 - \cos \varphi h_2}{\sin \varphi}
\]

can be interpreted geometrically. \(|d_1|\) equals the distance between the intersection point of the hyperplanes \( H(\hat{A}_0) \) and \( H(\hat{A}_k) \) in the \((e_{n-1}, e_n)\)-plane and the intersection point of the ray \( RH^+n_i \) and the hyperplane \( H(\hat{A}_0) \) for \( i = 1 \) resp. the hyperplane \( H(\hat{A}_k) \) for \( i = 2 \), cf. figure 4.

**Figure 4:**

![Figure 4](image)

**Third Rotation:**

Intuitively it is clear that functions \( \Lambda_{i,j,k,Z}(h_1, h_2, \varphi) \) are to be symmetric in the arguments \( h_1 \) und \( h_2 \). In order to realize this suggestion formally by the representation of the functions \( \Lambda_{i,j,k,Z}(h_1, h_2, \varphi) \) we rotate the vectors \( \hat{c}_{k+1}, \ldots, \hat{c}_{n+k} \) simultaneously resp. both of its last coordinates by the angle \( \pi - \varphi \) counterclockwise in the \((e_{n-1}, e_n)\)-plane. Because of (1.3) \( Z(\hat{A}_k) \) remains unchanged while performing the described rotation. Defining the matrix

\[
R_3 := \begin{pmatrix}
Id_{n-2} & 0 \\
0 & \tilde{R}
\end{pmatrix},
\]

where \( Id_{n-2} \) represents the identity in \( \mathbb{R}^{n-2} \) and

\[
\tilde{R} = \begin{pmatrix}
-\cos \varphi & \sin \varphi \\
\sin \varphi & \cos \varphi
\end{pmatrix}
\]
holds:

\[ R_3 \hat{c}_t = \begin{cases} 
(\hat{c}_t, d_2, h_2) & \text{for } \ell = k + 1, \ldots, n \\
(\hat{c}_t, \frac{c^{(n)} - \cos \varphi h_2}{\sin \varphi}, h_2) & \text{for } \ell = n + 1, \ldots, n + k
\end{cases} \]

We substitute

\[ \hat{c}_t^{(p)} = \begin{cases} 
(c^{(p-1)}_t \sin \varphi + \cos \varphi h_2) & p = n; \ \ell = n + 1, \ldots, n + k \\
(c^{(p)}_t) & p = 1, \ldots, n - 2; \ \ell = n + 1, \ldots, n + k \\
(c^{(p)}_t) & p = 1, \ldots, n; \ \ell = 1, \ldots, n
\end{cases} \]

and finally reach a symmetric form of \( \Lambda_{i,j,k,z}(h_1, h_2, \varphi) \), which we summarize in

**Theorem 3.2:**
Under the prepositions of Proposition 3.1 \( e_Z(k, i, j) \) satisfies (3.20) for \( k = 0 \) and (3.17) for \( k = 1, \ldots, n \), where in the latter case

\[ \Lambda_{i,j,k,z}(h_1, h_2, \varphi) = \frac{\omega_n |\omega_{n-1}|}{\sin^{2-k} \varphi} \int_{K_{1,i,j}}^{(n+k)} \int_{K_{2,i,j}}^{(k)} \lambda_{k,z}(c_1, \ldots, c_{n+k}) \, d\mu_k, \]

with

\[ K_{1,i,j} := \begin{cases} 
(0, d_2) & \text{if } i = 1 \\
[d_2, \infty) & \text{if } i = 2
\end{cases}, \quad K_{2,i,j} := \begin{cases} 
(0, d_1) & \text{if } j = 1 \\
[d_1, \infty) & \text{if } j = 2
\end{cases} \]

\[ \lambda_{k,z}(c_1, \ldots, c_{n+k}) := \left| \text{det} \begin{bmatrix} C_0 \\ h_1 \hat{c}_T \end{bmatrix} \right| Z \left( \begin{bmatrix} C_0 \\ h_1 \hat{c}_T \end{bmatrix} \right) \left| \text{det} \begin{bmatrix} C_k \\ h_2 \hat{c}_T \end{bmatrix} \right| Z \left( \begin{bmatrix} C_k \\ h_2 \hat{c}_T \end{bmatrix} \right), \]

and

\[ d\mu_k := \prod_{\ell=1}^{k+n} f(c_\ell) \prod_{\ell=1}^{k} dc_{(n-1)}^{(n-1)} \prod_{\ell=n+1}^{n+k} dc_{(n-1)}^{(n-1)} \prod_{\ell=1}^{n+k} d\hat{c}_\ell \]

The matrices \( \overline{C}_t \) take the form

\[ \overline{C}_0 = \begin{pmatrix} \overline{c}_1 & \ldots & \overline{c}_k & \overline{c}_{k+1} & \ldots & \overline{c}_n \\
d_1 & \ldots & d_1 \end{pmatrix} \]

and

\[ \overline{C}_k = \begin{pmatrix} \overline{c}_{k+1} & \ldots & \overline{c}_n & \overline{c}_{n+1} & \ldots & \overline{c}_{n+k} \\
d_2 & \ldots & d_2 \end{pmatrix} \]
whereas $\xi = (1, \ldots, 1)^T \in \mathbb{R}^n$.

