A reduction algorithm for integer multiple objective linear programs

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We consider a multiple objective linear program (MOLP) \( \max \{ Cx | Ax = b, x \in \mathbb{N}_0^p \} \) where \( C = (c_{ij}) \) is the \( p \times n \)-matrix of \( p \) different objective functions \( z_i(x) = c_{i1}x_1 + \ldots + c_{in}x_n, i = 1, \ldots, p \) and \( A \) is the \( m \times n \)-matrix of a system of \( m \) linear equations \( a_{k1}x_1 + \ldots + a_{kn}x_n = b_k, k = 1, \ldots, m \) which form the set of constraints of the problem. All coefficients are assumed to be natural numbers or zero. The set \( M \) of admissable solutions is given by \( M = \{ x | Ax = b, x \in \mathbb{N}_0^n \} \). An efficient solution \( \overline{x} \) is an admissable solution such that there exists no other admissable solution \( x' \) with \( C\overline{x} < Cx' \). The efficient solutions play the role of optimal solutions for the MOLP and it is our aim to determine the set of all efficient solutions.

From the \( p \) different objective function we generate a new parametric objective function \( f \) which has the property to preserve the canonical order on \( (\mathbb{N}_0^n; \leq_{lex}) \). Therefore we can use \( f \) to find all efficient solutions in a lexicographic order. An efficient solution which is already found will be eliminated by some constraints to get the next efficient solution.

This is a theoretical approach to solve the problem because of the high complexity of the procedures involved. But we have already shown in [] that for small bicriteria integer linear programs it is applicable. This approach generalizes the results in [] to the multicriteria case by introducing an objective function \( f \) which is strictly monotone for two different orderings.

1 The scaling

We determine for every objective function \( z_i(x) \) the minimal value \( a_i \) and the maximal value \( b_i \). We denote the difference by \( d_i = b_i - a_i + 1 \). If for some objective function \( z_i(x) \) we have \( d_i = 1 \) then we may drop this objective function because it is unnecessary as a criterion for the decision. Without a loss of generality we assume that \( d_i = 1 \) then we may drop this objective function because it is unnecessary as a criterion for the decision. Without a loss of generality we assume that \( d_i > 1 \) and introduce a new parametrix
objective function $f : \mathbb{N}_0^n \to \mathbb{N}_0^n$

$$f(x) = \sum_{i=1}^p \left( \prod_{j=1}^i d_j \right)^{-1} z_i(x)$$

We write $z_i(x) = c_i(x)$ where $c_i = (c_{i1}, ..., c_{im})$, and have $f(x) = f(x) = \sum_{i=1}^p \left( \prod_{j=1}^i d_j \right)^{-1} c_i(x) = \frac{1}{d_1} c_1 x + \frac{1}{d_1 d_2} c_2 x + ... + \frac{1}{d_1 ... d_p} c_p x$. We consider the canonical order $\leq$ on $\mathbb{N}_0^p$ which is defined componentwise. $(a_1, ..., a_p) \leq (b_1, ..., b_p)$ if and only if $a_i \leq b_i$ for every $i = 1, ..., p$. The criterion space of the MOLP is given by $Z = \{ z \in \mathbb{N}_0^p | z = Cx, x \in \mathbb{N} \}$

under the above hypothesis. It is obvious that $(Z; \leq)$ is a suborder of $(\mathbb{N}_0^p; \leq)$. Furthermore the function $g : Z \to \mathbb{R}_0$ $g(z) = g(c_1 x, ..., c_n x) = f(x)$ is strictly monotone on $Z$ as

$$\left( \prod_{j=1}^i d_j \right)^{-1} > 0.$$

On the other hand let us consider the lexicographic order $<_{lex}$ on $\mathbb{N}_0^p$ which is defined in the following way: $(a_1, ..., a_p) <_{lex} (b_1, ..., b_p)$ if there exists $m \in \mathbb{R}_0$ with $0 \leq m < p$ such that $a_k = a_k$ for $k = 1, ..., m$ and $a_m < b_m$. The linear order $(\mathbb{N}_0^p; <_{lex})$ has the linear suborder $(Z; <_{lex})$. Furthermore we may consider $(\mathbb{R}; <)$ as a lexicographic order as well. Our aim is to show that $g : Z \to \mathbb{R}_0$ also preserves the lexicographic order. We will use the following

**Proposition 1.1**

Let $\Delta c_i d_i \in \mathbb{N}$ with $0 < \Delta c_i < d_i$ and $i = 1, ..., p$. If $a_i = \Delta c_i + \frac{a_i + 1}{d_{i+1}}$ for $i = 1, ..., p-1$ and $a_p = \Delta c_p$ for $i = p$ then we have $\frac{a_i}{d_i} < 1$ for $i = 1, ..., p$.

**Proof.** For $i = p$ we have $a_p < d_p$ and hence $\frac{a_p}{d_p} < 1$. Assume that we have proved $\frac{a_i}{d_i} < 1$ for some $i = 1, ..., p-1$. We have

$$a_i = \Delta c_i + \frac{a_i + 1}{d_i + 1} \leq d_i - 1 + \frac{d_i + 1}{d_i + 1} < d_i - 1 + 1 = d_i.$$
Hence we have \( \frac{a_i}{d_i} < 1 \).

**Theorem 1.2**

The function \( g : Z \to \mathbb{R}_0 \) defined by \( (z) = g(c_1 x, \ldots, c_p x) = f(x) \) preserves the lexicographic order.

Proof. Let \( z^1 <_{leq} z^2 \) and hence \( (c_1 x^1, \ldots, c_n x^1) \leq_{leq} (c_1 x^2, \ldots, c_n x^2) \). We have

\[
\left( \prod_{j=1}^{i} d_j \right)^{-1} c_i x^1 = \left( \prod_{j=1}^{i} d_j^{-1} \right) c_i x^2 \text{ for } i = 1, \ldots, m - 1
\]

and

\[
\left( \prod_{j=1}^{m} d_j \right)^{-1} c_m x^1 < \left( \prod_{j=1}^{m} d_j \right)^{-1} c_m x^2.
\]

It remains to show that

\[
\left| \sum_{i=m+1}^{p} \left( \prod_{j=1}^{m} d_j \right)^{-1} c_i (x^2 - x^1) \right| < \left( \prod_{j=1}^{m} d_j \right)^{-1} c_m (x^2 - x^1)
\]

or after a division that

\[
\sum_{i=m+1}^{p} \left( \prod_{j=m+1}^{i} d_j \right)^{-1} c_i |x^2 - x^1| < c_m (x^2 - x^1)
\]

(We also notice that \( 1 \leq c_m (x^2 - x^1) \) because we have only integers). For our convenience we put \( \Delta \bar{c}_i := |c_i (x^2 - x^1)| \)

\[
\sum_{i=m+1}^{p} \left( \prod_{j=m+1}^{i} d_j \right)^{-1} \Delta \bar{c}_i = \frac{\Delta c_{m+1}}{d_{m+1}} + \frac{\Delta c_{m+2}}{d_{m+1} \cdot d_{m+2}} + \cdots + \frac{\Delta c_{p-1}}{d_{m+1} \cdots d_{p-1}} + \frac{\Delta c_p}{d_{m+1} \cdots d_p} = \frac{1}{d_{m+1}}
\]
\[
(\Delta c_{m+1} + \frac{1}{d_{m+2}}(\Delta c_{m+2} + \ldots + \frac{1}{d_{p-1}}(\Delta c_{p-1} + \frac{1}{d_p} \Delta c).
\]

Using the proposition 1.1 we get

\[
= \frac{1}{d_{m+1}}(\Delta c_{m+1} + \frac{1}{d_{m+2}}(\Delta c_{m+2} + \ldots + \frac{1}{d_{p-1}}(\Delta c_{p-1} + \frac{a_p}{d_p}).
\]

\[
= \frac{1}{d_{m+1}}(\Delta c_{m+1} + \frac{1}{d_{m+2}}(\Delta c_{m+2} + \ldots + \frac{a_{p-1}}{d_{p-1}}) = \ldots
\]

\[
= \frac{a_{m+1}}{d_{m+1}} < 1 \leq c_m (x^2 - x^1)
\]

**Corollary 1.3**

The function \( f \) has the properties

(1.3.1) \( Cx^1 < Cx^2 \) implies \( f(x^1) < f(x^2) \)

(1.3.2) \( Cx^1 <_{\text{lex}} Cx^2 \) implies \( f(x^1) < f(x^2) \)

Assume that the admissible solution \( x^0 \in M \) is not efficient. Then there exists an admissible solution \( x' \) with \( Cx^0 < Cx' \). By (1.3.1) \( x^0 \) is not an optimal solution of \( \max\{f(x)|Ax = b, x \in \mathbb{N}_0^n\} \).

**Corollary 1.4**

If \( x^0 \) is an optimal solution of \( \max\{f(x)|Ax = b, x \in \mathbb{N}_0^n\} \) then \( x^0 \) is an efficient solution of \( \max\{Cx|Ax = b, x \in \mathbb{N}_0^n\} \)

### 2 The adaptation of constraints

Let \( x^0 \) be the efficient solution which is found as an optimal solution of the linear program \( \max\{f(x)|Ax = b, x \in \mathbb{N}_0^n\} \). Let \( f_0 = f(x^0) \) the optimal value
of the objective function $f$. Then we eliminate this efficient solution by the constraint $f(x) < f_0$.

We call a solution $x^1 \in M$ dominated by a solution $x^2 \in M$ if $C x^1 < C x^2$. We eliminate all solutions $x \in M$ which are dominated by $x^0$ with the constraint $y^i(C x - (x^0)) > 0$ then $x^0$ determinates no $x \in X$ for $y \in \mathbb{N}^n, (y \neq 0)$. By adding these constraints the set of admissible solutions changes.

**Lemma 2.1**

Let $x$ be the set of all admissible solutions. If for every $x \in X$ there is a vector $y \in \mathbb{N}^n$ with $y^i(C x - C x^0) > 0$ then $x^0$ dominates no $x \in X$.

**Proof.** If we have $y^i(C x - C x^0) = \sum_{i=1}^{p} ((C x)_i - (C x^0)_i) y_i > 0$ then there exists at least one index $i$ such that $(C x)_i - (C x^0)_i > 0$. It means that at least in one component $i$ the value of the new solution $x$ in the objective function is greater than the value of $x^0$. Hence $x$ will not be dominated by $x^0$.

Let $z^1 = C x^1, z^2 = C x^2, ... , z^j = C x^j$ be different efficient solutions for the problem $\max\{C x | A x = b, x \in \mathbb{N}_0\}$. Let $L = \{x^1, ..., x^j\}$ be the set of the efficient solutions which were found till now.

**Lemma 2.2**

Let $(C x)_i > 0$ for $i = 1, ..., p$ and $x \in X$. For $x^j \in L$ there exists $y^j \in \mathbb{N}^n$ with $y^j(C x - C x^j) > 0$ if and only if $(C x)_i - (C x^j)_i y_i^j > 0$ for $i = 1, ..., p$ and $\sum_{i=1}^{p} y_i^j \geq 1$.

**Proof.** If $y^j(C x - C x^0) > 0$ holds then we have $(C x - C x^0)_i > 0$ for at least one component $i_0$. We choose $y_{i_0}^j = 1$ and all other components $y_i^j = 0$. As $C x > 0$ we have $((C x)_i - (C x^j)_i) y_i^j > 0$ for every $i = 1, ..., p$ and $\sum_{i=1}^{p} y_i^j \geq 1$.

On the other hand from $\sum_{i=1}^{p} y_i^j \geq 1$ it follows that there is a vector $y_k^j = m \geq 1$ for some $k$. For this $k$ we have $((C x)_k - (C x^j)_k) y_k^j > 0$. Now we choose $y_k^j = 1$ and every other component $y_i^j = 0$ and we have $y^j(C x - C x^j) > 0$.

**Theorem 3.3**
Let $X$ be the set of all admissible solutions which fulfill the following constraints $Ax = b, f(x) < f(x^j), y^j(Cx - Cx^j) > 0$ for every $x^j \in L$ with $x \in \mathbb{N}_0^p, y^j \in \mathbb{N}_0^p$. If $x \neq 0$ then the linear program $\max\{f(x) | x \in X\}$ generates a new efficient solution. If $x = \emptyset$ then all efficient solutions have been already found in the list $L$.

Proof. The new solution $x'$ has the property that for every efficient solution $x^j$ of our list $L$ we have $f(x^j) > f(x')$. We have to show that $x'$ is efficient. If $x'$ is not efficient then either $x'$ is dominated by an element of the admissible set $x$ of the actual calculation or by an already eliminated element.

Case 1. $z^j = Cx'$ is dominated by $z = Cx$ of the actual admissible set $X$. Then we have $Cx > Cx'$ and as $f$ is strictly monotone $f(x) > f(x')$, a contradiction.

Case 2. $z^j$ is dominated by the already eliminated point $z^j$. But this contradicts the constraint $y^j(Cx - Cx^j) > 0$. Hence $x'$ is an efficient solution.

3 The reduction algorithm.

We use the notations of the preceding sections.

Step 1. Calculation of the objective function $f$ for $i = 1, ..., p$ do

\[ b_i = \max\{c_i x | Ax = b, x \in \mathbb{N}_0^p\} \]
\[ a_i = \min\{c_i x | Ax = b, x \in \mathbb{N}_0^p\} \]
\[ d_i = b_i - a_i + 1 \]
\[ f(x) = \sum_{i=1}^{p} \left( \prod_{j=1}^{i} d_j \right)^{-1} c_i x \]

Step 2. Initial solution $(z^1, x^1)$

$(z^1, x^1)$ is calculated by $\max\{f(x) | Ax = b, x \in \mathbb{N}_0^n\}$

$L := \{(z^1, x^1)\}$

$x^1 := \{x | Ax = b, x \in \mathbb{N}_0^n, \text{there is } y^1 \in \mathbb{N}_0^p \text{ with } (Cx)_j - z^1_j y^1_j \geq 1\}$
\( i := 1 \)

Step 3: Searching loop

for \( x^i \neq \emptyset \) do \((z^{i+1}, x^{i+1})\) is calculated by \( \max \{ f(x) | x \in x^i \} \)

\[ L := L u \{(z^{i+1}, x^{i+1})\} \]

\[ x^{i+1} = \{ x \in x^i : f(x) < f(x^{i+1}) \} \text{ if there is } y^{i+1} \in \mathbb{N}_0^p \text{ with } \sum_{j=1}^p y^{i+1}_j \geq 1 \text{ and } \\
(C x)_j - z^{i+1} y^{i+1}_j > 0 \text{ for every } j = 1, ..., p \}

\( i := i + 1 \)

Step 4. Output

for \( j = 1, ..., i \) do print \((z^j, x^j)\)

Theorem 3.1

The reduction algorithm finds every efficient solution of the integer multiple objective linear program

\[
\max \{ C x | A x = b, x \in \mathbb{N}_0^n \}
\]

Proof. Assume there is an efficient solution \( z^0 = C x^0 \) of \( \max \{ C x | A x = b, x \in \mathbb{N}_0^n \} \). Then there exists \( z^k, z^{k+1} \in L \) such that \( z^k >_{\text{lex}} z >_{\text{lex}} z^i \) and such that for \( z^i \in L \) we have either \( z^i >_{\text{lex}} \ z^k \) or \( z^{k+1} >_{\text{lex}} z^i \). We consider the \((k + 1)^{st}\) iteration of the searching loop in step 3. In this state \( z \) belongs to admissible set \( x \) as \( f(x^k) > f(x) \) and \( z \) is efficient by hypothesis. The algorithm found \( z^{k+1} \) as \( f(x^{k+1}) \geq f(x) \) holds in contradiction to \( z >_{\text{lex}} z^{k+1} \) and hence \( f(x) > f(x^{k+1}) \).
Literatur


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