Geometric Methods to Solve Max-Ordering Location Problems

Matthias Ehrgott* Horst W. Hamacher* Stefan Nickel*

Abstract

Location problems with Q (in general conflicting) criteria are considered. After reviewing previous results of the authors dealing with lexicographic and Pareto location the main focus of the paper is on max-ordering locations. In these location problems the worst of the single objectives is minimized. After discussing some general results (including reductions to single criterion problems and the relation to lexicographic and Pareto locations) three solution techniques are introduced and exemplified using one location problem class, each: The direct approach, the decision space approach and the objective space approach. In the resulting solution algorithms emphasis is on the representation of the underlying geometric idea without fully exploring the computational complexity issue. A further specialization of max-ordering locations is obtained by introducing lexicographic max-ordering locations, which can be found efficiently. The paper is concluded by some ideas about future research topics related to max-ordering location problems.

Contents

*Department of Mathematics, University of Kaiserslautern, POBox 3049, D-67653 Kaiserslautern, Germany. This research was partially supported by contract HA 1795/4-2 of the DFG.

1 Multicriteria Location Problems

In this paper we study location problems which are subject to Q - possibly conflicting - objective functions $f^1, \ldots, f^Q$. More precisely, any feasible location x (denoted $x \in \mathcal{F}$ in the following) is assigned a vector $f(x) = (f^1(x), \ldots, f^Q(x))$ in $\mathbb{R}^Q$. We will assume throughout that the single objectives are of the

$$\text{median type} \quad f^q(x) := \sum_{m=1}^{M} w_m^q d(Ex_m, x)$$

or of the

$$\text{center type} \quad f^q(x) := \max_{m=1}^{M} w_m^q d(Ex_m, x)$$

where $w_m^q (m = 1, \ldots, M;\ q = 1, \ldots, Q)$ are, by default, non-negative weights, Ex_m are existing facilities and $d(Ex_m, x)$ is a given distance function.

Location problems of this type have only been studied recently in their full generality (e.g. [HN93] and [HN96]) while only special cases with specific choices of $f^q, q = 1, \ldots, Q$ were considered previously ([Pla95]).
In order to find a "best" location $x^* \in \mathcal{F}$, i.e.

$$f(x^*) = \min_{x \in \mathcal{F}} f(x)$$

we need to be able to compare vectors in $\mathbb{R}^Q$. In this section we will briefly review some results for the lexicographic and component-wise ordering which will be helpful in dealing with max-ordering location problems discussed in detail in Section 2. Section 3 will then describe three solution strategies for solving max-ordering location problems, each exemplified in a specific problem class. The paper is concluded by summarizing the results and a discussion of related ongoing research.

If we compare vectors by the lexicographic ordering i.e.

$$f(x) <_{lex} f(y) \Leftrightarrow f^p(x) < f^p(y) \text{ for } p = \min\{q : f^q(x) \neq f^q(y)\},$$

the *lexicographic location problem* $\text{lexmin}_{x \in \mathcal{F}} f(x)$ can be solved by iteratively finding the set $\mathcal{F}^q$ of all optimal locations of the single objective location problem with respect to objective f^q and feasibility set

$$\mathcal{F}^{q-1} := \begin{cases} \mathcal{F} & \text{if } q = 1 \\ \text{argmin} \ \{f^{q-1}(x) : x \in \mathcal{F}^{q-2}\} & \text{if } 1 < q \leq Q \end{cases}$$

For planar location problems (i.e. $\mathcal{F} \subseteq \mathbb{R}^2, Ex_m \in \mathbb{R}^2, m = 1, \ldots, M$) such problems can be solved by using the theory of restricted location problems.

- If there exists an optimal location $x^q \in \text{argmin}\{f^q(x) : x \in \mathbb{R}^2\}$ such that $x^q \in \mathcal{F}^{q-1}$, then $\mathcal{F}^q := \text{argmin}\{f^q(x) : x \in \mathbb{R}^2\} \cap \mathcal{F}^{q-1}$.

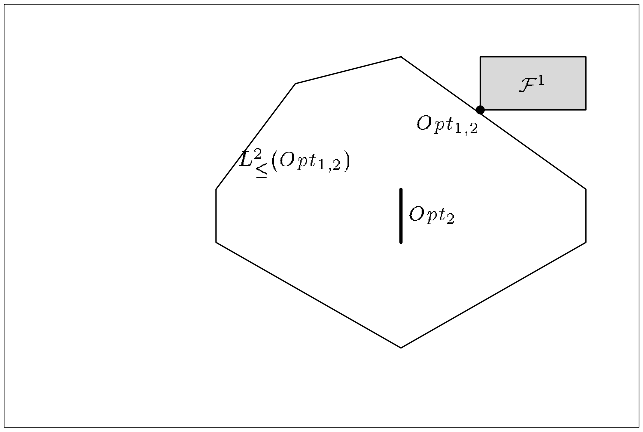

Figure 1.1: Lex location $Opt_{1,2}$ defined as the best location for f^2 in $\mathcal{F}^1$.

- Otherwise find a *level* z and a level set

$$L^q_\leq(z) := \{x \in \mathbb{R}^2 : f^q(x) \leq z\}$$

  such that $L^q_\leq(z) \cap \mathcal{F}^{q-1} \neq \emptyset$ and z is minimal with this property.

Obviously, the second part of this procedure is just a reformulation of the restricted location problem, but it is very useful to implement a geometric approach replacing the search methods already sketched in Francis and White ([FW74]) by polynomial time algorithms ([HN95], [Nic95]).
As an example we consider in Figure 1.1 $1/P/\bullet/l_1/2 - \sum_{lex}$. Following the classification of [Ham92], [Ham95]), [HN96] and [HN94] this is the problem of finding 1 new facility in the plane, with no special constraints, with respect to the rectilinear distance $d(Ex_m, x) = l_1(Ex_m, x) = |a_{m_1} - x_1| + |a_{m_2} - x_2|$, and 2 median type objective functions.
For network location problems (i.e. $\mathcal{F}$ = set of points in a graph, Ex_m = nodes of the graph, $m = 1, \ldots, M$, d = shortest path distance) the problem reduces to finding in a given finite set of vectors the subset of those which are lexicographically minimal. Obviously, this is trivial if the sequence $1, \ldots, Q$ of the objective functions is fixed. This can, however, also be done in polynomial time if lexicographically minimal vectors are sought for all permutations $\pi(1), \ldots, \pi(Q)$ ([HNL96]).

The latter observation becomes important as starting point to find all *Pareto locations*, i.e. minimizers of $f(x) = (f^1(x), \ldots, f^Q(x))$ with respect to the component-wise ordering. A Pareto location does not allow for another location $y \in \mathcal{F}$ such that $f(y) <_{comp} f(x)$, i.e.

$$f^q(y) \leq f^q(x) \quad \forall q = 1, \ldots, Q$$

and

$$f^p(y) < f^p(x) \quad \text{for at least one } p \in \{1, \ldots, Q\}.$$

(an *x-dominating location*).
Pareto locations can be characterized using level sets, which were already introduced above, and level curves defined by

$$L^q_=(z) := \{x \in \mathbb{R}^2 : f^q(x) = z\}.$$

Theorem 1.1. *Let $x \in \mathcal{F}$ be a feasible location and let $z^q := f^q(x) \quad \forall q = 1, \ldots, Q$. Then, x is a Pareto location if and only if*

$$\bigcap_{q=1}^{Q} L^q_=(z^q) = \bigcap_{q=1}^{Q} L^q_\leq(z^q). \tag{1.1}$$

The proof follows immediately by the definitions of Pareto locations, level sets and level curves. In the context of location problems it was first stated in [HN96]. Its usage in general multicriteria problems and its relation to other characterizations of Pareto solutions is described in [EHK+97].

If Opt_{lex} is the set of all lexicographic locations where we allow any change in the sorting of the objective functions i.e. $f \to f^\pi = (f^{\pi(1)}, \ldots, f^{\pi(Q)})$, and if Opt_{Par} is the set of all Pareto locations we get

Proposition 1.2.

$$Opt_{lex} \subseteq Opt_{Par}$$

Notice that one can easily find examples, where this inequality is strict.
For a large class of location problems, the set Opt_{Par} is obtained by "connecting" the locations in Opt_{lex} with each other. As example for this approach we consider the location problem of finding

- 1 new facility,
- in the plane,
- with no special constraints
- with respect to the polyhedral gauge distance defined by convex polyhedra P_m, containing Ex_m in its interior, respectively, and

$$d(Ex_m, x) = \gamma_{pol}(Ex_m, x) := \min\{\lambda > 0 : x - E_{x_m} \in \lambda P_m,\ m = 1, \ldots, M$$

and

- Pareto location objective composed of 2 median single objective functions f^1 and f^2.

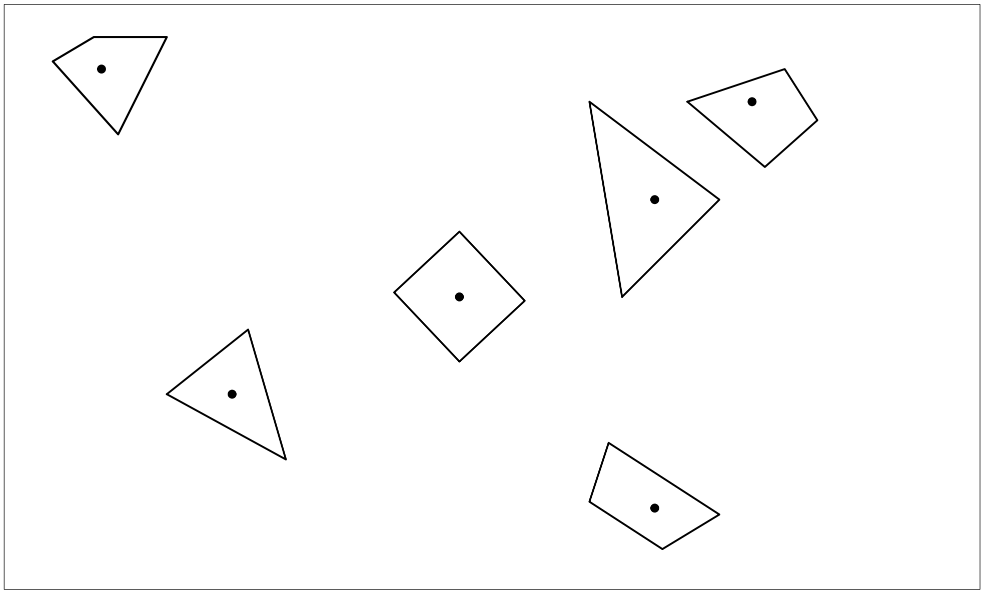

Figure 1.2: Six existing facilities with corresponding polyhedra P_m.

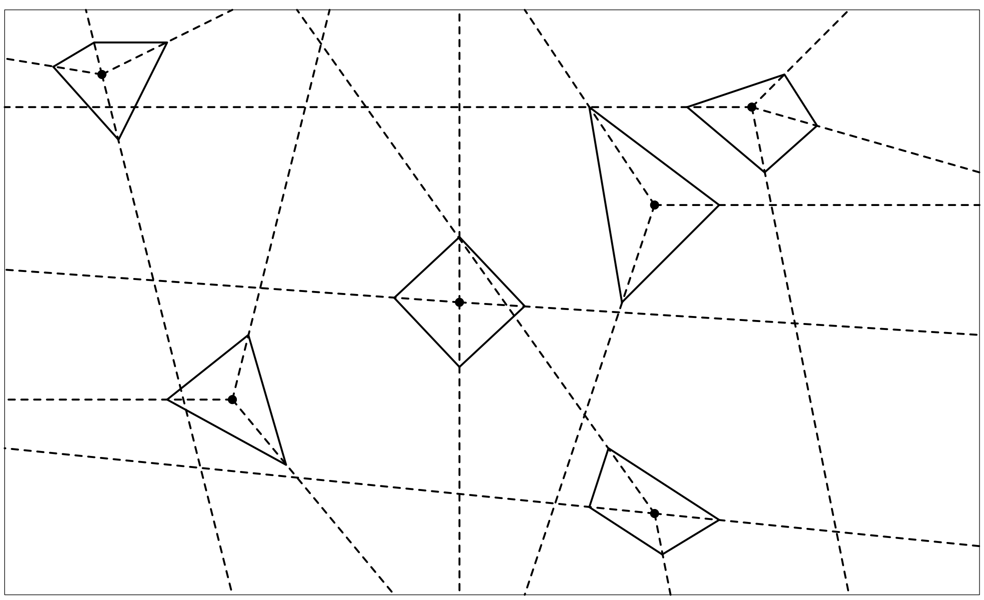

Figure 1.3: The grid graph generated by P_m.

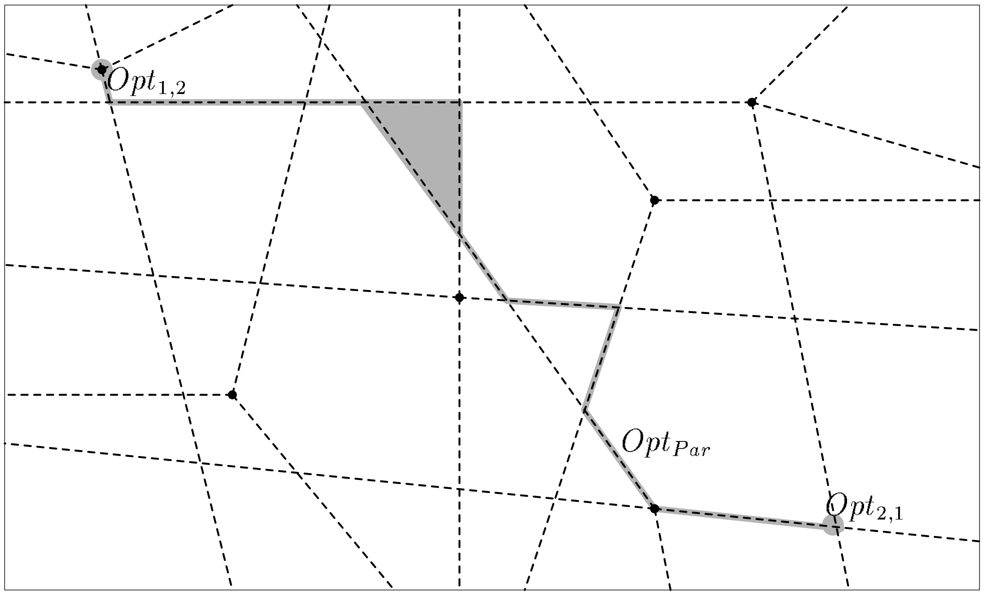

Figure 1.4: Lexicographic locations with connecting edges and cells constituting the set of Pareto locations.

(In short, using the above mentioned classification scheme, $1/P/\bullet/\gamma_{pol}/2-\sum_{Par}\cdot$.)
[Nic95] showed that the lexicographic locations can be computed in polynomial time. The Pareto locations are obtained by connecting these lexicographic locations by edges and cells of a grid graph which is defined by the half lines passing through Ex_m and each of the extreme points of $P_m, m=1,\ldots,M$ (see Figure 1.2 - 1.4).
The algorithm to find the correct edges and cells is based on Theorem 1.1 and the fact that the level curves are closed polygons which are linear in each cell of the grid graph.

2 Max-Ordering Location Problems

In this paper the main focus will be on max-ordering (MO) location problems which are optimal locations with respect to the max-ordering defined by

$$f(x) \leq_{MO} f(y) :\Leftrightarrow \max\{f^1(x),\ldots,f^Q(x)\} \leq \max\{f^1(y),\ldots,f^Q(y)\}$$

The problem $\min_{MO} f(x)$ which minimizes the worst of the objective values is used in conservative planning and robust optimization [KY97]. In location theory it has not been investigated in any depth so far. In general optimization it is also known as min-max optimization (e.g. [Du95] and references therein) or as max-linear optimization ([CHMM93], [HR94]) problem. We will use the notion of *max-ordering location problem* introduced in [Ham95] and [HN96] since the former two notions are ambiguous in the context of location problems. Obviously we can reformulate MO location problems as follows:

$$
\begin{aligned}
& \min_{x\in\mathcal{F}}{}_{MO}\ f(x) \\
\Leftrightarrow\ & \min_{x\in\mathcal{F}} \max\{f^1(x),\ldots,f^Q(x)\} \\
\Leftrightarrow\ & \min z \\
& \text{such that } f^q(x) \le z \quad \forall q=1,\ldots,Q \quad x\in\mathcal{F}
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow\ & \min z \\
& \text{such that } \mathcal{F}\cap\bigcap_{q=1}^{Q} L^q_{\le}(z) \ne \emptyset
\end{aligned} \tag{2.1}
$$

Similar to Theorem 1.1 we can therefore characterize MO locations by their level sets.

Theorem 2.1. *z_{MO} is the optimal objective value of a MO problem if and only if z_{MO} is the smallest value such that*

$$\mathcal{F}\cap\bigcap_{q=1}^{Q} L^q_{\le}(z_{MO}) \ne \emptyset. \tag{2.2}$$

In this case, the set (2.2) is Opt_{MO}, the set of all MO locations.

Reformulation (2.1) of the MO location problem indicates a close relation between MO location problems and single objective center problems.

Theorem 2.2. a) *single objective center problems are special cases of MO problems with $Q = M$ objective functions.*

b) *If in a given MO problem all single objectives are of the center type, then the MO location problem is equivalent to a single objective center problem.*

c) *$Opt_{MO}\cap Opt_{Par} \ne \emptyset$*

Proof:

a) Consider in (2.1) the special case

$$
\begin{aligned}
Q &:= M \\
f^q(x) &= w_q d(Ex_m, x)
\end{aligned}
$$

which is equivalent to the classical single-criterion center problem

$$\min_{x\in\mathcal{F}} \max\{w_1 d(Ex_1,x),\ldots,w_M d(Ex_m,m)\}.$$

b) If all single objectives are of the center type, i.e.

$$f^q(x) = \max_{m=1}^{M} w_m^q d(Ex_m, x) \quad \forall q = 1, \dots, Q,$$

then

$$\begin{aligned} \max_{q=1}^{Q} f^q(x) &= \max_{q=1}^{Q} \max_{m=1}^{M} w_m^q d(Ex_m, x) \\ &= \max_{m=1}^{M} \left(\max_{q=1}^{Q} w_m^q \right) d(Ex_m, x) \\ &= \max_{m=1}^{M} \tilde{w}_m d(Ex_m, x). \end{aligned}$$

c) Suppose $x \in Opt_{MO}$ and $x \notin Opt_{Par}$. Then there exists some $y \in Opt_{Par}$ which dominates x, i.e.

$$\begin{aligned} f^q(y) &\leq f^q(x) \quad \forall q = 1, \dots, Q \\ \text{and} \quad f^p(y) &< f^p(x) \text{ for some } p \in \{1, \dots, Q\} \end{aligned}$$

But then

$$\max\{f^1(y), \dots, f^Q(y)\} \leq \max\{f^1(x), \dots, f^Q(x)\}$$

such that $y \in Opt_{Par} \cap Opt_{MO}$.

□

Independent of any special structure the MO location problem may often be very simple to solve. This is, for instance, the case if one of the objective functions is decisively worse than the others.

Theorem 2.3. *Let Opt^q be the set of optimal locations for the single objective location problem $\min_{x \in \mathcal{F}} f^q(x)$ and let z^q be the corresponding optimal objective value, $q = 1, \dots, Q$. If there exists some $x^q \in Opt^q$ such that for all $p = 1, \dots, Q$*

$$f^p(x^q) \leq z^q \tag{2.3}$$

then x^q is an MO location with objective value $f(x^q) = z^q = z_{MO}$.

Proof: Obviously

$$f^q(x^q) = z^q \leq z_{MO} \leq \max_{p=1}^{Q} f^p(x^q). \tag{2.4}$$

The assumption of Theorem 2.3 implies that both inequalities above can be reversed such that we obtain $f^q(x^q) = z^q = z_{MO}$. □

If inequality (2.3) does not hold for all $p = 1, \dots, Q$ we can nevertheless use inequality (2.4) to obtain

$$\max_{q=1}^{Q} f^q(x^q) \leq z_{MO} \leq \min_{q=1}^{Q} \min_{x^q \in Opt^q} \max_{p=1}^{Q} f^p(x^q) \tag{2.5}$$

providing lower and upper bounds for the optimal MO objective value z_{MO}. Both bounds may be further improved by considering convex combinations

$$f(\lambda, x) = \sum_{q=1}^{Q} \lambda_q f^q(x)$$

of the objective functions $f^1, \ldots, f^Q$, where

$$\lambda \in \Lambda := \left\{ \lambda : \sum_{q=1}^{Q} \lambda_q = 1, \lambda_q \geq= 0 \right\}.$$

Since

$$f(\lambda, x) = \sum_{q=1}^{Q} \lambda_q \cdot f^q(x) \leq \left(\sum_{q=1}^{Q} \lambda_q \right) \max f^q(x)$$

holds for any location x we obtain for any MO location x_{MO}

$$\begin{aligned} \min_{x \in \mathcal{F}} f(\lambda, x) &\leq f(\lambda, x_{MO}) \\ &\leq \max_{q=1}^{Q} f^q(x_{MO}) = z_{MO} \end{aligned}$$

Hence

$$\max_{\lambda \in \Lambda} \min_{x \in \mathcal{F}} f(\lambda, x) \leq z_{MO} \tag{2.6}$$

which improves the lower bound of (2.5).

The set of Pareto locations, Opt_{Par} is often large and a decision has to be made, which of the Pareto locations to choose. On the other hand MO locations, which are preferable from the point of view of conservative planning, are not always Pareto locations. One possibility to solve this dilemma is making use of part c) of Theorem 2.2, which states that there are always solutions which are both MO and Pareto locations. For $Q > 2$ the idea of max-ordering can be iterated, i.e. among all MO solutions one is chosen, which minimizes the second largest objective value and so on. Let x be a location and $f(x)$ the corresponding objective value vector. Let $\mathrm{sort}(f(x))$ be a permutation of the components of $f(x)$ in nondecreasing order, i.e.

$$\mathrm{sort}(f(x))^1 \geq \ldots \geq \mathrm{sort}(f(x))^Q.$$

Location x is said to be *lexicographic max-ordering (lex-MO)* optimal, or a lex-MO location, if

$$\mathrm{sort}(f(x)) \leq_{lex} \mathrm{sort}(f(y)) \quad \forall y \in \mathcal{F}.$$

The set of lex-MO locations is denoted by Opt_{lex-MO}. Clearly, $\text{sort}(f(x))^1 = z_{MO}$ for a lex-MO location. Furthermore, suppose a lex-MO location x is dominated, then the sorted objective value vector would certainly be not lexicographically minimal. Thus we have (see also [Ehr97]):

Proposition 2.4.

$$Opt_{lex-MO} \subset (Opt_{Par} \cap Opt_{MO}) \tag{2.7}$$

The sorted objective value vector is the same for all lex-MO locations. Furthermore, lex-MO solutions can be axiomatically characterized by three properties which are appealing from a decision making point of view, see e.g. [Ehr97]. Lex-MO locations have already been investigated in the discrete case in [Ogr97]. We consider lex-MO location problems in the plane and on networks.

3 Solution strategies for MO location problems

In this section three strategies for solving MO location problems are presented and exemplified with a specific location problem, respectively.

3.1 Direct Approach: $1/P/\bullet/l_2^2/Q - \sum_{MO}$

In some cases it is possible to solve MO problems directly by just using the definition or reformulation (2.1). As an example we discuss $1/P/\bullet/l_2^2/Q - \sum_{MO}$, i.e. the problem of finding all max-ordering locations in the plane, where each of the Q objective functions is of the median type with squared Euclidean distances between existing and new facilities. Since each of the objective functions is of the form

$$f^q(x) = \sum_{m=1}^{M} w_m^q \left((a_{m_1} - x_1)^2 + (a_{m_2} - x_2)^2\right)$$

we obtain by differentiation or by finding the smallest z such that $L_{\leq}(z) = L_{=}(z) \neq \emptyset$ that

$$x^q = (x_1^q, x_2^q)$$

with

$$x_k^q = \frac{\sum w_m^q a_{m_k}}{\sum w_m^q}, k = 1, 2 \tag{3.1}$$

is the unique optimal location of $\min_{x \in \mathbb{R}^2} f^q(x)$ (see [FJW92], [LMW88], or [Ham95]). The same literature also features proofs that the level curves $L_{\leq}^q(z)$ are circles $C(x^q, r^q(z))$ centered at x^q with radius

$$r^q(z) = \sqrt{\frac{1}{\sum_{m=1}^{M} w_m^q} z - const(q)} \tag{3.2}$$

Consequently, the results of Section 1 imply that

$$Opt_{lex} = \{x^1, \ldots, x^Q\}$$

since in the first iteration of the solution algorithm $\mathcal{F}^{\pi(1)} = \{x^{\pi(1)}\}$ will always contain exactly one element, such that the following iterations become redundant. Moreover

$$Opt_{Par} = conv\{x^1, \ldots, x^Q\}$$

is the convex hull of the single-criterion optima, since criterion (1.1) can be satisfied for circles $L^q_=(z^q)$ and disks $L^q_\leq(z^q)$ exactly in points $x \in conv\{x^1, \ldots, x^Q\}$. In order to solve the MO problem $1/P/\bullet/l_2^2/Q - \sum_{MO}$ we use (3.2) to rewrite

$$\begin{aligned} f^q(x) = z &= w^q \cdot (r^q(z))^2 + const(q) \text{ with } w^q = \sum_{m=1}^{M} w^q_m \\ &= w^q \cdot l_2^2(x^q, x) + const(q). \end{aligned}$$

Hence the reformulation (2.1) of the MO problem implies

Theorem 3.1. *The MO location problem $1/P/\bullet/l_2^2/Q - \sum_{MO}$ is equivalent to the single objective center problem*

$$\min_{x \in \mathbb{R}^2} \max_{q=1}^{Q} \left(w^q l_2^2(x^q, x) + const(q) \right) \tag{3.3}$$

with respect to "existing" facilities x^q and weighted squared Euclidean plus constant distance functions.

In order to compute Opt_{MO} we can adapt the Elzinga/Hearn algorithm [EH72] or use the following algorithm which relies on Theorem 2.1

Algorithm 3.1. *(Solving $1/P/\bullet/l_2^2/Q - \sum_{MO}$)*

Input: $Ex_m = (a_{m_1}, a_{m_2}) \in \mathbb{R}^2, m = 1, \ldots, M$
$w^q_m, m = 1, \ldots, M;\ q = 1, \ldots, Q$

Output: Opt_{MO}

1. *Compute $x^q = (x^q_1, x^q_2)$ with (3.1)*
2. *If criterion (2.2) holds, output $Opt_{MO} = \{x^q\}$*
3. *For all triples $\{q_1, q_2, q_3\} \subseteq \{1, \ldots, Q\}$ compute the unique intersection point*

$$x_{q_1,q_2,q_3} = L^{q_1}_\leq(z) \cap L^{q_2}_\leq(z) \cap L^{q_3}_\leq(z)$$

with minimal z until $f^q(x_{q_1,q_2,q_3}) \leq z \quad \forall q = 1, \ldots, Q$. Output in the latter case

$$Opt_{MO} = \bigcap_{q=1}^{Q} L^q_\leq(z) = \{x_{q_1,q_2,q_3}\}.$$

Note that the validity of Algorithm 3.1 follows directly from Helly's Theorem, [Hel23].

3.2 Decision Space Approach: $1/P/\bullet/\gamma_{pol}/2 - \sum_{(lex-)MO}$

In the decision space approach we apply the following strategy: We use the level curve/level set characterization of Pareto locations (Theorem 1.1) to determine Opt_{Par} and use the fact that it is sufficient to search in Opt_{Par} for some MO location (Theorem 2.2(c)) to find some $x \in Opt_{MO}$ efficiently.
The approach is exemplified in the problem $1/P/\bullet/\gamma_{pol}/2-\sum_{MO}$ considered already at the end of Section 1 in its Pareto version. As we have seen there, Opt_{Par} is a chain connecting the two sets of lexicographically optimal locations by edges and cells of the grid graph (see Figure 3.1).
For this purpose consider the two sets $Opt_{1,2}$ and $Opt_{2,1}$ of lexicographic locations with respect to permutations $\pi(1) = 2$ and $\pi(1) = 1$, respectively. The following three cases may occur:

Case 1 $f^1(x_{1,2}) \geq f^2(x_{1,2}) \; \forall x_{1,2} \in Opt_{1,2}$.
Then

$$\max_{q=1,2} f^q(x) \geq f^1(x) \geq f^1(x_{1,2}) = \max_{q=1,2} f^q(x_{1,2}) \; \forall x \in \mathbb{R}^2$$

and

$$Opt_{Par} \cap Opt_{MO}(f^1, f^2) = Opt_{1,2} \, .$$

Case 2 $f^2(x_{2,1}) \geq f^1(x_{2,1}) \; \forall x_{2,1} \in Opt_{2,1}$.
By symmetry we obtain as in Case 1

$$Opt_{Par} \cap Opt_{MO}(f^1, f^2) = Opt_{2,1} \, .$$

Case 3 $\forall x = x_{1,2} \in Opt_{1,2}$ and $y = y_{1,2} \in Opt_{2,1}$

$$f^1(x) < f^2(x) \text{ and } f^2(y) < f^1(y) \tag{3.4}$$

Consider any path through Opt_{Par} connecting two endpoints $x_{1,2} \in Opt_{1,2}$ and $x_{2,1} \in Opt_{2,1}$ and consisting of edges of the grid graph (see Figure 3.1).

If $p(t)$ is a parametrization of the path with $p(0) = x_{1,2}$ and $p(1) = x_{1,2}, f^1(p(t))$ increases while $f^2(p(t))$ decreases. Therefore (3.4) implies the existence of a node x_{MO} of the grid graph such that $f^1(x_{MO}) = f^2(x_{MO})$ or of two adjacent nodes x and y such that (3.4) holds. In the latter case x_{MO} is the unique point in the line segment $[x, y]$ with $f^1(x_{MO}) = f^2(x_{MO})$. Notice that x_{MO} is easy to compute since $f^1(p(t))$ and $f^2(p(t))$ are linear on $[x, y]$ (see Figure 3.1). In both cases $x_{MO} \in Opt_{MO}$, and the whole set Opt_{MO} is obtained by Theorem 2.1.

Since the objective functions $\sum_{m=1}^{M} w_m d(x, x_m)$ and $\max_{m=1}^{M} w_m d(x, x_m)$ are strictly quasiconvex and continuous if $w_m \geq 0$ for all $m = 1 \ldots M$, we can apply a theorem from [Beh77] and get the following result.

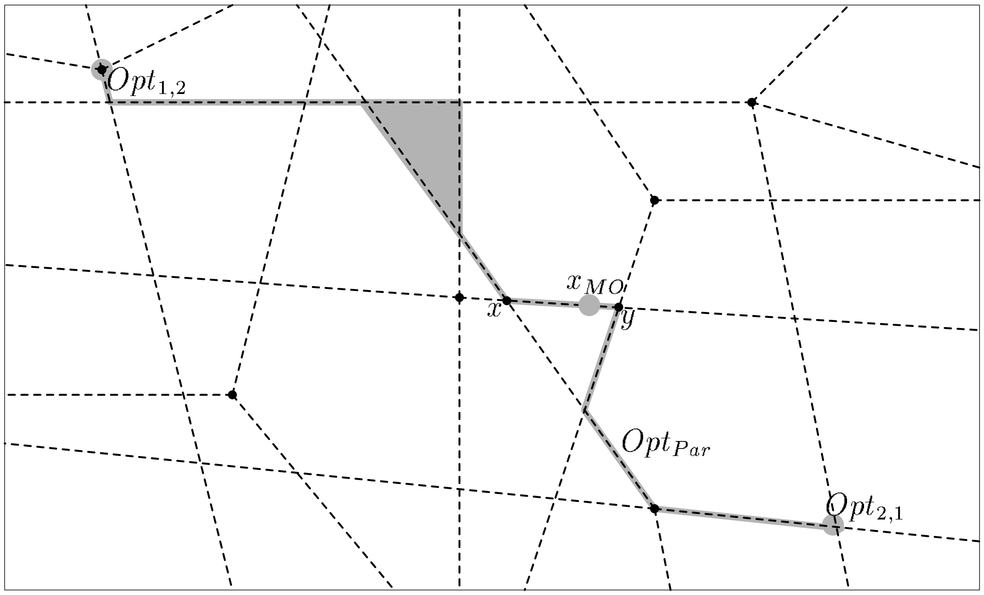

Figure 3.1: An example for x_{MO} on the Pareto chain.

Theorem 3.2. *If* $w_m^q \geq 0 \ \forall m = 1, \ldots, M; \ q = 1, \ldots Q$ *then there exists an index* $q^* \in \{1, \ldots, Q\}$ *such that* $f^{q^*}(x) = z_{MO}$ *for all* $x \in Opt_{MO}$.

Theorem 3.2 immediately implies a generic algorithm to solve lex-MO location problems in the plane.

Algorithm 3.2. *(Solving* $1/P/\bullet/\bullet/Q - \sum_{lex-MO}$*)*

Input: $Ex_m = (a_{m_1}, a_{m_2}) \in \mathbb{R}^2, m = 1, \ldots, M$
$w_m^q \geq 0, m = 1, \ldots, M, \ q \in \mathcal{Q} := \{1, \ldots, Q\}, \mathcal{F} = \mathbb{R}^2$

Output: Opt_{lex-MO}

1. *Find* Opt_{MO} *with respect to* $\mathcal{F}$ *and* $\mathcal{Q}$, *i.e. find all solutions of*

$$\min_{x \in \mathcal{F}} \max_{q \in \mathcal{Q}} f^q(x).$$

2. *Identify an index* q^* *according to Theorem 3.2.*

3. *Let* $\mathcal{F} := Opt_{MO}$ *and* $\mathcal{Q} = \mathcal{Q} \setminus \{q^*\}$.

4. *If* $\mathcal{Q} = \emptyset$ or $|Opt_{MO}| = 1$ *then STOP else goto 1.*

The determination of an index q^* has to be adapted to the problem at hand. Let us consider, for instance, $1/P/\bullet/\gamma_{pol}/Q - \sum_{lex-MO}$. The set of MO locations can be found as described above. Furthermore, it is known that the plane can be partitioned into cells, where the objective functions are linear, see [Nic95]. If the cell partition of Opt_{MO} is known we can exploit linearity of the objective functions to easily check which of the objectives is constant on Opt_{MO}.
The generic algorithm actually exhibits one important property of lex-MO locations: If the value of one objective function is known for an optimal solution ($f^{q^*}(x) = z_{MO}$ in this case) then only the remaining ones have to be considered for optimization where $f^{q^*}(x) = z_{MO}$ is used as an additional constraint. This property is called *reduction property* in [Ehr97]. The same reference also shows that, together with the fact that lex-MO solutions are always MO solutions, the reduction property is characteristic for lex-MO optimization problems.

3.3 Objective Space Approach: $1/\mathcal{G}/\bullet/d(\mathcal{G},\mathcal{V})/2 - \sum_{MO}$

In one way, this approach is similar to the one in subsection 3.2: First the set of Pareto locations is computed, albeit in this instance of a location problem without using level curves and level sets. But then the space of objective values is investigated to determine a MO location x_{MO}.

The network location problem which is used as an example to show the approach is defined as follows: $\mathcal{G}$ is an undirected graph with node set $\mathcal{V}$ and edge set $\mathcal{E}$. The nodes are identified with the existing facilities and $Q = 2$ sets of non-negative weights w_m^1, w_m^2, $m = 1, \ldots, M = |V|$ are given. The edges have lengths $l(e)$. The new facility x can be any *point* in the graph, i.e. either $x = v \in V$ is a node or $x = (e,t)$ lies on the interior of an edge $e = [v_i, v_j] \in E$ with distance

$$d(x,v) = \min\{d(v,v_i) + t \cdot l(e), (1-t)l(e) + d(v_j,v)\} \tag{3.5}$$

to node $v \in V$ (see Figure 3.2). Here $d(v,v_i)$ is the usual shortest path distance between nodes v_i and v.
In order to compute Opt_{Par} we follow the solution strategy developed in [HNL96].

1. Find for all edges $e = [v_i, v_j]$ the set $Opt_{Par}(e)$ of local Pareto locations, i.e. points in e which are not dominated by any other point in e.

2. Eliminate all local Pareto locations which are not globally Pareto, i.e. $x \in Opt_{Par}(e)$ is dominated by some $x' \in Opt_{Par}(e')$ with $e', e \in \mathcal{E}, e \neq e'$.

Part 1 of the solution strategy is implemented by analyzing the functions $f^1(x)$ and $f^2(x)$ where $x = (e,t)$ is parameterized using $t \in [0,1]$. Due to (3.5)

$$f^q(x) = \sum_{m=1}^{M} w_m^q d(x, v_m)$$

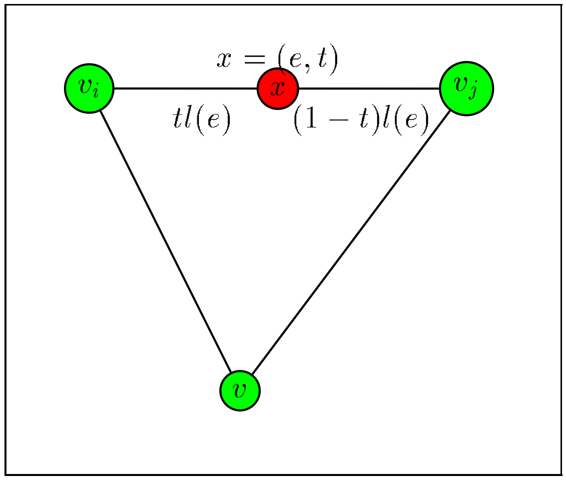

Figure 3.2: Distance between point $x = (e,t)$ on edge $e = [v_i, v_j]$ and node $v \in \mathcal{V}$.

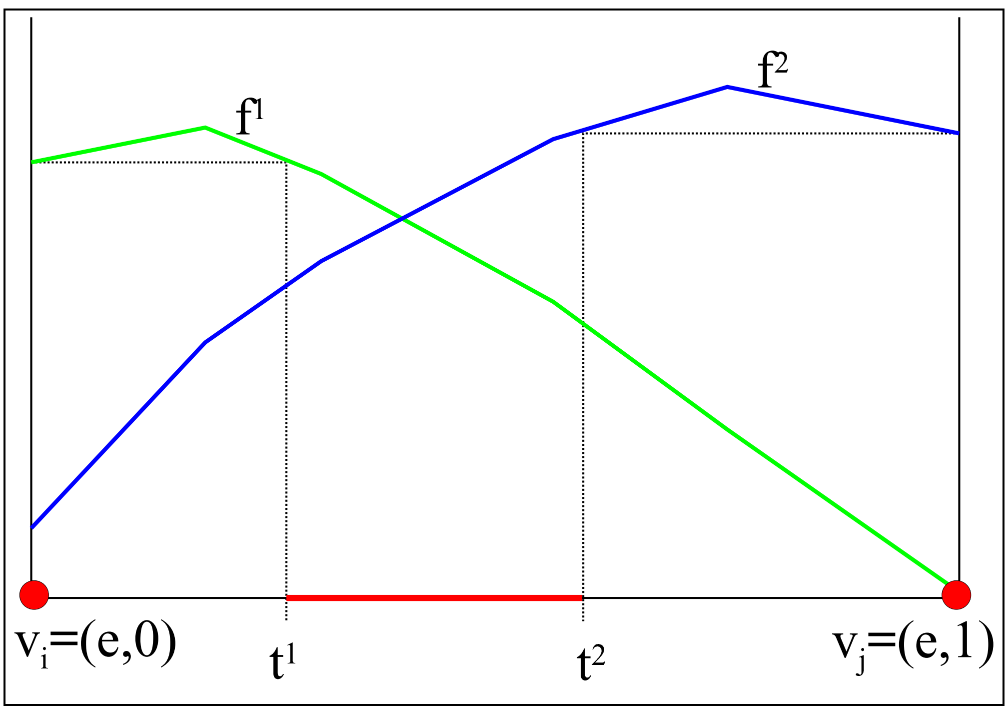

Figure 3.3: Example of local Pareto locations: $\{v_i, v_j\}$ and $\{(e,t) : t^1 < t < t^2\}$.

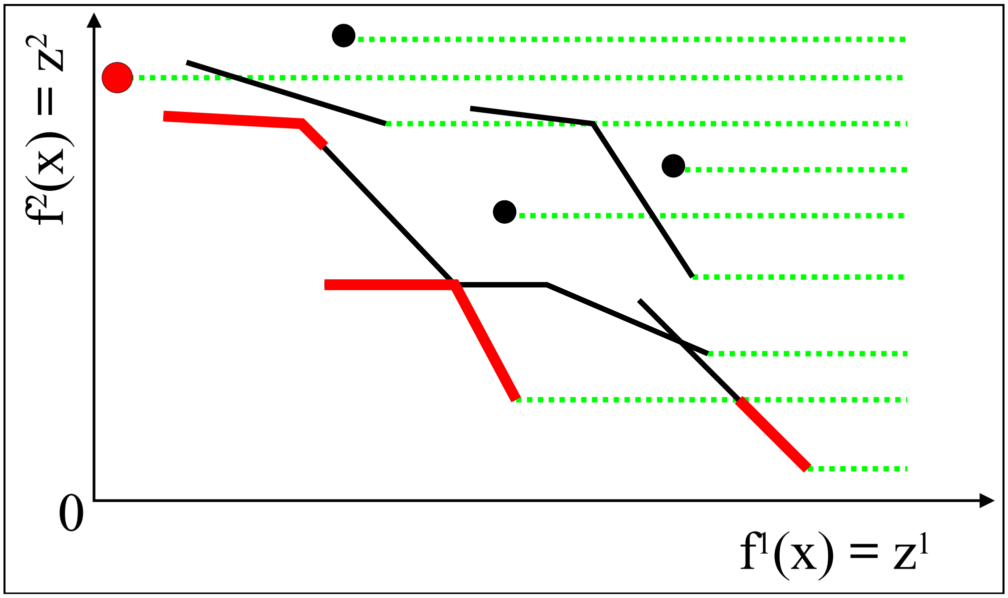

Figure 3.4: Example of the point set $\mathcal{Z}$ representing the vector values of local Pareto points of 5 edges and its lower envelope (in bold).

is piecewise linear and concave such that the local Pareto locations are easily identified (see Figure 3.3)
The objective values (z^1, z^2) associated with the set of local Pareto locations are represented in the objective space where the interior part $\{(e,t) : t^1 < t < t^2\}$ (provided it exists) corresponds to a piecewise linear concave function. Hence we can implement Part 2 of the solution strategy by computing the lower envelope of the points

$$\mathcal{Z} = \Big\{(z^1, z^2) \quad : \quad \begin{array}{l} z^1 = f^1(x) \text{ and } z^2 = f^2(x) \text{ and } x \text{ interior local Pareto point} \\ \text{or} \\ z^1 \geq f^1(x) \text{ and } z^2 = f^2(x) \text{ and } x \in V \text{ locally Pareto}\Big\} \end{array}$$

Using Hershberger's algorithm (see [Her89]) this can be done very efficiently in time bounded by $O(|\mathcal{V}||\mathcal{E}| \log |\mathcal{V}||\mathcal{E}|)$. The set Opt_{Par} of (global) Pareto locations is then the set of points in G mapped into the lower envelope (where only the left-most point of each horizontal piece of the envelope is used).

According to Theorem 2.2(c) it suffices to find in Opt_{Par} a location x with smallest MO value $\max\{f^1(x), f^2(x)\}$. This problem is easy to solve since the mapping of Pareto locations into $\mathcal{Z}$ is known: We just find the point in the set $\mathcal{Z}$ (excluding the redundant horizontal pieces) which is closest(with respect to the Tchebycheff distance) to the origin $(z^1, z^2) = (0, 0)$ (see Figure 3.5).
A generalization of this approach can be used to solve the same problem with Q single objectives, i.e. $1/\mathcal{G}/\bullet/d(\mathcal{V},\mathcal{G})/Q - \sum_{MO}$. The solution strategy of finding $Opt_{Par}(e)$ is

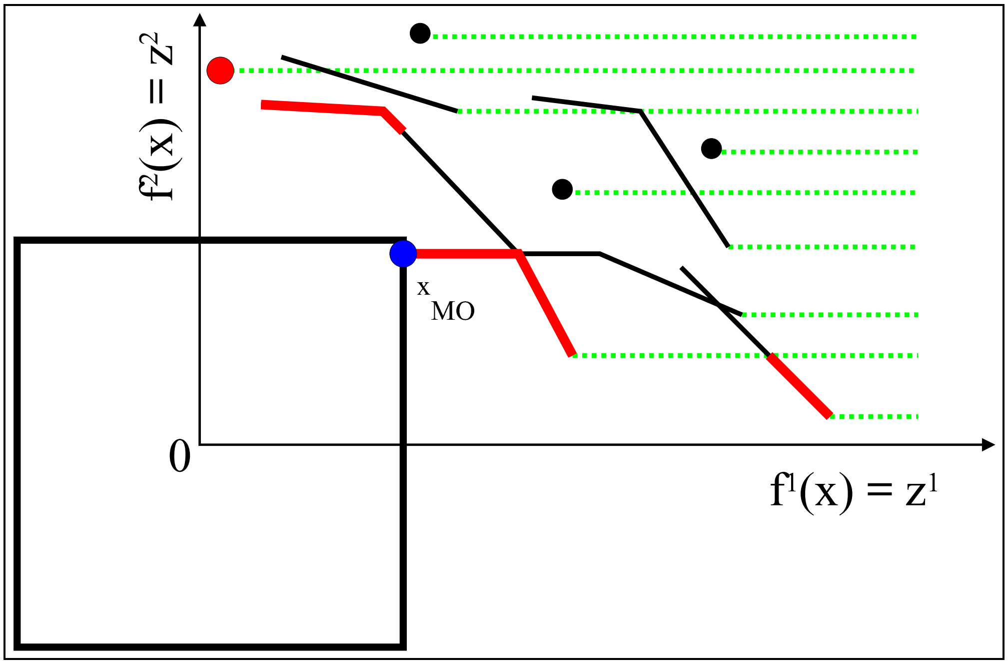

Figure 3.5: Value (z^1, z^2) of point $x_{MO} \in Opt_{Par}$ with minimal value $\|(z^1, z^2)\|_\infty$.

the same as in the case of $Q = 2$. The elimination of dominated local Pareto locations is done using 2-variable linear programs (see [HNL96]) applying Megido's algorithm [Meg82]. Finding Opt_{MO} is then equivalent to solving a distance problem in $\mathbb{R}^Q$ with respect to the Tchebycheff distance.

The same problem can be solved by a second objective space approach, which is the first step in a procedure to solve the lex-MO location problem $1/\mathcal{G}/\bullet/d(\mathcal{V},\mathcal{G})/Q - \sum_{lex-MO}$ described below. Note that the objective functions f^q are not quasiconvex here, so Theorem 3.2 cannot be applied and an index q^* such that $f^{q^*}(x) = z_{MO}$ for all $x \in Opt_{MO}$ need not exist. In general the set of MO locations may be composed of Q subsets with $f^q(x) = z_{MO}$, $f^j(x) \neq z_{MO}, j \neq Q$. An approach as presented in the planar case would then lead to an exponential algorithm considering all permutations π of $(1, \ldots, Q)$. However, we will show that the problem can be solved efficiently by an objective space approach.

As described in Section 3.3 the objective functions are concave on each edge of the network. In fact, they are piecewise linear with at most $|\mathcal{V}|$ breakpoints corresponding to bottleneck points, see [HNL96]. Therefore, on each edge, the objective functions can be determined in $O(Q|\mathcal{V}|\log|\mathcal{V}|)$.

Note that, due to (2.7), only edges possibly containing Pareto points have to be considered in the following. We first solve the max-ordering location problem.

Approach 1:

We compute the upper envelope of the objectives f^q on the edge $[v_i, v_j]$, see Figure 3.6. This can be done in $O(Q|\mathcal{V}|\log(\max|\mathcal{V}|, Q))$ time, because there are at most $Q|\mathcal{V}|$ line

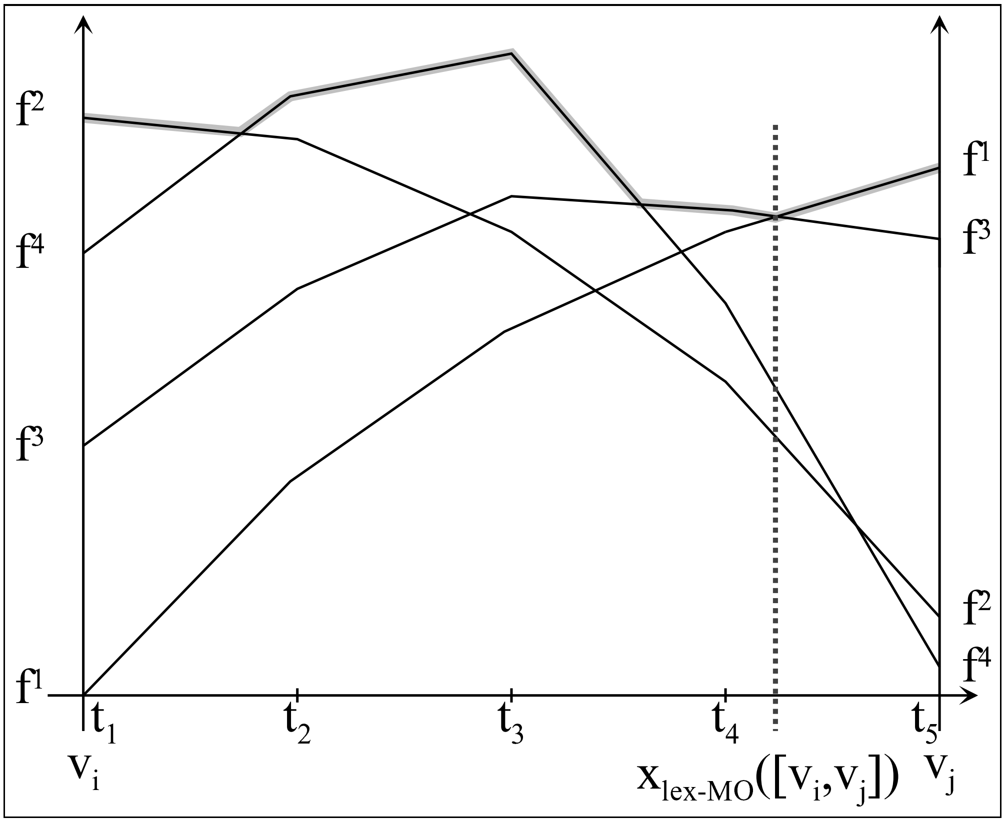

Figure 3.6: Objective functions on an edge $[v_i, v_j]$, upper envelope, and candidate $x_{MO}([v_i, v_j])$ for a lex-MO location.

segments, see [Her89].

Lemma 3.3. *On each edge there is at most one candidate for a lex-MO solution, which is at a breakpoint of the upper envelope.*

Proof: The lemma follows from concavity and piecewise linearity of the objectives. □

The (unique) minimizer of the upper envelope can now be found in $O(|\mathcal{V}|Q)$ time. Finally, the at most $|\mathcal{E}|$ candidates on the edges and the nodes have to be compared, i.e. their objective value vectors sorted and lexicographically compared. This requires $O(Q \log Q(|\mathcal{E}| + |\mathcal{V}|))$. In total this approach requires $O(|\mathcal{E}|Q \log(\max\{|\mathcal{V}|, Q\}))$ elementary operations.

Approach 2:
Since there exist at most $|\mathcal{V}|$ breakpoints of the objectives on each edge (the breakpoints can only occur at bottleneck points, and these are independent of the specific weights), we can subdivide this edge into smaller intervals and solve $\min_x \max_q f^q(x)$ on each of these intervals, where all objectives are linear. This is equivalent to solving

$$\begin{aligned} &\min z \\ z &\geq f^q((e,t)) \quad q = 1,\ldots,Q \\ t &\in [t_i, t_j] \end{aligned}$$

where $0 = t_1 \leq \ldots \leq t_K = 1$ and $K \leq |\mathcal{V}|$. These linear programs in two variables can be solved in time linear in the number of constraints, i.e. $O(Q)$, see [Meg82]. Finding the smallest of the optimal solutions is not worse than $O(|\mathcal{V}|)$ and the comparison of all the candidates is the same as in Approach 1. Hence we have a total of $O(\max\{Q|\mathcal{V}|, |\mathcal{E}|Q \log Q\})$ for this algorithm.

4 Conclusions and Further Research

In this paper basic results for lexicographic and Pareto location problems were reviewed. Max-ordering location problems were introduced and some general results were proved. Three solution strategies, the direct approach, the decision space approach, and the objective space approach where shown to find the set Opt_{MO} of MO locations efficiently.
An area of multicriteria location theory which is immediately motivated by this paper is one where vector-valued locations are subject to an additional norm. That is, we look for a location x such that

$$\|(f^1(x), \ldots, f^Q(x))\|$$

is minimized. MO problems are special cases of this, more general, model where the norm is the Tchebycheff norm. The objective space approach of Section 3.3 can obviously immediately be carried over to this more general approach. General results for these types of location problem are under research.
Another topic is the determination of other location problems which have the reduction property. As in lex-MO location problems in the planar case this property may be exploited to design efficient algorithms.

References

[Beh77] F.A. Behringer. Lexicographic quasiconcave multiobjective programming. *Zeitschrift für Operations Research*, 21:103–116, 1977.

[CHMM93] Sung-Jin Chung, H. W. Hamacher, Francesco Maffioli, and Katta G. Murty. A note on combinatorial optimization with max-linear objective functions. *Discrete Applied Mathematics*, 42:139–145, 1993.

[Du95] D.Z. Du. Minimax and its applications. In R. Horst and P.M. Pardalos, editors, *Handbook of Global Optimization*, pages 339–367. Kluwer Academic Publishers, 1995.

[EH72] D.J. Elzinga and D.W. Hearn. Geometrical solutions for some minimax location problems. *Transportation Science*, 6:389–394, 1972.

[EHK+97] M. Ehrgott, H.W. Hamacher, K. Klamroth, S. Nickel, A. Schöbel, and M.M. Wiecek. A note on the equivalence of balance points and Pareto solutions in multiple-objective programming. *Journal of Optimization Theory and Applications*, 92(1):209–212, 1997.

[Ehr97] M. Ehrgott. *Multiple Criteria Optimization – Classification and Methodology.* Shaker Verlag, Aachen, 1997.

[FJW92] R. L. Francis, L.F. McGinnis Jr., and J.A. White. *Facility Layout and Location: an Analytical Approach.* Prentice Hall, Engelwood Cliffs, N.J., 1992. 2nd Edition, International Series in Industrial and Systems Engineering.

[FW74] R.L. Francis and J.A. White. *Facility Layout and Location: an Analytical Approach.* Prentice Hall, Engelwood Cliffs, N.J., 1974.

[Ham92] H. W. Hamacher. Lectures on location problems. Technical report, University of Kaiserslautern, 1992. WS 92/93.

[Ham95] H.W. Hamacher. *Mathematische Lösungsverfahren für planare Standortprobleme.* Vieweg Verlag, Braunschweig, 1995. 171 pages.

[Hel23] E. Helly. On sets of convex bodies with common points (in German). *Jahrbuch der Deutschen Mathematiker Vereinigung*, 32:175–176, 1923.

[Her89] J. Hershberger. Finding the upper envelope of n line segments in o (n log n) time. *Information Processing Letters*, 33(4):169–174, 1989.

[HN93] H.W. Hamacher and S. Nickel. Median location probems with several objectives. *Studies in Locational Analysis*, 4:149–153, 1993.

[HN94] H.W. Hamacher and S. Nickel. Combinatorial algorithms for some 1-facility median problems in the plane. *European Journal of Operational Research*, 79:340–351, 1994.

[HN95] H.W. Hamacher and S. Nickel. Restricted planar location problems and applications. *Naval Research Logistics*, 42:967–992, 1995.

[HN96] H.W. Hamacher and S. Nickel. Multicriteria planar location problems. *European Journal of Operational Research*, 94:66–86, 1996.

[HNL96] H.W. Hamacher, S. Nickel, and M. Labbé. Multicriteria network location problems with sum objectives. Technical report, University of Kaiserslautern, Department of Mathematics, 1996. Report in Wirtschaftsmathematik No. 15, submitted to Networks.

[HR94] H.W. Hamacher and G. Ruhe. On spanning tree problems with multiple objectives. *Annals of Operations Research*, 52:209–230, 1994.

[KY97] P. Kouvelis and G. Yu. *Robust Discrete Optimization and Its Applications.* Kluwer, 1997.

[LMW88] R.F. Love, J.G. Morris, and G.O. Wesolowsky. *Facilities Location: Models and Methods.* North Holland, 1988.

[Meg82] N. Megiddo. Linear-time algorithms for linear programming in r^3 and related problems. *SIAM Journal on Computing*, 12:759–776, 1982.

[Nic95] S. Nickel. *Discretization of Planar Location Problems.* Shaker Verlag, Aachen, 1995.

[Ogr97] W. Ogryczak. On the lexicographic minimax approach to location problems. *European Journal of Operational Research*, 100:566–585, 1997.

[Pla95] F. Plastria. Continuous location problems. In Zvi Drezner, editor, *Facility Location. A Survey of Applications and Methods*, pages 225–260. Springer Verlag, 1995.