In the remaining part of the paragraph we concentrate on the quantities $G_{i,j}(h_1, h_2, \varphi)$ in order to obtain integral formulas like (2.7). Obviously, because of (3.16) and the easily proved equations

\begin{align}
G_{1,i}(h_1, h_2, \varphi) + G_{2,i}(h_1, h_2, \varphi) &= G_i(h_2), \\
G_{i,1}(h_1, h_2, \varphi) + G_{i,2}(h_1, h_2, \varphi) &= G_i(h_1)
\end{align}

for $h_1, h_2 \in [0, \infty)$, $\varphi \in (0, \pi)$ and $i \in \{1, 2\}$ it is enough to investigate $G_{1,1}$. Before we formulate our result on this issue we introduce a new variable $r$ by

\begin{align}
r = \sqrt{d_1^2 + h_1^2} = \sqrt{d_2^2 + h_2^2} = \frac{\sqrt{h_1^2 + h_2^2 - 2h_1h_2 \cos \varphi}}{\sin \varphi},
\end{align}

Geometrically, $r$ equals the distance between the origin and the intersection point of the hyperplanes containing the boundary simplices we are dealing with, cf. figure 4.

**Theorem 3.3:**

For $n \geq 2$, rotationally symmetric distributions with RDF $F$, $0 \leq h_1, h_2 < \infty$ and $\varphi \in (0, \pi)$ holds:

\begin{align}
G_{1,1}(h_1, h_2, \varphi) = 1 - \frac{1}{2}G_2(h_1) - \frac{1}{2}G_2(h_2) - A_1(h_1, h_2, \varphi) - A_2(h_1, h_2, \varphi),
\end{align}

where for $\ell \in \{1, 2\}$

\begin{align}
A_\ell(h_1, h_2, \varphi) := \text{sign}(d_\ell) \int_{h_\ell}^r g_{0,\ell}(t) \frac{h_\ell dt}{t \sqrt{t^2 - h_\ell^2}},
\end{align}

with $r = r(h_1, h_2, \varphi)$ and

\begin{align}
g_{0,2}(t) := \frac{|\omega_{n-2}|}{|\omega_n|(n-2)} \int_t^\infty \frac{(s^2 - t^2)(n-2)/2}{s^{n-2}} dF(s).
\end{align}

**Proof:**

We prove the theorem for $n \geq 3$ the result remaining valid for the easy special case $n = 2$. The weighted area $A_1(h_1, h_2, \varphi)$, cf. figure 5, satisfies:

\begin{align}
A_1(h_1, h_2, \varphi) = \int_0^{\arcsin \frac{h_1}{\cos \varphi}} \int_0^\infty g_{0,0}(t) dt d\psi.
\end{align}
with

\[(3.58) \quad g_{0,0}(t) := \frac{\omega_{n-2}}{s} \int \frac{(s^2 - t^2)^{(n-4)/2}}{s^{n-2}} F(s) \, ds \]

being the radial density of \( G_{1,1} \) for a point with polar coordinates \((t, \psi)\) in the \((e_n, e_{n-1})\)-plane.

\[ \text{Figure 5:} \]

The variable \( r \) is seen as a function of variables \( h_1, h_2, \varphi \). Using

\[(3.59) \quad \frac{d}{dt} g_{0,2}(t) = -t g_{0,0}(t) , \quad g_{0,2}(\infty) = 0 , \]

we have

\[(3.60) \quad A_1(h_1, h_2, \varphi) = \int_0^{\arcsin \frac{4h_1}{t}} g_{0,2}\left(\frac{h_1}{\cos \psi}\right) d\psi . \]

By substituting

\[(3.61) \quad \frac{h_1}{\cos \psi} = t , \quad d\psi = \frac{h_1 dt}{t \sqrt{t^2 - h_1^2}} , \]

we reach the desired

\[(3.62) \quad A_1(h_1, h_2, \varphi) := \text{sign}(d_1) \int_{h_1}^{r} g_{0,2}(t) \frac{h_1 dt}{t \sqrt{t^2 - h_1^2}} . \]

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The corresponding representation for the weighted area $A_2(h_1, h_2, \varphi)$ is established analogously. By use of (3.52) the statement (3.53) follows.

Finally we give a useful

**Corollary 3.4:**

*Under the prepositions of Theorem 3.3 holds:*

\begin{align}
(3.63) \quad 1 - G_2(h_1) - G_2(h_2) & \leq G_{1,1}(h_1, h_2, \varphi) \leq \min(G_1(h_1), G_1(h_2)) \\
\text{and} \quad (3.64) \quad G_{1,1}(h_1, h_2, \varphi_1) & \leq G_{1,1}(h_1, h_2, \varphi_2) \quad \text{for} \quad \varphi_1 \geq \varphi_2
\end{align}

*Proof:*

While (3.63) being obvious we have:

\begin{align}
(3.65) \quad \frac{d}{d\varphi} G_{1,1}(h_1, h_2, \varphi) &= -\frac{d}{d\varphi} \sum_{\ell \in \{1, 2\}} A_{\ell}(h_1, h_2, \varphi) \\
&= -\frac{d}{d\varphi} (\arcsin \frac{d_1}{r} + \arcsin \frac{d_2}{r}) g_{0,2}(r) = -g_{0,2}(r) \leq 0.
\end{align}

*Concluding remarks:*

In a further article being published we will exploit the above developed theory on the first two moments of additive type variables systematically under slightly sharpened prepositions concerning the class of distributions and the function $Z$ in (1.3). As a first example we state without proof:

**Theorem 3.5:**

i) *For uniform distribution in the $n$-dimensional unit ball, $n \geq 2$, holds:*

\begin{align}
(3.66) \quad \text{Var}(Vol(\mathcal{P})) &= O(m^{-\frac{n-1}{n+1}}), \quad m \to \infty.
\end{align}

ii) *For uniform distribution on the $n$-dimensional unit sphere, $n \geq 2$, holds:*

\begin{align}
(3.67) \quad \text{Var}(Vol(\mathcal{P})) &= O(m^{-1}), \quad m \to \infty.
\end{align}
References:


