Structure and Construction of Instanton Bundles on $\mathbb{P}_3$

Thomas Nüßler

vom Fachbereich Mathematik der Universität Kaiserslautern zur
Verleihung des akademischen Grades Doktor der Naturwissenschaften
(Doktor rerum naturalium, Dr. rer. nat.) genehmigte Dissertation

1. Gutachter Prof. Dr. G. Trautmann
2. Gutachter Prof. Dr. J. Le Potier

Vollzug der Promotion: 9.Februar 1996

To my parents

I thank Professor Günther Trautmann for providing me with this problem, for supervising my thesis and for carefully reading the manuscript.

I would also like to express my gratitude to him for many years of encouragement and assistance without which my thesis research would have been impossible.

Furthermore I take the chance to thank all the other people who helped and encouraged me during the preparation of my dissertation, in particular Dr. Klaus Altmann, Dr. Bernd Kreußler and Dr. Hans Schönemann.

Contents

Preface

Mathematicians have studied the moduli problem for vector bundles for a long time. Even though there are now many results there is still a lot of work to be done. It is not known in all cases whether moduli spaces of mathematical instanton bundles are smooth or irreducible. In this thesis I restrict my interest to rank 2 mathematical instantons on $\mathbb{P}_3$. For instanton bundles $\mathcal{E}$ $(H^1\mathcal{E}(-2) = 0)$ with fixed second Chern class ≤ 4 Barth proved the irreducibility and Le Potier verified the smoothness of the moduli space. But for higher second Chern classes only results for 't Hooft bundles, i.e.:instanton bundles with at least one linear section, exist so far. Recently M. Skiti and independently P. Rao proved the smoothness of the moduli space along a stratum, found by Ch. Peskine, of dimension $6c_2 + 2$, see [27], [26] and [9].

The first chapter is a joint work with G.Trautmann and already published as T. Nüßler - G. Trautmann, Multiple Koszul Structures on Lines and Instanton Bundles, Int. Jour. of Math. Vol.5, No 3, 373-388, 1994. This part is almost self contained. The results of the first chapter are used in the remaining paragraphs of this paper. Furthermore I have added to the published version a new section which describes the splitting of a 't Hooft bundle restricted to line in $\mathbb{P}_3$.

In this chapter it is proved that the moduli scheme $MI(n)$ of mathematical instanton bundles over $\mathbb{P}_3$ with second Chern class n is smooth at bundles $\mathcal{E}$ with $h^0\mathcal{E}(1) \neq 0$. By a result in [8] $h^0\mathcal{E}(1) \leq 2$. In case $h^0\mathcal{E}(1) = 2$ of special 't Hooft bundles the smoothness is a result of A. Hirschowitz-M.S. Narasimhan in [17]. This case is included in my result. In the remaining general situation $h^0\mathcal{E}(1) = 1$, the reduced zero scheme Z_{red} of the unique section is a disjoint union of lines, but the scheme Z itself has nilpotent structure in general. We study these nilpotent structures, which turn out to have resolutions

$$0 \to n\mathcal{O}(-2) \to 2n\mathcal{O}(-1) \to n\mathcal{O} \to \mathcal{O}_Z \to 0$$

given by nice regular matrices, and which are self-dual, see 1.1. I call such structures of Koszul type or Koszul structures. They are exactly the primitive structures of type $\mathcal{O}_\ell$ in the notation of C. Banica- O. Forster, [6], see Corollary 1.9.

In proposition 1.14 I show that a multiple structure X on a line, which admits a resolution of the above type with unspecified matrices, and which satisfies $\omega_X = \mathcal{O}_X(-2)$, is already a Koszul structure given by special matrices as in 1.2. It follows from this that the zero scheme Z of the unique section of $\mathcal{E}(1)$ as above is a disjoint union of Koszul structure. This enables me to prove $H^1\mathcal{N}_Z^\vee(1) = 0$ for the conormal sheaf of Z, which in turn implies $Ext^2(\mathcal{E}, \mathcal{E}(-1)) = 0$ and $Ext^2(\mathcal{E}, \mathcal{E}) = 0$, and hence smoothness of $MI(n)$ at $\mathcal{E}$.

An additional consideration describes the splitting behavior of an instanton bundle with a linear section along the support of a Koszul structure. It generalizes a result of Hartshorne [16], 9.1 and 9.11, for instantons with second Chern class equal to 3.

In chapter 2 I am going to explain an inductive method for the construction of vector bundles as the cohomology of monads which is used on the computer to produce explicit monads with the computer algebra systems SINGULAR and Macaulay. The first applications were Beilinson-I-monads for mathematical instanton bundles. For those examples the vanishing for the group $Ext^2(\mathcal{E}, \mathcal{E}(-1))$ and hence $Ext^2(\mathcal{E}, \mathcal{E})$ can be proved with the computer. Thus the isomorphism classes of such bundles are smooth points of their Maruyama scheme.

In the rest of this paper a certain class of instantons is studied which admits not only Beilinson type monads but NC-type monads. This means they are the cohomology of

$$0 \rightarrow m\mathcal{O}(-1) \stackrel{S}{\rightarrow} (m+1)\mathcal{E}_a \stackrel{T}{\rightarrow} m\mathcal{O}(1) \rightarrow 0$$

where $\mathcal{E}_a$ denotes a null correlation bundle.

The chapter "Complements on null correlation bundles" explains why the quadric $t^\vee \circ J \circ s$ defined by

$$0 \rightarrow \mathcal{O}(-1) \stackrel{s}{\rightarrow} \mathcal{E}_a \stackrel{t^\vee \circ J}{\rightarrow} \mathcal{O}(1) \rightarrow 0$$

where $\mathcal{E}_a$ is a null correlation bundle, $s, t \in H^0\mathcal{E}_a(1)$ and J is induced by $\mathcal{E}_a \simeq \mathcal{E}_a^\vee \otimes det(\mathcal{E}_a)$ (see also 3.6) corresponds to the exterior product $s \wedge t$. With this result it is possible to calculate $t^\vee \circ J \circ s$ on a computer algebra system which has the structure of the exterior algebra implemented. Moreover the proof of the result mentioned gives a nice geometric interpretation of the dependency loci of two sections $s, t \in H^0\mathcal{E}_a(1)$.

The concrete examples of NC-type monads suggested a series of propositions on the dimensions of the cohomology group $H^0\mathcal{E}(1)$ for instantons of NC-type $\mathcal{E}$ which are proven in an abstract way.

The situation is well understood for special 't Hooft bundles $\mathcal{E}$ where two independent linear sections exist. The paper focuses now on special 't Hooft bundles which are instanton of NC-type. There is a result on this class of bundles concerning the matrices in the monad. More precisely, if $\mathcal{E}$ is an instanton of NC-type admitting the monad

$$0 \rightarrow m\mathcal{O}(-1) \stackrel{S}{\rightarrow} (m+1)\mathcal{E}_a \stackrel{T}{\rightarrow} m\mathcal{O}(1) \rightarrow 0$$

then $h^0\mathcal{E}(1) = 2$ is equivalent to the fact that there exist a two dimensional subspace $H_2 \subset H_a^{\mathcal{E}}(1)$ such that the matrix S has entries in this subspace H_2 only.

It is well known [8] that the condition $h^0\mathcal{E}(1) = 2$ implies that the special 't Hooft bundle $\mathcal{E}$ corresponds to a Poncelet pair (S, C) where C is a conic and S is a Poncelet curve of degree $c_2(\mathcal{E})$ with respect to C. It is now shown in theorem 5.16 and theorem 5.17 that a special 't Hooft bundle admits a monad of NC-type if and only if the curve S contains the polar line of a as a component. This result leads to the descriptions of the moduli for special 't Hooft bundles of NC-type, see 5.18. It sets up a correspondence between such a bundle and a pair (ℓ, L) where ℓ is the polar line and L is a basepoint free pencil on ℓ of degree $c_2(\mathcal{E})/2$.

Last but not least there is a normal form given for the NC-type monads with defines a special 't Hooft bundle, see 5.35 ff.

All **prerequisites** and **definitions** frequently used throughout this paper are listed in the next chapter pp.4.

0.1 Notation and basic prerequisites

- A "#" in a diagram indicates that it is commutative.
- k denotes an algebraically closed ground field of characteristic 0.
- $\mathbb{P}_n := \mathbb{P}V$ denotes the projective n-space of a fixed $n+1$-dimensional k-vector space V. Furthermore I use the notation $\mathbb{P}_n^\vee := \mathbb{P}V^\vee$ where $V^\vee$ is the dual space of V
- $V(s)$ is the **zero scheme of a section** s of any sheaf.
- $\mathcal{O}(d)$ denotes the invertible sheaf of degree d on $\mathbb{P}_3$, Ω^p the locally free sheaf of differential p-forms, $\omega = \Omega^3$ the dualizing sheaf.
- The terms vector bundle and locally free sheaf are used synonymously.
- It is well known and easily proved it by looking at the fibres, that the map

$$\begin{array}{ccc} \Lambda^q V & \to & Hom(\mathbb{P}V, \Omega^{p+q}(p+q), \Omega^p(p)) \\ a & \mapsto & (\Omega^{p+q}(p+q) \stackrel{\lrcorner a}{\to} \Omega^p(p)) \end{array}$$

is an isomorphism. The morphism $\lrcorner a$ is fibrewise defined by

$$\Lambda^{p+q}(V/\langle x\rangle)^\vee \quad \stackrel{\lrcorner a}{\to} \quad \Lambda^p(V/\langle x\rangle)^\vee \ ,$$

where $\lrcorner a$ denotes the contraction with the element $a \in \Lambda^q V$ which comes from the duality pairing.

- Therefore there exists the evaluation map

$$\Lambda^q V \otimes \Omega^{p+q}(p+q) \to \Omega^p(p)$$

- For any n-dimensional vector space V there exists a canonical isomorphism

$$\Lambda^q V \quad \to \quad (\Lambda^{n-q} V)^\vee \otimes \Lambda^n V$$

which corresponds to the canonical pairing

$$\Lambda^q V \otimes \Lambda^{n-q} V \quad \to \quad \Lambda^n V$$

If one chooses now an isomorphism

$$\alpha : \quad \Lambda^n V \quad \to \quad k$$

then one gets an isomorphism, depending on α,

$$J(\alpha) : \quad \Lambda^q V \quad \to \quad (\Lambda^{n-q} V)^\vee \ ,$$

which is no longer canonical but uniquely defined up to a non zero scalar. This morphism translates the contraction above into a wedge product:

$$\begin{array}{ccc} (\Lambda^{n-p}V)^\vee & \xrightarrow{\lrcorner a} & (\Lambda^{n-p-q}V)^\vee \\ J(\alpha)\downarrow & \# & \downarrow J(\alpha) \\ \Lambda^p V & \xrightarrow{\wedge a} & \Lambda^{p+q}\ .V \end{array}$$

In the sequel I shall use the notation J instead of $J(\alpha)$. The latter notation is more comfortable in explicit computations

- For a vector bundle $\mathcal{E}$ of rank r and first Chern class c_1 there exits an analogous isomorphism:

$$\Lambda^q\mathcal{E} \quad\rightarrow\quad (\Lambda^{r-q}\mathcal{E})^\vee \otimes \Lambda^r\mathcal{E}$$

which is defined by the canonical pairing

$$\Lambda^q\mathcal{E} \otimes \Lambda^{r-q}\mathcal{E} \quad\rightarrow\quad \Lambda^r\mathcal{E}\ .$$

If one chooses now an isomorphism

$$\alpha:\quad \Lambda^r\mathcal{E} \quad\rightarrow\quad \mathcal{O}(c_1)$$

then we get an isomorphism, depending on α,

$$J(\alpha):\quad \Lambda^q\mathcal{E} \quad\rightarrow\quad (\Lambda^{r-q}\mathcal{E})^\vee(c_1)$$

which is no longer canonical but uniquely defined up to a non zero scalar. This morphism translates now a contraction

$$(\Lambda^{r-p}\mathcal{E})^\vee \quad\rightarrow\quad (\Lambda^{r-p-q}\mathcal{E})^\vee$$

into a wedge product:

$$\begin{array}{ccc} (\Lambda^{r-p}\mathcal{E})^\vee & \longrightarrow & (\Lambda^{r-p-q}\mathcal{E})^\vee \\ J(\alpha)\downarrow & \# & \downarrow J(\alpha) \\ \Lambda^p\mathcal{E}(-c_1) & \longrightarrow & \Lambda^{p+q}\mathcal{E}(-c_1) \end{array}$$

For $\Lambda^r\mathcal{E}$ I shall also use the notation $det(\mathcal{E})$, if no rank of the vector bundle is specified. As in the case of vector spaces I shall use the notation J instead of $J(\alpha)$

- If $\mathcal{F}$ is a coherent sheaf on $I\!P_3$ we use the abbreviations $\mathcal{F}(d) = \mathcal{F} \otimes \mathcal{O}(d)$, $H^i\mathcal{F} = H^i(I\!P_3, \mathcal{F})$, $h^i\mathcal{F} = dim H^i\mathcal{F}$, and $m\mathcal{F} = k^m \otimes \mathcal{F}$, where for a vector space E the symbol $E \otimes \mathcal{O}$ denotes the sheaf of sections of the trivial vector bundle with fibre E and $E \otimes \mathcal{F} = (E \otimes \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{F}$.

- We use the **Euler sequence**

$$0 \;\to\; \Omega^1(1) \;\to\; V^\vee \otimes \mathcal{O} \;\to\; \mathcal{O}(1) \;\to\; 0$$

and the derived sequences in its Koszul complex

$$0 \;\to\; \Omega^p(p) \;\to\; \Lambda^p V^\vee \otimes \mathcal{O} \;\to\; \Omega^{p-1}(p) \;\to\; 0$$

- A **mathematical instanton** bundle $\mathcal{E}$ on $\mathbb{P}_3$ is a rank-2 locally free sheaf with first Chern class $c_1(\mathcal{E}) = 0$ and vanishing conditions $h^0\mathcal{E} = 0$ and $h^1\mathcal{E}(-2) = 0$. Since $c_1(\mathcal{E}) = 0$ and rank$\mathcal{E} = 2$ the condition $h^0\mathcal{E} = 0$ is the stability condition, see [4], [25]. It is common to call instantons with at least one linear section ($h^0\mathcal{E}(1) \geq 1$) as 't Hooft-bundles and those with two linear sections ($h^0\mathcal{E}(1) = 2$) as special 't Hooft-bundles. An instanton bundle with $c_2(\mathcal{E}) = 1$ is named a **null correlation bundle**.
For null correlation bundles the Beilinson-I-monad, see next paragraph, degenerates to a short exact sequence:

$$\begin{array}{ccccccccc} 0 & \to & \Omega^3(3) & \overset{a}{\to} & \Omega^1(1) & \to & \mathcal{E}_a & \to & 0 \\ & & & & \nearrow & & & & \\ & & \downarrow \alpha & & a' & & & & \\ & & \mathcal{O}(-1) & & & & & & \end{array}$$

where $a \in \Lambda^2 V$, $a' \in \Lambda^2 V^\vee$ and α is a non canonical isomorphism between Ω^3 and $\mathcal{O}(-4)$. In [25] Theorem 4.3.4 it is shown that $MI(1)$, the moduli space of null correlation bundles over $\mathbb{P}_3 \simeq \mathbb{P}(V)$, is isomorphic to the complement

$$\mathbb{P}_5 \setminus G(2,4) \simeq \mathbb{P}(\Lambda^2 V) \setminus G(2,4)$$

where $G(2,4)$ is the Grassmannian of lines in $\mathbb{P}_3$, Plücker embedded in $\mathbb{P}(\Lambda^2 V)$

- An important tool used to study vector bundles are **monads** which were so named by Horrocks. A monad $M^\bullet$ is a complex of vector bundles

$$0 \;\to\; A \;\overset{a}{\to}\; B \;\overset{b}{\to}\; C \;\to\; 0$$

which is exact at A and C, but not at B. Furthermore a is supposed to define a subbundle of B, not only a subsheaf. A monad has a so called **display**. This is the induced commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & A & \longrightarrow & K & \longrightarrow & E & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & A & \longrightarrow & B & \overset{a}{\longrightarrow} & Q & \to & 0 \\ & & & & \downarrow b & & \downarrow & & \\ & & & & C & = & C & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

$$0 \;\rightarrow\; K \;\rightarrow\; B \;\rightarrow\; A \;\rightarrow\; 0$$

will be called the **first display sequence** and

$$0 \;\rightarrow\; A \;\rightarrow\; K \;\rightarrow\; E \;\rightarrow\; 0$$

the **second display sequence**.
Here $K = Ker(b)$ and $Q = Coker(a)$and E is the cohomology bundle.
I will use the term of **self dual monad** in the sequel if a monad is of type:

$$0 \;\rightarrow\; A \;\stackrel{a}{\rightarrow}\; B \;\stackrel{a^{\vee}\circ J}{\rightarrow}\; A^{\vee} \;\rightarrow\; 0 \qquad (0.1.1)$$

where J is a symplectic or symmetric isomorphism. This implies in addition that $(a^{\vee} \circ J)^{\vee} = -a^{\vee}$. There is a famous existence theorem due to Beilinson [3]. See also [25] pp240. This can be used to prove the existence of monads for instanton bundles [25] pp252 with second Chern class n. There exist in general two types of monads for an instanton $\mathcal{E}$ on $\mathbb{P}_3$ with second Chern class $c_2(\mathcal{E}) = n$. The **Beilinson-I-monad**:

$$0 \;\rightarrow\; n\mathcal{O}(-1) \;\rightarrow\; 2(n+2)\mathcal{O} \;\rightarrow\; n\mathcal{O}(1) \;\rightarrow\; 0$$

and the **Beilinson-II-monad**:

$$0 \;\rightarrow\; n\Omega^3(3) \;\stackrel{M}{\rightarrow}\; n\Omega^1(1) \;\stackrel{B}{\rightarrow}\; (2n-2)\mathcal{O} \;\rightarrow\; 0$$

For more details see page 14 and page 17

- For any vector bundle of rank n the 'duality' formula $(*)$

$$\Lambda^q\mathcal{E}^{\vee} \;\stackrel{\simeq}{\rightarrow}\; \Lambda^{n-q}\mathcal{E} \otimes det\mathcal{E}^{\vee}$$

gives us for $\mathcal{E} = \Omega^q$

$$\Omega^{q\vee} \;\stackrel{\simeq}{\rightarrow}\; \Omega^{n-q} \otimes \Omega^{n\vee}$$

Hence $\Omega^3(3) \simeq \mathcal{O}(-1)$ where the isomorphism is given by the canonical class of $\mathbb{P}_3$. There are now two ways to interpret the left arrow of the Beilinson-II-monad:

$$\begin{array}{ccc} n\Omega^3(3) & \xrightarrow{\lrcorner a} & n\Omega^1(1) \\ \simeq\searrow & \# & \nearrow a' \\ & n\mathcal{O}(-1) & \end{array}$$

where $n := c_s(\mathcal{E})$ and a' corresponds to a under the isomorphism $\Lambda^2 V^{\vee} \simeq \Lambda^2 V$.

- In this paper I shall study instanton bundles possessing not only monads of Beilinson type I & II which are built from line bundles $\mathcal{O}(i)$ and cotangent bundles $\Omega^j(i)$ but also monads containing direct sums of a null correlation bundles. Hence it is convenient to have the following notation throughout this paper.

I will call an instanton bundle of rank 2 an

- **instanton of NC-type** if it has a monad

$$0 \rightarrow (m-1)\mathcal{O}(-1) \rightarrow m\mathcal{E}_a \rightarrow (m-1)\mathcal{O}(1) \rightarrow 0$$

  where $\mathcal{E}_a$ is a null correlation bundle

I shall use the notation of NC-type not only for the instanton bundle but also for it's monad.

1 Instanton bundles with one linear section

1.1 Multiple extensions and Koszul structures on lines

1.1 By a multiple structure on a line ℓ in $\mathbb{P}_3$ we understand a subscheme X of $\mathbb{P}_3$ whose underlying reduced subscheme is the line ℓ. An n-fold extension of the line ℓ is defined by induction as follows. A 1-fold extension is the line ℓ and an n-fold extension of ℓ is a subscheme X which has a subscheme $X' \subset X$ with exact sequence

$$0 \to \mathcal{O}_\ell \to \mathcal{O}_X \to \mathcal{O}_{X'} \to 0 \tag{1.1.2}$$

and s.t. X' is an $(n-1)$-fold extension of ℓ. Then X has multiplicity n, i.e. its Hilbert polynomial is $\mathcal{X}\mathcal{O}_X(d) = nd + n$, and X is Cohen-Macaulay. By adding resolutions of $\mathcal{O}_\ell$ and $\mathcal{O}_{X'}$ in $\mathbb{P}_3$ we obtain a resolution

$$0 \to n\mathcal{O}(-2) \xrightarrow{A'} 2n\mathcal{O}(-1) \xrightarrow{A} n\mathcal{O} \to \mathcal{O}_X \to 0. \tag{1.1.3}$$

It is easy to see by the induction process that the matrices A and A' can be given the following triangular block form

$$A' = \begin{pmatrix} Z' & & & & \\ A'_{21} & Z' & & & \\ \vdots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \\ A'_{n1} & \cdots & \cdots & A'_{n,n-1} & Z' \end{pmatrix} \qquad A = \begin{pmatrix} Z & & & & \\ A_{21} & Z & & & \\ \vdots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \\ A_{n1} & \cdots & \cdots & A_{n,n-1} & Z \end{pmatrix}$$

where the block matrices are as follows. For any two forms $x, y \in V^\vee$ we define their Koszul matrices by $K(x,y) = \binom{x}{y}$ and $K'(x,y) = (-y, x)$. We choose a decomposition $V^\vee = H^0\mathcal{I}_\ell(1) \oplus H$ with $H \cong H^0\mathcal{O}_\ell(1)$. Then $Z = K(z_2, z_3)$ for a basis $z_2, z_3 \in H^0\mathcal{I}_\ell(1)$ and $A_{\mu\nu} = K(a_{\mu\nu}, b_{\mu\nu})$ for some $a_{\mu\nu}, b_{\mu\nu} \in H$, and $Z', A'_{\mu\nu}$ are the corresponding second Koszul matrices. In [20], §5, n-fold extension sheaves $\mathcal{L}$ of $\mathcal{O}_\ell(1)$ had been considered. If $\mathcal{L} = \mathcal{O}_X(1)$ then $\mathcal{L} \otimes \mathcal{O}_\ell = \mathcal{O}_\ell(1)$ is locally free and $\mathcal{L}$ is generated by two sections, using Nakayama's Lemma. From [20], Proposition 5.7. we obtain

Proposition 1.2 *Let X be an n-fold extension of ℓ. Then $\mathcal{O}_X$ has a resolution*

$$0 \to n\mathcal{O}(-2) \xrightarrow{A'} 2n\mathcal{O}(-1) \xrightarrow{A} n\mathcal{O} \to \mathcal{O}_X \to 0 \tag{1.1.4}$$

where A and A' are given as

$$A' = \begin{pmatrix} Z' & & & & & \\ W' & Z' & & & & \\ A'_3 & W' & Z' & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \\ A'_n & \cdots & \cdots & A'_3 & W' & Z' \end{pmatrix} \qquad A = \begin{pmatrix} Z & & & & & \\ W & Z & & & & \\ A_3 & W & Z & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \\ A_n & \cdots & \cdots & A_3 & W & Z \end{pmatrix}$$

in which $Z = K(z_2, z_3)$ as before, $W = K(z_0, z_1)$ for a basis $z_0,\ z_1 \in H \cong H^0\mathcal{O}_\ell(1)$ and $A_\mu = K(a_\mu, b_\mu)$ for some $a_\mu, b_\mu \in H$.

Definition 1.3 *a) If X is an n-fold extension of a line ℓ we also call X or $\mathcal{O}_X$ a Koszul structure of multiplicity n or an n-***Koszul structure on** *ℓ.*

b) As in [6] a Cohen–Macaulay multiple structure X on ℓ is called primitive *if it is locally contained in a smooth surface.*

Proposition 1.4 *An n-Koszul structure X on ℓ is primitive and has dualizing sheaf $\omega_X = \mathcal{O}_X(-2)$. In particular X is a locally complete intersection (l.c.i.).*

Proof: By the resolution type X must be Cohen-Macaulay. Its ideal is the Fitting ideal of n-minors of the matrix A in 1.2. It contains n-forms of the form

$$z_0^\mu z_1^\nu (z_0 z_3 - z_1 z_2) + f_{\mu\nu}, \qquad \mu + \nu = n - 2,$$

where $f_{\mu\nu} \in I_\ell^2$, where I_ℓ is the ideal of ℓ in the coordinate ring.

If $z_0 \neq 0$ then the local ideal of X contains

$$z_3 - z_1 z_2 + f_{n-2,0}$$

and if $z_1 \neq 0$ the local ideal of X contains

$$z_2 - z_0 z_3 + f_{0,n-2}.$$

Since both functions define locally a smooth surface, X is primitive. Applying the functor $\mathcal{H}om_{\mathcal{O}}(-, \omega)$ to the resolution of $\mathcal{O}_X$, where ω is the dualizing sheaf Ω^3 of $\mathbb{P}_3$, we obtain by $\omega_X = \mathcal{E}xt^2_{\mathcal{O}}(\mathcal{O}_X, \omega)$ and $\mathcal{E}xt^i_{\mathcal{O}}(\mathcal{O}_X, \omega) = 0$ for $i \neq 2$ and by the self-duality, up to a symplectic form, of the resolution that $\omega_X \cong \mathcal{O}_X(-2)$.

Remark 1.5 *The matrices $A_\mu, \mu \geq 2$, in the resolution of a Koszul structure yield moduli for fixed n. It is easy to see that X is contained in the quadric*

$$z_0 z_3 - z_1 z_2 = 0 \quad \textit{if and only if} \quad A_\mu = 0 \quad \textit{for} \quad \mu \geq 2$$

This is the case if the corresponding n-instanton bundle $\mathcal{E}$ satisfies $h^0\mathcal{E}(1) = 2$, see [8], §3.

Remark 1.6 *For an n-Koszul structure X there are extension sequences*

$$0 \to \mathcal{O}_{X_m} \to \mathcal{O}_X \to \mathcal{O}_{X_{n-m}} \to 0$$

for any $1 \leq m < n$ where X_m and X_{n-m} are again Koszul structures of multiplicity m and $n-m$. This follows directly from the shape of the resolution matrices.

1.7 The infinitesimal filtration of an n-Koszul structure

Let $\ell^{(m)}$ denote the m-th infinitesimal neighborhood of ℓ with ideal sheaf $\mathcal{I}_\ell^{m+1}, m \geq 0$. We consider the filtration $\ell = Y_0 \subset Y_1 \subset ... \subset Y_{n-1} = X$ of an n-Koszul structure X, where $Y_m = X \cap \ell^{(m)}$.
Since $\mathcal{I}_X$ contains $\mathcal{I}_\ell^n$, as follows directly by computing the Fitting ideal of A, we have $X \subset \ell^{(n-1)}$. By [6], §2, the Y_μ are again primitive structures on ℓ of multiplicity $\mu + 1$, and the embeddings $Y_{\mu-1} \hookrightarrow Y_\mu$ are described by exact sequences

$$0 \to \mathcal{L}^{\otimes\mu} \to \mathcal{O}_{Y_\mu} \to \mathcal{O}_{Y_{\mu-1}} \to 0 \qquad (1.1.5)$$

where $\mathcal{L}$ is an invertible sheaf on the reduced line ℓ.

Lemma 1.8 *Let X be an n-Koszul structure on ℓ. Then the intersections Y_μ are again Koszul structures on ℓ and $\mathcal{L} = \mathcal{O}_\ell$.*

Proof: Since X is an n-fold extension there is an exact sequence

$$0 \to \mathcal{O}_\ell \to \mathcal{O}_X \to \mathcal{O}_{X'} \to 0$$

where X' is an $(n-1)$-Koszul structure, $X' \subset X$. Therefore $\mathcal{I}_\ell^{n-1} \subset \mathcal{I}_{X'}$ and hence $\mathcal{I}_{Y_{n-2}} = \mathcal{I}_X + \mathcal{I}_\ell^{n-1} \subset \mathcal{I}_{X'}$. From this we have an exact sequence

$$0 \to \mathcal{C} \to \mathcal{O}_{Y_{n-2}} \to \mathcal{O}_{X'} \to 0$$

in which the kernel $\mathcal{C}$ must be a 0-dimensional sheaf since both Y_{n-2} and X' have multiplicity $n-1$. But since Y_{n-2} is Cohen-Macaulay, $\mathcal{C} = 0$, i.e. $Y_{n-1} = X'$. Now we can finish the proof by induction on n.

Corollary 1.9 *1) An n-Koszul structure X has a unique filtration $\ell = X_1 \subset \ldots \subset X_n = X$ by μ-Koszul structures $X_\mu = X \cap \ell^{(\mu+1)}$ with exact sequences*

$$0 \to \mathcal{O}_\ell \to \mathcal{O}_{X_{\mu+1}} \to \mathcal{O}_{X_\mu} \to 0.$$

2) The n-Koszul structures on ℓ are exactly the primitive structures of type $\mathcal{O}_\ell$ in the sense of [6],1.12

1.10 Conormal sequence

Let X be an n-Koszul structure. By Lemma 1.7 and [6], proposition 1.1.3.3, the canonical homomorphism $\mathcal{N}_X^\vee \otimes \mathcal{O}_\ell \to \mathcal{N}_\ell^\vee$ of the conormal sheaves has kernel and cokernel equal to $\mathcal{O}_\ell$, i.e. we have an exact sequence

$$0 \to \mathcal{O}_\ell \to \mathcal{N}_X^\vee \otimes \mathcal{O}_\ell \to \mathcal{N}_\ell^\vee \to \mathcal{O}_\ell \to 0. \tag{1.1.6}$$

Since $\mathcal{N}_\ell^\vee \cong 2\mathcal{O}_\ell(-1)$ the sheaf $\mathcal{N}_X^\vee \otimes \mathcal{O}_\ell$ is an extension of $\mathcal{O}_\ell(-2)$ by $\mathcal{O}_\ell$ on the reduced line. Since the group $Ext^1_\ell(\mathcal{O}_\ell(-2), \mathcal{O}_\ell) \cong H^1\mathcal{O}_\ell(2) = 0$, this extension is trivial. Therefore

$$\mathcal{N}_X^\vee \otimes \mathcal{O}_\ell \cong \mathcal{O}_\ell \oplus \mathcal{O}_\ell(-2). \tag{1.1.7}$$

Proposition 1.11 *Let X be an n-Koszul structure on ℓ. Then $H^1\mathcal{N}_X^\vee(1) = 0$.*

Proof: By induction on n; the statement being trivial for $n = 1$. For $n > 1$ we are given exact sequences

$$0 \to \mathcal{O}_\ell \to \mathcal{O}_X \to \mathcal{O}_Y \to 0 \tag{1.1.8}$$

and

$$0 \to \mathcal{I}_X \to \mathcal{I}_Y \to \mathcal{O}_\ell \to 0 \tag{1.1.9}$$

with Y an $(n-1)$-Koszul structure. Dualizing (1.1.8) and using
$\omega_X = \mathcal{O}_X(-2), \quad \omega_Y = \mathcal{O}_Y(-2)$ and $\omega_\ell = \mathcal{O}_\ell(-2)$ we also obtain the exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_X \to \mathcal{O}_\ell \to 0. \tag{1.1.10}$$

If we tensor (1.1.8) and (1.1.10) by the locally free $\mathcal{O}_X$- module $\mathcal{N}_X^\vee$ we get the exact sequences

$$0 \to \mathcal{N}_X^\vee \otimes \mathcal{O}_\ell \to \mathcal{N}_X^\vee \to \mathcal{N}_X^\vee \otimes \mathcal{O}_Y \to 0 \tag{1.1.11}$$

$$0 \to \mathcal{N}_X^\vee \otimes \mathcal{O}_Y \to \mathcal{N}_X^\vee \to \mathcal{N}_X^\vee \otimes \mathcal{O}_\ell \to 0. \tag{1.1.12}$$

Tensoring (1.1.9) by $\mathcal{O}_Y$ gives us the exact sequence

$$\mathcal{T}or_1^{\mathcal{O}}(\mathcal{O}_\ell, \mathcal{O}_Y) \to \mathcal{N}_X^\vee \otimes \mathcal{O}_Y \to \mathcal{N}_Y^\vee \to \mathcal{O}_\ell \to 0. \tag{1.1.13}$$

Using the resolution of $\mathcal{O}_Y$ as a Koszul structure we easily find

$\mathcal{T}or_1^{\mathcal{O}}(\mathcal{O}_\ell, \mathcal{O}_Y) \cong 2\mathcal{O}_\ell(-1)$. Now we proceed calculating the group $H^1\mathcal{N}_X^\vee(1)$. From (1.1.11) for the Koszul structure Y with substructure Z and the induction hypothesis we get

$$0 = H^1\mathcal{N}_Y^\vee(1) = H^1\mathcal{N}_Y^\vee \otimes \mathcal{O}_Z(1) \tag{1.1.14}$$

because $\mathcal{N}_Y^\vee \otimes \mathcal{O}_\ell = \mathcal{O}_\ell \oplus \mathcal{O}_\ell(-2)$. Hence from (1.1.12) for Y we get the surjection

$$H^0\mathcal{N}_Y^\vee(1) \to H^0\mathcal{N}_Y^\vee \otimes \mathcal{O}_\ell(1) \to 0. \tag{1.1.15}$$

Now split (1.1.13) into two exact sequences

$$2\mathcal{O}_\ell(-1) \to \mathcal{N}_X^\vee \otimes \mathcal{O}_Y \to \mathcal{A} \to \mathcal{O} \tag{1.1.16}$$

$$0 \to \mathcal{A} \to \mathcal{N}_Y^\vee \xrightarrow{\pi} \mathcal{O}_\ell \to 0 \tag{1.1.17}$$

The surjection π factors through

$$\mathcal{N}_Y^\vee \otimes \mathcal{O}_\ell = \mathcal{O}_\ell \oplus \mathcal{O}_\ell(-2) \to \mathcal{O}_\ell,$$

whose first component α must be nonzero, hence an isomorphism. From (1.1.16) we get

$$H^1\mathcal{N}_X^\vee \otimes \mathcal{O}_Y(1) = H^1\mathcal{A}(1)$$

and from (1.1.17) and (1.1.15) the exact diagram

$$\begin{array}{ccccccc} h^0\mathcal{N}_Y^\vee(1) & \to & H^0\mathcal{O}_\ell(1) & \to & H^1\mathcal{A}(1) & \to & H^1\mathcal{N}_Y^\vee(1) \\ \downarrow & \approx\nearrow \alpha & & & & & \\ H^0(\mathcal{N}_Y^\vee \otimes \mathcal{O}_\ell(1)) & & & & & & \\ \downarrow & & & & & & \\ 0 & & & & & & \end{array}$$

By induction hypothesis and the induced isomorphism α we conclude that $H^1\mathcal{A}(1) = 0$. By (1.1.11) again for X we finally have $H^1\mathcal{N}_X^\vee(1) = H^1\mathcal{N}_X^\vee \otimes \mathcal{O}_Y(1) = H^1\mathcal{A}(1) = 0$.

1.2 The Beilinson resolution

For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}_3$ there is a Beilinson complex

$$0 \to C^{-3}(\mathcal{F}) \to C^{-2}(\mathcal{F}) \to \ldots \to C^{0}(\mathcal{F}) \to \ldots \to C^{3}(\mathcal{F}) \to 0$$

which is exact except at $C^0(\mathcal{F})$ and has $\mathcal{F}$ as cohomology at $C^0(\mathcal{F})$, see [2]. The sheaves of the complex are given by

$$C^p(\mathcal{F}) = \bigoplus_{i-j=p} H^i(\mathcal{F} \otimes \Omega^j(j)) \otimes \mathcal{O}(-j)$$

If X is a 1-dimensional subscheme the only nonzero terms for $\mathcal{O}_X(-1)$ are

$$\begin{array}{rcll} C^{-3} & = & H^0(E_X^3) \otimes \mathcal{O}(-3) & \\ C^{-2} & = & H^0(E_X^2) \otimes \mathcal{O}(-2) & \oplus \quad H^1(E_X^3) \otimes \mathcal{O}(-3) \\ C^{-1} & = & H^0(E_X^1) \otimes \mathcal{O}(-1) & \oplus \quad H^1(E_X^2) \otimes \mathcal{O}(-2) \\ C^{0} & = & H^0(E_X^0) \otimes \mathcal{O} & \oplus \quad H^1(E_X^1) \otimes \mathcal{O}(-1) \\ C^{1} & = & & \quad H^1(E_X^0) \otimes \mathcal{O} \end{array}$$

where $E_X^j = \Omega^j(j-1) \otimes \mathcal{O}_X$. The following Lemma is a direct consequence.

Lemma 1.12 *Let X be a multiple structure on a line ℓ. Then the following conditions are equivalent:*

(i) $\mathcal{O}_X$ has a resolution $0 \to n\mathcal{O}(-2) \xrightarrow{B} 2n\mathcal{O}(-1) \xrightarrow{A} n\mathcal{O} \to \mathcal{O}_X \to 0$

(ii) $h^0 E_X^j = 0$ *and* $h^1 E_X^3 = n, \quad h^1 E_X^2 = 2n, \quad h^1 E_X^1 = n, \quad h^1 E_X^0 = 0.$

The conditions in 1.12 do not yet imply that $\mathcal{O}_X$ is a Koszul structure. The additional condition needed for that is self duality.

Lemma 1.13 *Let a multiple structure X on a line ℓ satisfy the condition of Lemma 1.12. Then the following conditions are equivalent:*
(i) *The resolution (B, A), is self-dual, i.e. (B, A) is isomorphic to (A^t, B^t)*
(ii) $\omega_X \cong \mathcal{O}_X(-2)$

Proof: Since X is Cohen-Macaulay of dimension 1 the dual of the above resolution of $\mathcal{O}_X$ with respect to $\mathcal{H}om_{\mathcal{O}}(-, \omega)$ gives us the resolution

$$0 \to n\mathcal{O}(-4) \xrightarrow{A^t} 2n\mathcal{O}(-3) \xrightarrow{B^t} n\mathcal{O}(-2) \to \omega_X \to 0.$$

Therefore (i) implies (ii). If conversely $\omega_X \cong \mathcal{O}_X(-2)$ then the two Beilinson resolutions (which are determined by the cohomology of the sheaves) must be isomorphic.

Now we can prove

Proposition 1.14 *Any multiple structure X on a line ℓ with a resolution*

$$0 \to n\mathcal{O}(-2) \to 2n\mathcal{O}(-1) \to n\mathcal{O} \to \mathcal{O}_X \to 0$$

and satisfying $\omega_X = \mathcal{O}_X(-2)$ is an n-Koszul structure.

Proof: By Lemma 1.12 the given resolution is the Beilinson resolution. It shows that $\mathcal{O}_X$ is Cohen-Macaulay. Since $\ell \subset X$ we have an exact sequence

$$0 \to \mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_\ell \to 0 \tag{1.2.18}$$

whose dual becomes

$$0 \to \omega_\ell \to \omega_X \to \mathcal{E}xt^2_{\mathcal{O}}(\mathcal{J}, \omega) \to 0.$$

Twisting by 2 and using $\omega_X(2) \cong \mathcal{O}_X$, we get the exact sequence

$$0 \to \mathcal{O}_\ell \to \mathcal{O}_X \to \mathcal{O}_Y \to 0 \tag{1.2.19}$$

in which $\mathcal{O}_Y$ is the quotient structure. We proceed now by **induction** on the multiplicity. For this it is enough to show that Y again has a Beilinson resolution

$$0 \to (n-1)\mathcal{O}(-2) \to (2n-2)\mathcal{O}(-1) \to (n-1)\mathcal{O} \to \mathcal{O}_Y \to 0 \tag{1.2.20}$$

and satisfies $\omega_Y = \mathcal{O}_Y(-2)$.

To show that we first remark that Y is again Cohen-Macaulay: The dual of (1.2.19) becomes

$$0 \to \omega_Y(2) \to \mathcal{O}_X \xrightarrow{\pi} \mathcal{O}_\ell \to \mathcal{E}xt^3_{\mathcal{O}}(\mathcal{O}_Y, \omega) \to 0,$$

and since π is the original surjection, the sheaf $\mathcal{E}xt^3$ is zero, which implies that $\mathcal{O}_Y$ has no 0-dimensional torsion. We also have the exact sequence

$$0 \to \omega_Y(2) \to \mathcal{O}_X \to \mathcal{O}_\ell \to 0. \tag{1.2.21}$$

Sequence (1.2.19) yields the exact sequences

$$0 \to E^j_\ell \to E^j_X \to E^j_Y \to 0$$

where as above $E^j = \Omega^j(j-1)$. Since $\Omega^1(1) \otimes \mathcal{O}_\ell = \Omega^1_\ell(1) \oplus 2\mathcal{O}_\ell$ we obtain

$$E^j_\ell \cong \begin{cases} \mathcal{O}_\ell(-1) & j=0 \\ \mathcal{O}_\ell(-2) \oplus 2\mathcal{O}_\ell(-1) & j=1 \\ 2\mathcal{O}_\ell(-2) \oplus \mathcal{O}_\ell(-1) & j=2 \\ \mathcal{O}_\ell(-2) & j=3. \end{cases}$$

Since $h^1 E^0_\ell = 0$ we get $h^0 E^0_Y = h^0 \mathcal{O}_Y(-1) = 0$. If z is a linear form, which is not a zero divisor for $\mathcal{O}_\ell$ we get injections $\mathcal{O}_Y(-d-1) \hookrightarrow \mathcal{O}_Y(-d)$ for $d \geq 0$, because $\mathcal{O}_Y$ is Cohen-Macaulay. Therefore $h^0 \mathcal{O}_Y(-d) = 0$ for $d \geq 0$. This implies $h^0 E^j_Y = 0$, using the standard Koszul resolutions of Ω^j. Now we get the exact sequences

$$0 \to H^1 E^j_\ell \to H^1 E^j_X \to H^1 E^j_Y \to 0.$$

Since $h^1 E^0_X = 0$, also $h^1 E^0_Y = 0$. On the other hand $h^1 E^1_\ell = 1$, $h^1 E^2_\ell = 2$, $h^1 E^3_\ell = 1$. Hence $h^1 E^1_Y = n-1$, $h^1 E^2_Y = 2n-2$, $h^1 E^3_Y = n-1$. By Lemma 1.12 $\mathcal{O}_Y$ has a resolution (1.2.20). Dualizing this we get a resolution

$$0 \to (n-1)\mathcal{O}(-2) \to (2n-2)\mathcal{O}(-1) \to (n-1)\mathcal{O} \to \omega_Y(2) \to 0.$$

In order to show that $\omega_Y \cong \mathcal{O}_Y(-2)$ we let $\mathcal{L}$ denote the torsion free part of $\omega_Y(2) \otimes \mathcal{O}_\ell$, such that

$$\omega_Y(2) \otimes \mathcal{O}_\ell = \mathcal{T} \oplus \mathcal{L}.$$

$\mathcal{L}$ is a vector bundle on ℓ which is generated by global sections coming from $\omega_Y(2)$, hence $\mathcal{L} = \oplus \mathcal{O}(a_i)$ with $a_i \geq 0$. On the other hand sequence (1.2.21) gives us a surjection $\mathcal{T}or_1(\mathcal{O}_\ell, \mathcal{O}_\ell) \to \omega_Y(2) \otimes \mathcal{O}_\ell$, hence a surjection $2\mathcal{O}_\ell(-1) \to \mathcal{L}$. It follows that $\mathcal{L} = \mathcal{O}_\ell(a)$ is of rank 1. Now let $p \in \ell \backslash Supp(\mathcal{T})$. Since $\mathcal{O}_\ell(a)$ is globally generated by induced sections of $\omega_Y(2)$, there is one section $\mathcal{O} \to \omega_Y(2)$ which generates $\mathcal{O}_\ell(a)$ at p under

$$\mathcal{O} \to \omega_Y(2) \to \omega_Y(2) \otimes \mathcal{O}_\ell \to \mathcal{O}_\ell(a).$$

By Nakayama's Lemma this section generates $\omega_Y(2)$ at p. Therefore $\mathcal{O} \to \omega_Y(2)$ is generically surjective. It induces a sequence

$$0 \to \mathcal{C}' \to \mathcal{O}_Y \to \omega_Y(2) \to \mathcal{C}'' \to 0$$

where $\mathcal{C}''$ is 0-dimensional. Since both $\mathcal{O}_Y$ and $\omega_Y(2)$ have the same Hilbert polynomial, also $\mathcal{C}'$ is 0-dimensional, hence $\mathcal{C}' = 0$. We thus have shown that $\mathcal{O}_Y \cong \omega_Y(2)$, which completes the proof of proposition 1.14.

1.3 Instanton bundles with one linear section

A mathematical instanton bundle on $\mathbb{P}_3$ is a stable rank 2 vector bundle $\mathcal{E}$ with first Chern class $c_1(\mathcal{E}) = 0$ and vanishing condition $h^1\mathcal{E}(-2) = 0$, see [5]. The stability condition implies $n = c_2(\mathcal{E}) > 0$. Here I recall more specificially the properties of the matrices M and B in the Beilinson-II-monad, see page 7. It is wellknown that $\mathcal{E}$ is the cohomology of a Beilinson complex

$$0 \to n\Omega^3(3) \xrightarrow{M} n\Omega^1(1) \xrightarrow{B} (2n-2)\mathcal{O} \to 0 \tag{1.3.22}$$

in which M and B are induced by linear maps

$$k^n \xrightarrow{M} k^n \otimes \wedge^2 V, \qquad k^n \xrightarrow{B} k^{2n-2} \otimes V.$$

The conditions for M, B to define an instanton bundle are:

(i) M is symmetric

(ii) the induced sequence

$$k^n \otimes V \xrightarrow{\wedge M} k^n \otimes \wedge^3 V \xrightarrow{\wedge B} k^{2n-2} \otimes \wedge^4 V \to 0$$

is exact

(iii) $k^{2n-2} \xrightarrow{B^t} k^n \otimes V$ satisfies $Im(B^t) \cap (k^n \otimes v) = 0$ for any nonzero $v \in V$

see [8], section 1. We let $MI(n)$ denote the open subscheme of the Maruyama scheme $M(2;0,n,0)$ of all semi–stable coherent sheaves on $\mathbb{P}_3$ of rank 2 and Chern classes $(c_1, c_2, c_3) = (0, n, 0)$ whose closed points are the isomorphism classes of mathematical instanton bundles. Up to now it is not known whether $MI(n)$ is smooth and irreducible for all n. $MI(n)$ is smooth at $\mathcal{E}$ if $Ext^2(\mathcal{E}, \mathcal{E}) = 0$. There are reasons to believe that the stronger condition $Ext^2(\mathcal{E}, \mathcal{E}(-1)) = 0$ holds for any $\mathcal{E} \in MI(n)$. Indeed this is true for the so–called special $'tHooft$ instanton bundles characterized by $h^0\mathcal{E}(1) = 2$, see [8]. This was shown in [17], or can easily be derived from the normal form of B in [8]. We are going to show that $Ext^2(\mathcal{E}, \mathcal{E}(-1)) = 0$ also holds for any $\mathcal{E} \in MI(n)$ satisfying $h^0\mathcal{E}(1) = 1$. Note that by [8] $h^0\mathcal{E}(1) \leq 2$ for any $\mathcal{E} \in MI(n)$. In the following we assume $n \geq 3$, since for $n = 2$ always $h^0\mathcal{E}(1) = 2$.

1.15 By the general **Serre construction**, see [16] , rank 2 bundles can be constructed on $\mathbb{P}_3$ from l.c.i. intersection curves X s.t. the dualizing sheaf ω_X is the restriction of a line bundle on $\mathbb{P}_3$. In particular, if X is an n–Koszul structure on a line ℓ with $\omega_X = \mathcal{O}_X(-2)$, we have

$$\begin{aligned} H^0\omega_X(2) = Hom_{\mathcal{O}_X}(\mathcal{O}_X(-2), \omega_X) &= Ext^2_{\mathcal{O}}(\mathcal{O}_X(-2), \omega_{\mathbb{P}_3}) \\ &= Ext^1(\mathcal{I}_X(-2), \omega_{\mathbb{P}_3}) \\ &= Ext^1(\mathcal{I}_X(1), \mathcal{O}(-1)) \end{aligned}$$

using Grothendieck isomorphism [1]5.2 and (proof of 5.4.iii). A section s of $\omega_X(2)$ thus defines an extension $\mathcal{F}$,

$$0 \to \mathcal{O}(-1) \to \mathcal{F} \to \mathcal{I}_X(1) \to 0$$

which is locally free if s defines an isomorphism $\mathcal{O}_X \cong \omega_X(2)$. If not, $\mathcal{F}$ will be singular along the zero scheme $Z(s) \subset X$, see [5]. for an example. In our case one easily verifies, using the resolution of $\mathcal{O}_X$, that

$$c_1\mathcal{F} = 0,\ c_2\mathcal{F} = n,\ c_3\mathcal{F} = 0,\ h^0\mathcal{F} = 0,\ h^1\mathcal{F}(-2) = 0$$

and $h^0\mathcal{F}(1) = 1$. Hence, if $\mathcal{F}$ is a bundle it belongs to $MI(n)$. We can now prove the converse:

Proposition 1.16 *Let $\mathcal{E}$ be an instanton bundle with second Chern class $n-1 \geq 3$ such that $h^0\mathcal{E}(1) = 1$, and let X be the zero scheme of the non–zero section s of $\mathcal{E}(1)$. then*

(i) X_{red} is a disjoint union of lines $\ell_1, \ldots, \ell_m$;

(ii) X is a disjoint union of Koszul structures $X_1, \ldots, X_m$ on $\ell_1, \ldots, \ell_m$ respectively.

Proof:

(1) Since $h^0\mathcal{E} = 0$, X is a 1–dimensional l.c.i., and we are given the exact sequence

$$0 \to \mathcal{O}(-2) \xrightarrow{s} \mathcal{E}(-1) \to \mathcal{O} \to \mathcal{O}_X \to 0. \qquad (1.3.23)$$

Its dual sequence becomes

$$0 \to \mathcal{O} \xrightarrow{s} \mathcal{E}(1) \to \mathcal{O}(2) \to \omega_X \to 0,$$

where we use $\mathcal{E} \cong \mathcal{E}^\vee$, which follows from $\wedge^2\mathcal{E} \cong \mathcal{O}$. Hence $\omega_X = \mathcal{O}_X(-2)$.

(2) Statement (i) was proved in [8], 1.3, using the monad description of $\mathcal{E}$ as above. Therefore, X is a disjoint union of possibly multiple structures X_μ on ℓ_μ. In order to prove (ii) we can calculate the dimensions of the cohomology groups of $\mathcal{O}_X$ from (1.3.23) using (1.3.22). By (1.3.22) (for $n-1$)

$$\begin{aligned} h^0\mathcal{E}(d) &= 0 \quad \text{for } d \leq 0 \\ h^1\mathcal{E}(d) &= 0 \quad \text{for } d \leq -2 \\ h^2\mathcal{E}(d) &= 0 \quad \text{for } d \geq -2 \\ h^3\mathcal{E}(d) &= 0 \quad \text{for } d \geq -4 \\ h^2\mathcal{E}(-3) &= h^1\mathcal{E}(-1) = n-1 \\ h^1\mathcal{E} &= 2n-4. \end{aligned}$$

From (1.3.23) we get $H^1\mathcal{O}_X(-1) = H^2\mathcal{I}_X(-1) = 0$ because $H^2\mathcal{E}(-2) = 0$, and similarly $H^0\mathcal{O}_X(-1) = 0$. From this it is easy to see that $H^0E_X^j = H^0\mathcal{O}_X \otimes \Omega^j(j-1) = 0$ for $0 \le j \le 3$. It remains to calculate $h^1E_X^j$. We have the exact sequence from the Euler sequence

$$0 \to H^1\mathcal{E} \otimes \Omega^1(-1) \to \underset{\substack{\| \\ 0}}{4H^1\mathcal{E}(-2)} \to H^1\mathcal{E}(-1) \to H^2\mathcal{E} \otimes \Omega^1(-1) \to 0$$

and from (1.3.23)

$$\begin{array}{ccccccccc} H^1\mathcal{I}_X \otimes \Omega^1 & \to & H^1\Omega^1 & \to & H^1\mathcal{O}_X \otimes \Omega^1 & \to & H^2\mathcal{I}_X \otimes \Omega^1 & \to & H^2\Omega^1 \\ \| & & & & & & \| & & \| \\ H^1\mathcal{E} \otimes \Omega^1(-1) & & & & & & H^2\mathcal{E} \otimes \Omega^1(-1) & & 0 \\ \| & & & & & & \| & & \\ 0 & & & & & & H^1\mathcal{E}(-1) & & \end{array}$$

hence $h^1E_X^1 = 1 + (n-1) = n$. Similarly, we obtain

$$H^1E_X^2 = H^2\mathcal{I}_X \otimes \Omega^2(1) = H^2(\mathcal{E} \otimes \Omega^2)$$

with $h^2(\mathcal{E} \otimes \Omega^2) = 2(n-1) + 2 = 2n$, and

$$\begin{array}{ccccc} & & H^1E_X^3 & & \\ & & \| & & \\ 0 \to H^2\mathcal{E}(-3) \to & & H^2\mathcal{I}_X \otimes \Omega^3(2) & & \to H^3\Omega^3 \to 0. \end{array}$$

These data imply by Lemma 1.12 that $\mathcal{O}_X$ has a Beilinson resolution

$$0 \to n\mathcal{O}(-2) \to 2n\mathcal{O}(-1) \to n\mathcal{O} \to \mathcal{O}_X \to 0.$$

In order to show that each of the X_μ has a resolution of the same type we proceed as in the proof of Proposition 2.3. Assume that $mult(X_1) \ge 2$, say. There is a sequence

$$0 \to \mathcal{O}_{\ell_1} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

with $Y \subset X$ satisfying (i), (ii) of Proposition 2.1, and $Y = Y_1 \dot\cup X_2 \dot\cup \ldots \dot\cup X_m$. By induction over n we may assume that $Y_1, X_2, \ldots, X_m$ are Koszul structures. Then also X_1 is a Koszul structure. In particular, the above resolution of $\mathcal{O}_X$ can be replaced by the direct sum of the Beilinson resolutions of $\mathcal{O}_{X_\mu}$.

Theorem 1.17 *Let $\mathcal{E}$ be an instanton bundle on $\mathbb{P}_3$. If $h^0\mathcal{E}(1) > 0$ then $Ext^2(\mathcal{E}, \mathcal{E}(-1)) = Ext^2(\mathcal{E}, \mathcal{E}) = 0$.*

Proof:
By [8] $h^0\mathcal{E}(1) \le 2$. If $h^0\mathcal{E}(1) = 2$ this follows from [17]. If $h^0\mathcal{E}(1) = 1$ we obtain an exact sequence (1.3.23) where $X = X_1 \dot\cup \ldots \dot\cup X_m$ is a disjoint union of Koszul

structures X_μ on lines ℓ_μ. By Proposition 1.8 $H^1\mathcal{N}_X^\vee(1) = \oplus H^1\mathcal{N}_{X_\mu}^\vee(1) = 0$, where $\mathcal{N}_X^\vee$ denotes the conormal sheaf of X.

The sequence (1.3.23) induces the exact sequence

$$Ext^2(\mathcal{I}_X(1), \mathcal{E}(-1)) \to Ext^2(\mathcal{E}, \mathcal{E}(-1)) \to \begin{array}{c} H^2\mathcal{E} \\ \| \\ 0 \end{array}$$

On the other hand, we obtain

$$\begin{array}{lll} & Ext^2(\mathcal{I}_X(1), \mathcal{E}(-1)) & \\ = & Ext^3(\mathcal{O}_X, \mathcal{E}(-2)) & \text{since } H^2\mathcal{E}(-2) = H^3\mathcal{E}(-2) = 0 \\ = & H^1(\mathcal{E}xt^2(\mathcal{O}_X, \mathcal{E}(-2)) & \text{by Leray's spectral sequence} \\ = & H^1(\mathcal{E}xt^2(\mathcal{O}_X, \mathcal{O}(-2)) \otimes \mathcal{E}) & \text{since } \mathcal{E} \text{ is locally free} \\ = & H^1(\mathcal{E}xt^2(\mathcal{O}_X(-2), \omega_{\mathbb{P}_3}) \otimes \mathcal{E}) & \\ = & H^1(\omega_X(2) \otimes \mathcal{E}) & \text{by Grothendieck duality} \\ = & H^1(\mathcal{E} \otimes \mathcal{O}_X) & \text{since } \omega_X(2) = \mathcal{O}_X. \end{array}$$

If we tensor sequence (1.3.23) by $\mathcal{O}_X$ we get the exact sequence

$$\mathcal{O}_X(-1) \to \mathcal{E} \otimes \mathcal{O}_X \to \mathcal{N}_X^\vee(1) \to 0$$

and this implies

$$H^1(\mathcal{E} \otimes \mathcal{O}_X) = H^1\mathcal{N}_X^\vee(1) = 0.$$

This proves the vanishing of $Ext^2(\mathcal{E}, \mathcal{E}(-1))$. To show that also $Ext^2(\mathcal{E}, \mathcal{E}) = 0$ we consider a plane $P \subset \mathbb{P}_3$ and the restriction sequence

$$0 \to \mathcal{E}(-1) \to \mathcal{E} \to \mathcal{E} \otimes \mathcal{O}_P \to 0.$$

In the induced exact sequence

$$Ext^2(\mathcal{E}, \mathcal{E}(-1)) \to Ext^2(\mathcal{E}, \mathcal{E}) \to Ext^2(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_P)$$

also the last group vanishes, because it is Serre–dual to $H^0(\mathcal{E} \otimes \mathcal{E} \otimes \mathcal{O}_P(-3)) = 0$. Since $Ext^2(\mathcal{E}, \mathcal{E})$ is the group of obstructions to smoothness of the Maruyama scheme at stable points, see [19] and [29], MI(n) is smooth at $\mathcal{E}$.

1.4 Jump phenomena

I determined for the conormal bundle of a Koszul structure its restriction to a line contained in the support of the Koszul structure. If the Koszul structure is $V(s)$, the zero locus of a linear section of an instanton bundle $\mathcal{E}$, then the restricted conormal bundle is isomorphic to $\mathcal{E}_{|\ell}(-1)$, see 1.1.7, page 12. Hence I know without further effort the splitting behaviour of this rank-2-instanton bundle $\mathcal{E}$ on a line contained the support of the multiple Koszul structure $V(s)$, see proposition 1.18. This provides a generalization of a proposition of Hartshorne [16]9.11, which proves the splitting behaviour of $\mathcal{E}$ on lines in $\mathbb{P}_3$, to 't Hooft bundles with general $c_2(\mathcal{E})$. The generalization to the case of special 't Hooft bundles of arbitrary second Chern class $c_2(\mathcal{E})$ was done in [8]. For 't Hooft bundles with at most one linear section ($h^0\mathcal{E}(1) = 1$) I obtain a slightly different result. First I present a lemma which is used in the proof of the result.

Proposition 1.18 *Let $\mathcal{E}$ be a 't Hooft bundle, hence there is one linear section $s \in H^0\mathcal{E}(1)$ whose zero locus is denoted by $Y := V(s)$. Then for $\mathcal{E}_{|\ell}$, the bundle $\mathcal{E}$ restricted to a line $\ell \subset \mathbb{P}_3$, we have:*

(1) If ℓ does not intersect Y, then

$$\mathcal{E}_{|\ell} \simeq \mathcal{O}_\ell \oplus \mathcal{O}_\ell \quad \text{or} \quad \mathcal{E}_{|\ell} \simeq \mathcal{O}_\ell(1) \oplus \mathcal{O}_\ell(-1) \ .$$

(2) If ℓ intersects Y with multiplicity 1, then

$$\mathcal{E}_{|\ell} \simeq \mathcal{O}_\ell \oplus \mathcal{O}_\ell \ .$$

(3) If ℓ intersects Y with multiplicity $m \geq 2$, then

$$\mathcal{E}_{|\ell} \simeq \mathcal{O}_\ell(-m+1) \oplus \mathcal{O}_\ell(m-1) \ .$$

(4) If $\ell \subset Y_i$ where Y_i is a component of $V(s)$ then it follows that

$$\begin{array}{rcl} if\ \ell = Y_i\ i.e\ Y_i\ is\ reduced & \Longrightarrow & \mathcal{E}_{|\ell} \simeq \mathcal{O}_\ell \oplus \mathcal{O}_\ell \\ if\ \ell \subsetneqq Y_i\ i.e.\ Y_i\ is\ non\ reduced & \Longrightarrow & \mathcal{E}_{|\ell} \simeq \mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell(1) \end{array}$$

Proof:
For any $s \in H^0(\mathcal{E}(1))$ one has the exact sequence:

$$\begin{array}{ccccccccccc} 0 & \rightarrow & \mathcal{O}(-1) & \stackrel{s}{\rightarrow} & \mathcal{E} & & \rightarrow & \mathcal{O}(1) & \rightarrow & \mathcal{O}_Y & \rightarrow & 0 \\ & & & & & \searrow & \nearrow & & & & & \\ & & & & & \mathcal{I}_Y(1) & & & & & & \\ & & & & \nearrow & & \searrow & & & & & \\ & & & & 0 & & & 0 & & & & \end{array}$$

I restrict now this sequence to the line ℓ. i.e.: I tensor the whole sequence with $\mathcal{O}_\ell$:

$$\begin{array}{ccccccccc} \mathcal{T}or_1^{\mathcal{O}}(\mathcal{I}_Y(1),\mathcal{O}_\ell) \to & \mathcal{O}_\ell(-1) & \stackrel{s_{|\ell}}{\to} & \mathcal{E}_{|\ell} & \to & \mathcal{O}_\ell(1) & \to & \mathcal{O}_{Y\cap\ell} & \to 0 \\ & & & & \searrow \quad \nearrow & & & & \\ & & & & \mathcal{I}_{Y\cap\ell|\ell}(1) & & & & \end{array}$$

where $Y\cap\ell$ is a finite scheme of multiplicity m. $\mathcal{T}or_1^{\mathcal{O}}(\mathcal{I}_Y(1),\mathcal{O}_\ell)$ vanishes because it is a sheaf of finite length which can not be mapped to a locally free sheaf $\mathcal{O}_\ell(-1)$. Hence I obtain two short exact sequences:

$$0 \to \mathcal{T}or_1(\mathcal{O}_Y,\mathcal{O}_\ell) \to \mathcal{I}_{Y\cap\ell|\ell}(1) \to coker \to 0$$

$$0 \to coker \to \mathcal{O}_\ell \to \mathcal{O}_{Y\cap\ell} \to 0$$

Thus $coker \simeq \mathcal{O}_\ell(-m+1)$ and $\mathcal{T}or_1^{\mathcal{O}}(\mathcal{O}_Y,\mathcal{O}_\ell)$ is a sheaf of finite length m. Any exact sequence

$$0 \to \mathcal{T} \to \mathcal{F} \to \mathcal{G} \to 0$$

where $\mathcal{T}$ is a sheaf of finite length and $\mathcal{G}$ locally free splits. This follows from the fact that $Ext^1(\mathcal{T},\mathcal{G}) \simeq H^1\mathcal{H}om(\mathcal{T},\mathcal{G}) = 0$. $H^1\mathcal{H}om(\mathcal{T},\mathcal{G})$ vanishes because $\mathcal{H}om(\mathcal{T},\mathcal{G})$ has finite support. Therefore $\mathcal{I}_{Y\cap\ell|\ell}(1) \simeq \mathcal{O}_\ell(-m+1)\oplus k^m$. From this reasoning one gets now the exact sequence below.

$$0 \to \mathcal{O}_\ell(-1) \stackrel{s_{|\ell}}{\to} \mathcal{E}_{|\ell} \to \mathcal{O}_\ell(-m+1)\oplus k^m \to 0$$

This sequence provides now an exact cohomology sequence of the following kind.

$$0 \to H^0\mathcal{O}_\ell(-1) \to H^0\mathcal{E}_{|\ell} \to H^0\mathcal{O}_\ell(-m+1)\oplus k^m \to H^1\mathcal{O}_\ell(-1) \to 0$$

This proves the isomorphism

$$H^0\mathcal{E}_{|\ell} \simeq H^0(\mathcal{O}_\ell(a)\oplus\mathcal{O}_\ell(-a)) \simeq H^0\mathcal{O}_\ell(a)\oplus H^0\mathcal{O}_\ell(-a) \simeq H^0\mathcal{O}_\ell(-m+1)\oplus k^m$$

For $m\geq 2$ it follows :

$$H^0\mathcal{O}_\ell(a) \simeq H^0\mathcal{O}_\ell(m-1)\oplus k^m \simeq k^m$$

therefore $a = m-1$. This proves case (3).
For $m = 0,1$ both cases $\mathcal{E}_{|\ell} \simeq \mathcal{O}_\ell\oplus\mathcal{O}_\ell$ or $\mathcal{E}_{|\ell} \simeq \mathcal{O}(1)\oplus\mathcal{O}(-1)$ can occur. Hence case (1) is true. I can exclude in case $m=1$ the splitting $\mathcal{E}_{|\ell} \simeq \mathcal{O}_\ell(1)\oplus\mathcal{O}\ell(-1)$ because the exact sequence

$$0 \to \mathcal{O}_\ell(-1) \to \mathcal{O}_\ell(1)\oplus\mathcal{O}_\ell(-1) \to \mathcal{O}_\ell\oplus k \to 0$$

is impossible. So case (2) is verified.

The case (4) is due to the fact that

$$\mathcal{E}_{|\ell} \simeq \mathcal{N}_{Y_i}^{\vee}(-1)_{|\ell}$$

where Y_i is the component of Y supported on ℓ Now $\mathcal{N}_{Y_i}^{\vee}(-1)_{|\ell}$ can be $\mathcal{O}_\ell\oplus\mathcal{O}_\ell$ or $\mathcal{O}_\ell(1)\oplus\mathcal{O}_\ell(-1)$ if Y_i is reduced or not, see 1.1.7 on page 12 #

Remark 1.19 For special 't Hooft bundles it is possible to determine the splitting behaviour for all possible lines with this methods because one uses not only the zero locus of one section but for a whole pencil of sections. It is shown in [8] that the pencil of sections defines the bundle uniquely. We see later in chapter 5 the connection between special 't Hooft bundles and their Poncelet curves which was introduced in [17] see also [8]. For 't Hooft bundles there exist only one section. The section determines the bundle only up to an extension class in $Ext^1(\mathcal{I}_Y(1), \mathcal{O}(-1))$. We shall see later that, if $\mathcal{E}$ is a special 't Hooft bundle, then the multiple Koszul structure in 1.18 case 2 are the points in $\mathbb{P}\Lambda^2 V^\vee$ in which the Poncelet curve $S(\mathcal{E})$ intersects the conic $C(\mathcal{E}) = P(\mathcal{E}) \cap G(2,4)$.

2 An inductive construction of instanton bundles

2.1 Introduction

In this chapter I shall describe an algorithm constructing explicitly selfdual monads in the meaning of page 7. This algorithm is applied to two cases: The construction of Beilinson-I-monads and monads of NC-type. I am going to use this algorithm for two purposes, the construction of examples of instanton bundles with the help of the computer algebra systems SINGULAR and Macaulay and the abstract determination of the dimension of the vector spaces of linear section for certain families of instanton bundles of NC-type.

Always, the natural question arises whether isomorphism classes of bundles are smooth points of their moduli space. This is the matter for all examples constructed in this chapter. As already mentioned on page 20 it is shown in [29] that the obstruction group to the smoothness of the Maruyama scheme at stable points $[\mathcal{E}]$ is $Ext^2(\mathcal{E},\mathcal{E})$. It is verified on page 20 that for the vanishing of $Ext^2(\mathcal{E},\mathcal{E})$ it is enough to prove that $H^2\mathcal{E}nd(\mathcal{E})(-1) \simeq Ext^2(\mathcal{E},\mathcal{E}(-1)) = 0$. For these examples here it is checked by explicit computation and in fact they are all smooth points of the Maruyama scheme.

2.2 Remarks on Beilinson-I-monads

Let $\mathbb{P}_3 = P(V)$. A Beilinson-I-monad for instantons $\mathcal{E}$ on $\mathbb{P}_3$ is a complex

$$0 \quad \to \quad n\mathcal{O}(-1) \quad \overset{a}{\to} \quad (2n+2)\mathcal{O} \quad \overset{b}{\to} \quad n\mathcal{O}(1) \quad \to \quad 0$$

with the following properties:

(1) Let a be the induced map $a : k^n \to k^{2n+2} \otimes V^\vee$ then

$$k^n \quad \overset{a(v)}{\to} \quad k^{2n+2}$$

is injective for all $v \in V$

(2) Let b be the induced map $b : k^{2n+2} \to k^n \otimes V^\vee$ then

$$k^{2n+2} \quad \overset{b(v)}{\to} \quad k^n$$

is surjective for all $v \in V$

such that $\mathcal{E}$ is the cohomology of the complex.

Remark 2.1 The subbundle conditions (1) and (2) can be checked easily by verifying that the vanishing locus of the Fitting ideal of the maximal minors of A an B looked upon as forms on $\mathbb{P}_3$ is empty.

A monad is **selfdual** if it is of the following kind:

$$0 \;\to\; A \;\xrightarrow{a}\; B \;\xrightarrow{a^\vee \circ J}\; A^\vee \;\to\; 0$$

where A and B are vector bundles and J is a symplectic isomorphism $J : B \to B^\vee$; (i.e.: $J^\vee = -J$).

It is known, that Beilinson-I-monads for instantons can be brought into a selfdual form, c.f. [25] pp 280.

Lemma 2.2 *Let $\mathcal{E}$ be a rank-2 bundle with $c_1(\mathcal{E}) = 0$ and being the cohomology bundle of the monad*

$$0 \;\to\; n\mathcal{O}(-1) \;\xrightarrow{a}\; (2n+2)\mathcal{O} \;\xrightarrow{b}\; n\mathcal{O}(1) \;\to\; 0$$

and

$$j : \;\; \mathcal{E} \;\to\; \mathcal{E}^\vee$$

be the duality pairing given by $\mathcal{E} \simeq \mathcal{E}^\vee \otimes det(\mathcal{E})$ is symplectic. This pairing is in fact $\mathcal{E} \simeq \mathcal{E}^\vee$ after a choice of an isomorphism

$$\alpha : \;\; det(\mathcal{E}) \;\to\; \mathcal{O}\ .$$

and is symplectic. Then there exist morphisms

$$a' : \;\; n\mathcal{O}(-1) \;\to\; (2n+2)\mathcal{O}$$

$$b' : \;\; (2n+2)\mathcal{O} \;\to\; n\mathcal{O}(1)$$

$$J : \;\; (2n+2)\mathcal{O} \;\to\; (2n+2)\mathcal{O}$$

such that

$$0 \;\to\; n\mathcal{O}(-1) \;\xrightarrow{a'}\; (2n+2)\mathcal{O} \;\xrightarrow{b'}\; n\mathcal{O}(1) \;\to\; 0$$

is a monad for $\mathcal{E}$ and such that

$$J^\vee = -J \text{ and } b' = a'^\vee \circ J$$

2.3 The algorithm

Before I present the algorithm in a more abstract setting I want to give an easy example for a Beilinson-I-monad which can still be done by hand. It is an example of a monad which yields an instanton $\mathcal{E}$ with $c_2(\mathcal{E}) = 3$ as its cohomology:

$$0 \;\to\; 3\mathcal{O}(-1) \;\xrightarrow{S}\; 8\mathcal{O} \;\xrightarrow{S^\vee \circ J}\; 3\mathcal{O}(1) \;\to\; 0 \;\; (*)\ .$$

After an appropriate base change of $8\mathcal{O}$ J is given by the matrix $\left(\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right)$. Let $(x_0, \cdots, x_3)$ be a basis of $V^\vee \simeq H^0\mathcal{O}(1)$

Step 1 I choose an element $s_0 \in 8H^0\mathcal{O}(1)$ randomly. In our example:

$$s_0 := (x_0, x_1, 0, 0, x_2, x_3, 0, 0)$$

This is the first row of the matrix S in the monad above. Hence:

$$S^\vee \circ J := s_0^\vee \circ J = (-x_2, -x_3, 0, 0, x_0, x_1, 0, 0)^\vee$$

where "$\vee$" indicates the transpose. Let $\mathcal{K}_1$ be the kernel of $s_0^\vee \circ J$:

$$0 \rightarrow \mathcal{K}_1 \rightarrow 8\mathcal{O} \overset{s_0^\vee \circ J}{\rightarrow} \mathcal{O}(1) \rightarrow 0$$

and let K_1 be defined as $K_1 := H^0\mathcal{K}_1$ which is the set of syzygies of $s_0^\vee \circ J$.

Step 2 In the second step I repeat the procedure of the first step, but chose now a tuple s_1 in K_1, for example

$$s_1 := (0, 0, x_0, x_1, 0, 0, x_2, x_3)\ .$$

Then

$$s_1^\vee \circ J = (0, 0, -x_2, -x_3, 0, 0, x_0, x_1)^\vee$$

Therefore we have obtained a monad

$$0 \rightarrow 2\mathcal{O}(-1) \overset{S'}{\rightarrow} 8\mathcal{O} \overset{S'^\vee \circ J}{\rightarrow} 2\mathcal{O}(1) \rightarrow 0$$

with $S' = \binom{s_0}{s_1}$ and $S'^\vee \circ J = (s_0^\vee \circ J, s_1^\vee \circ J)$.
Now it is time to verify that S is a subbundle, i.e. it has constant rank in all points of $\mathbb{P}_3$. This is done by showing that the scheme defined by all 2×2-minors has codimension 4 in $\mathbb{P}_3$. It remains now as before to determine $K_2 := H^0\mathcal{K}_2$ which is the space of linear relations of $S'^\vee = (s_0^\vee \circ J, s_1^\vee \circ J)$ where $\mathcal{K}_2$ is defined by:

$$0 \rightarrow \mathcal{K}_2 \rightarrow 8\mathcal{O} \overset{S'^\vee \circ J}{\rightarrow} 2\mathcal{O}(1) \rightarrow 0$$

Step 3 In a last step completely analogous to the previous one I finish the construction by concatenating with the element $s_2 \in K_2$, for example

$$s_2 := (-x_3, -x_2, 0, 0, 0, 0, -x_1, -x_0)\ .$$

Hence we get a subsheaf $S\ :\ 3\mathcal{O}(-1) \rightarrow 8\mathcal{O}$ given by

$$S := \begin{pmatrix} x_0 & x_1 & 0 & 0 & x_2 & x_3 & 0 & 0 \\ 0 & 0 & x_0 & x_1 & 0 & 0 & x_2 & x_3 \\ -x_3 & x_2 & 0 & 0 & 0 & 0 & -x_1 & x_0 \end{pmatrix}$$

and a self dual monad. Again one has to check whether S defines a subbundle.

Step 4 Now smoothness of $MI(3)$ at $[\mathcal{E}]$ can be checked using the monad $(*)$. We see later that it is enough to verify that the operator $(S^\vee \circ J \otimes id \mid id \otimes S^\vee \circ J)$ has no cokernel, see 2.6.

It is now the turn to present the algorithm. I use here an algorithmic notation in a meta-language not showing the details of the computer program, but presenting the idea of the algorithm. Our goal is to achieve the monad:

$$0 \quad \to \quad mA \quad \stackrel{a}{\to} \quad nB \quad \stackrel{a^\vee \circ J}{\to} \quad mA^\vee \quad \to \quad 0$$

where A and B are bundles.

The Algorithm:

$S := 0$; `/*` S `is the zero matrix */`
$K_0 := nmHom(A,B) \simeq nmH^0(A^\vee \otimes B)$;
$i := 0$;
`WHILE` $i < m$ `DO`
  `CHOOSE` $s_i \in K_0$; `/* It may be randomly chosen */`
  $S := \binom{S}{S_i}$; `/* The new` S `is the concatenation of` S `and` s_i `*/`
  `/*Hence` S `is the map` $(i+1)A \to nB$ `*/`
  `Compute` $S^\vee \circ J$; `/* ''`$\vee$`'' indicates the transpose*/`
  `/*Hence` $S^\vee \circ J$ `is the map` $nB \to i+1A^\vee$ `*/`
  `/*`$\mathcal{K}_{i+1}$ `is the kernel:` $0 \quad \to \quad \mathcal{K}_{i+1} \quad \to \quad nB \quad \stackrel{S^\vee \circ J}{\to} \quad (i+1)A^\vee \quad \to \quad 0$ `*/`
  $K_{i+1} := H^0\mathcal{K}_{i+1}$;
  `/*Check that` S `is a subbundle:*/`
    `COMPUTE the` $F_{i+1}S := (i+1) \times (i+1) - minors$ `of` S;
    `IF` $codimension(V(F_{i+1}S)) \neq 4$ `THEN END`
    `/*`$V(F_{i+1}S)$ `is the zero locus of the` $i+1^{th}$ `Fitting ideal*/`
  `i:=i+1;`
  `RETURN`
`/*Check the smoothness of the moduli space i.e */`
$coker(S^\vee \circ J \otimes id \mid id \otimes S^\vee \circ J) = 0$?
`END`

Remark 2.3 The crucial point of this algorithm is the computation of the space of linear relations in the module $Hom(mA, nB^\vee)$ for the matrix $S^\vee \circ J$. The matrix has entries in $Hom(B, A^\vee)$. Hence a pairing

$$Hom(A, B^\vee) \times Hom(B, A^\vee) \quad \to \quad Hom(A, A^\vee)$$

must be explained. Moreover the computer algebra system has to be able to handle this structure. We are here not in the general situation but construct Beilinson-I-monads. Hence the multiplicative structure is obvious:

$$0 \quad \to \quad n\mathcal{O}(-1) \quad \stackrel{S}{\to} \quad (2n+2)\mathcal{O} \quad \stackrel{S^\vee \circ J}{\to} \quad n\mathcal{O}(1) \quad \to \quad 0$$

where J is the symplectic structure defined by the matrix $\left(\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right)$. Therefore a multiplicative structure

$$H^0\mathcal{O}(1) \otimes H^0\mathcal{O}(1) \quad \to \quad H^0\mathcal{O}(2) \simeq k^{10}$$

is the multiplication of two polynomials. In a next step in chapter 4 I am going to study NC-type monads. For them the multiplication needs more consideration.

Lemma 2.4 *The rank-2 vector bundles on $\mathbb{P}_3$ constructed inductively as the cohomology of Beilinson-I-monads by this algorithm starting with a $4(m+1)\mathcal{O}$ as a middle term are instanton bundles i.e: $H^1\mathcal{E}(-2) = 0$.*

Proof:
The inductively constructed monad

$$0 \rightarrow n\mathcal{O}(-1) \rightarrow (2n+2)\mathcal{O} \rightarrow n\mathcal{O}(1) \rightarrow 0 \ ,$$

twisted by -2, has the display:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \rightarrow & n\mathcal{O}(-3) & \longrightarrow & \mathcal{K}(-2) & \longrightarrow & \mathcal{E}(-2) & \rightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \rightarrow & n\mathcal{O}(-3) & \longrightarrow & (2n+2)\mathcal{O}(-2) & \longrightarrow & Coker & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & n\mathcal{O}(-1) & = & n\mathcal{O}(-1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Obviously we have the isomorphisms

$$H^1\mathcal{E}(-2) \simeq H^1\mathcal{K}(-2) \simeq H^1\mathcal{O}(-2) = 0$$

because $H^j\mathcal{O}_i(-i) = 0$ for all $i > 0$ and $j \in \{0, 1, 2\}$. #

I list now the examples and some of their properties which did occur.

monad type	middle term	$c_2(\mathcal{E})$	$dim H^0\mathcal{E}(1)$	smooth point of $MI(c_2(\mathcal{E}))$
Beilinson-I	generic	5	0	yes
	splits off $\bigoplus_i \mathcal{E}_{a_i}$	5	0,1	yes
	splits off $(m+1)\mathcal{E}_a$	5	1,2	yes

"To split off" a generalized null correlation bundle means, that there exists a splitting as in proposition 4.5 for the Beilinson-I monad.

2.4 How to compute smoothness for a point in the moduli space ?

As already mentioned on page 20 it is shown in [29] that the obstruction group to the smoothness of the Maruyama scheme at stable points is $Ext^2(\mathcal{E},\mathcal{E})$. It is well known how one can determine the dimension of $H^2\mathcal{E}nd(\mathcal{E})$ for vector bundles $\mathcal{E}$ which are the cohomology of a monad. This result which can be found at many places, see for instance [25], requires the vanishing of certain cohomology groups. Unfortunately this is not for all kinds of monads fulfilled. I will return to this fact later when I study NC-type-monads, see page 66. Here I want to recall the result on $H^2\mathcal{E}nd(\mathcal{E})(-1)$ which is valid for Beilinson-I-monads

$$0 \;\rightarrow\; (2m+1)\mathcal{O}(-1) \;\rightarrow\; 4(m+1)\mathcal{O} \;\rightarrow\; (2m+1)\mathcal{O}(1) \;\rightarrow\; 0\ .$$

Remark 2.5 The vanishing of $H^2\mathcal{E}nd(\mathcal{E})(-1)$ is a stronger condition and implies $H^2\mathcal{E}nd(\mathcal{E})=0$, see 1.3, bottom.

Lemma 2.6 *Let*

$$M:\; 0 \;\rightarrow\; m\mathcal{O}(-1) \;\xrightarrow{a}\; n\mathcal{O} \;\xrightarrow{a^\vee\circ J}\; m\mathcal{O}(1) \;\rightarrow\; 0$$

be a selfdual Beilinson-I-monad on $\mathbb{P}_3$ with cohomology $\mathcal{E}$. We have for the mapping

$$d_0:\; Hom(m\mathcal{O}(-1),n\mathcal{O})\times Hom(n\mathcal{O},m\mathcal{O}(1)) \;\rightarrow\; Hom(m\mathcal{O}(-1),m\mathcal{O}(1))$$

defined by the matrix $(id\otimes a^\vee\circ J|a^\vee\circ J\otimes id)\in Mat(2mn\times m^2,\mathcal{O}(1))$

$$coker(d_0)=H^2\mathcal{E}nd(\mathcal{E})\ .$$

Proof:
First I remark that $\mathcal{E}nd(\mathcal{E})\simeq\mathcal{E}\otimes\mathcal{E}$ because $\mathcal{E}$ is locally free and selfdual. Let $D^{\bullet,\bullet}$ the double complex below

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & m\mathcal{O}(-1)\otimes m\mathcal{O}(-1) & \rightarrow & m\mathcal{O}(-1)\otimes n\mathcal{O} & \rightarrow & m\mathcal{O}(-1)\otimes m\mathcal{O}(1) & \rightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & n\mathcal{O}\otimes m\mathcal{O}(-1) & \rightarrow & n\mathcal{O}\otimes n\mathcal{O} & \rightarrow & n\mathcal{O}\otimes m\mathcal{O}(1) & \rightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle a^\vee\circ J\otimes id} & & \\
0 & \rightarrow & m\mathcal{O}(1)\otimes m\mathcal{O}(-1) & \rightarrow & m\mathcal{O}(1)\otimes n\mathcal{O} & \xrightarrow{id\otimes a^\vee\circ J} & m\mathcal{O}(1)\otimes m\mathcal{O}(1) & \rightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The associated total complex $K^\bullet$ is of the form:

$$0 \rightarrow K^{-2} \rightarrow K^{-1} \xrightarrow{S} K^0 \xrightarrow{T} K^1 \rightarrow K^2 \rightarrow 0$$

where

$$\begin{aligned} K^{-2} &= m\mathcal{O}(-1) \otimes m\mathcal{O}(-1) \\ K^{-1} &= m\mathcal{O}(-1) \otimes n\mathcal{O} \oplus n\mathcal{O} \otimes m\mathcal{O}(-1) \\ K^0 &= m\mathcal{O}(-1) \otimes m\mathcal{O}(1) \oplus n\mathcal{O} \otimes n\mathcal{O} \oplus m\mathcal{O}(1) \otimes m\mathcal{O}(-1) \\ K^1 &= n\mathcal{O} \otimes m\mathcal{O}(1) \oplus m\mathcal{O}(1) \otimes n\mathcal{O} \\ K^2 &= m\mathcal{O}(1) \otimes m\mathcal{O}(1) \end{aligned}$$

The complex $K^\bullet$ is exact except at K^0 where we have

$$ker(T)/im(S) = \mathcal{E}nd(\mathcal{E})$$

Fix a $k \in \mathbb{Z}$. Consider the first and second spectral sequences of hypercohomology both abutting in $\boldsymbol{H}^\bullet(\mathbb{P}_3, K^\bullet(k))$

$$\begin{aligned} {}'E_2^{p,q} &= H^p(\mathcal{H}^q(\mathbb{P}_3, K^\bullet(k))) \\ {}''E_2^{p,q} &= H^p(H^q(\mathbb{P}_3, K^\bullet(k))) \end{aligned}$$

Because $\mathcal{H}(K^\bullet) \simeq \mathcal{H}(K^0) \simeq \mathcal{E}nd(\mathcal{E})$ the first spectral sequence degenerates in the E_1 term.
Thus $H^*\mathcal{E}nd(\mathcal{E}) \simeq \boldsymbol{H}^*(\mathbb{P}_3, K^\bullet)$.

p ↑

0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	H^0K^0	H^0K^1	H^0K^2

→ q

The d_0-differentials are horizontal and the d_1-differentials go two steps to the left and one step down. $H^0K^2/d_0(H^0K^1)$ maps surjective on $H^2\mathcal{E}nd(\mathcal{E})$

$$H^0K^2/d_0(H^0K^1) \rightarrow H^2\mathcal{E}nd(\mathcal{E}) \rightarrow 0$$

because H^1K^1, H^2K^0 and H^3K^1 are zero and $GR(H^2\mathcal{E}nd(\mathcal{E})) \simeq \bigoplus_{i+j=2} E_{i,j}^\infty$ is a Quotient of $\bigoplus_{i+j=2} H^iK^j/d_0(H^iK^{j-1})$ by a general property of the local-global spectral sequence. Finally I want to determine the operator d_0 explicitly in terms of the matrix a. We have the diagram:

$$\begin{array}{ccc} & & n\mathcal{O}\otimes m\mathcal{O}(1) \\ & & \Big\downarrow a^\vee\circ J\otimes id \\ m\mathcal{O}(1)\otimes n\mathcal{O} & \xrightarrow{id\otimes a^\vee\circ J} & m\mathcal{O}(1)\otimes m\mathcal{O}(1) \end{array}$$

Hence, the map in the double complex is:

$$m\mathcal{O}(1)\otimes n\mathcal{O}\oplus n\mathcal{O}\otimes m\mathcal{O}(1) \quad \overset{(id\otimes a^\vee\circ J|a^\vee\circ J\otimes id)}{\rightarrow} \quad m\mathcal{O}(1)\otimes m\mathcal{O}(1)$$

Therefore the morphism d_0 is represented by the matrix: $(id\otimes a^\vee\circ J|a^\vee\circ J\otimes id)$ which is the concatenation of $(id\otimes a^\vee\circ J)$ and $(a^\vee\circ J\otimes id)$.

Remark 2.7 As before, let $K^\bullet$ be the total complex of the double complex $D^{\bullet,\bullet}$ as in the proof of lemma 2.6 and $\mathcal{E}nd\mathcal{E}$ is the cohomology of the complex $K^\bullet$. In fact it is only used that sufficiently many cohomology groups of the sheaves of complex $K^\bullet$ vanish such that $H^2\mathcal{E}nd\mathcal{E}$ is the cokernel of d_0, i.e.:

$$H^0K^1 \quad \overset{d_0}{\rightarrow} \quad H^0K^2 \quad \rightarrow \quad H^2\mathcal{E}nd\mathcal{E} \quad \rightarrow \quad 0$$

At the beginning of chapter 2.4 it is mentioned that $H^2\mathcal{E}nd\mathcal{E}(-1)=0$ is sufficient for the vanishing of $H^2\mathcal{E}nd\mathcal{E}$. $H^2\mathcal{E}nd\mathcal{E}(-1)$ can be computed analogously to the case of $H^2\mathcal{E}nd\mathcal{E}$. The only difference occuring is that the operator $d_0=(a^\vee\circ J\otimes id|id\otimes -a^\vee\circ J)$ is no longer a morphism

$$m\mathcal{O}(1)\otimes n\mathcal{O}\oplus n\mathcal{O}\otimes m\mathcal{O}(1) \quad \rightarrow \quad m\mathcal{O}(1)\otimes m\mathcal{O}(1)\ ,$$

but a morphism

$$m\mathcal{O}(1)\otimes n\mathcal{O}(-1)\oplus n\mathcal{O}\otimes m\mathcal{O} \quad \rightarrow \quad m\mathcal{O}(1)\otimes m\mathcal{O}\ .$$

For this situation we obtain the following lemma.

Lemma 2.8 *Let*

$$M:\ 0 \quad \rightarrow \quad m\mathcal{O}(-1) \quad \overset{a}{\rightarrow} \quad n\mathcal{O} \quad \overset{a^\vee\circ J}{\rightarrow} \quad m\mathcal{O}(1) \quad \rightarrow \quad 0\ \cdot$$

be a selfdual Beilinson-I-monad on $\mathbb{P}_3$ *with cohomology* $\mathcal{E}$. *We have for the mapping*

$$d_0:\quad Hom(m\mathcal{O}(-1),n\mathcal{O}(-1))\times Hom(n\mathcal{O},m\mathcal{O}) \quad \rightarrow \quad Hom(m\mathcal{O}(-1),m\mathcal{O})$$

defined by the matrix $(id\otimes a^\vee\circ J|a^\vee\circ J\otimes id)\in Mat(2mn\times m^2,\mathcal{O}(1))$

$$coker(d_0)=H^2\mathcal{E}nd\mathcal{E}(-1)\ .$$

Sketch of the proof :
$\mathcal{E}nd\mathcal{E}(-1)$ is the cohomology of the complex $K^\bullet(-1)$ and $H^2\mathcal{E}nd\mathcal{E}(-1)$ is the cokernel:

$$H^0K^1(-1) \quad \overset{d_0}{\rightarrow} \quad H^0K^2(-1) \quad \rightarrow \quad H^2\mathcal{E}nd\mathcal{E}(-1) \quad \rightarrow \quad 0$$

#

2.5 Examples of Beilinson-I-monads

2.5.1 Program example 1

The instanton bundle $\mathcal{E}$ with $c_2(\mathcal{E}) = 5$ constructed in this computation is an example for an instanton bundle of NC-type with $h^0\mathcal{E}(1) = 2$. Moreover its monad is selfdual in the meaning of page 7. I compute not a NC-type monad with cohomology $\mathcal{E}$, but a Beilinson-I-monad for the instanton $\mathcal{E}$. Hence such a monad splits, see proposition 4.3.

$$0 \quad \to \quad 3\mathcal{O}(-1) \oplus 2\mathcal{O}(-1) \quad \overset{\binom{A}{\tilde{\phi}'}}{\to} \quad 12\mathcal{O} \quad \overset{(A^\vee \circ J, \tilde{\phi}'^\vee \circ J)}{\to} \quad 3\mathcal{O}(1) \oplus 2\mathcal{O}(1) \quad \to \quad 0 \quad (*)$$

where J is the symplectic form $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and

$$0 \quad \to \quad 3\mathcal{O}(-1) \quad \overset{A}{\to} \quad 12\mathcal{O} \quad \overset{A^\vee \circ J}{\to} \quad 3\mathcal{O}(1) \quad \to \quad 0 \quad (**)$$

defines the direct sum of null correlation bundles, $3\mathcal{E}_a$. In the comments to the Macaulay program output I will refer to the steps in the explicit explanations of the algorithm, see 25. The commentaries I entered later in a different font: sans serif. I skipped parts of the output which are not essential for the understanding of the computation. $x[0], \cdots, x[3]$ are the coordinates of $\mathbb{P}_3$.

```
1% ring r
! characteristic (if not 31991)      ?
! number of variables                ? 4
!   4 variables, please              ? x[0]-x[3]
! variable weights (if not all 1)    ?
! monomial order (if not rev. lex.)  ?
  largest degree of a monomial       : 512
```

ker is A of the sequence $(*)$. I start the program with the matrix ker but in fact this matrix is already generated by the Macaulay script, which provides also random matrices. The same statement is valid for the matrix coker which is $A^\vee \circ J$. So the sequence $(**)$ is now fixed.

```
1% type ker
-x[2] -x[3] 0     0     0     0     x[0] x[1] 0    0    0    0
0     0     -x[2] -x[3] 0     0     0    0    x[0] x[1] 0    0
0     0     0     0     -x[2] -x[3] 0    0    0    0    x[0] x[1]

1% res ker rkernew
computation complete after degree 0
```

```
1% betti rkernew
total:      3    12    18    12     3
--------------------------------------
   -1:      3    12    18    12     3

1% type coker
x[0] 0    0
x[1] 0    0
0    x[0] 0
0    x[1] 0
0    0    x[0]
0    0    x[1]
x[2] 0    0
x[3] 0    0
0    x[2] 0
0    x[3] 0
0    0    x[2]
0    0    x[3]

1% mult ker coker result
1% type result
0 0 0
0 0 0
0 0 0
```

I prove now that ker defines a subbundle. This follows from the fact that the zero locus of the Fitting ideal of the coker is zero dimensional. But from the explicit ideal, minor below, we see that the zero locus is the point $(0,0,0,0) \in \mathbb{C}^4$ with multiplicity 4

```
1% wedge coker 3 minor
1% flatten minor mnrid

1% std mnrid rmnrid
computation complete after degree 4

1% codim rmnrid
component 1:
[4] 1 1 1 1
codimension : 4

1% copy coker newcoker
```

res is a random element in K_1 chosen, see page 25 step (1) and concatenated with the old $S =$ coker, it provides the new $S =$ newcoker.

```
1% mat rand <try.18
! number of rows    ? 18
! number of columns ? 1
! (1,1) ? 1
! (2,1) ? 0
! (3,1) ? 0
! (4,1) ? 0
! (5,1) ? 0
! (6,1) ? 0
! (7,1) ? 0
! (8,1) ? 0
! (9,1) ? 0
! (10,1) ? 0
! (11,1) ? 0
! (12,1) ? 0
! (13,1) ? 0
! (14,1) ? 0
! (15,1) ? 0
! (16,1) ? 0
! (17,1) ? 0
! (18,1) ? 1

1% mult rkernew.2 rand res
1% concat newcoker res

1% type newcoker
x[0] 0    0    -x[3]
x[1] 0    0    x[2]
0    x[0] 0    0
0    x[1] 0    0
0    0    x[0] 0
0    0    x[1] 0
x[2] 0    0    0
x[3] 0    0    0
0    x[2] 0    0
0    x[3] 0    0
0    0    x[2] -x[1]
0    0    x[3] x[0]
```

I prove now that ker defines a subbundle. This follows from the fact that the zero locus of the Fitting ideal of the 4×4 minors of newcoker is zero dimensional. But from the explicit ideal, minor below, we see that the zero locus is the point $(0,0,0,0) \in \mathbb{C}^4$ with multiplicity 4.

```
1% wedge newcoker 4 newminor
```

```
1% flatten newminor inewminor

1% std inewminor sinewminor
computation complete after degree 6
1% codim sinewminor
component 1:
[4] 1 1 1 1
codimension : 4
```

The next step contains the computation of the monad arrow S from $S^\vee \circ J$.

```
1% submat newcoker phinew
! rows ? 1..6
! columns ? 1..4

1% type phinew
x[0] 0    0    -x[3]
x[1] 0    0    x[2]
0    x[0] 0    0
0    x[1] 0    0
0    0    x[0] 0
0    0    x[1] 0

1% submat newcoker psinew
! rows ? 7..12
! columns ? 1..4

1% type psinew
x[2] 0    0    0
x[3] 0    0    0
0    x[2] 0    0
0    x[3] 0    0
0    0    x[2] -x[1]
0    0    x[3] x[0]

1% smult psinew -1 npsinew
1% transpose phinew phinewt
1% transpose npsinew npsinewt
1% concat npsinewt phinewt
1% copy npsinewt kernew

1% type kernew
-x[2] -x[3] 0     0     0     0     x[0]  x[1] 0    0    0    0
0     0     -x[2] -x[3] 0     0     0     0    x[0] x[1] 0    0
0     0     0     0     -x[2] -x[3] 0     0    0    0    x[0] x[1]
```

```
0      0      0      0      x[1]  -x[0] -x[3] x[2] 0     0     0     0
```

```
1% mult kernew newcoker res

1% type res
0 0 0 0
0 0 0 0
0 0 0 0
0 0 0 0
```

rkernew.2 is the set of generators for all relations of kernew, where kernew is the transpose of newcoker with respect to J. elem is the set of generators for all linear relations in rkernew.2.

```
1% res kernew rkernew
computation complete after degree 2

1% betti rkernew
total:      4     12     16     12     4
--------------------------------------
   -1:      4     12     11      4     1
    0:      -      -      -      -     -
    1:      -      -      5      8     3

1% submat rkernew.2 elem
! rows ? 1..12
! columns ? 1..11

1% type elem
0     0     0     0     0     0     0     x[0] 0     0     -x[3]
0     0     0     0     0     0     0     x[1] 0     0     x[2]
0     0     x[1] 0     x[0] 0     0     0     0     -x[3] 0
0     0     0     x[1] 0     x[0] 0     0     0     x[2]  0
0     0     0     0     0     0     -x[3] 0     x[0] 0     0
0     0     0     0     0     0     x[2]  0     x[1] 0     0
0     0     0     0     0     0     -x[1] x[2] 0     0     0
0     0     0     0     0     0     x[0]  x[3] 0     0     0
0     -x[1] 0     0     x[2] x[3] 0     0     0     0     0
0     x[0]  x[2] x[3] 0     0     0     0     0     0     0
-x[1] 0     0     0     0     0     0     0     x[2] 0     0
x[0]  0     0     0     0     0     0     0     x[3] 0     0
```

res is a random element in elem chosen, see page 25 step (1) and concatenated with the old $S=$ coker it provides the new $S=$ newcoker.

```
1% mat rand <try.11
[252k]! number of rows    ? 11
! number of columns ? 1
! (1,1) ? 0
! (2,1) ? 0
! (3,1) ? 0
! (4,1) ? 0
! (5,1) ? 0
! (6,1) ? 0
! (7,1) ? 1
! (8,1) ? 0
! (9,1) ? 0
! (10,1) ? 1
! (11,1) ? 0

1% mult elem rand res
1% copy newcoker newcokerold
1% concat newcoker res

1% type newcoker
x[0] 0    0    -x[3] 0
x[1] 0    0    x[2]  0
0    x[0] 0    0     -x[3]
0    x[1] 0    0     x[2]
0    0    x[0] 0     -x[3]
0    0    x[1] 0     x[2]
x[2] 0    0    0     -x[1]
x[3] 0    0    0     x[0]
0    x[2] 0    0     0
0    x[3] 0    0     0
0    0    x[2] -x[1] 0
0    0    x[3] x[0]  0
```

I prove now that coker defines a quotient bundle. This follows from the fact that the zero locus of the Fitting ideal of the 5×5 minors is zero dimensional. But from the explicit ideal, minor below, we see that the zero locus is the point $(0,0,0,0) \in k^4$ with multiplicity 4.

```
1% wedge newcoker 5 newminor
1% flatten newminor inewminor

1% std inewminor sinewminor
computation complete after degree 9

1% codim sinewminor
```

```
component 1:
[4] 1 1 1 1
codimension : 4

1% transpose newcoker newcokert
1% res newcokert rnewcokert
computation complete after degree 7

1% betti rnewcokert
total:      5    12    14    12     5
--------------------------------------
   -1:      5    12     7<--- -     -
    0:      -     -     -     -     -
    1:      -     -     -     -     -
    2:      -     -     -     -     -
    3:      -     -     7    12     5
```

$\mathcal{E}$ is now an instanton bundle with $h^0\mathcal{E}(1) = 2$ because according to the resolution of the kernel bundle $\mathcal{K}$ has 7 dimensional space of linear sections, this is indicated by the arrow $<$— above.

In a last step I check the smoothness of the moduli space $MI(5)$ in the point $[\mathcal{E}]$, see 2.8. The operator d_0, which is called B here, is a morphism

$$k^{120} \quad \to \quad k^{25} \otimes V^\vee$$

Hence B is surjective if and only if $ker(B) \subset k^{120}$ is 20 dimensional. The position of this dimension of $ker(B)$ in the output of the betti command is indicated by the arrow $<$—. These are the constant relations in the betti diagram

```
1% tensor newcokert newcokert B
[315k]

1% res B Bres 5
computation complete after degree -1
elapsed time :  9 seconds

11% betti Bres
total:     25   120   170   100    25
--------------------------------------
   -3:      -     -    20<--- -     -
   -2:     25   120   150   100    25

1% exit
```

2.5.2 Program example 2

The instanton bundle $\mathcal{E}$ with $c_2(\mathcal{E}) = 5$ constructed in this computation is an example for an instanton bundle of NC-type with $h^0\mathcal{E}(1) = 1$. Moreover its monad is selfdual in the meaning of page 7. I compute not a NC-type monad with cohomology $\mathcal{E}$, but a Beilinson-I-monad for the instanton $\mathcal{E}$. Hence such a monad splits, see proposition 4.3.

$$0 \to 3\mathcal{O}(-1) \oplus 2\mathcal{O}(-1) \stackrel{\binom{A}{\phi'}}{\to} 12\mathcal{O} \stackrel{(A^\vee \circ J, \tilde{\phi}'^\vee \circ J)}{\to} 3\mathcal{O}(1) \oplus 2\mathcal{O}(1) \to 0 \quad (*)$$

where J is the symplectic form $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and

$$0 \to 3\mathcal{O}(-1) \stackrel{A}{\to} 12\mathcal{O} \stackrel{A^\vee \circ J}{\to} 3\mathcal{O}(1) \to 0 \quad (**)$$

defines the direct sum of null correlation bundles $\mathcal{E}_{a_1} \oplus \mathcal{E}_{a_2} \oplus \mathcal{E}_{a_3}$ In the comments to the program output I will refer to the steps in the explicit explanations of the algorithm, see 25. The commentaries I added later are in a different font: sans serif. I skipped parts of the output which are not essential for the understanding of the computation. $x[0], \cdots, x[3]$ are the coordinates of $\mathbb{P}_3$.

```
1% ring r
! characteristic (if not 31991)      ?
! number of variables                ? 4
!   4 variables, please              ? x[0]-x[3]
! variable weights (if not all 1)    ?
! monomial order (if not rev. lex.)  ?
  largest degree of a monomial       : 512
```

ker is A of the sequence $(*)$. I start the program with the matrix ker but in fact this matrix is already generated by the Macaulay script, which provides also random matrices. The same statement is valid for the matrix coker which is $A^\vee \circ J$. So the sequence $(**)$ is now fixed.

```
1% type ker
-x[2] -x[3] 0           0           0           0           x[0] x[1] 0
0     0     -x[0]+x[2] -x[1]+x[3] 0           0           0    0    x[0]+x[2]
0     0     0           0           -x[0]+x[3] -x[1]+x[2] 0    0    0

0         0         0
x[1]+x[3] 0         0
0         x[0]+x[3] x[1]+x[2]

1% res ker rkernew
computation complete after degree 0
```

```
1% betti rkernew
total:      3     12     18     12      3
--------------------------------------
   -1:      3     12     18     12      3

1% type coker
x[0] 0         0
x[1] 0         0
0    x[0]+x[2] 0
0    x[1]+x[3] 0
0    0         x[0]+x[3]
0    0         x[1]+x[2]
x[2] 0         0
x[3] 0         0
0    x[0]-x[2] 0
0    x[1]-x[3] 0
0    0         x[0]-x[3]
0    0         x[1]-x[2]

1% mult ker coker result

1% type result
0 0 0
0 0 0
0 0 0
```

I prove now that ker defines a subbundle. This follows from the fact that the zero locus of the Fitting ideal of the coker is zero dimensional. But from the explicit ideal, minor below, we see that the zero locus is the point $(0,0,0,0) \in \mathbb{C}^4$ with multiplicity 4

```
1% wedge coker 3 minor
1% flatten minor mnrid

1% std mnrid rmnrid
computation complete after degree 4

1% codim rmnrid
component 1:
[4] 1 1 1 1
codimension : 4

1% copy coker newcoker
```

res is a random element in K_1 chosen, see page 25 step (1) and concatenated with the old $S =$ coker it provides the new $S =$ newcoker.

```
1% mat rand <try.18
! number of rows    ? 18
! number of columns ? 1
! (1,1) ? 1
! (2,1) ? 0
! (3,1) ? 1
! (4,1) ? 0
! (5,1) ? 0
! (6,1) ? 1
! (7,1) ? 0
! (8,1) ? 1
! (9,1) ? 0
! (10,1) ? 0
! (11,1) ? 1
! (12,1) ? 0
! (13,1) ? 0
! (14,1) ? 0
! (15,1) ? 0
! (16,1) ? 0
! (17,1) ? 0
! (18,1) ? 1


1% mult rkernew.2 rand res

1% concat newcoker res

1% type newcoker
x[0] 0         0         -x[3]
x[1] 0         0         x[2]
0    x[0]+x[2] 0         0
0    x[1]+x[3] 0         x[0]+x[2]
0    0         x[0]+x[3] 2x[2]
0    0         x[1]+x[2] x[0]-x[3]
x[2] 0         0         -x[1]
x[3] 0         0         x[0]
0    x[0]-x[2] 0         -2x[3]
0    x[1]-x[3] 0         x[0]+x[2]
0    0         x[0]-x[3] 0
0    0         x[1]-x[2] x[0]-x[3]
```

I prove now that ker defines a subbundle. This follows from the fact that the zero locus of the Fitting ideal of the 4×4 minors of newcoker is zero dimensional. But from the explicit ideal, minor below, we see that the zero locus is the point $(0,0,0,0) \in \mathbb{C}^4$ with multiplicity 4.

```
1% wedge newcoker 4 newminor
1% flatten newminor inewminor
[252k]
1% std inewminor sinewminor
computation complete after degree 6

1% codim sinewminor
component 1:
[4] 1 1 1 1
codimension : 4
```

The next step contains the computation of the monad arrow S from $S^\vee \circ J$.

```
1% submat newcoker phinew
! rows ? 1..6
! columns ? 1..4

1% type phinew
x[0] 0         0         -x[3]
x[1] 0         0         x[2]
0    x[0]+x[2] 0         0
0    x[1]+x[3] 0         x[0]+x[2]
0    0         x[0]+x[3] 2x[2]
0    0         x[1]+x[2] x[0]-x[3]

1% submat newcoker psinew
! rows ? 7..12
! columns ? 1..4

1% type psinew
x[2] 0         0         -x[1]
x[3] 0         0         x[0]
0    x[0]-x[2] 0         -2x[3]
0    x[1]-x[3] 0         x[0]+x[2]
0    0         x[0]-x[3] 0
0    0         x[1]-x[2] x[0]-x[3]

1% smult psinew -1 npsinew
```

```
1% transpose phinew phinewt

1% transpose npsinew npsinewt

1% concat npsinewt phinewt

1% copy npsinewt kernew

1% type kernew
-x[2] -x[3] 0           0           0           0           x[0]  x[1] 0
0     0     -x[0]+x[2] -x[1]+x[3] 0           0           0     0    x[0]+x[2]
0     0     0           0           -x[0]+x[3] -x[1]+x[2] 0     0    0
x[1]  -x[0] 2x[3]       -x[0]-x[2] 0           -x[0]+x[3] -x[3] x[2] 0

0         0         0
x[1]+x[3] 0         0
0         x[0]+x[3] x[1]+x[2]
x[0]+x[2] 2x[2]     x[0]-x[3]

1% mult kernew newcoker res
[315k]
1% type res
0 0 0 0
0 0 0 0
0 0 0 0
0 0 0 0
```

rkernew.2 is the set of generators for all relations of kernew, where kernew is the transpose of newcoker with respect to J. elem is the set of generators for all linear relations in rkernew.2.

```
1% res kernew rkernew
computation complete after degree 1

1% betti rkernew
total:      4    12    16    12     4
--------------------------------------
   -1:      4    12     8     -     -
    0:      -     -     8    12     4

1% submat rkernew.2 elem
! rows ? 1..12
! columns ? 1..9

1% type elem
```

```
0         0         0         0         -x[3] x[3]
0         0         0         0         x[2]  -x[0]-x[2]
0         0         0         x[0]+x[2] 0     15995x[1]
0         0         x[0]+x[2] x[1]+x[3] 0     x[1]+15995x[2]+x[3]
2x[2]     x[0]+x[3] 0         0         0     -x[1]-15995x[2]
x[0]-x[3] x[1]+x[2] 0         0         0     x[0]+x[1]+x[2]+15995x[3]
0         0         0         0         -x[1] -x[3]
0         0         0         0         x[0]  0
0         0         -2x[3]    x[0]-x[2] 0     15995x[1]
0         0         x[0]+x[2] x[1]-x[3] 0     x[1]-15995x[2]-x[3]
0         x[0]-x[3] 0         0         0     15995x[2]
x[0]-x[3] x[1]-x[2] 0         0         0     x[1]-x[2]-15995x[3]

x[3]                     x[0] -2x[2]x[3]-4x[3]2
-x[0]-x[2]               x[1] 2x[2]2-2x[1]x[3]+4x[2]x[3]
15995x[1]+15995x[3]      0    2x[1]x[2]+2x[1]x[3]-x[2]x[3]
x[0]+x[2]                0    x[1]2-6x[1]x[2]+3x[2]2+x[1]x[3]-4x[2]x[3]
0                        0    2x[2]2-x[2]x[3]
0                        0    -3x[1]x[2]-3x[2]2+2x[1]x[3]+x[3]2
-x[3]                    x[2] 0
0                        x[3] -2x[3]2
-15995x[1]+15994x[3]     0    2x[1]x[2]+2x[1]x[3]-x[2]x[3]
x[2]                     0    x[1]2-6x[1]x[2]-x[2]2-x[1]x[3]+4x[2]x[3]
0                        0    2x[2]2-x[2]x[3]
0                        0    -3x[1]x[2]+3x[2]2+2x[1]x[3]-4x[2]x[3]+x[3]2
```

res is a random element in elem chosen, see page 25 step (1) and concatenated with the old $S =$ coker it provides the new $S =$ newcoker.

```
1% mat rand <try.9
! number of rows    ? 9
! number of columns ? 1
! (1,1) ? 0
! (2,1) ? 0
! (3,1) ? 0
! (4,1) ? 0
! (5,1) ? 0
! (6,1) ? 0
! (7,1) ? 1
! (8,1) ? 1
! (9,1) ? 0

1% mult elem rand res

1% copy newcoker newcokerold
```

```
1% concat newcoker res

1% type newcoker
x[0] 0         0         -x[3]     x[0]+x[3]
x[1] 0         0         x[2]      -x[0]+x[1]-x[2]
0    x[0]+x[2] 0         0         15995x[1]+15995x[3]
0    x[1]+x[3] 0         x[0]+x[2] x[0]+x[2]
0    0         x[0]+x[3] 2x[2]     0
0    0         x[1]+x[2] x[0]-x[3] 0
x[2] 0         0         -x[1]     x[2]-x[3]
x[3] 0         0         x[0]      x[3]
0    x[0]-x[2] 0         -2x[3]    -15995x[1]+15994x[3]
0    x[1]-x[3] 0         x[0]+x[2] x[2]
0    0         x[0]-x[3] 0         0
0    0         x[1]-x[2] x[0]-x[3] 0
```

I prove now that coker defines a quotient bundle. This follows from the fact that the zero locus of the Fitting ideal of the 5×5 minors is zero dimensional. But from the explicit ideal, minor below, we see that the zero locus is the point $(0,0,0,0) \in k^4$ with multiplicity 4.

```
1% wedge newcoker 5 newminor
1% flatten newminor inewminor

1% std inewminor sinewminor
computation complete after degree 7

1% codim sinewminor
component 1:
[4] 1 1 1 1
codimension : 4

1% transpose newcoker newcokert

1% res newcokert rnewcokert
computation complete after degree 9
elapsed time :  1 second

1% betti rnewcokert
total:      5    12    15    13     5
--------------------------------------
   -1:      5    12     6<--- -     -
    0:      -     -     -     -     -
    1:      -     -     5     4     1
```

```
   2:         -      -      4      9      4
```

$\mathcal{E}$ is now an instanton bundle with $h^0\mathcal{E}(1) = 1$ because according to the resolution of the kernel bundle $\mathcal{K}$ has 6 dimensional space of linear sections, this is indicated by the arrow $<$— above.

In a last step I check the smoothness of the moduli space $MI(5)$ in the point $[\mathcal{E}]$, see 2.8. The operator d_0, which is called B here, is a morphism

$$k^{120} \quad \rightarrow \quad k^{25} \otimes V^{\vee}$$

Hence B is surjective if and only if $ker(B) \subset k^{120}$ is 20 dimensional. The position of this dimension of $ker(B)$ in the output of the betti command is indicated by the arrow $<$— . These are the constant relations in the betti diagram

```
1% tensor newcokert newcokert B

1% res B Bres 5
computation complete after degree -1
elapsed time :  9 seconds

1% betti Bres
total:     25   120   170   100    25
--------------------------------------
   -3:      -     -    20<--- -     -
   -2:     25   120   150   100    25

1% exit
```

2.5.3 Program example 3

The instanton bundle $\mathcal{E}$ with $c_2(\mathcal{E}) = 5$ constructed in this computation is an example for an instanton bundle of NC-type with $h^0\mathcal{E}(1) = 0$. Moreover its monad is selfdual in the meaning of page 7. I compute not a NC-type monad with cohomology $\mathcal{E}$, but a Beilinson-I-monad for the instanton $\mathcal{E}$. Hence such a monad splits, see proposition 4.3.

$$0 \rightarrow 3\mathcal{O}(-1) \oplus 2\mathcal{O}(-1) \overset{\binom{A}{\phi'}}{\rightarrow} 12\mathcal{O} \overset{(A^\vee \circ J, \tilde{\phi}'^\vee \circ J)}{\rightarrow} 3\mathcal{O}(1) \oplus 2\mathcal{O}(1) \rightarrow 0 \quad (*)$$

where J is the symplectic form $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and

$$0 \rightarrow 3\mathcal{O}(-1) \overset{A}{\rightarrow} 12\mathcal{O} \overset{A^\vee \circ J}{\rightarrow} 3\mathcal{O}(1) \rightarrow 0 \quad (**)$$

defines the direct sum of null correlation bundles $\mathcal{E}_{a_1} \oplus \mathcal{E}_{a_2} \oplus \mathcal{E}_{a_3}$ In the comments to the program output I will refer to the steps in the explicit explanations of the algorithm, see 25. The commentaries I added later are in a different font: sans serif. I skipped parts of the output which are not essential for the understanding of the computation. $x[0], \cdots, x[3]$ are the coordinates of $\mathbb{P}_3$.

```
1% ring r
! characteristic (if not 31991)      ?
! number of variables                ? 4
!   4 variables, please              ? x[0]-x[3]
! variable weights (if not all 1)    ?
! monomial order (if not rev. lex.)  ?
  largest degree of a monomial       : 512
```

ker is A of the sequence $(*)$. I start the program with the matrix ker but in fact this matrix is already generated by the Macaulay script, which provides also random matrices. The same statement is valid for the matrix coker which is $A^\vee \circ J$. So the sequence $(**)$ is now fixed.

```
1% type ker
-x[2] -x[3] 0           0           0           0           x[0] x[1] 0
0     0     -x[0]+x[2] -x[1]+x[3] 0           0           0    0    x[0]+x[2]
0     0     0           0           -x[0]+x[3] -x[1]+x[2] 0    0    0

0         0         0
x[1]+x[3] 0         0
0         x[0]+x[3] x[1]+x[2]

1% res ker rkernew
computation complete after degree 0
```

```
1% betti rkernew
total:      3    12    18    12     3
------------------------------------
   -1:      3    12    18    12     3

1% type coker
x[0] 0         0
x[1] 0         0
0    x[0]+x[2] 0
0    x[1]+x[3] 0
0    0         x[0]+x[3]
0    0         x[1]+x[2]
x[2] 0         0
x[3] 0         0
0    x[0]-x[2] 0
0    x[1]-x[3] 0
0    0         x[0]-x[3]
0    0         x[1]-x[2]

1% mult ker coker result

1% type result
0 0 0
0 0 0
0 0 0
```

I prove now that ker defines a subbundle. This follows from the fact that the zero locus of the Fitting ideal of the coker is zero dimensional. But from the explicit ideal, minor below, we see that the zero locus is the point $(0,0,0,0) \in \mathbb{C}^4$ with multiplicity 4

```
1% wedge coker 3 minor
1% flatten minor mnrid

1% std mnrid rmnrid
computation complete after degree 4

1% codim rmnrid
component 1:
[4] 1 1 1 1
codimension : 4

1% copy coker newcoker
```

res is a random element in K_1 chosen, see page 25 step (1) and concatenated with the old $S =$ coker it provides the new $S =$ newcoker.

```
1% random 18 1 rand

1% mult rkernew.2 rand res

1% concat newcoker res

1% type newcoker
x[0] 0         0         12060x[0]+12767x[1]+10808x[3]
x[1] 0         0         234x[0]+9084x[1]-10808x[2]
0    x[0]+x[2] 0         -15779x[0]+10988x[1]-15779x[2]-12780x[3]
0    x[1]+x[3] 0         -1275x[0]-8981x[1]-9498x[2]-8981x[3]
0    0         x[0]+x[3] -14476x[0]+13508x[1]+12503x[2]-14476x[3]
0    0         x[1]+x[2] 2390x[0]+5758x[1]+5758x[2]+3395x[3]
x[2] 0         0         -2749x[1]+12060x[2]+234x[3]
x[3] 0         0         2749x[0]+12767x[2]+9084x[3]
0    x[0]-x[2] 0         -15779x[0]+4086x[1]+15779x[2]+14859x[3]
0    x[1]-x[3] 0         5627x[0]-8981x[1]+7419x[2]+8981x[3]
0    0         x[0]-x[3] -14476x[0]-940x[1]-6725x[2]+14476x[3]
0    0         x[1]-x[2] -15153x[0]+5758x[1]-5758x[2]-9173x[3]
```

I prove now that coker defines a quotient bundle. This follows from the fact that the zero locus of the Fitting ideal of the 4×4 minors is zero dimensional. But from the explicit ideal, minor below, we see that the zero locus is the point $(0,0,0,0) \in \mathbb{C}^4$ with multiplicity 4.

```
1% wedge newcoker 4 newminor
1% flatten newminor inewminor

1% std inewminor sinewminor
computation complete after degree 5

1% codim sinewminor
component 1:
[4] 1 1 1 1
codimension : 4
```

The next step contains the computation of the monad arrow S from $S^\vee \circ J$.

```
1% submat newcoker phinew
! rows ? 1..6
```

```
! columns ? 1..4

1% type phinew
x[0] 0         0         12060x[0]+12767x[1]+10808x[3]
x[1] 0         0         234x[0]+9084x[1]-10808x[2]
0    x[0]+x[2] 0         -15779x[0]+10988x[1]-15779x[2]-12780x[3]
0    x[1]+x[3] 0         -1275x[0]-8981x[1]-9498x[2]-8981x[3]
0    0         x[0]+x[3] -14476x[0]+13508x[1]+12503x[2]-14476x[3]
0    0         x[1]+x[2] 2390x[0]+5758x[1]+5758x[2]+3395x[3]

1% submat newcoker psinew
! rows ? 7..12
! columns ? 1..4

1% type psinew
x[2] 0         0         -2749x[1]+12060x[2]+234x[3]
x[3] 0         0         2749x[0]+12767x[2]+9084x[3]
0    x[0]-x[2] 0         -15779x[0]+4086x[1]+15779x[2]+14859x[3]
0    x[1]-x[3] 0         5627x[0]-8981x[1]+7419x[2]+8981x[3]
0    0         x[0]-x[3] -14476x[0]-940x[1]-6725x[2]+14476x[3]
0    0         x[1]-x[2] -15153x[0]+5758x[1]-5758x[2]-9173x[3]

1% smult psinew -1 npsinew

1% transpose phinew phinewt

1% transpose npsinew npsinewt

1% concat npsinewt phinewt

1% copy npsinewt kernew

1% type kernew
-x[2]                        -x[3]
0                            0
0                            0
2749x[1]-12060x[2]-234x[3] -2749x[0]-12767x[2]-9084x[3]

0                                       0
-x[0]+x[2]                              -x[1]+x[3]
0                                       0
15779x[0]-4086x[1]-15779x[2]-14859x[3] -5627x[0]+8981x[1]-7419x[2]-8981x[3]

0                                    0
0                                    0
```

```
-x[0]+x[3]                            -x[1]+x[2]
14476x[0]+940x[1]+6725x[2]-14476x[3] 15153x[0]-5758x[1]+5758x[2]+9173x[3]

x[0]                          x[1]
0                             0
0                             0
12060x[0]+12767x[1]+10808x[3] 234x[0]+9084x[1]-10808x[2]

0                                         0
x[0]+x[2]                                 x[1]+x[3]
0                                         0
-15779x[0]+10988x[1]-15779x[2]-12780x[3] -1275x[0]-8981x[1]-9498x[2]-8981x[3]

0                                         0
0                                         0
x[0]+x[3]                                 x[1]+x[2]
-14476x[0]+13508x[1]+12503x[2]-14476x[3] 2390x[0]+5758x[1]+5758x[2]+3395x[3]

1% mult kernew newcoker res

1% type res
0 0 0 0
0 0 0 0
0 0 0 0
0 0 0 0
```

rkernew.2 is the set of generators for all relations of kernew, where kernew is the transpose of newcoker with respect to J. elem is the set of generators for all linear relations in rkernew.2.

```
1% res kernew rkernew
computation complete after degree 1

1% betti rkernew
total:      4    12    16    12     4
--------------------------------------
   -1:      4    12     8     -     -
    0:      -     -     8    12     4

1% submat rkernew.2 elem
! rows ? 1..12
! columns ? 1..8

1% type elem
8241x[0]-15389x[1]+13892x[3]          6373x[0]-13062x[1]-8874x[3]
```

```
-15387x[0]-4712x[1]-13892x[2]        10003x[0]+15400x[1]+8874x[2]
9138x[0]+9138x[2]+7358x[3]           -7727x[0]-14070x[1]-7727x[2]+12737x[3]
15123x[0]-8515x[1]+7765x[2]-8515x[3] -9078x[0]+7037x[1]-3894x[2]+7037x[3]
3559x[0]-2561x[1]+8450x[2]+3559x[3]  x[0]+x[3]
x[0]-4579x[1]-4579x[2]-11010x[3]     x[1]+x[2]
8241x[2]-15387x[3]                   6373x[2]+10003x[3]
-15389x[2]-4712x[3]                  -13062x[2]+15400x[3]
9138x[0]-9138x[2]+9103x[3]           -7727x[0]-14070x[1]+7727x[2]-1098x[3]
15123x[0]-8515x[1]+7765x[2]+8515x[3] -9078x[0]+7037x[1]-7745x[2]-7037x[3]
3559x[0]-2561x[1]+8448x[2]-3559x[3]  x[0]-x[3]
x[0]-4579x[1]+4579x[2]-5888x[3]      x[1]-x[2]

13978x[0]+2285x[1]-14377x[3]      -1527x[0] -15062x[0]-1194x[1]-13507x[3]
-1527x[0]+1576x[1]+14377x[2]      -1527x[1] 5586x[0]+2486x[1]+13507x[2]
347x[0]-9471x[1]+347x[2]+7762x[3] x[0]+x[2] 0
x[0]+2932x[1]+14759x[2]+2932x[3]  x[1]+x[3] 0
0                                 0         0
0                                 0         0
13978x[2]-1527x[3]                -1527x[2] -x[1]-15062x[2]+5586x[3]
2285x[2]+1576x[3]                 -1527x[3] x[0]-1194x[2]+2486x[3]
347x[0]-9471x[1]-347x[2]+7760x[3] x[0]-x[2] 0
x[0]+2932x[1]+1710x[2]-2932x[3]   x[1]-x[3] 0
0                                 0         0
0                                 0         0

-4455x[0]-9874x[1]+8094x[3]         5443x[0]-3574x[1]+14045x[3]     x[0]
14750x[0]+4304x[1]-8094x[2]         -10464x[0]-11602x[1]-14045x[2]  x[1]
-9333x[1]+267x[3]                   -9387x[1]-5443x[3]              0
3280x[1]-9600x[2]+3280x[3]          x[0]+4132x[1]-3943x[2]+4132x[3] 0
-7755x[1]+10230x[2]                 0                               0
x[0]-14813x[1]-14813x[2]+14007x[3]  0                               0
-4455x[2]+14750x[3]                 5443x[2]-10464x[3]              x[2]
-9874x[2]+4304x[3]                  -3574x[2]-11602x[3]             x[3]
-9333x[1]+267x[3]                   -9386x[1]-5444x[3]              0
3280x[1]+9066x[2]-3280x[3]          4132x[1]+14830x[2]-4132x[3]     0
-7754x[1]+10229x[2]                 0                               0
-14813x[1]+14813x[2]-2475x[3]       0                               0
```

res is a random element in elem chosen, see page 25 step (1) and concatenated with the old $S =$ coker it provides the new $S =$ newcoker.

```
1% random 8 1 rand

1% mult elem rand res
```

```
1% copy newcoker newcokerold

1% concat newcoker res

1% type newcoker
x[0] 0         0         12060x[0]+12767x[1]+10808x[3]
x[1] 0         0         234x[0]+9084x[1]-10808x[2]
0    x[0]+x[2] 0         -15779x[0]+10988x[1]-15779x[2]-12780x[3]
0    x[1]+x[3] 0         -1275x[0]-8981x[1]-9498x[2]-8981x[3]
0    0         x[0]+x[3] -14476x[0]+13508x[1]+12503x[2]-14476x[3]
0    0         x[1]+x[2] 2390x[0]+5758x[1]+5758x[2]+3395x[3]
x[2] 0         0         -2749x[1]+12060x[2]+234x[3]
x[3] 0         0         2749x[0]+12767x[2]+9084x[3]
0    x[0]-x[2] 0         -15779x[0]+4086x[1]+15779x[2]+14859x[3]
0    x[1]-x[3] 0         5627x[0]-8981x[1]+7419x[2]+8981x[3]
0    0         x[0]-x[3] -14476x[0]-940x[1]-6725x[2]+14476x[3]
0    0         x[1]-x[2] -15153x[0]+5758x[1]-5758x[2]-9173x[3]

-5416x[0]+5025x[1]+4981x[3]
5708x[0]-3914x[1]-4981x[2]
-8714x[0]-1337x[1]-8714x[2]+10631x[3]
10697x[0]-429x[1]-1271x[2]-429x[3]
9322x[0]+5730x[1]-5387x[2]+9322x[3]
3457x[0]+15052x[1]+15052x[2]+14574x[3]
11496x[1]-5416x[2]+5708x[3]
-11496x[0]+5025x[2]-3914x[3]
-8714x[0]+10030x[1]+8714x[2]+604x[3]
-670x[0]-429x[1]-9964x[2]+429x[3]
9322x[0]-15950x[1]-1990x[2]-9322x[3]
-6854x[0]+15052x[1]-15052x[2]-7197x[3]
```

I prove now that ker defines a subbundle. This follows from the fact that the zero locus of the Fitting ideal of the 5×5 minors of newcoker is zero dimensional. But from the explicit ideal, minor below, we see that the zero locus is the point $(0, 0, 0, 0) \in k^4$ with multiplicity 4.

```
1% wedge newcoker 5 newminor
1% flatten newminor inewminor

1% std inewminor sinewminor
computation complete after degree 6

1% codim sinewminor
component 1:
[4] 1 1 1 1
```

```
codimension : 4

1% transpose newcoker newcokert

1% res newcokert rnewcokert
computation complete after degree 7
elapsed time :  2 seconds

1% betti rnewcokert
total:       5    12    20    20     7
--------------------------------------
   -1:       5    12     5<--- -     -
    0:       -     -     -     -     -
    1:       -     -    15    20     7
```

$\mathcal{E}$ is now an instanton bundle with $h^0\mathcal{E}(1) = 0$ because according to the resolution of the kernel bundle $\mathcal{K}$ has only a 5-dimensional space of linear sections, this is indicated by the arrow ¡— above.

In a last step I check the smoothness of the moduli space $MI(5)$ in the point $[\mathcal{E}]$, see 2.8. The operator d_0, which is called B here, is a morphism

$$k^{120} \quad \to \quad k^{25} \otimes V^{\vee}$$

Hence B is surjective if and only if $ker(B) \subset k^{120}$ is 20 dimensional. The position of this dimension of $ker(B)$ in the output of the betti command is indicated by the arrow ¡—. These are the constant relations in the betti diagram

```
1% tensor newcokert newcokert B
[1259k]
1% transpose B Bt

1% res B Bres 5
computation complete after degree -1

11% betti Bres
total:      25   120   170   100    25
--------------------------------------
   -3:       -     -    20<--- -     -
   -2:      25   120   150   100    25

1% exit
```

3 Complements on null correlation bundles

This chapter is preparatory for the next one. Proposition 3.9 I shows that the two products of two sections $s, t \in H^0\mathcal{E}_a(1)$ defined in 3.6 and 3.8 are isomorphic. This means the diagram 3.9 commutes and B in this diagram is an isomorphism.

3.1 Digression on conics in the Grassmannian of lines $G(2,4)$, and reguli in the projective 3-fold

We can describe the relations between conics C in $G(2,4)$ and quadrics Q in $\mathbb{P}_3$ in terms of the incidence diagram:

$$\begin{array}{ccc} \mathbb{F} & \xrightarrow{p} & G(2,4) \\ {\scriptstyle q}\downarrow & & \\ \mathbb{P}_3 & & \end{array}$$

where $\mathbb{F}$ is the incidence variety of points and lines in $P_3 := \mathbb{P}(V)$ and q, p are the natural projections. If we restrict p to the conic $C \subset G(2,4)$ we get:

$$\begin{array}{ccc} p^{-1}(C) & \longrightarrow & C \\ \downarrow & & \\ Q := qp^{-1}(C) & & \end{array}$$

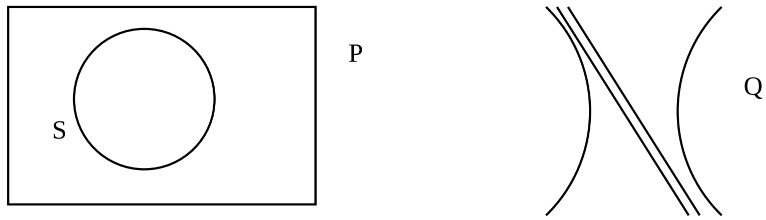
such that $Q := qp^{-1}(C)$ is a quadric surface. ($p^{-1}(C) \to C$ is a $\mathbb{P}_1$-bundle and q is one-to-one on $p^{-1}(C)$)

Definition 3.1 (associated quadric) *The quadric $Q := qp^{-1}(C)$ in the incidence diagram above is called the associated quadric to the conic C. One can also say that the quadric Q is "swept out" by the linear system L parameterized by the conic C.*

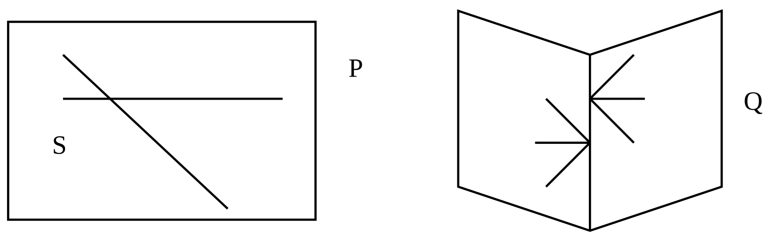
Remark 3.2 Classically the pair (Q, L) is called a regulus.

A description including the singular cases is given in [22]. We can restrict ourselves to the cases A,D and G of the classification in [22] for which the unique plane $P \subset \mathbb{P}\Lambda^2 V$ containing C is not contained in $G(2,4)$ because in our case the plane contains always the point $\langle a \rangle \in \mathbb{P}\Lambda^2 V \setminus G(2,4)$, representing the null correlation bundle $[\mathcal{E}_a]$. Therefore $C = G(2,4) \cap P$, see 3.4. Hence I list only type A, D and G of the classification of reguli in [22]:

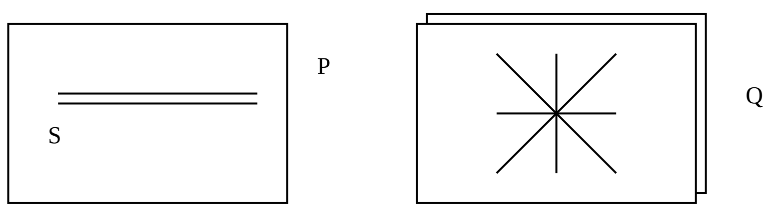
A a smooth quadric with q one-to-one on $p^{-1}(C)$



D two planes intersecting in a line



G a double plane blown up in one point



Remark 3.3 (1) Observe that in the general classification given in [22] the conic is not unique for quadrics but it is for reguli.

(2) It is well known that any smooth quadric in $\mathbb{P}_3$ is uniquely determined by three lines lying on it.

(3) Following the last remark we can construct the quadric associated to a smooth conic C by choosing three points on C.

3.2 Equivalence of pairings

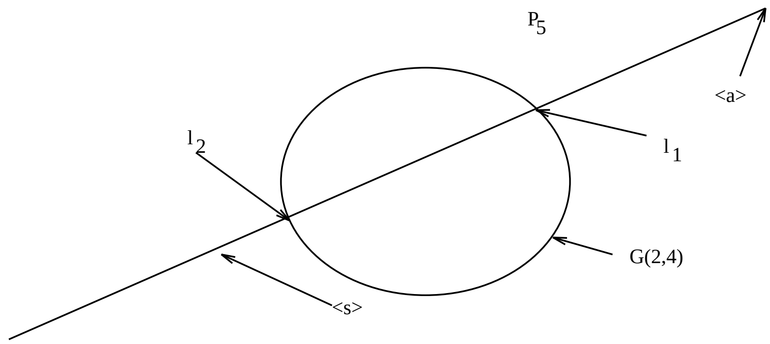
In this paragraph I apply now the remark on conics in $G(2,4)$ to the already announced equivalence of pairings, defined in 3.6 and 3.8. First the geometric meaning of a linear section of a null correlation bundle is explained. For the following the definitions and prerequisites about duality in 0.1. First I want to describe the zero locus of a linear section of a null correlation bundle. By 0.1 one has the exact diagram:

$$\begin{array}{ccccccccc} 0 & \to & \Omega^3(3) & \xrightarrow{a} & \Omega^1(1) & \to & \mathcal{E}_a & \to & 0 \\ & & & & \nearrow & & & & \\ & & \downarrow \alpha & & a' & & & & \\ & & \mathcal{O}(-1) & & & & & & \end{array}$$

Thus I obtain the cohomology sequence:

$$\begin{array}{ccccccccc} 0 & \to & \Lambda^4 V^\vee & \to & \Lambda^2 V^\vee & \to & H^0\mathcal{E}_a(1) & \xrightarrow{\lrcorner a} & 0 \\ & & \wr\wr & & \wr\wr & \nearrow & & & \\ & & k & \xrightarrow{a} & \Lambda^2 V & & & & \end{array}$$

Hence $\Lambda^2 V/\langle a\rangle \simeq H^0\mathcal{E}_a(1)$. Thus a section $\bar{s} \in H^0\mathcal{E}_a(1)$ corresponds to a line $\overline{\langle a\rangle\langle s\rangle}$ in $\mathbb{P}\Lambda^2 V$. where s is a representative of $\bar{s}$. This is visualize in the picture below



Moreover $\langle a\rangle$ is the point in the moduli space $\mathbb{P}\Lambda^2 V \setminus G(2,4) \simeq MI(1)$ corresponding

to $\mathcal{E}_a$. Summarizing we have:

Lemma 3.4 *Let $\mathcal{E}_a$ be a null correlation bundle, then*

(1) a section $\bar{s} \in H^0\mathcal{E}_a(1)$ defines a line in $\mathbb{P}\Lambda^2 V$ through $\langle a\rangle$ and vice versa.

(2) The zero locus $V(\bar{s})$ of $\bar{s}$ is the union of the two lines in $\mathbb{P}_3$ defined by the points of intersection ℓ_1 and ℓ_2 in $\overline{\langle a\rangle\langle s\rangle} \cap G(2,4)$.

Remark 3.5 This lemma above is a special case of [8][lemma 1.4]

I want to define now a "pairing" of two sections of $\mathcal{E}(1)$ in two different ways and shall show that they are isomorphic.

Let $\mathcal{E}_a$ be a null correlation bundle. If we choose an isomorphism $\alpha : det(\mathcal{E}_a(1)) \to \mathcal{O}(2)$ we obtain a morphism $J(\alpha)$

$$\mathcal{E}_a(1) \otimes \mathcal{E}_a(1) \quad \xrightarrow{J} \quad det(\mathcal{E}_a(1)) \quad \xrightarrow{\alpha} \quad \mathcal{O}(2)$$

where J is the canonical pairing.

Definition 3.6 (definition 1) *Let $\mathcal{E}_a$ be a null correlation bundle and $s, t \in H^0(\mathcal{E}_a(1))$ linear sections. Let $t^\vee \circ J \circ s$ be the composition:*

$$\mathcal{O} \xrightarrow{s} \mathcal{E}_a(1) \xrightarrow{J} \mathcal{E}_a^\vee(1) \xrightarrow{t^\vee} \mathcal{O}(2)$$

This composition induces a pairing

$$\begin{array}{rccc} \bar{A}: & H^0\mathcal{E}_a(1) \otimes H^0\mathcal{E}_a(1) & \to & H^0\mathcal{O}(2) \\ & s \otimes t & \mapsto & t^\vee \circ J \circ s \end{array}$$

Before I can continue with the second definition I recall some facts about the cup product, see Godement [11]pp.255 . I want to mention only the results for the special case needed to define the multiplication.

Let $\mathcal{E}$ and $\mathcal{F}$ locally free sheaves. The cup product in degree 0 is defined as follows

$$\begin{array}{rccc} \cup: & H^0(X, \mathcal{E}) \otimes H^0(X, \mathcal{F}) & \to & H^0(X, \mathcal{E} \otimes \mathcal{F}) \\ & s \otimes t & \mapsto & (x \to s_x \otimes t_x) \end{array}$$

where s_x and t_x are the germs of s and t in the stalks $\mathcal{E}_x$and $\mathcal{F}_x$. Note that this cup product is neither surjective nor injective in general.

I fix here the notation for fibres used in the sequel. Let $x = \langle v \rangle \in \mathbb{P}(V)$, then:

$$\begin{array}{rcl} E_x & := & \mathcal{E}_{a,x}/m_x\mathcal{E}_{a,x} \\ \langle v \rangle & \simeq & \mathcal{O}_x(-1)/m_x\mathcal{O}(-1) \\ \langle v \rangle^\vee & \simeq & \mathcal{O}_x(1)/m_x\mathcal{O}(1) \\ \langle v \rangle^{\vee\otimes 2} & \simeq & \mathcal{O}_x(2)/m_x\mathcal{O}(2) \end{array}$$

where m_x is the maximal ideal in the local ring $\mathcal{O}_x$.

Notation 3.7 Let $\mathcal{E}$ a vector bundle and $s \in H^0\mathcal{E}(1)$ a section. In the sequel I denote by $s(x)$ the class of the germ $s_x \in \mathcal{E}_x$ in $\mathcal{E}_x/m_x\mathcal{E}_x$.

Let $\alpha : \Lambda^2\mathcal{E}_a(1) \to \mathcal{O}(2)$ be an isomorphism, hence there is an isomorphism $\alpha(x) : \Lambda^2 E_x \to k$. Then I define the duality pairing on the fibres E_x by:

$$\begin{array}{rcc} E_x \otimes E_x & \to & k \\ \xi \otimes \eta & \mapsto & J(x)(\xi)(\eta) := \alpha(x)(\xi \wedge \eta) \end{array}$$

where $\alpha(x)$ is the restriction of

$$\alpha : \quad \Lambda^2\mathcal{E}_a(1) \quad \to \quad \mathcal{O}(2)$$

to the fibre E_x. I am now prepared to give the second definition.

Definition 3.8 (definition 2) *Let $\mathcal{E}_a$ be a null correlation bundle, $\alpha : det(\mathcal{E}_a(1)) \to \mathcal{O}(2)$ and $s, t \in H^0\mathcal{E}_a(1)$, then there exists a pairing B:*

$$\begin{array}{rccl} B: & H^0\mathcal{E}_a(1) \otimes H^0\mathcal{E}_a(1) & \to & H^0\mathcal{O}(2) \\ & s \otimes t & \mapsto & B(s \otimes t) \end{array}$$

where B is defined as the composition:

$$\begin{array}{ccccccc} H^0(\mathcal{E}_a(1) \otimes H^0\mathcal{E}_a(1)) & \stackrel{\cup}{\to} & H^0(\mathcal{E}_a(1) \otimes \mathcal{E}_a(1)) & \to & H^0\Lambda^2\mathcal{E}_a(1) & \stackrel{\alpha}{\to} & H^0\mathcal{O}(2) \\ s \otimes t & \mapsto & (x \to s_x \otimes t_x) & \mapsto & (x \to s_x \wedge t_x) & \mapsto & (x \to \alpha(s_x \wedge t_x)) \end{array}$$

I denote by $c_\alpha q(s,t) := (x \to \alpha(s_x \wedge t_x))$ where c_α indicates the scalar factor caused by the choice of α.

The next proposition will justify the third definition of the product which is the most convenient for explicit calculations.

Proposition 3.9 *Under the assumptions made above we obtain :*

(1) Both maps A and B as in definition 3.6 and 3.8 factor over $\Lambda^2 H^0(\mathcal{E}_a(1))$ as follows:

$$\begin{array}{ccc} H^0\mathcal{E}_a(1) \otimes H^0\mathcal{E}_a(1) & \longrightarrow & H^0\mathcal{O}(2) \\ & \searrow \quad \# \quad {}^{\Psi}\nearrow & \\ & \Lambda^2 H^0\mathcal{E}_a(1) & \end{array}$$

(2) $t^\vee \circ J \circ s = c_\alpha q(s,t) = \Psi(s \wedge t)$ as in definition 3.8

Proof:
The claim (1) is obvious by the universal property of the wedge product for anticommutative maps.

for (2) it is sufficient to show that $t(x)^\vee \circ J(x) \circ s(x) = c_\alpha q(s,t)(x)$ is valid. The sequence in 3.6 restricted to the fibres looks as follows:

$$\langle v \rangle \stackrel{s(x)}{\to} E_x \stackrel{J(x)}{\to} E_x^\vee \stackrel{t(x)^\vee}{\to} \langle v \rangle^\vee$$

By twisting with $\langle v \rangle^\vee$ I obtain:

$$\begin{array}{ccccccc} \langle v \rangle \otimes \langle v \rangle^\vee & \stackrel{s(x)\otimes id}{\to} & E_x \otimes \langle v \rangle^\vee & \stackrel{J(x)\otimes id}{\to} & E_x^\vee \otimes \langle v \rangle^\vee & \stackrel{t(x)^\vee \otimes id}{\to} & \langle v \rangle^{\vee\otimes 2} \\ 1 \mapsto v \otimes v^\vee & \mapsto & s(x)(v) \otimes v^\vee & \mapsto & J(x)(s(x)(v)) \otimes v^\vee & \mapsto & t^\vee(x)(J(x)(s(x) \\ & & & & & & (v))) \otimes v^\vee \end{array}$$

Note that:

$$t^\vee(x)(J(x)(s(x)(v))) \otimes v^\vee = J(x)(s(x)(v))t(x) \otimes v^\vee \in \langle v \rangle^{\vee\otimes 2}$$

and $\langle v \rangle \otimes \langle v \rangle^{\vee} \simeq k$. Hence the evaluation map yields

$$J(x)(s(x)(v)t(x) \otimes v^{\vee})v^{\otimes 2} = J(x)(s(x)(v))t(x)v = \alpha_x(s(x)(v) \wedge t(x)(v))$$

where $t^{\vee}(x)$ is the map defined by $u(x) \circ t(x)$ for $u(x) \in E_x^{\vee}$:

$$\begin{array}{ccc} \langle v \rangle & \stackrel{t(x)}{\rightarrow} & E_x \\ & \searrow & \downarrow u(x) \\ & & k \end{array}$$

#

$Q(s,t)$ describes the dependency locus of s and t:

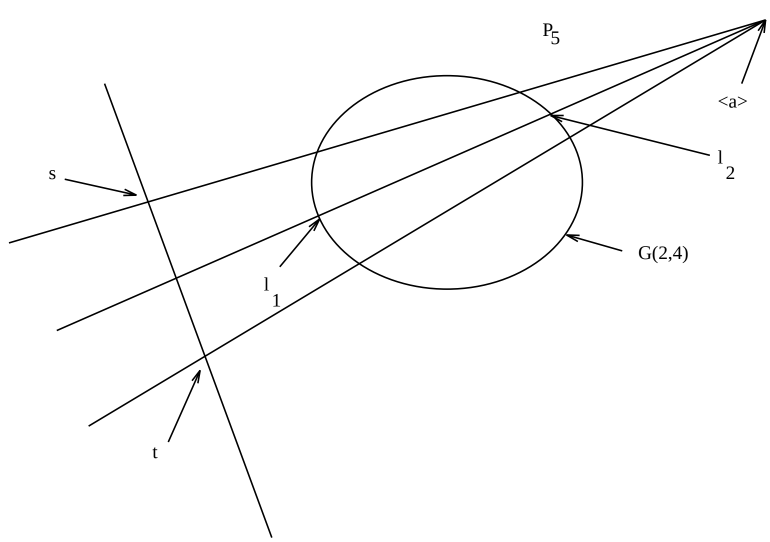
Lemma 3.10 *Let $Q(s,t)$ be the zero locus of the quadric $q(s,t)$ defined by $V((x \to s_x \wedge t_x)) = V((x \to \lambda s_x + \nu t_x))$,where $\lambda, \nu \neq (0,0)$, then:*

$$V(s \wedge t) = Q(s,t)$$

Proof: I proceed now showing both inclusions separately

(1) $V(s \wedge t) \subset Q(s,t)$
Let $x \in V(s \wedge t)$ hence $\lambda s(x) + \nu t(x) = 0 \Longrightarrow s(x) \wedge t(x) = 0 \Longrightarrow q(s,t)(x) = 0$. Therefore the first inclusion is valid

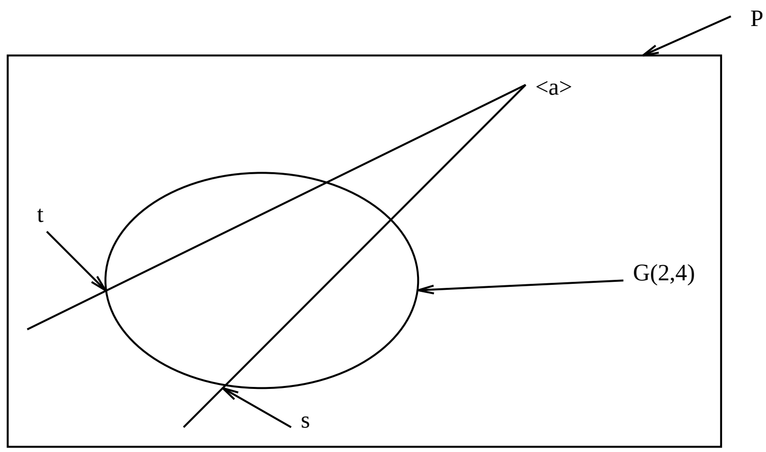
(2) $V(s_x \wedge t_x) \supset Q(s,t)$
Let $x \in Q(s,t)$ then by 3.4 there exist two lines l_1 and l_2 in $\mathbb{P}_3$ with $x \in l_1 \cup l_2$. The two representatives of s and t define a line which intersects the line $\overline{\langle l_1 \rangle \langle l_2 \rangle}$ one in a point $\lambda s(x) + \nu t(x)$ for appropriate λ and ν. Hence $x \in V(\lambda s + \nu t))$. I visualize this in the picture below



#

3.3 The geometric interpretation of the multiplication

We have already seen in 3.4 that a section of $\mathcal{E}_a(1)$ defines a line in $\mathbb{P}(\Lambda^2 V)$ containing the point $\langle a\rangle$ which represents the null correlation bundle. Two sections s and t together with the point $\langle a\rangle$ define a plane $P \subset \mathbb{P}\Lambda V^2$ which is not contained in $G(2,4)$ because $\langle a\rangle$ is indecomposable. Therefore the intersection $P \cap G(2,4)$ is a conic. According to definition 3.1 we have the associated quadric in $\mathbb{P}_3$ which is the dependency quadric of s and t, and according to lemma 3.10 it is $Q(s,t)$.



The next proposition is the key for the pairings defined in 3.6, 3.8 to be isomorphic to the wedge product.

Proposition 3.11

$$\Psi : \quad \Lambda^2 H^0(\mathcal{E}(1)) \quad \rightarrow \quad H^0(\mathcal{O}(2))$$

with

$$\Psi(s \wedge t) = t^\vee \circ J \circ s = c_\alpha q(s,t)$$

is an isomorphism

Proof:
It is sufficient to prove that there exist 10 planes in $\mathbb{P}\Lambda^2 V$ containing $\langle a\rangle$ such that the equations of their associated quadrics form a basis of $S^2 V^\vee$.

I fix now a basis $e_0, \cdots, e_3$ of V and its dual basis of coordinate functions $Z_0, \cdots, Z_3$. Hence I get a basis $\{e_i \wedge e_j\}_{i\neq j}$ of $\Lambda^2 V$ and the Plücker coordinates $\{P_{ij} = Z_i \wedge Z_j\}_{i\neq j}$ forming a basis of $(\Lambda^2 V)^\vee$. There are several possible orderings for the Plücker coordinates. I choose once and for all the lexicographical ordering:

$$(P_{01}, P_{02}, P_{03}, P_{12}, P_{13}, P_{23})$$

I recall now that the pairing $A = B : H^0\mathcal{E}_a(1) \otimes H^0\mathcal{E}_a(1) \to H^0\mathcal{O}(2)$ factor not only over $\Lambda^2 H^0\mathcal{E}_a(1)$ but over the affine cone over the Grassmannian of lines in $H^0\mathcal{E}_a(1)$, $G(2,5) \subset I\!P\Lambda^2 H^0\mathcal{E}_a(1)$, because the image of

$$\wedge : \quad H^0\mathcal{E}_a(1) \otimes H^0\mathcal{E}_a(1) \quad \to \quad \Lambda^2 H^0\mathcal{E}_a(1)$$

consist of decomposable elements. Hence a vector $s \wedge t$ with $0 \neq s, t \in H^0\mathcal{E}_a(1)$ represents a line in $I\!P\Lambda^2 V/\langle a\rangle$ in other words a plane in $I\!P\Lambda^2 V$ which contains the point $\langle a\rangle$. I fix now without loss of generality once and for all the point $\langle a\rangle \notin G(2,4)$ to be the point in $I\!P\Lambda^2 V$ with the Plücker coordinates:

$$(1:0:0:0:0:1)$$

Obviously this point is not contained in $G(2,4) = V(P_{01}P_{23} - P_{02}P_{13} + P_{12}P_{03})$. After this choice of $\langle a\rangle$ I can list now the 10 planes as follows. I choose 5 points in general position in $I\!P\Lambda^2 V$ different from $\langle a\rangle$ given by their Plücker coordinates.

$$\begin{array}{ll} s_1 & (1:0:0:0:0:0) \\ s_2 & (0:1:0:0:0:0) \\ s_3 & (0:0:1:0:0:0) \\ s_4 & (0:0:0:1:0:0) \\ s_5 & (0:0:0:0:1:0) \end{array}$$

The lines $\overline{\langle s_i\rangle\langle a\rangle}$ $i \in \{1, \cdots, 5\}$ which intersect only in $\langle a\rangle$ define as in proposition 3.4 linear independent sections $\bar{s}_i \quad i \in \{1, \cdots, 5\} \in H^0\mathcal{E}_a(1)$. Hence the 10 vectors $\bar{s}_i \wedge \bar{s}_j$ $i, j \in \{1, \cdots, 5\}$, $i \neq j$ define 10 planes in $I\!P\Lambda^2 V$ which I shall denote by H_{ij} $i, j \in \{1, \cdots, 5\}$. I list now these planes together with their equations in the Plücker coordinates of $I\!P\Lambda^2 V$ and their associated quadrics, see 3.1, given as an element in $H^0\mathcal{O}(2)$. Obviously the 10 quadrics in the list below are independent.

Plane	Equations of the plane	Quadric in $I\!P_3$
H_{12}	$P_{03} = P_{12} = P_{13} = 0$	Z_0Z_2
H_{13}	$P_{02} = P_{12} = P_{13} = 0$	Z_0Z_3
H_{14}	$P_{02} = P_{03} = P_{13} = 0$	Z_1Z_2
H_{15}	$P_{02} = P_{03} = P_{12} = 0$	Z_1Z_3
H_{23}	$P_{12} = P_{13} = P_{01} - P_{23} = 0$	Z_1^2
H_{24}	$P_{03} = P_{13} = P_{01} - P_{23} = 0$	Z_3^2
H_{25}	$P_{03} = P_{12} = P_{01} - P_{23} = 0$	$Z_0Z_1 + Z_2Z_3$
H_{34}	$P_{02} = P_{13} = P_{01} - P_{23} = 0$	$Z_0Z_1 - Z_2Z_3$
H_{35}	$P_{02} = P_{12} = P_{01} - P_{23} = 0$	Z_2^2
H_{45}	$P_{02} = P_{03} = P_{01} - P_{23} = 0$	Z_0^2

It remains now to compute the associated quadric to the planes to finish the proof. It takes too much time and is not further alluding to do the computation in all 10 cases. Thus I shall prove this lemma in two exemplaric cases: H_{12} and H_{25}.

3.12 (Plane 25) A Quadric Q in $\mathbb{P}_3$ can be defined by a symmetric 4×4 matrix $A(Q)$ modulo k^* via the equation:

$$(Z_0, Z_1, Z_2, Z_3) \begin{pmatrix} a_{0,0} & \cdots & a_{3,0} \\ \vdots & & \vdots \\ a_{0,3} & \cdots & a_{3,3} \end{pmatrix} \begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} = a_{0,0}Z_0^2 + \cdots + a_{3,3}Z_3^2 = 0$$

I determine now the associated quadric to the conic C by taking three points on C representing 3 lines on Q the determine Q completely. First I collect the ideals in the coordinate ring needed.

$$\begin{aligned} \mathcal{I}_{P_{025}} &:= \{P_{03}, P_{12}, P_{01} - P_{23}\} \\ \mathcal{I}_C &:= \{P_{03}, P_{12}, P_{01} - P_{23}, P_{01}^2 - P_{02}P_{13}\} \end{aligned}$$

I choose now three points on C defined by their ideals in the coordinate ring are:

$$\begin{aligned} \mathcal{I}_{P_1} &:= (P_{01}, P_{03}, P_{12}, P_{13}, P_{23}) & P_1 &= (0:1:0:0:0:0) \in \mathbb{P}_5 \\ \mathcal{I}_{P_2} &:= (P_{01}, P_{02}, P_{03}, P_{12}, P_{23}) & P_2 &= (0:0:0:0:1:0) \in \mathbb{P}_5 \\ \mathcal{I}_{P_3} &:= (P_{01} - P_{23}, P_{01} - P_{02}, P_{03}, P_{12}, P_{02} - P_{13}) & P_3 &= (1:1:0:0:1:1) \in \mathbb{P}_5 \end{aligned}$$

The point P_1 is the vector $e_0 \wedge e_2$ which represents the line $e_0 + \lambda e_2 \subset \mathbb{P}_3$ which has the coordinates $(1, 0, \lambda, 0)$.
The fact that the line with the parameterization $(1, 0, \lambda, 0)^t$ lies on the quadric determined by a matrix $A(Q)$ is equivalent to:

$$(1, 0, \lambda, 0) \begin{pmatrix} a_{0,0} & \cdots & a_{3,0} \\ \vdots & & \vdots \\ a_{0,3} & \cdots & a_{3,3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \lambda \\ 0 \end{pmatrix} = a_{0,0} + 2\lambda a_{0,2} + \lambda^2 a_{2,2} = 0$$

Hence $a_{0,0} = a_{0,2} = a_{2,2} = 0$. Analogously the point P_2 gives $a_{1,1} = a_{1,3} = a_{3,3} = 0$. To proceed we must compute the coordinate of the line defined by the third point P_3:

Claim 3.13 *The incidence diagram for lines and points in* $\mathbb{P}_3$ *is:*

$$\begin{array}{ccc} p^{-1}(C) & \longrightarrow & C \\ \big\downarrow & & \\ Q := qp^{-1}(C) & & \end{array}$$

then it follows for the third point $P_3 \in \Lambda^2 V$ *with coordinates* $(1:1:0:0:1:1)$ *the line* $\ell := pq^{-1}(P_3) \in \mathbb{P}_3 \subset \mathbb{P}_3$ *has the ideal* $\mathcal{I}_\ell = (Z_0 - Z_3, Z_1 - Z_2)$.

Proof:
This results from the equation below. It is easy to guess what the coordinates of the

desired line $pq^{-1}(P_3)$ if one takes into account that the Pücker coordinates are the 2×2 minors of the embedding of $\mathbb{P}_1$ onto the line in $\mathbb{P}_3$

$$\begin{array}{ccc} \mathbb{P}_1 & \to & \mathbb{P}_3 \\ (\mu : \lambda) & \mapsto & (\mu : \lambda) \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

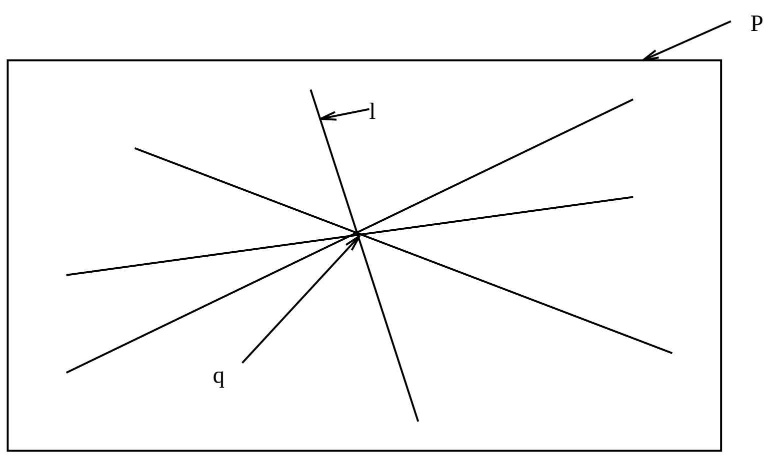
#

Therefore $a_{0,3} = a_{1,3} = 0$ and $a_{0,1} = a_{2,3} = 1$ by the same reasoning a s for the points P_1 and P_2. Thus the quadric defined by the plane P_{136} is the smooth hypersurface with the equation

$$\mathcal{I}_Q = (Z_0 Z_1 + Z_2 Z_3)$$

The singular cases (D,G) in the classification 3.1 are easier to handle if one remembers that any line $\mathbb{P}_1 \subset G(2,4)$ is a Schubert cycle $\sigma(q,P)$ of all lines in a plane P intersecting in a point $q \in P$.

$$\sigma(q,P) := \{l \mid q \in l \supset P \subset \mathbb{P}_3\}$$



3.14 (Plane 12)

$$C_{12} := G(2,4) \cap H_{12} \Longrightarrow \mathcal{I}_C := \{P_{03}, P_{12}, P_{13}, P_{01}P_{23}\}$$

Thus C_{123} is reducible and consists of two lines:

$$C = \ell_1 \cup \ell_2 \qquad \mathcal{I}_{\ell_1} = \{P_{03}, P_{12}, P_{13}, P_{23}\} \quad \mathcal{I}_{\ell_2} = \{P_{03}, P_{12}, P_{13}, P_{01}\}$$

which intersect in P and $\mathcal{I}_P = \{P_{01}, P_{03}, P_{12}, P_{13}, P_{23}\}$.

The line ℓ_1 is now the Schubert cycle $\sigma(q,P)$ of all lines in the plane P containing the point q, where $\mathcal{I}_P = (Z_0)$ and $\mathcal{I}_q = (Z_0, Z_1, Z_2)$ because all lines parameterized by l_1 are linear combinations of the lines in $\mathbb{P}_3$ defined as $aZ_0 + bZ_1 = 0$ and $cZ_0 + dZ_2 = 0$. This follows from the fact that

$$\mathcal{O}_{\mathbb{P}(\Lambda^2 V)}/\mathcal{I}_{\ell_1} \simeq \mathcal{O}_{\ell_1} \simeq k[P_{01}, P_{02}].$$

Analogously the plane for the Schubert cycle ℓ_2 is defined by $\mathcal{I}_P = (Z_2)$ and $\mathcal{I}_q = (Z_0, Z_2, Z_3)$. Thus we observe that the desired quadric is given by

$$\mathcal{I}_Q = (Z_0 Z_2)$$

The remaining eight cases can be solved in a completely analogous way. Hence there are 10 planes in $\mathbb{P}\Lambda^2 V$ containing $\langle a \rangle$ which give 10 independent quadrics. #

Remark 3.15 As we have seen in the proof above, two conics cut out by two planes having a point in common can not determine the same quadric in $\mathbb{P}_3$. which is not obvious. For instance a smooth conic in $\mathbb{P}_3$ has two rulings.

Remark 3.16 Thus we can compute the syzygies of a morphism

$$(m+1)\mathcal{E} \to n\mathcal{O}(1)$$

in the exterior algebra. This is very helpful if one wants to construct vector bundles explicitly with the aid of the computer algebra system SINGULAR. [24]. Moreover we can use this algebra structure to prove conjectures on the dimension of cohomology groups for certain classes of instanton bundles. This will be done in the next chapter.

4 Instantons of NC-type

4.1 Summary of chapter 4

In this chapter instantons $\mathcal{E}$ of NC-type are built using the algorithm introduced in 2.3. As already mentioned in the chapter 2 the multiplication in the algorithm need some further explanations. Most of these preparations were done in the chapter "Complements on null correlation bundles".

Unfortunately the criterion used in lemma 2.6 in chapter 2 to validate for instantons which a defined by Beilinson-I-monads that they are smooth points of the moduli scheme does not work for NC-type monads. This problem can be solved by a comparison theorem between NC-type and Beilinson-I-monads. Hence the smoothness question for NC-type-monads can be returned to case of Beilinson-I-monads.

The chapter proceeds with three program examples. I determine in these examples the dimension of $H^0\mathcal{E}(1)$ for the constructed instanton bundles $\mathcal{E}$. These explicit computations suggested propositions on the dimensions of $H^0\mathcal{E}(1)$ not only for the examples but also for whole families.

4.2 The algorithmic construction of instantons of NC-type

I recall the notion of a selfdual NC-type-monad first, see also p.6.
A **selfdual NC-type-monad** is a complex

$$0 \quad \rightarrow \quad m\mathcal{O}(-1) \quad \stackrel{S}{\rightarrow} \quad (m+1)\mathcal{E}_a \quad \stackrel{S^\vee \circ J}{\rightarrow} \quad m\mathcal{O}(1) \quad \rightarrow \quad 0$$

where $\mathcal{E}_a$ is a null correlation bundle and J is a symplectic pairing:

$$J: \quad (m+1)\mathcal{E}_a \quad \stackrel{\simeq}{\rightarrow} \quad (m+1)\mathcal{E}_a^\vee$$

The complex satisfies the following "subbundle" condition:

$$Im(a) \cap (k^{m+1} \otimes \sigma) = 0 \ \forall \text{ nonzero } \sigma \in \Lambda^2 V/\langle a \rangle \simeq H^0\mathcal{E}_a(1)$$

Remark 4.1 (1) This subbundle condition is a consequence of lemma 5.2:
If the rank of a is not constant, then there exist a linear combination $t = (t_1, \cdots, t_{m+1})$ of rows of S such that $V(t) \neq \emptyset$. The matrix S factors over the kernel bundle $\mathcal{K}$:

$$\begin{array}{ccc} m\mathcal{O}(1) & \longrightarrow & \mathcal{K} \\ & {}^{S}\searrow \quad \# \quad \swarrow & \\ & (m+1)\mathcal{E}_a & \end{array}$$

Hence, according to lemma 5.2, there exists a $\sigma \in H^0\mathcal{E}_a(1)$ and a vector $\alpha := (a_1, \cdots, a_{m+1}) \in k^{m+1}$ such that

$$t = (a_1, \cdots, a_{m+1}) \otimes \sigma \ .$$

(2) For the computations with SINGULAR a general linear combination of rows of the matrix a is represented in a basis $s_1, \cdots, s_5$ of $H^0\mathcal{E}_a(1)$, see 4.7, i.e.:

$$(s_1, \cdots, s_5) \circ A = (t_1, \cdots, t_{m+1})$$

If there exists such a linear combination as in (1) $t = \alpha \otimes \sigma$ with $\alpha \in k^{m+1}$ and $\sigma \in H^0\mathcal{E}_a(1)$, then all 2×2-minors of the coefficient matrix A must vanish.

I apply now the results of the chapter "Complements on null correlation bundles" on the multiplication to direct sums of a null correlation bundle $(m+1)\mathcal{E}_a$. Let A be the pairing

$$\begin{array}{rccc} A: & H^0\mathcal{E}_a(1) \otimes H^0\mathcal{E}_a(1) & \to & H^0\mathcal{O}(2) \\ & t \otimes u & \mapsto & t \circ J \circ u^\vee \end{array}$$

defined in the definition 3.6. This pairing extends linearly to direct sums of $H^0\mathcal{E}_a(1)$.

$$\begin{array}{rccc} \bar{A}: & \bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1) \otimes \bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1) & \to & H^0\mathcal{O}(2) \\ & \sum_{i=1}^{m+1} t_i \otimes \sum_{i=1}^{m+1} u_i & \mapsto & \sum_{i=1}^{m+1} t_i \circ J \circ u_i^\vee \end{array}$$

Analogously the wedge product

$$\begin{array}{rccc} \wedge: & H^0\mathcal{E}_a(1) \otimes H^0\mathcal{E}_a(1) & \to & \Lambda^2 H^0\mathcal{E}_a(1) \\ & t \otimes u & \mapsto & t \wedge u \end{array}$$

extends to

$$\begin{array}{rccc} \bar{\wedge}: & \bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1) \otimes \bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1) & \to & \Lambda^2 H^0\mathcal{E}_a(1) \\ & \sum_{i=1}^{m+1} t_i \otimes \sum_{i=1}^{m+1} u_i & \mapsto & \sum_{i=1}^{m+1} t_i \wedge u_i \end{array}$$

It is already known by proposition 3.11 that the pairings $\wedge$ and A are linearly isomorphic via Ψ. Therefore $\bar{A}$ and $\bar{\wedge}$ are isomorphic too:

$$\begin{array}{ccc} \bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1) \otimes \bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1) & \xrightarrow{\bar{A}} & H^0\Lambda^2\mathcal{E}_a(1) \simeq H^0\mathcal{O}(2) \\ & \bar{\wedge}\searrow \quad \# \quad \nearrow\Psi & \\ & \Lambda^2 H^0\mathcal{E}_a(1) & \end{array}$$

Proposition 4.2 The pairing

$$\begin{array}{rccc} \bar{A}: & \bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1) \otimes \bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1) & \to & H^0\mathcal{O}(2) \\ & \sum_{i=1}^{m+1} t_i \otimes \sum_{i=1}^{m+1} u_i & \mapsto & \sum_{i=1}^{m+1} t_i \circ J \circ u_i^\vee \end{array}$$

can be computed as

$$\begin{array}{rccc} \bar{\wedge}: & \bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1) \otimes \bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1) & \to & \Lambda^2 H^0\mathcal{E}_a(1) \\ & \sum_{i=1}^{m+1} t_i \otimes \sum_{i=1}^{m+1} u_i & \mapsto & \sum_{i=1}^{m+1} t_i \wedge u_i \end{array}$$

by the computer algebra SINGULAR.

The algorithm 2.3 introduced in the chapter 2 for Beilinson-I-monads is in fact more general. It requires only a multiplicative pairing which can be handled by a computer algebra system. Therefore it is possible to use the algorithm 2.3 for the construction of NC-type-monads.

4.3 The correspondence between NC-type monads and Beilinson-I-monads

For monads of NC-type one cannot apply lemma 2.6 to prove the **smoothness of the moduli space**. Nevertheless there exists a correspondence for an instanton $\mathcal{E}$ between its NC-type monad an the Beilinson-I monad. I shall describe this explicitly, so it is always possible to use the Beilinson-I monad to prove the smoothness of $MI(c_2(\mathcal{E}))$ in $[\mathcal{E}]$. This I want to do in this paragraph.

First I recall the display of the Beilinson-I monad

$$0 \rightarrow (m+1)\mathcal{O}(-1) \stackrel{A}{\rightarrow} 4(m+1)\mathcal{O} \stackrel{B}{\rightarrow} (m+1)\mathcal{O}(1) \rightarrow 0$$

of a direct sum of null correlation bundles $(m+1)\mathcal{E}_a$. This is in principle the "tool" which connects a Beilinson-I monad with the corresponding null correlation type monad for an instanton bundle $\mathcal{E}$.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \rightarrow & (m+1)\mathcal{O}(-1) & \longrightarrow & \mathcal{K}_0 & \longrightarrow & (m+1)\mathcal{E}_a & \rightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \rightarrow & (m+1)\mathcal{O}(-1) & \longrightarrow & 4(m+1)\mathcal{O} & \longrightarrow & Coker & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & (m+1)\mathcal{O}(1) & = & (m+1)\mathcal{O}(1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

I start now with a NC-type monad

$$0 \rightarrow m\mathcal{O}(-1) \stackrel{\phi}{\rightarrow} (m+1)\mathcal{E}_a \stackrel{\psi}{\rightarrow} m\mathcal{O}(1) \rightarrow 0$$

and construct in two steps the associated Beilinson-I monad:
step 1:

I proceed with the aid of the top row of the monad above.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & (m+1)\mathcal{O}(-1) & = & (m+1)\mathcal{O}(-1) & & & & \\
 & & \Big\downarrow & & \Big\downarrow A & & & & \\
0 & \to & (2m+1)\mathcal{O}(-1) & \xrightarrow{\binom{A}{\phi'}} & \mathcal{K}_0 & \longrightarrow & m\mathcal{O}(1) & \to & 0 \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\| & & \\
0 & \to & m\mathcal{O}(-1) & \xrightarrow{\phi} & (m+1)\mathcal{E}_a & \longrightarrow & m\mathcal{O}(1) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

The diagram above is not exact, the middle and the bottom rows only are monads. The morphisms

$$
\begin{array}{rccc}
\phi : & m\mathcal{O}(-1) & \to & (m+1)\mathcal{E}_a \\
\psi : & (m+1)\mathcal{E}_a & \to & m\mathcal{O}(-1)
\end{array}
$$

lift to morphisms:

$$
\begin{array}{rccc}
\phi' : & m\mathcal{O}(-1) & \to & \mathcal{K}(-1) \\
\psi' : & \mathcal{K}_0 & \to & m\mathcal{O}(-1)
\end{array}
$$

because the obstruction $Ext^1(m\mathcal{O}(-1), m\mathcal{O}(-1))$ vanishes.

$$
\begin{aligned}
0 \to Hom(m\mathcal{O}(-1), (m+1)\mathcal{O}(-1)) \xrightarrow{A} Hom(m\mathcal{O}(-1), \mathcal{K}_0) \to \\
\to Hom(m\mathcal{O}(-1), (m+1)\mathcal{E}_a) \to 0 \qquad (4.3.24) \\
0 \to Hom((m+1)\mathcal{E}_a, m\mathcal{O}(1)) \to Hom(\mathcal{K}_0, m\mathcal{O}(1)) \xrightarrow{B} \\
\xrightarrow{B} Hom((m+1)\mathcal{O}(-1), m\mathcal{O}(1)) \qquad (4.3.25)
\end{aligned}
$$

step 2:

The next step is a pullback diagram.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \rightarrow & (2m+1)\mathcal{O}(-1) & \longrightarrow & \mathcal{K}_0 & \xrightarrow{\psi'} & m\mathcal{O}(1) & \rightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \rightarrow & (2m+1)\mathcal{O}(-1) & \xrightarrow{\binom{B}{\tilde{\psi}'}} & 4(m+1)\mathcal{O} & \longrightarrow & (2m+1)\mathcal{O}(1) & \rightarrow & 0 \\
 & & & & \downarrow B & & \downarrow & & \\
 & & & & (m+1)\mathcal{O}(1) & = & (m+1)\mathcal{O}(1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Again the diagram above is not exact, the middle and the top rows are monads.

$$\phi' : \quad m\mathcal{O}(-1) \quad \rightarrow \quad \mathcal{K}_0(-1)$$

$$\psi' : \quad \mathcal{K}_0 \quad \rightarrow \quad m\mathcal{O}(-1)$$

lift to morphisms:

$$\tilde{\phi}' : \quad m\mathcal{O}(-1) \quad \rightarrow \quad 4(m+1)\mathcal{O}$$
$$\tilde{\psi}' : \quad 4(m+1)\mathcal{O} \quad \rightarrow \quad m\mathcal{O}(-1)$$

because the obstruction $Ext^1((2m+1)\mathcal{O}(1), (m+1)\mathcal{O}(1))$ vanishes.

$$0 \rightarrow Hom((m+1)\mathcal{O}(1), (2m+1)\mathcal{O}(1))) \xrightarrow{B} Hom(4(m+1)\mathcal{O}, (2m+1)\mathcal{O}(1)) \rightarrow$$
$$\rightarrow Hom(\mathcal{K}_0, (2m+1)\mathcal{O}(1) \rightarrow 0 \qquad (4.3.26)$$
$$0 \rightarrow Hom((2m+1)\mathcal{O}(1), \mathcal{K}_0) \rightarrow Hom((2m+1)\mathcal{O}(1), 4(m+1)\mathcal{O})) \xrightarrow{A}$$
$$\xrightarrow{A} Hom((2m+1)\mathcal{O}(1), (m+1)\mathcal{O}(1)) \qquad (4.3.27)$$

$\mathcal{K}_0$ is a pullback.

Thus a Beilinson-I-monad

$$0 \quad \rightarrow \quad (2m+1)\mathcal{O}(-1) \quad \xrightarrow{\binom{A}{\tilde{\phi}'}} \quad 4(m+1)\mathcal{O} \quad \xrightarrow{(B,\tilde{\psi}')} \quad (2m+1)\mathcal{O}(1) \quad \rightarrow \quad 0$$

is constructed from a given NC-type-monad

$$0 \quad \rightarrow \quad m\mathcal{O}(-1) \quad \xrightarrow{\phi} \quad (m+1)\mathcal{E}_a \quad \xrightarrow{\psi} \quad m\mathcal{O}(1) \quad \rightarrow \quad 0 \ .$$

One should remark that the lifting $\tilde{\phi}'$ and $\tilde{\psi}'$ are not unique. Moreover both monads have isomorphic cohomology bundles.

Proposition 4.3 *Let $(m+1)\mathcal{E}_a$ a direct sum of null correlation bundles defined by the Beilinson-I-type-monad*

$$0 \rightarrow (m+1)\mathcal{O}(-1) \stackrel{A}{\rightarrow} 4(m+1)\mathcal{O} \stackrel{B}{\rightarrow} (m+1)\mathcal{O}(1) \rightarrow 0$$

and $\mathcal{E}$ and instanton of NC-type with monad

$$0 \rightarrow m\mathcal{O}(-1) \stackrel{\phi}{\rightarrow} (m+1)\mathcal{E}_a \stackrel{\psi}{\rightarrow} m\mathcal{O}(1) \rightarrow 0$$

then there exist a Beilinson-I-type monad

$$0 \rightarrow (2m+1)\mathcal{O}(-1) \stackrel{\binom{A}{\tilde{\phi}'}}{\rightarrow} 4(m+1)\mathcal{O} \stackrel{(B,\tilde{\psi}')}{\rightarrow} (2m+1)\mathcal{O}(1) \rightarrow 0$$

for $\mathcal{E}$ and $\tilde{\phi}'$ and $\tilde{\psi}'$ are liftings of the map ψ and ϕ in the NC-type-monads to maps in $Hom(m\mathcal{O}(-1), 4m\mathcal{O})$ and $Hom(4m\mathcal{O}, m\mathcal{O}(1))$ respectively.

The proposition 4.3 shows now how to decide whether an instanton given as the cohomology of a monad of NC-type is a smooth point of the moduli space $MI(c_2(\mathcal{E}))$.

Proposition 4.4 *Let $\mathcal{E}$ be an instanton bundle defined as the cohomology of a monad of NC-type*

$$0 \rightarrow m\mathcal{O}(-1) \stackrel{\phi}{\rightarrow} (m+1)\mathcal{E}_a \stackrel{\psi}{\rightarrow} m\mathcal{O}(1) \rightarrow 0 \cdot$$

Then $[\mathcal{E}] \in MI(c_2(\mathcal{E}))$ is a smooth point if and only if the cokernel

$$coker\left(\begin{pmatrix} B \\ \tilde{\psi}' \end{pmatrix} \otimes id \mid id \otimes (B, \tilde{\psi}')\right) = 0$$

where $A, \tilde{\phi}'$ and $\tilde{\psi}'$ are defined by the associated Beilinson-I-monad according to proposition 4.3:

$$0 \rightarrow (2m+1)\mathcal{O}(-1) \stackrel{\binom{A}{\tilde{\phi}'}}{\rightarrow} 4(m+1)\mathcal{O} \stackrel{(B,\tilde{\psi}')}{\rightarrow} (2m+1)\mathcal{O}(1) \rightarrow 0$$

vanishes.

Moreover there exists a lifting of a Beilinson-I monad which has a splitting in the sense of proposition 4.5 to a NC-type monad. We have already seen that the monad arrows of the NC- type-monad lifts properly with respect to f. One sees from the sequence 4.3.24 on page 69 that

$$\tilde{\psi}' : \ 4(m+1)\mathcal{O} \rightarrow m\mathcal{O}(-1)$$

lifts to morphism

$$\psi' : \ \mathcal{K}_0 \rightarrow m\mathcal{O}(-1)$$

if and only if $\tilde{\psi}'$ is not in the image of A. This can be attained easily by a simple base change of $(2m+1)\mathcal{O}(1)$ which keeps the subvector space $(m+1)\mathcal{O}(1)$ invariant. Analogously

$$\psi' : \quad \mathcal{K}_0 \quad \to \quad m\mathcal{O}(-1)$$

lifts to

$$\psi : \quad (m+1)\mathcal{E}_a \quad \to \quad m\mathcal{O}(-1)$$

iff it is not in the image of $B^\vee$ again a base change of $(2m+1)\mathcal{O}(1)$ remedies the situation. Hence we have a converse of the proposition 4.3.

Proposition 4.5 *Let $\mathcal{E}$ be an instanton bundle being the cohomology of a Beilinson-I-monad where the left and right monad arrows decompose as follows*

$$0 \to (m+1)\mathcal{O}(-1) \oplus m\mathcal{O}(-1) \stackrel{\binom{A}{\tilde{\phi}'}}{\to} 4(m+1)\mathcal{O} \stackrel{(B,\tilde{\psi}')}{\to} (m+1)\mathcal{O}(1) \oplus m\mathcal{O}(1) \to 0 \ ,$$

where A, $\tilde{\phi}'$ are subbundles. Then $\mathcal{E}$ is the cohomology of a NC-type monad.

$$0 \quad \to \quad m\mathcal{O}(-1) \quad \stackrel{\phi}{\to} \quad (m+1)\mathcal{E}_a \quad \stackrel{\psi}{\to} \quad m\mathcal{O}(1) \quad \to \quad 0$$

where $\tilde{\phi}'$ and $\tilde{\psi}'$ are appropriate lifts of ϕ and ψ respectively.

As I already mentioned before it is much more pleasant to work with self dual monads in the meaning of page 7. For any instanton $\mathcal{E}$ the Beilinson-I-monad can be chosen to be self dual, see lemma 2.2. This cannot be shown for NC-type monads.

Remark 4.6 Albeit only direct sums of null correlation bundles $(m+1)\mathcal{E}_a$ are mentioned in this chapter 4.3 these result are valid in the more general situation where $(m+1)\mathcal{E}_a$ is replaced by any cohomology bundles of a monad of type:

$$0 \quad \to \quad (m+1)\mathcal{O}(-1) \quad \stackrel{A}{\to} \quad 4(m+1)\mathcal{O} \quad \stackrel{B}{\to} \quad (m+1)\mathcal{O}(1) \quad \to \quad 0$$

4.4 Examples of NC-type-monads

4.4.1 Program example 4

The instanton bundle $\mathcal{E}$ constructed in this program output is an example for an instanton of NC-type. Moreover its monad is selfdual in the meaning of 0.1.1. i.e.: it has a monad of type

$$0 \rightarrow 3\mathcal{O}(-1) \overset{S}{\rightarrow} 4\mathcal{E}_a \overset{S^\vee \circ J}{\rightarrow} 3\mathcal{O}(1) \rightarrow 0$$

Its second Chern class $c_2(\mathcal{E}) = 7$. The null correlation bundle $\mathcal{E}_a$ is here not specified, hence I constructed a whole family where the null correlation bundle $\mathcal{E}_a$ and hence the x_i vary.

The first row of the matrix S consists of 4 independent vectors which is the maximal possible number. Thus for generic examples $h^0\mathcal{E}(1) = 1$ by proposition 4.11. This result is reproved by explicit computation.

In the comments to the program listing I shall refer to the steps in the explicit explanations of the algorithm, see 25. The commentaries I entered later are in a different font: sans serif. I skipped parts of the output which are not essential for the understanding of the computation. The SINGULAR program code is echoed in the lines which start with "Versuch.unab.44".

```
Versuch.unab.44   1> ring r=101,(s(1..4)),(c,dp);
Versuch.unab.44   2> alternating=1;
Versuch.unab.44   3> option(protocoll);
Versuch.unab.44   4> <"lib";
Versuch.unab.44   5> ideal m=s(1),s(2),s(3),s(4);
```

m is the first row of S, all s_i are independent.

```
Versuch.unab.44   6> m;
m[1]=s(1)
m[2]=s(2)
m[3]=s(3)
m[4]=s(4)

Versuch.unab.44   7> //----------------------------------------
```

m1 is the set of generators for the vector space of the linear relations of first row.

```
Versuch.unab.44   8. module m1=syz(m);
```

```
Versuch.unab.44   9> m1;
m1[1]=[0,0,0,s(4)]
m1[2]=[0,0,s(4),s(3)]
m1[3]=[0,0,s(3)]
m1[4]=[0,s(4),0,s(2)]
m1[5]=[0,s(3),s(2)]
m1[6]=[0,s(2)]
m1[7]=[s(4),0,0,s(1)]

m1[8]=[s(3),0,s(1)]
m1[9]=[s(2),s(1)]
m1[10]=[s(1)]
```

$s(1)$=m2 is a random element in K_1 chosen, see page 25 step (2) and after concatenation with the old S and one obtains the new S =m3.

```
Versuch.unab.44  10> matrix mat1=randmat(ncols(m1),1);
Versuch.unab.44  12. mat1;
mat1[1,1]=-16
mat1[2,1]=45
mat1[3,1]=4
mat1[4,1]=10
mat1[5,1]=-42
mat1[6,1]=4
mat1[7,1]=-41
mat1[8,1]=15
mat1[9,1]=-48
mat1[10,1]=41

Versuch.unab.44  13> matrix m2=matrix(m1)*mat1;
Versuch.unab.44  15. matrix m3[4][2];
Versuch.unab.44  16> m3[1..4,1]=m2;
Versuch.unab.44  17> m3[1..ncols(m),2]=m;
```

m4 is the set of all relations of $S^\vee \circ J$ =m3t, where m3t is the transpose of m3. m5 is the set of all linear relations in m4.

```
Versuch.unab.44  19. matrix m3t=transp(m3);
Versuch.unab.44  20> m3t;
m3t[1,1]=41*s(1)-48*s(2)+15*s(3)-41*s(4)
m3t[1,2]=-48*s(1)+4*s(2)-42*s(3)+10*s(4)
m3t[1,3]=15*s(1)-42*s(2)+4*s(3)+45*s(4)
m3t[1,4]=-41*s(1)+10*s(2)+45*s(3)-16*s(4)
```

```
m3t[2,1]=s(1)
m3t[2,2]=s(2)
m3t[2,3]=s(3)
m3t[2,4]=s(4)

Versuch.unab.44  25. module m4=syz(module(m3t));
Versuch.unab.44  26> m4=std(m4);

Versuch.unab.44  29. module null;
Versuch.unab.44  30> module m5=jet(m4,1)+null;
Versuch.unab.44  31> m5;
m5[1]=[s(4),50*s(2)+41*s(3)-29*s(4),41*s(2)-9*s(3)-45*s(4),s(1)-29*s(2)-
      45*s(3)-23*s(4)]
m5[2]=[s(3)-16*s(4),17*s(2)+14*s(3),s(1)+14*s(2)+12*s(3)+4*s(4),-16*s(1)+
      4*s(3)+20*s(4)]
m5[3]=[s(2)+27*s(3)+12*s(4),s(1)-42*s(2)+7*s(3)+s(4),27*s(1)+7*s(2)
      +28*s(3)-35*s(4),12*s(1)+s(2)-35*s(3)-5*s(4)]
m5[4]=[s(1)-43*s(2)+43*s(3)+41*s(4),-43*s(1)-8*s(2)+6*s(3)+40*s(4),
      43*s(1)+6*s(2)-9*s(3)+45*s(4),41*s(1)+40*s(2)+45*s(3)-14*s(4)]
```

$s(2)$=m6 is a random element in K_2 chosen, see page 25 step (3) and after concatenation with the old S and one obtains the new S =m7.

```
Versuch.unab.44  32> matrix mat2=randmat(ncols(matrix(m5)),1);

Versuch.unab.44  34. mat2;
mat2[1,1]=-30
mat2[2,1]=1
mat2[3,1]=8
mat2[4,1]=-13

Versuch.unab.44  35> matrix m6=matrix(m5)*mat2;
Versuch.unab.44  38. matrix m7[4][3];
Versuch.unab.44  39> m7[1..4,1..2]=m3;
Versuch.unab.44  40> m7[1..4,3]=m6;
Versuch.unab.44  42. m7t=transp(m7);
Versuch.unab.44  43> m7t;
m7t[1,1]=41*s(1)-48*s(2)+15*s(3)-41*s(4)
m7t[1,2]=-48*s(1)+4*s(2)-42*s(3)+10*s(4)
m7t[1,3]=15*s(1)-42*s(2)+4*s(3)+45*s(4)
m7t[1,4]=-41*s(1)+10*s(2)+45*s(3)-16*s(4)
m7t[2,1]=s(1)
m7t[2,2]=s(2)
m7t[2,3]=s(3)
```

```
m7t[2,4]=s(4)
m7t[3,1]=-13*s(1)-39*s(2)-39*s(3)+22*s(4)
m7t[3,2]=-39*s(1)+2*s(2)-26*s(3)-46*s(4)
m7t[3,3]=-39*s(1)-26*s(2)+17*s(3)-16*s(4)
m7t[3,4]=22*s(1)-46*s(2)-16*s(3)+44*s(4)
```

m8 is the set of all relations of $S^{\vee} \circ J =$m7t, where m7t is the transpose of m7. m9 is the set of all linear relations in m8. Moreover m9 is a set of generators for $H^0\mathcal{K}(1)$, where $\mathcal{K}$ is the kernel bundle in the monad display:

$$0 \to 3\mathcal{O}(-1) \xrightarrow{S} \mathcal{K} \to \mathcal{E} \to 0$$

```
Versuch.unab.44  44> module m8=syz(module(m7t));
Versuch.unab.44  45> m8=std(m8);
Versuch.unab.44  47. //-----------------------------------------
Versuch.unab.44  48. module m9=jet(m8,1)+null;
Versuch.unab.44  49> m9;
m9[1]=[s(4),50*s(2)+41*s(3)-29*s(4),41*s(2)-9*s(3)-45*s(4),s(1)-29*s(2)-
      45*s(3)-23*s(4)]
m9[2]=[s(3)-16*s(4),17*s(2)+14*s(3),s(1)+14*s(2)+12*s(3)+4*s(4),-16*s(1)+
      4*s(3)+20*s(4)]
m9[3]=[s(2)+27*s(3)+12*s(4),s(1)-42*s(2)+7*s(3)+s(4),27*s(1)+7*s(2)
      +28*s(3)-35*s(4),12*s(1)+s(2)-35*s(3)-5*s(4)]
m9[4]=[s(1)-43*s(2)+43*s(3)+41*s(4),-43*s(1)-8*s(2)+6*s(3)+40*s(4),
      43*s(1)+6*s(2)-9*s(3)+45*s(4),41*s(1)+40*s(2)+45*s(3)-14*s(4)]
```

m9 has 4 independent linear relations of $S^{\vee} \circ J =$m7t, hence $h^0\mathcal{K}(1) = 4$ and $h^0\mathcal{E}(1)$ must be 1. $\mathcal{K}$ is the kernel bundle in the monad display.

$$0 \to 3\mathcal{O}(-1) \xrightarrow{S} \mathcal{K} \to \mathcal{E} \to 0$$

In the next step I check that the matrix S=m7 defines a subbundle i.e. there is no linear combination of rows of S which vanishes see the **subbundle condition** remark 4.1. along a subscheme i.e.: there exist $\alpha \in k^4$ and $t \in H^0\mathcal{E}_a(1)$ such that the linear combinations of rows is $\alpha \otimes t$. The vector m10 below is a general linear combination.

```
Versuch.unab.44  51. int col=nrows(m7);
Versuch.unab.44  52> ring r8=101,(s(1..4),b(1..col)),(c,dp);
Versuch.unab.44  53> map f=r,s(1),s(2),s(3),s(4);
Versuch.unab.44  54> int j;
Versuch.unab.44  55> matrix i[1][col];
Versuch.unab.44  56> for( j=1;j<=col;j=j+1){
```

```
Versuch.unab.44  57.    i[1,j]=b(j);
Versuch.unab.44  58.    }
Versuch.unab.44  59> i;
i[1,1]=b(1)
i[1,2]=b(2)
i[1,3]=b(3)

Versuch.unab.44  61. matrix m10[1][1]=i*f(m7);
Versuch.unab.44  62> m10=transp(m10);
Versuch.unab.44  63> m10;
m10[1,1]=41*s(1)*b(1)-48*s(2)*b(1)+15*s(3)*b(1)-41*s(4)*b(1)+s(1)*b(2)-
         13*s(1)*b(3)-39*s(2)*b(3)-39*s(3)*b(3)+22*s(4)*b(3)
m10[2,1]=-48*s(1)*b(1)+4*s(2)*b(1)-42*s(3)*b(1)+10*s(4)*b(1)+s(2)*b(2)-
         39*s(1)*b(3)+2*s(2)*b(3)-26*s(3)*b(3)-46*s(4)*b(3)
m10[3,1]=15*s(1)*b(1)-42*s(2)*b(1)+4*s(3)*b(1)+45*s(4)*b(1)+s(3)*b(2)-
         39*s(1)*b(3)-26*s(2)*b(3)+17*s(3)*b(3)-16*s(4)*b(3)
m10[4,1]=-41*s(1)*b(1)+10*s(2)*b(1)+45*s(3)*b(1)-16*s(4)*b(1)+s(4)*b(2)+
         22*s(1)*b(3)-46*s(2)*b(3)-16*s(3)*b(3)+44*s(4)*b(3)

Versuch.unab.44  64> matrix m11;
Versuch.unab.44  65> matrix m12;
Versuch.unab.44  66> module m10m=module(m10);
Versuch.unab.44  67> vector v=m10m[1];
Versuch.unab.44  68> v;
[41*s(1)*b(1)-48*s(2)*b(1)+15*s(3)*b(1)-41*s(4)*b(1)+s(1)*b(2)-
13*s(1)*b(3)-39*s(2)*b(3)-39*s(3)*b(3)+22*s(4)*b(3),-48*s(1)*b(1)+
4*s(2)*b(1)-42*s(3)*b(1)+10*s(4)*b(1)+s(2)*b(2)-39*s(1)*b(3)+
2*s(2)*b(3)-26*s(3)*b(3)-46*s(4)*b(3),15*s(1)*b(1)-42*s(2)*b(1)+
4*s(3)*b(1)+45*s(4)*b(1)+s(3)*b(2)-39*s(1)*b(3)-26*s(2)*b(3)+
17*s(3)*b(3)-16*s(4)*b(3),-41*s(1)*b(1)+10*s(2)*b(1)+45*s(3)*b(1)
-16*s(4)*b(1)+s(4)*b(2)+22*s(1)*b(3)-46*s(2)*b(3)-16*s(3)*b(3)+
44*s(4)*b(3)]
Versuch.unab.44  69> coef2(v,s(1)*s(2)*s(3)*s(4),m11,m12);
Versuch.unab.44  70> //coefs:
Versuch.unab.44  71. m11;
```

m11 is the matrix which represents a vector in $k^4 \otimes H^0\mathcal{E}_a(1)$ if I work with the representation introduced in 4.9.

```
m11[1,1]=41*b(1)+b(2)-13*b(3)
m11[1,2]=-48*b(1)-39*b(3)
m11[1,3]=15*b(1)-39*b(3)
m11[1,4]=-41*b(1)+22*b(3)
m11[2,1]=-48*b(1)-39*b(3)
m11[2,2]=4*b(1)+b(2)+2*b(3)
```

```
m11[2,3]=-42*b(1)-26*b(3)
m11[2,4]=10*b(1)-46*b(3)
m11[3,1]=15*b(1)-39*b(3)
m11[3,2]=-42*b(1)-26*b(3)
m11[3,3]=4*b(1)+b(2)+17*b(3)
m11[3,4]=45*b(1)-16*b(3)
m11[4,1]=-41*b(1)+22*b(3)
m11[4,2]=10*b(1)-46*b(3)
m11[4,3]=45*b(1)-16*b(3)
m11[4,4]=-16*b(1)+b(2)+44*b(3)

Versuch.unab.44  72> // zu den monomen:
Versuch.unab.44  73. m12;
m12[1,1]=s(1)
m12[1,2]=s(2)
m12[1,3]=s(3)
m12[1,4]=s(4)
m12[2,1]=s(1)
m12[2,2]=s(2)
m12[2,3]=s(3)
m12[2,4]=s(4)
m12[3,1]=s(1)
m12[3,2]=s(2)
m12[3,3]=s(3)
m12[3,4]=s(4)
m12[4,1]=s(1)
m12[4,2]=s(2)
m12[4,3]=s(3)
m12[4,4]=s(4)

Versuch.unab.44  74> ring r3=101,(b(1..col)),(c,dp);
Versuch.unab.44  75> map f1=r8,0,0,0,0,b(1),b(2),b(3);
Versuch.unab.44  76> matrix m13=wedge(f1(m11),2);
Versuch.unab.44  77> ideal id=m13;

Versuch.unab.44  79> id=std(id);
Versuch.unab.44  80> hilb(id);
//Hilbert function:
// 0    1
// 1    3
// 4
Versuch.unab.44  81> degree(id);
//dimension 0
//multiplicity 4
Versuch.unab.44  82> id=minbase(id);
```

According to remark 4.1 it suffices now that the zero locus of the Fitting ideal of 2×2 minors is zero dimensional. From the generators of the ideal id one sees that the zero locus is the point $(0,0,0) \in \mathbb{C}^3$ with multiplicity 4. But this is the trivial linear combination, so we are done.

```
Versuch.unab.44  83> id;
id[1]=b(1)^2
id[2]=b(1)*b(2)
id[3]=b(2)^2
id[4]=b(1)*b(3)
id[5]=b(2)*b(3)
id[6]=b(3)^2

Versuch.unab.44  84> quit;
```

4.4.2 Program example 5

The instanton bundle $\mathcal{E}$ constructed in this program output is an other example for an instanton of NC-type. Moreover its monad is selfdual in the meaning of page 7. i.e.: it has a monad of type

$$0 \quad \rightarrow \quad 3\mathcal{O}(-1) \quad \stackrel{S}{\rightarrow} \quad 4\mathcal{E}_a \quad \stackrel{S^\vee \circ J}{\rightarrow} \quad 3\mathcal{O}(1) \quad \rightarrow \quad 0$$

Its second Chern class $c_2(\mathcal{E}) = 7$. The null correlation bundle $\mathcal{E}_a$ is here not specified, hence I constructed a whole family where the null correlation bundle $\mathcal{E}_a$ hence the x_i vary.

The first row of the matrix S consists of 3 independent vectors which is one less than the maximal possible number. Thus for generic examples $h^0\mathcal{E}(1) = 0$ by proposition the remark 4.5.2.

In the comments to the program code I will refer to the steps in the explicit explanations of the algorithm, see 25. The commentaries I entered later are in a different font: sans serif. I skipped parts of the output which are not essential for the understanding of the computation. The SINGULAR program code is echoed in the lines which start with "Versuch.unab.43".

```
Versuch.unab.43   1> ring r=101,(s(1..4)),(c,dp);
Versuch.unab.43   2> alternating=1;
Versuch.unab.43   3>
Versuch.unab.43   4> <"lib";
Versuch.unab.43   5> ideal m=s(1),s(2),s(3);
```

m is the first row of S, where $s(1), s(2), s(3)$ are independent vectors.

```
Versuch.unab.43   6> m;
m[1]=s(1)
m[2]=s(2)
m[3]=s(3)
Versuch.unab.43   7> //--------------------------------------------------
```

m1 is the set of all linear relations of first row.

```
Versuch.unab.43   8. module m1=syz(m),s(1)*gen(4),s(2)*gen(4),s(3)*gen(4);
Versuch.unab.43   9> m1;}
m1[1]=[0,0,s(3)]
m1[2]=[0,s(3),s(2)]
```

```
m1[3]=[0,s(2)]
m1[4]=[s(3),0,s(1)]
m1[5]=[s(2),s(1)]
m1[6]=[s(1)]
m1[7]=[0,0,0,s(1)]
m1[8]=[0,0,0,s(2)]
m1[9]=[0,0,0,s(3)]
```

$s(1) =$m2 is a random element in K_1 chosen, see page 25 step (2) and concatenated with the old S and one obtains the new $S =$m3.

```
Versuch.unab.43  10> matrix mat1=randmat(ncols(m1),1);
Versuch.unab.43  12. mat1;
mat1[1,1]=2
mat1[2,1]=36
mat1[3,1]=-10
mat1[4,1]=48
mat1[5,1]=40
mat1[6,1]=21
mat1[7,1]=-23
mat1[8,1]=-30
mat1[9,1]=14
Versuch.unab.43  13> matrix m2=matrix(m1)*mat1;
Versuch.unab.43  14> matrix m3[nrows(m2)][2];
Versuch.unab.43  15> m3[1..nrows(m2),1]=m2;
Versuch.unab.43  16> m3[1..ncols(m),2]=m;
Versuch.unab.43  17> matrix m3t=transp(m3);
```

$m4$ is the set of all relations of $S^\vee \circ J =$m3t, where $m3t$ is the transpose of $m3$. $m5$ is the set of all linear relations in $m4$.

```
Versuch.unab.43  18> m3t;
m3t[1,1]=21*s(1)+40*s(2)+48*s(3)
m3t[1,2]=40*s(1)-10*s(2)+36*s(3)
m3t[1,3]=48*s(1)+36*s(2)+2*s(3)
m3t[1,4]=-23*s(1)-30*s(2)+14*s(3)
m3t[2,1]=s(1)
m3t[2,2]=s(2)
m3t[2,3]=s(3)
m3t[2,4]=0
Versuch.unab.43  19> module m4=syz(module(m3t));
Versuch.unab.43  20> m4=std(m4);
```

```
Versuch.unab.43  22. module null;
Versuch.unab.43  23> module m5=jet(m4,1)+null;
Versuch.unab.43  24> m5;
m5[1]=[0,0,0,s(1)-47*s(2)-5*s(3)]
m5[2]=[0,s(3),s(2)+46*s(3),46*s(2)-34*s(3)]
m5[3]=[0,s(2)-16*s(3),-16*s(2)+9*s(3),43*s(2)-31*s(3)]
m5[4]=[s(3),0,s(1)-26*s(3),-19*s(1)+24*s(3)]
m5[5]=[s(2)+32*s(3),s(1)+44*s(2)+32*s(3),32*s(1)+32*s(2)-20*s(3),
       -36*s(1)-49*s(2)+8*s(3)]
m5[6]=[s(1)+44*s(2)+32*s(3),44*s(1)+17*s(2)-6*s(3),32*s(1)-6*s(2)+14*s(3),
       -49*s(1)-35*s(2)+48*s(3)]
```

$s(2) =$m6 is a random element in K_2 chosen, see page 25 step (3) and concatenated with the old S and one obtains the new $S =$m7.

```
Versuch.unab.43  25> matrix mat2=randmat(ncols(matrix(m5)),1);
Versuch.unab.43  27. mat2;
mat2[1,1]=-19
mat2[2,1]=29
mat2[3,1]=-12
mat2[4,1]=-8
mat2[5,1]=8
mat2[6,1]=5
Versuch.unab.43  28> matrix m6=matrix(m5)*mat2;
Versuch.unab.43  29> matrix m7[nrows(m2)][3];
Versuch.unab.43  30> m7[1..nrows(m2),1..2]=m3;
Versuch.unab.43  31> m7[1..nrows(m2),3]=m6;
Versuch.unab.43  32> m7=transp(m7t);
Versuch.unab.43  33> m7;
m7t[1,1]=21*s(1)+40*s(2)+48*s(3)
m7t[1,2]=40*s(1)-10*s(2)+36*s(3)
m7t[1,3]=48*s(1)+36*s(2)+2*s(3)
m7t[1,4]=-23*s(1)-30*s(2)+14*s(3)
m7t[2,1]=s(1)
m7t[2,2]=s(2)
m7t[2,3]=s(3)
m7t[2,4]=0
m7t[3,1]=5*s(1)+26*s(2)+4*s(3)
m7t[3,2]=26*s(1)+21*s(2)+43*s(3)
m7t[3,3]=4*s(1)+43*s(2)+31*s(3)
m7t[3,4]=4*s(1)+33*s(2)-3*s(3)
```

m8 is the set of all relations of $S^{\vee} =$7t, where m7t is the transpose of m7. m9 is the set of all linear relations in m. Moreover m9 is a set of generators for $H^0\mathcal{K}(1)$, where $\mathcal{K}$ is the kernel bundle in the monad display:

$$0 \rightarrow 3\mathcal{O}(-1) \xrightarrow{S} \mathcal{K} \rightarrow \mathcal{E} \rightarrow 0$$

```
Versuch.unab.43  34> module m8=syz(module(m7));
Versuch.unab.43  35> m8=std(m8);
Versuch.unab.43  36> //-------------------------------------------------
Versuch.unab.43  37. module m9=jet(m8,1)+null;
Versuch.unab.43  38> m9;
m9[1]=[s(3),-3*s(2)-23*s(3),s(1)-23*s(2)+14*s(3),11*s(1)+43*s(2)-43*s(3)]
m9[2]=[s(2)-42*s(3),s(1)-5*s(2)+35*s(3),-42*s(1)+35*s(2)-22*s(3),
       -44*s(1)-5*s(2)+45*s(3)]
m9[3]=[s(1)-3*s(2)+43*s(3),-3*s(1)-38*s(2)-14*s(3),43*s(1)-14*s(2)+16*s(3),
       27*s(1)-19*s(2)]
```

m9 has 3 independent linear relations of $S^\vee \circ J$ =m7t, hence $h^0\mathcal{K}(1) = 3$ and $h^0\mathcal{E}(1)$ must be 0 $\mathcal{K}$ is the kernel bundle in the monad display.

$$0 \rightarrow 3\mathcal{O}(-1) \xrightarrow{S} \mathcal{K} \rightarrow \mathcal{E} \rightarrow 0$$

In the next step I check that the matrix S=m7 defines a subbundle i.e. there is now linear combination of rows of S which vanishes, see the **subbundle condition** remark 4.1. along a subscheme i.e.: there exist $\alpha \in k^4$ and $t \in H^0\mathcal{E}_a(1)$ such that the linear combinations of rows is $\alpha \otimes t$. The vector m10 below is a general linear combination

```
Versuch.unab.43  53. int col=nrows(m7);
Versuch.unab.43  54> ring r8=101,(s(1..4),b(1..col)),(c,dp);
Versuch.unab.43  55> map f=r,s(1),s(2),s(3),s(4);
Versuch.unab.43  56> int j;
Versuch.unab.43  57> matrix i[1][col];
Versuch.unab.43  58> for( j=1;j<=col;j=j+1){
Versuch.unab.43  59.    i[1,j]=b(j);
Versuch.unab.43  60.    }
Versuch.unab.43  61> i;
i[1,1]=b(1)
i[1,2]=b(2)
i[1,3]=b(3)
Versuch.unab.43  62> matrix m10[1][1]=i*f(m7);
Versuch.unab.43  63> m10=transp(m10);
Versuch.unab.43  64> m10;
m10[1,1]=21*s(1)*b(1)+40*s(2)*b(1)+48*s(3)*b(1)+s(1)*b(2)+5*s(1)*b(3)+
         +26*s(2)*b(3)+4*s(3)*b(3)
m10[2,1]=40*s(1)*b(1)-10*s(2)*b(1)+36*s(3)*b(1)+s(2)*b(2)+26*s(1)*b(3)+
```

```
         +21*s(2)*b(3)+43*s(3)*b(3)
m10[3,1]=48*s(1)*b(1)+36*s(2)*b(1)+2*s(3)*b(1)+s(3)*b(2)+4*s(1)*b(3)+
         +43*s(2)*b(3)+31*s(3)*b(3)
m10[4,1]=-23*s(1)*b(1)-30*s(2)*b(1)+14*s(3)*b(1)+4*s(1)*b(3)+33*s(2)*b(3)-
         -3*s(3)*b(3)
Versuch.unab.43  65> matrix m11;
Versuch.unab.43  66> matrix m12;
Versuch.unab.43  67> module m10m=module(m10);
Versuch.unab.43  68> vector v=m10m[1];
Versuch.unab.43  69> v;
[21*s(1)*b(1)+40*s(2)*b(1)+48*s(3)*b(1)+s(1)*b(2)+5*s(1)*b(3)+26*s(2)*b(3)+
+4*s(3)*b(3),40*s(1)*b(1)-10*s(2)*b(1)+36*s(3)*b(1)+s(2)*b(2)+26*s(1)*b(3)+
+21*s(2)*b(3)+43*s(3)*b(3),48*s(1)*b(1)+36*s(2)*b(1)+2*s(3)*b(1)+s(3)*b(2)+
+4*s(1)*b(3)+43*s(2)*b(3)+31*s(3)*b(3),-23*s(1)*b(1)-30*s(2)*b(1)+
+14*s(3)*b(1)+4*s(1)*b(3)+33*s(2)*b(3)-3*s(3)*b(3)]
Versuch.unab.43  70> coef(v,s(1)*s(2)*s(3)*s(4),m11,m12);
Versuch.unab.43  71> //coefs:
Versuch.unab.43  72. m11;
```

m11 is the matrix which represents a vector in $k^4 \otimes H^0\mathcal{E}_a(1)$ if I work with the representation introduced in 4.9.

```
m11[1,1]=21*b(1)+b(2)+5*b(3)
m11[1,2]=40*b(1)+26*b(3)
m11[1,3]=48*b(1)+4*b(3)
m11[2,1]=40*b(1)+26*b(3)
m11[2,2]=-10*b(1)+b(2)+21*b(3)
m11[2,3]=36*b(1)+43*b(3)
m11[3,1]=48*b(1)+4*b(3)
m11[3,2]=36*b(1)+43*b(3)
m11[3,3]=2*b(1)+b(2)+31*b(3)
m11[4,1]=-23*b(1)+4*b(3)
m11[4,2]=-30*b(1)+33*b(3)
m11[4,3]=14*b(1)-3*b(3)
Versuch.unab.43  73> // zu den monomen:
Versuch.unab.43  74. m12;
m12[1,1]=s(1)
m12[1,2]=s(2)
m12[1,3]=s(3)
m12[2,1]=s(1)
m12[2,2]=s(2)
m12[2,3]=s(3)
m12[3,1]=s(1)
m12[3,2]=s(2)
m12[3,3]=s(3)
```

```
m12[4,1]=s(1)
m12[4,2]=s(2)
m12[4,3]=s(3)
Versuch.unab.43  75> ring r3=101,(b(1..col)),(c,dp);
Versuch.unab.43  76> map f1=r8,0,0,0,0,b(1),b(2),b(3);
Versuch.unab.43  77> matrix m13=wedge(f1(m11),2);
Versuch.unab.43  78> ideal id=m13;
Versuch.unab.43  79> id=std(id);
Versuch.unab.43  80> hilb(id);

//         1 t^0
//        -6 t^2
//         8 t^3
//        -3 t^4

//         1 t^0
//         3 t^1

// codimension = 3
// degree      = 4
Versuch.unab.43  81> degree(id);
// codimension = 3
// dimension   = 0
// degree      = 4
Versuch.unab.43  82> id=minbase(id);
```

According to remark 4.1 it suffices now that the zero locus of the Fitting ideal of 2×2 minors is zero dimensional. From the generators of the ideal id one sees that the zero locus is the point $(0,0,0) \in \mathbb{C}^3$ with multiplicity 4. But this is the trivial linear combination, so we are done.

```
Versuch.unab.43  83> id;
id[1]=b(1)^2
id[2]=b(1)*b(2)
id[3]=b(2)^2
id[4]=b(1)*b(3)
id[5]=b(2)*b(3)

id[6]=b(3)^2

Versuch.unab.43  84> quit;
```

4.4.3 Program example 6

The instanton bundle $\mathcal{E}$ constructed in this program output is an other example for an instanton of NC-type. Moreover its monad is selfdual in the meaning of page 7. i.e.: it has a monad of type

$$0 \quad \rightarrow \quad 3\mathcal{O}(-1) \quad \stackrel{S}{\rightarrow} \quad 4\mathcal{E}_a \quad \stackrel{S^\vee \circ J}{\rightarrow} \quad 3\mathcal{O}(1) \quad \rightarrow \quad 0$$

Its second Chern class $c_2(\mathcal{E}) = 7$. The null correlation bundle $\mathcal{E}_a$ is here not specified, hence I constructed a whole family where the null correlation bundle $\mathcal{E}_a$ hence the x_i vary.

The first row of the matrix S consists of 2 independent vectors. Hence $\mathcal{E}$ is a special 't Hooft bundle by proposition 4.31 Thus $h^0\mathcal{E}(1) = 2$.

In the comments to the program code I will refer to the steps in the explicit explanations of the algorithm, see 25. The commentaries I entered later re in a different font: sans serif. I skipped parts of the output which are not essential for the understanding of the computation. The SINGULAR program code is echoed in the lines which start with "Versuch.unab.42".

```
Versuch.unab.42   1> ring r=101,(s(1..2)),(c,dp);
Versuch.unab.42   2> alternating=1;
Versuch.unab.42   3>
Versuch.unab.42   4> <"lib";
Versuch.unab.42   5> ideal m=s(1),s(2);
```

m is the first row of where $s(1)$ and $s(2)$ are independent in $H^0\mathcal{E}_a(1)$.

```
Versuch.unab.42   6> m;
m[1]=s(1)
m[2]=s(2)
Versuch.unab.42   7> //-----------------------------------------------
```

m1 is the set of generators for the vector space of the linear relations of first row.

```
Versuch.unab.42   8. module m1=syz(m),s(1)*gen(3),s(2)*gen(3),s(1)*gen(4),
                                s(2)*gen(4);

Versuch.unab.42   9> m1;
m1[1]=[0,s(2)]
m1[2]=[s(2),s(1)]
m1[3]=[s(1)]
```

```
m1[4]=[0,0,s(1)]
m1[5]=[0,0,s(2)]
m1[6]=[0,0,0,s(1)]
m1[7]=[0,0,0,s(2)]
```

$s(1)$=m2 is a random element in K_1 chosen, see page 25 step (2) and after concatenation with the old S and one obtains the new S =m3.

```
Versuch.unab.42  10> matrix mat1=randmat(ncols(m1),1);
Versuch.unab.42  12. mat1;
mat1[1,1]=6
mat1[2,1]=15
mat1[3,1]=46
mat1[4,1]=-50
mat1[5,1]=35
mat1[6,1]=-21
mat1[7,1]=43
Versuch.unab.42  13> matrix m2=matrix(m1)*mat1;
Versuch.unab.42  15. matrix m3[nrows(m2)][2];
Versuch.unab.42  16> m3[1..nrows(m2),1]=m2;
Versuch.unab.42  17> m3[1..ncols(m),2]=m;
```

m4 is the set of all relations of $S^\vee \circ J$ =m3t, where m3t is the transpose of m3. m5 is the set of all linear relations in m4.

```
Versuch.unab.42  19. matrix m3t=transp(m3);
Versuch.unab.42  20> m3t;
m3t[1,1]=46*s(1)+15*s(2)
m3t[1,2]=15*s(1)+6*s(2)
m3t[1,3]=-50*s(1)+35*s(2)
m3t[1,4]=-21*s(1)+43*s(2)
m3t[2,1]=s(1)
m3t[2,2]=s(2)
m3t[2,3]=0
m3t[2,4]=0
Versuch.unab.42  21> module m4=syz(module(m3t));
Versuch.unab.42  22> m4=std(m4);
Versuch.unab.42  24. //-----------------------------------------------------
Versuch.unab.42  25. module null;
Versuch.unab.42  26> module m5=jet(m4,1)+null;
Versuch.unab.42  27> m5;
m5[1]=[0,0,0,s(1)+22*s(2)]
m5[2]=[0,0,s(2),-27*s(1)]
m5[3]=[0,0,s(1),-29*s(1)]
```

```
m5[4]=[0,s(2),-30*s(2)]
m5[5]=[s(2),s(1)-20*s(2),23*s(1)+3*s(2),-40*s(1)]
m5[6]=[s(1)-20*s(2),-20*s(1)-4*s(2),3*s(1)+41*s(2)]
```

$s(2)$=m6 is a random element in K_2 chosen, see page 25 step (3) and after concatenation with the old S and one obtains the new S =m7

```
Versuch.unab.42  28> matrix mat2=randmat(ncols(matrix(m5)),1);
Versuch.unab.42  30. mat2;
mat2[1,1]=-25
mat2[2,1]=-3
mat2[3,1]=30
mat2[4,1]=29
mat2[5,1]=-30
mat2[6,1]=11

Versuch.unab.42  31> matrix m6=matrix(m5)*mat2;
Versuch.unab.42  32> m6;
m6[1,1]=11*s(1)-48*s(2)
m6[2,1]=-48*s(1)-21*s(2)
m6[3,1]=-21*s(1)-7*s(2)
m6[4,1]=-18*s(1)-45*s(2)
Versuch.unab.42  33> matrix m7[4][3];
Versuch.unab.42  34> m7[1..4,1..2]=m3;
Versuch.unab.42  35> m7[1..4,3]=m6;
Versuch.unab.42  36> m7=transp(m7t);
Versuch.unab.42  37> m7;
m7t[1,1]=46*s(1)+15*s(2)
m7t[1,2]=15*s(1)+6*s(2)
m7t[1,3]=-50*s(1)+35*s(2)
m7t[1,4]=-21*s(1)+43*s(2)
m7t[2,1]=s(1)
m7t[2,2]=s(2)
m7t[2,3]=0
m7t[2,4]=0
m7t[3,1]=11*s(1)-48*s(2)
m7t[3,2]=-48*s(1)-21*s(2)
m7t[3,3]=-21*s(1)-7*s(2)
m7t[3,4]=-18*s(1)-45*s(2)
```

m8 is the set of all relations of $S^\vee \circ J$ =m7t, where m7t is the transpose of m7. m9 is the set of all linear relations in m8. Moreover m9 is a set of generators for $H^0\mathcal{K}(1)$, where $\mathcal{K}$ is the kernel bundle in the monad display:

$$0 \;\to\; 3\mathcal{O}(-1) \;\xrightarrow{S}\; \mathcal{K} \;\to\; \mathcal{E} \;\to\; 0$$

```
Versuch.unab.42  38> module m8=syz(module(m7t));
Versuch.unab.42  39> m8=std(m8);
Versuch.unab.42  40> //--------------------------------------------
Versuch.unab.42  41. module m9=jet(m8,1)+null;
Versuch.unab.42  42> m9;
m9[1]=[0,0,s(2),23*s(1)-11*s(2)]
m9[2]=[0,0,s(1)+31*s(2),-29*s(1)+32*s(2)]
m9[3]=[0,s(2),-19*s(2),-31*s(2)]
m9[4]=[s(2),s(1)+43*s(2),23*s(1)+32*s(2),-40*s(1)]
m9[5]=[s(1)-20*s(2),-20*s(1)-22*s(2),3*s(1)-25*s(2)]
```

m9 has 5 independent linear relations of $S^\vee \circ J$ =m7t. Hence $h^0\mathcal{K}(1) = 5$ and $h^0\mathcal{E}(1)$ must be 2. $\mathcal{K}$ is the kernel bundle in the monad display.

$$0 \rightarrow 3\mathcal{O}(-1) \xrightarrow{S} \mathcal{K} \rightarrow \mathcal{E} \rightarrow 0$$

In the next step I check that the matrix S=m7 defines a subbundle i.e. there is now linear combination of rows of S which vanishes, see the **subbundle condition** remark 4.1. along a subscheme i.e.: there exist $\alpha \in k^4$ and $t \in H^0\mathcal{E}_a(1)$ such that the linear combinations of rows is $\alpha \otimes t$. The vector m10 below is a general linear combination.

```
Versuch.unab.42  52. int col=nrows(m7);
Versuch.unab.42  53> ring r8=101,(s(1..2),b(1..col)),(c,dp);
Versuch.unab.42  54> map f=r,s(1),s(2);
Versuch.unab.42  55> int j;
Versuch.unab.42  56> matrix i[1][col];
Versuch.unab.42  57> for( j=1;j<=col;j=j+1){
Versuch.unab.42  58.    i[1,j]=b(j);
Versuch.unab.42  59.    }
Versuch.unab.42  60> i;
i[1,1]=b(1)
i[1,2]=b(2)
i[1,3]=b(3)
Versuch.unab.42  62. matrix m10[1][1]=i*f(m7);
Versuch.unab.42  63> m10=transp(m10);
Versuch.unab.42  64> m10;
m10[1,1]=46*s(1)*b(1)+15*s(2)*b(1)+s(1)*b(2)+11*s(1)*b(3)-48*s(2)*b(3)
m10[2,1]=15*s(1)*b(1)+6*s(2)*b(1)+s(2)*b(2)-48*s(1)*b(3)-21*s(2)*b(3)
m10[3,1]=-50*s(1)*b(1)+35*s(2)*b(1)-21*s(1)*b(3)-7*s(2)*b(3)
m10[4,1]=-21*s(1)*b(1)+43*s(2)*b(1)-18*s(1)*b(3)-45*s(2)*b(3)
Versuch.unab.42  65> matrix m11;
Versuch.unab.42  66> matrix m12;
Versuch.unab.42  67> module m10m=module(m10);
Versuch.unab.42  68> vector v=m10m[1];
```

```
Versuch.unab.42  69> v;
[46*s(1)*b(1)+15*s(2)*b(1)+s(1)*b(2)+11*s(1)*b(3)-48*s(2)*b(3),
 15*s(1)*b(1)+6*s(2)*b(1)+s(2)*b(2)-48*s(1)*b(3)-21*s(2)*b(3),
 -50*s(1)*b(1)+35*s(2)*b(1)-21*s(1)*b(3)-7*s(2)*b(3),
 -21*s(1)*b(1)+43*s(2)*b(1)-18*s(1)*b(3)-45*s(2)*b(3)]
Versuch.unab.42  70> coef(v,s(1)*s(2),m11,m12);
Versuch.unab.42  71> //coefs:
Versuch.unab.42  72. m11;
```

m11 is the matrix which represents a vector in $k^4 \otimes H^0\mathcal{E}_a(1)$ if I work with the representation introduced in 4.9.

```
m11[1,1]=46*b(1)+b(2)+11*b(3)
m11[1,2]=15*b(1)-48*b(3)
m11[2,1]=15*b(1)-48*b(3)
m11[2,2]=6*b(1)+b(2)-21*b(3)
m11[3,1]=-50*b(1)-21*b(3)
m11[3,2]=35*b(1)-7*b(3)
m11[4,1]=-21*b(1)-18*b(3)
m11[4,2]=43*b(1)-45*b(3)
Versuch.unab.42  73> // zu den monomen:
Versuch.unab.42  74. m12;
m12[1,1]=s(1)
m12[1,2]=s(2)
m12[2,1]=s(1)
m12[2,2]=s(2)
m12[3,1]=s(1)
m12[3,2]=s(2)
m12[4,1]=s(1)
m12[4,2]=s(2)
Versuch.unab.42  75> ring r3=101,(b(1..col)),(c,dp);
Versuch.unab.42  76> map f1=r8,0,0,b(1),b(2),b(3);
Versuch.unab.42  77> matrix m13=wedge(f1(m11),2);
Versuch.unab.42  78> ideal id=m13;
Versuch.unab.42  79> id;
id[1]=-50*b(1)^2-49*b(1)*b(2)+b(2)^2+35*b(1)*b(3)-10*b(2)*b(3)-10*b(3)^2
id[2]=-37*b(1)^2-35*b(1)*b(2)+2*b(1)*b(3)+7*b(2)*b(3)-26*b(3)^2
id[3]=-30*b(1)^2+43*b(1)*b(2)-12*b(1)*b(3)-45*b(2)*b(3)-46*b(3)^2
id[4]=-17*b(1)^2-50*b(1)*b(2)-18*b(1)*b(3)-21*b(2)*b(3)+4*b(3)^2
id[5]=-37*b(1)^2+21*b(1)*b(2)-42*b(1)*b(3)+18*b(2)*b(3)-36*b(3)^2
id[6]=b(1)^2-12*b(1)*b(3)-11*b(3)^2
Versuch.unab.42  80> id=std(id);
Versuch.unab.42  81> hilb(id);
//          1 t^0
//         -6 t^2
```

```
//          8 t^3
//         -3 t^4
//          1 t^0
//          3 t^1
// codimension = 3
// degree      = 4
Versuch.unab.42  82> degree(id);
// codimension = 3
// dimension   = 0
// degree      = 4
Versuch.unab.42  83> id=minbase(id);
```

According to remark 4.1 it suffices now that the zero locus of the Fitting ideal of 2×2 minors is zero dimensional. From the generators of the ideal id one sees that the zero locus is the point $(0,0,0) \in \mathbb{C}^3$ with multiplicity 4. But this is the trivial linear combination, so we are done.

```
Versuch.unab.42  84> id;
id[1]=b(1)^2
id[2]=b(1)*b(2)
id[3]=b(2)^2
id[4]=b(1)*b(3)
id[5]=b(2)*b(3)
id[6]=b(3)^2
Versuch.unab.42  85> quit;
```

4.5 The dimensions of $H^0\mathcal{E}(1)$

In this chapter I shall determine the dimension of the space of linear sections $H^0\mathcal{E}(1)$ for several families of instanton bundles $\mathcal{E}$ of NC-type on $\mathbb{P}_3$, which are the cohomology bundles of symmetric monads in the meaning of page 7

$$0 \;\rightarrow\; m\mathcal{O}(-1) \;\stackrel{S}{\rightarrow}\; (m+1)\mathcal{E}_a \;\stackrel{S^\vee\circ J}{\rightarrow}\; m\mathcal{O}(10 \;\rightarrow\; 0$$

These families are constructed according to the algorithm, see page 25. All these results were suggested by explicit computations, which are listed in the previous chapter 4.4. First the pairings

$$\begin{array}{rcll} \bar{A}: & \bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1)\otimes\bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1) & \rightarrow & H^0\mathcal{O}(2) \\ & \sum_{i=1}^{m+1} t_i\otimes\sum_{i=1}^{m+1} u_i & \mapsto & \sum_{i=1}^{m+1} t_i\circ J\circ u_i^\vee \end{array}$$

and

$$\begin{array}{rcll} \bar{\wedge}: & \bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1)\otimes\bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1) & \rightarrow & \Lambda^2 H^0\mathcal{E}_a(1) \\ & \sum_{i=1}^{m+1} t_i\otimes\sum_{i=1}^{m+1} u_i & \mapsto & \sum_{i=1}^{m+1} t_i\wedge u_i \end{array}$$

which are isomorphic are expressed in a basis of $H^0\mathcal{E}_a(1)$:

Notation 4.7 (1) Once and for all in this chapter I fix a basis of $H^0\mathcal{E}_a(1)$:

$$s_1,\cdots,s_5$$

(2) $H_k\subset H^0\mathcal{E}_a(1)$ is the linear subspace spanned by $s_1,\cdots,s_k$, $1\le k\le 5$.

(3) Then $\{s_k\wedge s_j\}_{k,j}$ $1\le k<j\le 5$ is a basis of $\Lambda^2 H^0\mathcal{E}_a(1)$. I work with it for the rest of the paragraph.

(4) An element t of $\bigoplus^{m+1} H^0\mathcal{E}_a(1)$ is a tuple $t=(t_1,\cdots,t_{m+1})$, where

$$t_i=\sum_{j=1}^{5} a_{ji}s_j$$

(5) For future use I fix the notation for two more elements, $u=(u_1,\cdots,u_{m+1})$ and $v=(v_1,\cdots,v_{m+1})$:

$$u_i=\sum_{j=1}^{5} b_{ji}s_j,\;\; v_i=\sum_{j=1}^{5} c_{ji}s_j$$

With this notation I can describe the pairings $\bar{A}$ and $\bar{\wedge}$ with respect to the given basis. The operator $\bar{A}$ is in the basis $s_1,\cdots,s_5$:

$$\begin{array}{rcll} \bar{A}: & \bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1)\otimes\bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1) & \rightarrow & H^0\mathcal{O}(2) \\ & u\otimes t & \mapsto & \sum_{1\le j<k\le 5}\left(\sum_{i=1}^{m+1}(a_{ji}b_{ki}-b_{ji}a_{ki})\right) s_j\circ J\circ s_k^\vee \end{array}$$

and $\bar{\wedge}$ is given by:

$$\begin{array}{rcll} \bar{\wedge}: & \bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1)\otimes\bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1) & \rightarrow & \Lambda^2 H^0\mathcal{E}_a(1) \\ & u\otimes t & \mapsto & \sum_{1\le j<k\le 5}\left(\sum_{i=1}^{m+1}(a_{ji}b_{ki}-b_{ji}a_{ki})\right) s_j\wedge s_k \end{array}$$

Remark 4.8 The vanishing condition $\sum_{i=1}^{m+1}(a_{ji}b_{ki} - b_{ji}a_{ki}) = 0$ is bilinear in the coefficients a_{ji} and b_{ki} and hence the kernel

$$ker(t) \quad \rightarrow \quad \bigoplus^{m+1} H^0\mathcal{E}_a(1) \quad \stackrel{\bar{\wedge} t}{\rightarrow} \quad \Lambda^2 H^0\mathcal{E}_a(1)$$

for a fixed element t is a linear subspace in $\bigoplus^{m+1} H^0\mathcal{E}_a(1)$.

4.9 The computation of the dimension of $H^0\mathcal{E}_a(1)$ uses the following decomposition, with respect to the fixed basis $s_1, \cdots, s_5$ for $H^0\mathcal{E}_a(1)$. A general element $t := (t_1, \cdots, t_{m+1}) \in k^{m+1} \otimes H^0\mathcal{E}_a(1) \simeq k^{m+1} \otimes k^5$, which is also a morphism

$$\mathcal{O}(-1) \quad \stackrel{t}{\rightarrow} \quad (m+1)\mathcal{E}_a \ ,$$

has a representation as:

$$t = s_1(a_{11}, \cdots, a_{1m+1}) + \cdots + s_5(a_{51}, \cdots, a_{5m+1})$$

which looks in matrix notation as:

$$(s_1, \cdots, s_5) \circ A = (t_1, \cdots, t_{m+1})$$

where A is the matrix:

$$A := \begin{pmatrix} a_{11} & \cdots & a_{1m+1} \\ \vdots & & \vdots \\ a_{51} & \cdots & a_{5m+1} \end{pmatrix}$$

Hence one observes that these elements form the image of $(s_1, \cdots, s_5)$ under the morphism

$$\begin{array}{ccc} H^0\mathcal{E}_a(1) \otimes Mat(5, m+1; k) & \rightarrow & k_{m+1} \otimes H^0\mathcal{E}_a(1) \\ (s, A) & \mapsto & s \circ A \end{array}$$

In the sequel we will study not only the space of all matrices $Mat(5, m+1; k)$ but also the action of subvector spaces of $Mat(5, m+1; k)$.
Note that for $m+1 = 5$ the elements of maximal rank in $Mat(5, m+1; k)$ form the group $Gl(5)$ an the image is the orbit of $(s_1, \cdots, s_5)$ under the group action. This decomposition of an element in $\bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1)$ is the "tool" to reduce computations to linear algebra over a field k.

With this convention I compute now $H^0\mathcal{E}(1)$. I want to distinguish three cases which have different dimensions for $H^0\mathcal{E}(1)$. These cases differ by the maximal number of independent vectors of $H^0\mathcal{E}_a(1)$ occuring in the rows of the matrix S in the selfdual NC-type monad:

$$0 \quad \rightarrow \quad m\mathcal{O}(-1) \quad \stackrel{S}{\rightarrow} \quad (m+1)\mathcal{E}_a \quad \stackrel{S^\vee \circ J}{\rightarrow} \quad m\mathcal{O}(1) \quad \rightarrow \quad 0$$

4.5.1 Case 1: The number of independent vectors in a row of S is maximal

This result was suggested by the **example 4**, page 73.

4.10 Let $\mathcal{E}$ be an instanton bundle defined by:

$$0 \rightarrow m\mathcal{O}(-1) \stackrel{S}{\rightarrow} (m+1)\mathcal{E}_a \stackrel{S^\vee \circ J}{\rightarrow} m\mathcal{O}(1) \rightarrow 0$$

where the matrix $S \in Mat(m, m+1; H^0\mathcal{E}_a(1))$ can be chosen as:

$$S := \begin{pmatrix} s_1, \cdots, s_{m+1} \\ * \end{pmatrix}$$

where $s_1, \cdots, s_{m+1}$ are $m+1 \leq 5$ independent elements of the basis chosen once and for all in 4.7 in $H^0\mathcal{E}_a(1)$. This means there exists at least one row with $m+1$ independent entries in $H^0\mathcal{E}_a(1)$. I shall denote for a fixed $\langle a \rangle$ the family of instantons with this property by Y^0_{a1}.

In this paragraph the following proposition will be proven by a sequence of lemmata and propositions.

Proposition 4.11 *Let $\mathcal{E}$ be an instanton bundle of NC-type with monad*

$$0 \rightarrow m\mathcal{O}(-1) \stackrel{S}{\rightarrow} (m+1)\mathcal{E}_a \stackrel{S^\vee \circ J}{\rightarrow} m\mathcal{O}(1) \rightarrow 0$$

where S is the matrix

$$S := \begin{pmatrix} s_1, \cdots, s_{m+1} \\ * \end{pmatrix}$$

then $h^0\mathcal{E}(1) \geq 1$, i.e.: $\mathcal{E}$ is a 't Hooft bundle.

Remark 4.12 (1) Note that in case 1 always $1 \leq m \leq 4$

(2) $c_2(\mathcal{E}) = 2m + 1$

The next proposition is essential for the proof of the proposition 4.11. One can describe in more detail the structure of the matrices in the representation 4.9 and furthermore use the result to determine the dimension of $H^0\mathcal{E}(1)$.

Proposition 4.13 *Let A be as in 4.9 and $m \leq 4$ Then :*

(1) If the matrix A represents an element in $ker(s)$, where $s = (s_1, \cdots, s_{m+1}) \in k^{m+1} \otimes H^0\mathcal{E}_a(1)$, then $A \in Mat(m+1, m+1; k)$ is a symmetric matrix i.e: $A = A^\vee$

(2) If $v, u, t \in \ker(s)$, $v, u \in \ker(t)$ and moreover u and v are represented by invertible matrices with distinct eigenvalues then $u \in \ker(v)$. In other words in this situation ker *is a symmetric, reflexive and transitive relation for all elements in* $\ker(t)$

Proof:

(1) Let

$$t_i = \sum a_{ji} s_j \text{ and } u_i = \sum b_{ji} s_j \in H^0 \mathcal{E}_a(1)$$

Then by page 92

$$t \bar{\wedge} u = \sum_{\substack{j<k \\ j,k \in \{1,\cdots,5\}}} \left(\sum_{i=1}^{m+1} (a_{ji} b_{ki} - b_{ji} a_{ki}) \right) s_j \wedge s_k \ (*)$$

If we assume now that $t = s$ the coefficients in $t_i = \sum a_{ji} s_j$ become $a_{kj} = \delta_{kj}$. If I put this into the equation $(*)$ and thus

$$s \bar{\wedge} u = \sum_{1 \le j < k \le 5} \left(\sum_{i=1}^{m+1} (a_{ji} b_{ki} - b_{ji} a_{ki}) \right) s_j \wedge s_k = \sum_{1 \le j < k \le 5} (b_{kj} - b_{jk}) s_j \wedge s_k = 0 \ .$$

The $\{s_j \wedge s_k\}_{1 \le j < k \le 5}$ however form a basis of $\Lambda^2 H^0 \mathcal{E}_a(1)$. Hence I obtain 10 linear equations for all $1 \le j < k \le 5$:

$$\boxed{b_{kj} = b_{jk}}$$

Therefore the matrix B as in 4.9 is symmetric.

(2) Let now

$$\begin{aligned} s_i = \sum \delta_{ji} s_j &\iff s \circ Id = s \\ t_i = \sum a_{ji} s_j &\iff s \circ A = t \\ u_i = \sum b_{ji} s_j &\iff s \circ B = u \\ v_i = \sum c_{ji} s_j &\iff s \circ C = v \end{aligned}$$

Note that $A = A^\vee, B = B^\vee$ and $C = C^\vee$ by the first part of the proposition 4.13 and δ_{ji} is the Kronecker symbol.
The rest of the proof follows from the two lemmata 4.17 and 4.18 presented below.

#

Corollary 4.14 *Let $m+1 < 5$ and $\mathcal{E}$ as in 4.10, then $t \in ker(s)$, as in the proposition 4.13, has only entries in H_{m+1}, i.e.: $t \in k^{m+1} \otimes H_{m+1}$, where H_{m+1} is generated by $s_1, \cdots, s_{m+1}$.*

Proof:
As in the proposition t is represented a $(m+1)\times 5$ matrix B. $j\leq m+1<k$, then it follows from $(*)$:

$$s\bar{\wedge}t=\sum_{1\leq j<k\leq 5}\left(\sum_{i=1}^{m+1}(\delta_{ji}b_{ki}-b_{ji}\delta_{ki})\right)s_j\wedge s_k=$$

$$=\sum_{1\leq j<k\leq m+1}(b_{jk}-b_{kj})s_j\wedge s_k+\sum_{remaining\ j,k}b_{kj}s_j\wedge s_k=0$$

$$\Longrightarrow b_{kj}=0\forall\ j<k\ (j,k)\notin\{1,\cdots,m+1\}\times\{1,\cdots,m+1\}$$

#

Corollary 4.15 *Hence the condition in 4.10 is equivalent to* $S\in Mat(m,m{+}1;H_{m+1})$.

Remark 4.16 It is very important for the calculations which shall follow to stress the fact that the 10 equations $\sum_{i=1}^{m+1}(a_{ji}b_{ki}-b_{ji}a_{ki})=0,\ 1\leq j<k\leq 5$ in the expression $(*)$ are the **commutator relations** for the two matrices A and B. i.e:

$$[A,B]=0\quad\Longleftrightarrow\quad\sum_i a_{ji}b_{ki}-b_{ji}a_{ki}=0\ \forall\ 1\leq j<k\leq 5 \tag{4.5.28}$$

In the next step I use a lemma on commuting matrices which should be well known. Nevertheless, I found no reference so I'll put it here.

Lemma 4.17 *Let A,B and C be symmetric matrices in $GL(m+1)$ with scalar entries and assume A has distinct eigenvalues. If A,B and A,C commute, then C and B are commuting matrices:*

$$[A,B]=[A,C]=0\Longrightarrow[B,C]=0$$

$[*,*]$ *denotes the commutator of two matrices.*

In other words one can say that all such matrices commuting with such an A given, form a commutative algebra:

$$\{B\in GL(m)|[A,B]=0,\text{A has distinct eigenvalues}\}$$

Proof: (of the lemma)
A is symmetric and all eigenvalues are distinct. I can find an invertible orthogonal matrix P ($P^\vee=P^{-1}$) such that

$$A=P^{-1}A'P\text{ with }A'=\begin{pmatrix}a_1&&\\&\ddots&\\&&a_{m+1}\end{pmatrix}\quad\text{and } a_j\neq a_i\ \forall\ i\neq j$$

See for instance [18]§23.9 .
Now define $B' := PBP^{-1}$ and $C' := PCP^{-1}$.
I remark here that it is necessary to know that B' and C' are again symmetric, which is used in the next lemma. Hence

$$P^{-1}A'B'P = P^{-1}A'PP^{-1}B'P = AB = BA = P^{-1}B'PP^{-1}A'P = P^{-1}B'A'P$$

Thus $A'B' = B'A'$ and analogously $A'C' = C'A'$ and $C'B' = B'C'$ because B' and C' are diagonal by the next lemma. Hence we are done because

$$CB = P^{-1}C'PP^{-1}B'P = P^{-1}C'B'P = P^{-1}B'C'P = P^{-1}B'PP^{-1}C'P = BC$$

is valid. #

Lemma 4.18 *We keep now the assumptions and notations of the previous lemma and assume in addition that all eigenvalues of A hence of A' are distinct. If $A'B' = B'A'$ then B' is a diagonal matrix.*

Proof:
For all $j < k \leq m+1$ we have the commutator equation:

$$\begin{aligned} [A'B'] = 0 \qquad & \Longleftrightarrow \qquad \sum_{i=1}^{m+1} (a'_{ji}b'_{ki} - b'_{ji}a'_{ki}) = 0 \ \forall\ j < k \\ a'_{ji} \overset{\Longleftrightarrow}{=} 0\ \forall\ i \neq j \qquad & a'_{jj}b'_{kj} - b'_{jk}a'_{kk} = 0 \\ B' \overset{\Longleftrightarrow}{=} B'^{\vee} \qquad & (a'_{jj} - a'_{kk})b'_{kj} = 0 \\ a'_{jj} \overset{\Longleftrightarrow}{\neq} a'_{kk}\ \forall\ k \neq j \qquad & b'_{jk} = 0\ \forall\ k \neq j \end{aligned} \tag{4.5.29}$$

Hence B' is diagonal. #

Eigenvalues can be zero from now on. I treat now the case of matrices with eigenvalues of multiplicity > 1 next. In this case lemma 4.18 does not apply. But there is still an argument which allows me to compute the dimension of the kernels. The proof of lemma 4.18 is violated at step 4.5.29. Let a_i be an eigenvalue with multiplicity $m_i \geq 1$. Here I use again the argument that B' as in lemma 4.18 is still symmetric if I apply an orthogonal base change $P^{-1} = P^{\vee}$.

Lemma 4.19 *Let E_{a_i} be set of all indices with eigenvalue a_i. The commutator equations $\sum_{i=1}^{m+1}(a'_{ji}b'_{ki} - b'_{ji}a'_{ki}) = 0$ for all $j < k \in E_{a_i}\ \forall\ i$ are trivial. i.e.: They are automatically satisfied. Their total number is at least*

$$\kappa := \sum_{i \in \{1,..,m+1\}} \frac{1}{m_i} \frac{(m_i - 1)m_i}{2}$$

Notice that distinct eigenvalues do not contribute to κ.

Proof:
I compute now analogously to lemma 4.18 $[A'B']$, so I obtain:

$$\sum_{i=1}^{m+1}(a'_{ji}b'_{ki} - b'_{ji}a'_{ki}) = \underbrace{(a'_{jj} - a'_{kk})}_{=0} b'_{kj} = 0$$

for $a'_{jj} = a'_{kk}$. Hence it is no condition on the b'_{kj}'s. #

We can visualize the matrices contained in the normalizer of A' as :

$$\begin{pmatrix} \ddots & & & & \\ & \underbrace{\begin{array}{c|} * \end{array}}_{m_i} & & & \\ & & a_{k+1} & & \\ & & & \ddots & \\ & & & & a_{m+1} \end{pmatrix}$$

Lemma 4.20 *Let now B' and C' in the normalizer of A', A' as in lemma 4.19. Then the number of commutator equations is again at most κ which is the number in the previous lemma.*

Proof:
This is again a trivial consequence of the equations:

$$\sum_{i=1}^{m+1}(a'_{ji}b'_{ki} - b'_{ji}a'_{ki}) = 0 \; \forall \; j < k$$

#

Proof: of the proposition 4.11
Let

$$0 \to m\mathcal{O}(-1) \to (m+1)\mathcal{E}_a \to m\mathcal{O}(1) \to 0$$

be the monad for an instanton bundle of NC-type. The second and the first display sequences for the monad are:

$$0 \to m\mathcal{O}(-1) \to \mathcal{K} \to \mathcal{E} \to 0$$

and

$$0 \to \mathcal{K} \to (m+1)\mathcal{E}_a \to m\mathcal{O}(1) \to 0$$

It is now sufficient to collect the results from the previous lemmata 4.17, 4.18, 4.19 & 4.20. $\binom{m+1}{2}$ is the maximum number of linear equations which define the kernel of an element $u \in \bigoplus^{m+1} H^0\mathcal{E}_a(1)$. If lemma 4.18 is valid, the vector space $ker(v)$ with $v \in ker(u)$ and $u \in ker(s)$ is isotropic. Hence

$$h^0\mathcal{E}(1) \geq (m+1)^2 - 2\binom{m+1}{2} - m = 1$$

If lemma 4.18 is not valid one must add the number of "trivial relations" in lemma 4.19 which is greater than κ and subtract the number of additional equations in lemma 4.20 which is less than κ. Therefore we have the inequality:

$$h^0\mathcal{E}(1) \geq (m+1)^2 - 2\binom{m+1}{2} - \kappa + \kappa - m = 1$$

#

Remark 4.21 I will prove in theorem 5.16 that for instanton bundles of NC-type $h^0\mathcal{E}(1) = 2$ **if and only if** there are only two independent sections of $H^0\mathcal{E}_a(1)$ in the matrix S in the monad

$$0 \rightarrow m\mathcal{O}(-1) \stackrel{S}{\rightarrow} (m+1)\mathcal{E}_a \stackrel{S^\vee \circ J}{\rightarrow} m\mathcal{O}(1) \rightarrow 0$$

Hence those sheaves constructed from three or more independent sections having $h^0\mathcal{E}(1) = 2$ cannot be locally free.

4.5.2 Case 2: The number of independent vectors in a row of S is at most the number of columns -1

This result was suggested by the **example 5**, see 4.4.2.

Let $\mathcal{E}$ be an instanton bundle which has a monad of type as follows

$$0 \rightarrow m\mathcal{O}(-1) \stackrel{S}{\rightarrow} (m+1)\mathcal{E}_a \stackrel{S^\vee \circ J}{\rightarrow} m\mathcal{O}(1) \rightarrow 0$$

where S is the matrix :

$$S := \begin{pmatrix} s_1, \cdots, s_m, 0 \\ * \end{pmatrix}$$

and $s_1, \cdots, s_m$ for $3 \leq m \leq 5$ are elements of the basis chosen once an for all in 4.7 for $H^0\mathcal{E}_a(1)$.

Notation 4.22 I shall denote the family of instantons with this property above by Y^0_{a2}

Remark 4.23 Necessarily $m + 1 \leq 6$ because $H^0\mathcal{E}_a(1) \simeq k^5$. Hence only the cases $1 \leq m \leq 5$ can occur. The cases $m = 1, 2$ have different behaviour and are not treated here. Actually for $m = 1$ the NC-type-monad defines a null correlation bundle and $m = 2$ is subsumed under case 3.

One can describe in more detail the structure of the matrices in the representation 4.9 and furthermore use this result to determine the dimension of $H^0\mathcal{E}(1)$. In this paragraph the following will be proven by a sequence of lemmata and propositions.

Proposition 4.24 *Let $\mathcal{E}$ be an instanton bundle with second Chern class $c_2(\mathcal{E}) = 7$ (i.e.: $m = 3$) which is of NC-type with monad*

$$0 \quad \to \quad 3\mathcal{O}(-1) \quad \stackrel{S}{\to} \quad 4\mathcal{E}_a \quad \stackrel{S^\vee \circ J}{\to} \quad 3\mathcal{O}(1) \quad \to \quad 0$$

where S is the matrix :

$$S := \begin{pmatrix} s_1, s_2, s_3, 0 \\ * \end{pmatrix}$$

and s_1, s_2, s_3 are elements of the basis chosen once an for all in 4.7 in $H^0\mathcal{E}_a(1)$, and u is an element in $ker(s)$ represented, see 4.9 by the matrix B as in lemma 4.25 Then

- *if $b_{4,4} = 0$ then $h^0\mathcal{E}(1) \geq 0$*
- *if $b_{4,4} \neq 0$ then $h^0\mathcal{E}(1) \geq 1$*

Lemma 4.25 *If $s = (s_1, \cdots, s_m, 0)$, then for all elements $u \in ker(s)$ u is represented, see 4.9, by the matrix:*

$$(s_1, \cdots, s_5) \circ \left(\begin{array}{ccc|c} & & & b_{1,m+1} \\ & B_m & & \vdots \\ & & & b_{4,m+1} \\ \hline 0 & \cdots & 0 & b_{m+1,m+1} \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & b_{5,m+1} \end{array}\right) = (u_1, \cdots, u_{m+1}) = u$$

Proof:
Case 1: $j, k < m + 1$

$$0 = \sum_{\substack{i=1 \\ j<k}}^{m+1} (a_{ji} b_{ki} - a_{ki} b_{ji}) = b_{kj} - b_{jk}$$

Hence $b_{kj} = b_{jk} \ \forall \ k, j \in \{1, \cdots, m\}$
Therefore the submatrix B_m of B is symmetric:

$$B_m = B_m^\vee$$

Remaining case:
Without any loss of generality choose j and k as follows:
$j < m + 1$ and $k \geq m + 1$

$$0 = \sum_{\substack{i=1 \\ j<m+1}} a_{ji} b_{m+1,i} - a_{m+1,i} b_{ji} = a_{jj} b_{m+1,j} = b_{j,m+1}$$

Hence $b_{kj} = 0 \; \forall \; j < m+1$ and $k \geq m+1$ because $a_{kk} = 0 \; \forall \; k \geq m+1$ and $a_{jj} = 1 \; \forall \; j < m+1$ by definition. #

Next is a simple but important corollary to the proof above.

Corollary 4.26 *If there exists $k \geq m+1$ such that $b_{m+1,k} \neq 0$ then there exist a row of the matrix S with $m+1$ independent vectors and I am back to the first case , see 4.10*

Remark 4.27 Hence from now on we can assume that $S \in Mat(m, m+1; H_m)$, where H_m is generated by $s_1, \cdots, s_m$.

Second induction step:

Lemma 4.28 *Let $u, v \in \ker(s)$ then $v \in \ker(u)$ and $u \in \ker(v)$ iff the following condition is fulfilled:*
If $j < k < m+1$, then $\sum_{i=1}^{m+1}(a_{ji}b_{ki} - a_{ki}b_{ji}) = 0$. These are $\binom{m}{2}$ equations.

Proof:
The proof is an easy calculation in terms of the matrix as in the other proofs before. #

Proof: of the proposition 4.24
Here I discuss results in case $m+1 = 4$. I have constructed so far in lemma 4.28 two families of instanton bundles having a different dimensions of $H^0\mathcal{E}(1)$:

- For the first family $Y^0_{a\,1}$ which has the property that the matrix S has entries in H_4, corresponding to case 1, I remark first that the vector space of all linear relations of s, $ker(s)$, has generically dimension 10 in this case. To verify this I just count the entries in the matrix 4.25. Lemma 4.28 shows that for each generic $u \in ker(s)$ the kernel $ker(u)$ is generically 4 dimensional and $ker(u)$ is isotropic with respect to J by lemma 4.13. Thus

  $$h^0\mathcal{K}(1) \geq 10 - 6 = 4$$

  and

  $$h^0\mathcal{E}(1) \geq 4 - 3 = 1$$

  In fact the inequalities are equalities, because $h^0\mathcal{E}(1) = 2$ if and only if the first row of S contains only two independent elements of $H^0\mathcal{E}_a(1)$, see theorem 5.16.

- For the other family $Y^0_{a\,2}$ corresponding to case 2 where the matrix S has entries in H_3 the above consideration shows that $ker(s)$ has dimension 9. Part 2 of 4.28

explains why there are only $\binom{3}{2}$ linear equations left for the next induction steps, see lemma 4.28(2). Thus

$$h^0\mathcal{K}(1) \geq 9 - 3 - 3 = 3$$

and

$$h^0\mathcal{E}(1) \geq 3 - 3 = 0 \ .$$

#

Remark 4.29 In fact $h^0\mathcal{E}(1) = 2$ is impossible for bundles, because $h^0\mathcal{E}(1) = 2$ if and only if the first row of S contains only two independent elements of $H^0\mathcal{E}_a(1)$, see theorem 5.16. Therefore $H^0\mathcal{E}(1) = 0, 1$.

4.5.3 Case 3: Instanton bundles with two independent linear sections

This result was suggested by the **example 6**, 4.4.3.

Let now H_2 be the vector space spanned by s_1, s_2. In contrast to the two cases before these considerations are valid for **any odd second Chern numbers** hence arbitrary m and also for non symmetric monads.

Proposition 4.30 *Let $\mathcal{E}$ be an instanton bundle of NC-type with $c_2(\mathcal{E}) = 2m + 1$ having a monad*

$$0 \quad \rightarrow \quad m\mathcal{O}(-1) \quad \stackrel{T}{\rightarrow} \quad (m+1)\mathcal{E}_a \quad \stackrel{S^\vee \circ J}{\rightarrow} \quad m\mathcal{O}(1) \quad \rightarrow \quad 0$$

where $S \in Mat(m, m+1; H_2)$ and $H_2 \subset H^0\mathcal{E}_a(1)$ as above, then

$$h^0\mathcal{E}(1) = 2$$

Lemma 4.31

$$\begin{aligned} h^0\mathcal{K}_m(1) &\geq 2(m+1) - m = m + 2 \\ h^0\mathcal{E}(1) &= h^0\mathcal{K}_m(1) - m \geq 2 \end{aligned}$$

Proof: We know from 4.8 that $H^0\mathcal{K}(1)$, which is the space of all linear relations of $S^\vee \circ J$ is a linear subspace in $k^{m+1} \otimes H_2 \simeq k^{2(m+1)}$. The pairing "degenerates" to a simple bilinear form because H_2 is two dimensional and hence $\Lambda^2 H_2 \simeq k$. Thus we are now in the situation of linear algebra over a field k. The matrix $S^\vee \circ J$ is a m by $m+1$ matrix hence

$$h^0\mathcal{K}_m(1) \geq 2(m+1) - m = m + 2$$

Let

$$0 \rightarrow m\mathcal{O}(-1) \rightarrow (m+1)\mathcal{E}_a \xrightarrow{S} m\mathcal{O}(1) \rightarrow 0$$

be the monad for an instanton bundle of NC-type. The second display sequence for the monad is:

$$0 \rightarrow m\mathcal{O}(-1) \rightarrow \mathcal{K} \rightarrow \mathcal{E} \rightarrow 0$$

#

Proof of the proposition 4.30
$\mathcal{E}$ is a vector bundle, thus $h^0\mathcal{E}(1) \leq 2$ by [8]2 so the propositions follows immediately.
#

Remark 4.32 In theorem 5.16 the converse of proposition 4.30 will be proven.

4.5.4 Summary

Now I summarize the results of the previous 3 sections: Let $\mathcal{E}$ be an instanton bundle of NC-type with second Chern class $c_2(\mathcal{E}) = 2m+1$, then

The max. number S of independent: vectors in a row	$c_2 = 5$ $m+1=3$	$c_2 = 7$ $m+1=4$	$c_2 = 9$ $m+1=5$
$H^0\mathcal{E}_a(1)$	-	-	$h^0\mathcal{E}(1) = 1$
4	-	$h^0\mathcal{E}(1) = 1$	?
3	$h^0\mathcal{E}(1) = 1$	$h^0\mathcal{E}(1) \geq 0$	?
2	$h^0\mathcal{E}(1) = 2$	$h^0\mathcal{E}(1) = 2$	$h^0\mathcal{E}(1) = 2$

For $m+1 > 5$ it's only known, that if the matrix S has entries in a twodimensional subspace then $h^0\mathcal{E}(1) = 2$.

Remark 4.33 If I consider not only bundles constructed by monads with one null correlation bundle $\mathcal{E}_a$ but with the middle term consisting of a direct sum of different null correlation bundles

$$0 \rightarrow m\mathcal{O}(-1) \rightarrow \bigoplus_{i=1}^{m+1} \mathcal{E}_{a_i} \rightarrow m\mathcal{O}(1) \rightarrow 0$$

then there exist examples of instanton bundles which have no linear section $h^0\mathcal{E}(1) = 0$. Those examples were constructed using a random generator.

4.6 Moduli problems for NC-type bundles and a question of irreducibility

This is a remark on the Grassmannian of isotropic vector spaces and the moduli space of instanton bundles of NC-type. We have seen from the monad

$$0 \quad \to \quad m\mathcal{O}(1) \quad \xrightarrow{S} \quad (m+1)\mathcal{E}_a \quad \xrightarrow{S^\vee \circ J} \quad m\mathcal{O}(1) \quad \to \quad 0$$

which shows explicitly the duality pairing between $(m+1)\mathcal{E}$ and $(m+1)\mathcal{E}^\vee$, that an instanton of NC-type is defined by a matrix S or $S^\vee$ which define an m- dimensional isotropic subspaces in $k^{m+1} \otimes H^0\mathcal{E}_a(1)$ which is isotropic under the pairing

$$\begin{array}{llll} \bar{A}: & \bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1) \otimes \bigoplus_{i=1}^{m+1} H^0\mathcal{E}_a(1) & \to & H^0\mathcal{O}(2) \\ & \sum_{i=1}^{m+1} t_i \otimes \sum_{i=1}^{m+1} u_i & \mapsto & \sum_{i=1}^{m+1} u_i^\vee \circ J \circ t_i \end{array}$$

introduced on page 58. Unfortunately this pairing has its values not in a field k but in the vector space $H^0\mathcal{O}(2) \simeq S^2V^\vee$, which is 10-dimensional. Hence the subscheme $G(m, k^{m+1} \otimes H^0\mathcal{E}_a(1))^J$ of subvector spaces isotropic according to the above pairing in the Grassmannian $G(m, k^{m+1} \otimes H^0\mathcal{E}_a(1))$ is a much more complicated object than in the case of a normal pairing taking its values in a field. I denote now by $Y_a^0 \subset G(m, k^{m+1} \otimes H^0\mathcal{E}_a(1))^J$ the open subset of those isotropic subspaces which determine a subbundle as their image. This means that $Image(S) \cap \alpha \otimes t = 0$ for any $t \in H^0\mathcal{E}_a(1)$, see [8] p.326,ii. For a fixed null correlation bundle one has the following sequence of inclusions:

$$Y_a^0 \subset G(m, k^{m+1} \otimes H^0\mathcal{E}_a(1))^J \subset G(m, k^{m+1} \otimes H^0\mathcal{E}_a(1))$$

In the next step I define now the global object which describes the family of such subspaces where the null correlation bundle $\mathcal{E}_a$ varies. Let now $MI(1)$ be the moduli space of null correlation bundles on $\mathbb{P}_3$. This is a fine moduli space of stable bundles. Hence there exists the universal bundle $\mathbb{E}$ on $MI(1) \times \mathbb{P}_3$. Let π be the projection

$$\pi: \quad MI(1) \times \mathbb{P}_3 \quad \to \quad MI(1)$$

and let $\pi_*\mathbb{E}$ be the direct image of $\mathbb{E}$ with respect to π. $\pi_*\mathbb{E}$ restricted to the fibre $\pi^{-1}([\mathcal{E}_a])$ is $H^0\mathcal{E}_a(1)$, because base change is valid. One can study now inside the relative Grassmannian

$$\tilde{\pi}: \quad G(m, \pi_*(k^{m+1} \otimes \mathbb{E}(1)) \quad \to \quad MI(1)$$

the subfamily $\tilde{Y^0}$ which is has the fibres $Y_a^0 = \tilde{\pi}^{-1}([\mathcal{E}_a])$. Thus I obtain the diagram:

$$\begin{array}{ccc} \tilde{Y^0} & \longrightarrow & G(m, \pi_*(k^{m+1} \otimes \mathbb{E}(1)) \\ \downarrow & & \downarrow \tilde{\pi} \\ U \subset MI(2m+1) & & MI(1) \end{array}$$

The vertical map at the left hand side of the diagram, is the map which maps any monad determined by an $S \in \tilde{Y^0}$ to its cohomology bundle. Not all monads parameterized by

$\tilde{Y^0}$ determine non isomorphic instanton bundles. Isomorphic selfdual monads define the same cohomology bundle. The base change group $GL(m)$ on $m\mathcal{O}(-1)$ is already divided out, because I parameterize with $\tilde{Y^0}$ the subspaces in $k^{m+1} \otimes H^0\mathcal{E}_a(1)$ generated by the matrices S. It remains to quotient out the action of those $A \in Gl(m+1)$ which respect the pairing J

$$S^\vee \circ A \circ J \circ A^\vee \circ S = S^\vee \circ J \circ S$$

where

$$0 \quad \rightarrow \quad m\mathcal{O}(1) \quad \stackrel{S}{\rightarrow} \quad (m+1)\mathcal{E}_a \quad \stackrel{S^\vee \circ J}{\rightarrow} \quad m\mathcal{O}(1) \quad \rightarrow \quad 0$$

Thus A must commute with J. If j is the symplectic pairing, see page 5

$$j: \quad \mathcal{E}_a \otimes \mathcal{E}_a \quad \rightarrow \quad \mathcal{O}(2)$$

then J is in a suitable basis the matrix $\begin{pmatrix} j & & & \\ & j & 0 & \\ & 0 & \ddots & \\ & & & j \end{pmatrix}$ The matrix A does not interfere with j, which operates only on $H_k \subset H^0\mathcal{E}_a(1)$ therefore A is an element in the orthotogonal group $O(m+1)$ preserving J. This group has dimension $\frac{m(m+1)}{2}$.

I can compute now for special 't Hooft bundles of NC-type

$$0 \quad \rightarrow \quad m\mathcal{O}(1) \quad \stackrel{S}{\rightarrow} \quad (m+1)\mathcal{E}_a \quad \rightarrow \quad m\mathcal{O}(1) \quad \rightarrow \quad 0$$

the dimension of the stratum. It is sufficient to determine the dimension by computing the dimension of the space of all selfdual monads modulo the $O(m+1)$ action. The same method gives only an inequality for more general bundles of NC-type.

(1) The dimension of the moduli space of all null correlation bundles $MI(1)$ is 5.

(2) The subspaces $H_k \subset H^0\mathcal{E}_a(1)$ of dimension k, in which the matrix has its entries, are parameterized by $G(k,5)$ which is $(5-k)k$ dimensional.

(3) The Grassmannian of all m dimensional subspaces in $k^{m+1} \otimes H_k$ $G(m, k^{m+1} \otimes H_k)$ is $m((m+1)k - m)$ dimensional.

(4) The dimension of the Grassmannian of all isotropic subspace with respect to the pairing as on page 93, $G^s(m, k^{m+1} \otimes H_k)$, is a more delicate number, because the pairing takes values in a vector space not only in a field. Thus one can not expect this space to be irreducible. Nevertheless, if the matrix S has its entries in a 2 dimensional subspace $H_2 \subset H^0\mathcal{E}_a(1)$, then $\Lambda^2 H_2 \simeq k$. Hence the dimension of $G^s(m, k^{m+1} \otimes H_2)$ is $\dim G^s(m, k^{m+1} \otimes H_2) - \frac{m(m-1)}{2}$

(5) In a last step I must subtract the dimension of the orthotogonal group $O(m+1)$ which is $\frac{m(m+1)}{2}$

I conclude now:

Lemma 4.34 *The dimension of the stratum of all NC-type bundles with monad*

$$0 \rightarrow m\mathcal{O}(1) \stackrel{S}{\rightarrow} (m+1)\mathcal{E}_a \stackrel{S^\vee \circ J}{\rightarrow} m\mathcal{O}(1) \rightarrow 0$$

such that the matrix S has its entries in a 2 dimensional subspace $H_2 \subset H^0\mathcal{E}_a(1)$ has dimension $2m+11$.

#

Remark 4.35 This number will be computed in a different way on page 119.

Remark 4.36 Before I start with the question of irreducibility, I want to recall the situation for instantons of type with $c_2(\mathcal{E}) = 7$ with monad, see 4.5.2:

$$0 \rightarrow 3\mathcal{O}(-1) \stackrel{S}{\rightarrow} 4\mathcal{E}_a \stackrel{s^\vee \circ J}{\rightarrow} 3\mathcal{O}(1) \rightarrow 0$$

(1) Iff $S \in Mat(3,4;H_4)$ then $h^0\mathcal{E}(1) = 1$. $(Y^0_{a\,1})$

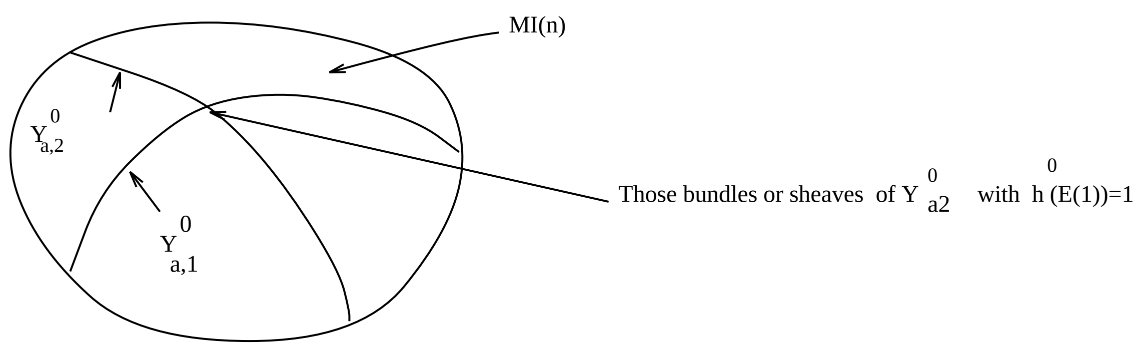
(2) Iff $S \in Mat(3,4;H_3)$ then $h^0\mathcal{E}(1) \geq 0$. $(Y^0_{a\,2})$

(3) Iff $S \in Mat(3,4;H_2)$ then $h^0\mathcal{E}(1) = 2$

where H_4, H_3, H_2 denote subvector spaces of $H^0\mathcal{E}_a(1)$ of dimension $4, 3, 2$.

Proposition 4.37 $\tilde{Y}^0$ *is not irreducible in general.*

I claim that in the case of $m = 3$ all fibres Y^0_a are reducible. Y^0_a contains two components $Y^0_{a\,1}$ and $Y^0_{a\,2}$ of dimensions ≥ 24 and ≥ 23 respectively, see 4.36, which can only intersect in those bundles or sheaves in the closure, $\overline{Y^0_{a\,2}}$, of $Y^0_{a\,2}$ which have $H^0\mathcal{E}(1) = 1$. Hence for all $[\mathcal{E}_a]$ the fibres Y^0_a can not be irreducible. This follows from the fact that a monad of type $Y^0_{a\,1}$ deforms into monad of type $Y^0_{a\,2}$, if one of the independent vectors $\{s_1, \cdots, s_4\}$ chosen in the beginning, see 4.7 specializes to zero. #



5 Special 't Hooft bundles of NC-type

5.1 Introduction

For special 't Hooft bundles of NC-type the situation is nice and allows a satisfactory geometrical description and interpretation. I develop a theory of special 't Hooft bundles which have a monad of NC-type

$$0 \;\to\; m\mathcal{O}(-1) \;\stackrel{A}{\to}\; (m+1)\mathcal{E}_a \;\stackrel{A'}{\to}\; m\mathcal{O}(1) \;\to\; 0$$

analogously to the theory of special 'tHooft bundles in general which is presented in [8]. Before I start with the new results of this chapter I am going to recall the "classical" results on special 't Hooft bundles which were published by G. Trautmann and W. Böhmer in [8]. From their point of view an instanton is the cohomology bundle of a Beilinson-II-monad

$$0 \;\to\; n\Omega^3(3) \;\stackrel{M}{\to}\; n\Omega^1(1) \;\stackrel{B}{\to}\; (2n-2)\mathcal{O} \;\to\; 0\;,$$

where $c_2(\mathcal{E}) =: n$ is the second Chern class of $\mathcal{E}$. In [8] they constructed a normal form for the Beilinson-II-monad of special 't Hooft bundles. In this normal form the matrix B is fixed, see page 127 for details. The matrix $M \in Mat(n, n, \Lambda^2 V)$ is symmetric and a product of a persymmetric matrix P and another fixed matrix R, see page 127. The columns of the matrix R define a basis of $H^0\mathcal{K}(1)$, the space of linear sections of the kernel bundle defined by the second display sequence of the monad

$$0 \;\to\; \mathcal{K} \;\to\; (2m+1)\Omega^1(1) \;\to\; \mathcal{O} \;\to\; 0\;.$$

Hence, if a monad is given in normal form with fixed matrices R and B, then $\mathcal{E}$ is defined by the persymmetric matrix P. Moreover they showed that a special 't Hooft bundle is completely determined by its pencil of linear sections $\mathbb{P}H^0\mathcal{E}(1)$. Actually two independent linear sections of a special 't Hooft bundle $\mathcal{E}$ define a smooth dependency quadric $Q \subset \mathbb{P}_3$, see [8]2.1. This quadric is attached with a ruling which contains the zero loci of all linear section $s \in H^0\mathcal{E}(1)$, see [8]2.1 proof 1. A quadric together with a ruling on it defines a conic in the Grassmannian $G(2,4)$ which was explained in the chapter "Complements on null correlation bundles", see page 55. In this way $\mathbb{P}H^0\mathcal{E}(1)$ determines a pencil $g^1_{2m+2}(\mathcal{E})$ of degree $2m+2$ on C, see page 55.

They give also a nice geometrical interpretation of special 't Hooft bundles in terms of classical geometry. They proved that there exists a curve S in the plane spanned by the conic C such that the tangents to the conic C of every pair of points of a divisor $D \in g^1_{2m+2}(\mathcal{E})$ intersect on S. This is called a Poncelet situation. This interpretation gives rise to their classification theorem, c.f. [8]4.7:
There is a bijection between the moduli space of special 't Hooft bundles $\mathcal{E}$ on $\mathbb{P}_3$ with $c_2(\mathcal{E}) = n$ and the set of pairs (P, L) where $P \subset \mathbb{P}\Lambda^2 V$ is a plane such that $P \cap G(2,4) = C$ is a regular conic and $L \subset PH^0\mathcal{O}_C(n+1)$ is a pencil without base points.

Albeit it is still an open problem to characterize instanton bundles of NC-type among all instantons, it is already possible to determine those special 't Hooft bundles which are of NC-type, i.e. which are the cohomology of a monad:

$$0 \rightarrow m\mathcal{O}(-1) \stackrel{S}{\rightarrow} (m+1)\mathcal{E}_a \stackrel{T^\vee \circ J}{\rightarrow} m\mathcal{O}(1) \rightarrow 0$$

and to describe their moduli.

In a first step special 't Hooft bundles of NC-type are distinguished among all instantons of NC-type by the matrices of their monads. It turns out that they are precisely those NC-type monads for which a two dimensional subspace H_2 in $H^0\mathcal{E}_a(1)$ exists such that all entries of the matrix S lie in H_2.

As for special 't Hooft bundles in general, there is a classification in terms of Poncelet situations. Special 't Hooft bundles of NC-type can be described geometrically among all special 't Hooft bundles by a property of their Poncelet curves S. In fact this curve splits into a line and a curve of degree one less. A further degeneration is possible as the example 5.12 will show. This gives rise to the following classification: Let $\mathbb{P}_3 = \mathbb{P}(V)$ and let U in $\mathbb{P}\Lambda^2 V \setminus G(2,4) \times G(2,5) \times G(2, 2m+1)$ be the open set of all triples $(\langle a \rangle, P, L)$ such that $\langle a \rangle \in \mathbb{P}\Lambda^2 V \setminus G(2,4)$, and P is a plane in $\mathbb{P}\Lambda^2 V$ containing $\langle a \rangle$ and $L \subset |H^0\mathcal{O}_C(\frac{2m+2}{2})|$ is a basepoint free pencil where C is the conic $C := P \cap G(2,4)$. Then there exists a finite map, generically 1 to 1, from U to the moduli space of all instantons of NC-type with $h^0\mathcal{E}(1) = 2$ and $c_2(\mathcal{E}) = 2m+1$. i.e.:

$$\begin{array}{ccc} U & \rightarrow & MI(2m+1) \\ (\langle a \rangle, P, L) & \mapsto & [\mathcal{E}] \end{array}$$

We see in contrast to [8] that the classification is no longer bijective but a finite map, generically 1 : 1. The phenomenon is caused by Poncelet curves which split into more than one line.

Last but not least the normal form problem is treated for the matrices in the NC-type-monads of special 't Hooft bundles of NC-type. This result is compared with the normal form found in [8].

5.2 The monad of special 't Hooft bundles of NC-type and sections of the kernel bundle

We have already seen in lemma 3.4 that the zero locus $V(s)$ of $s \in H^0\mathcal{E}_a(1) \simeq \Lambda^2 V / \langle a \rangle$ is the union of the two lines in $\mathbb{P}_3$ defined by the points of intersection ℓ_1 and ℓ_2 in $\overline{\langle a \rangle \langle s \rangle} \cap G(2,4) \subset \mathbb{P}\Lambda^2 V$.

I want to describe now the zero locus of a linear section of a special 't Hooft bundle of NC-type.

Notation 5.1 Let $\mathcal{E}$ be an instanton bundle such that $h^0\mathcal{E}(1) = 2$. Two independent sections $s, s' \in H^0\mathcal{E}(1) \simeq k^2$ define a smooth dependency quadric $Q \subset \mathbb{P}_3$, see [8]

1.4. The zero locus $V(s)$ for arbitrary $s \in H^0\mathcal{E}(1)$ consists of possibly non reduced "lines" supported on one of the rulings of Q. The quadric Q together with the ruling supporting s defines a conic $C(\mathcal{E}) \subset \mathbb{P}\bigwedge^2 V$ as mentioned on page 55. $P(\mathcal{E})$ is the plane spanned by $C(\mathcal{E})$ such that $P(\mathcal{E}) \cap G(2,4) = C(\mathcal{E})$. $g^1_{2m+2}(\mathcal{E})$ is the pencil on $C(\mathcal{E})$ induced by $H^0\mathcal{E}(1)$.

I describe now in more detail the zero locus of a linear section of a special 't Hooft bundle of NC-type.

Lemma 5.2 *Let $\mathcal{E}$ be an instanton bundle with $c_2(\mathcal{E}) = 2m+1$ of NC-type with monad*

$$0 \rightarrow m\mathcal{O}(-1) \rightarrow (m+1)\mathcal{E}_a \rightarrow m\mathcal{O}(1) \rightarrow 0$$

Let $\mathcal{K}$ be the kernel bundle defined by

$$0 \rightarrow \mathcal{K} \rightarrow (m+1)\mathcal{E}_a \rightarrow m\mathcal{O}(1) \rightarrow 0$$

and $\bar{s} \in H^0\mathcal{K}(1)$ a section such that $V(\bar{s}) \neq \emptyset$. Then there exists a section $\sigma \in H^0\mathcal{E}_a(1)$ and a vector $\alpha := (a_1, \cdots, a_{m+1}) \in k^{m+1}$ such that

$$\bar{s} = (a_1, \cdots, a_{m+1}) \otimes \sigma \text{ and } V(\bar{s}) = V(\sigma) = \ell \cup \ell' \ .$$

Proof: A general $\bar{s} \in H^0\mathcal{K}(1)$ is a vector $(\sigma_1, \ldots, \sigma_{m+1})$ with $\sigma_\nu \in H^0\mathcal{E}_a(1)$. I assume now that two σ_μ and σ_ν with $1 \leq \mu < \nu \leq m+1$ are independent vectors in $H^0\mathcal{E}_a(1)$. Hence, σ_μ and σ_ν intersect the conic $C(\mathcal{E})$ in different points. The 4 lines on which σ_μ and σ_ν vanish are disjoint because all 4 lines are contained in one ruling of the dependency quadric $Q(\mathcal{E})$ associated to $C(\mathcal{E})$. But this contradicts the assumption that $\emptyset \neq V(\bar{s}) = \bigcap_{1 \leq j \leq m+1} V(\sigma_j)$. Therefore all σ_j must be dependent. i.e. It $\exists$ a $\sigma \in H^0\mathcal{E}_a(1)$ and $\alpha_j \in k$ such that $\sigma_j = \alpha_j \sigma$. #

The next lemma explains how sections in $H^0\mathcal{E}(1)$ and $H^0\mathcal{K}(1)$ are related to each other.

Lemma 5.3 *Let $\mathcal{E}$ be an instanton bundle, $\mathcal{E} \in MI(2m+1)$ having at least one linear section, i.e. $h^0\mathcal{E}(1) \geq 1$, of NC-type with monad*

$$0 \rightarrow m\mathcal{O}(-1) \rightarrow (m+1)\mathcal{E}_a \rightarrow m\mathcal{O}(1) \rightarrow 0$$

and $\mathcal{K}$ be the kernel bundle defined by:

$$0 \rightarrow \mathcal{K} \rightarrow (m+1)\mathcal{E}_a \rightarrow m\mathcal{O}(1) \rightarrow 0$$

If $s \in H^0\mathcal{E}(1)$ and $\bar{s} \in H^0\mathcal{K}(1)$ such that $\bar{s} \rightarrow s$ by

$$H^0\mathcal{K}(1) \rightarrow H^0\mathcal{E}(1) \rightarrow 0 \ .$$

Then following is true:

(1) $supp(V(s)) = \bigcup_{\bar{s}\to s} supp(V(\bar{s}))$

(2) $\langle a\rangle \in P(\mathcal{E})$ *where* $P(\mathcal{E})$ *is the plane spanned by* $C(\mathcal{E})$*.*

Proof: Let X be an irreducible component of $V(s)$ and

$$0 \;\to\; \mathcal{I}_X \;\to\; \mathcal{O} \;\to\; \mathcal{O}_X \;\to\; 0$$

the associated sequence. One obtains from the second display sequence the short exact sequence:

$$0 \;\to\; mH^0\mathcal{O} \;\to\; H^0\mathcal{K}(1) \;\to\; H^0\mathcal{E}(1) \;\to\; 0\ .$$

I look now for those sections vanishing on X. If one tensors now the sequence of bundles

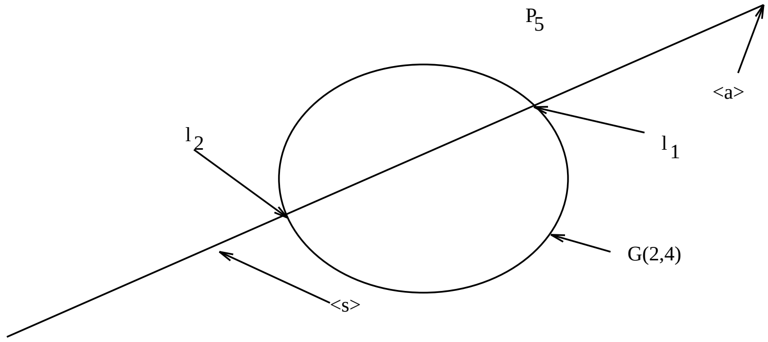
$$0 \;\to\; m\mathcal{O} \;\to\; \mathcal{K}(1) \;\to\; \mathcal{E}(1) \;\to\; 0$$

with the sequence of $\mathcal{O}_X$ one obtains the diagram below:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & mH^0\mathcal{I}_X & \longrightarrow & H^0\mathcal{K}(1)\otimes\mathcal{I}_X & \longrightarrow & H^0\mathcal{E}(1)\otimes\mathcal{I}_X & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & mH^0\mathcal{O} & \longrightarrow & H^0\mathcal{K}(1) & \longrightarrow & H^0\mathcal{E}(1) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & mH^0_X\mathcal{O} & \longrightarrow & H^0\mathcal{K}|_X(1) & \longrightarrow & H^0\mathcal{E}|_X(1) &
\end{array}$$

It is proved now with the help of this diagram that for every $s \in H^0\mathcal{E}(1)$ every irreducible component X of $V(s)$ is an irreducible component of $V(\bar{s})$ for an appropriate $\bar{s} \in H^0\mathcal{K}(1)$.
One can find by simple diagram chasing for every $s \in H^0\mathcal{E}(1)$ vanishing along X (i.e. $s \in H^0\mathcal{E}(1)\otimes\mathcal{I}_X$ a $t' \in H^0\mathcal{K}(1)\otimes\mathcal{I}_X$. Hence $supp(V(s)) \subset \bigcup_{\bar{s}\to s} supp(V(\bar{s}))$. Hence "$\subset$" is valid. For the other inclusion "$\supset$" it suffices to show: Let $V(\bar{s}) = \ell \cup \ell'$. If $\ell \subset V(s)$ then $\ell' \subset V(s)$. This follows easily from the same kind of diagram as above but for the line ℓ' instead of the irreducible component $X \subset V(s)$. This proves (1) of the lemma.

It remains to show part (2) of the lemma. Let s be a linear section of the bundle $\mathcal{E}$. In the first part of the lemma it was shown that there exists a section $\bar{s} \in H^0\mathcal{K}(1)$ such that $\bar{s} \to s$ and $V(\bar{s}) = \ell \cup \ell' \subset V(s)$. Thus it must be of form $\bar{s} = \alpha \otimes \sigma$ where $\sigma \in H^0\mathcal{E}_a(1)$ and $\alpha \in k^{m+1}$, see lemma 5.3. The bundle $\mathcal{E}$ is special 't Hooft, therefore both lines $\ell, \ell' \subset V(s)$ correspond to points on a conic $C(\mathcal{E}) \subset \mathbb{P}\Lambda^2 V$ and $P(\mathcal{E}) \subset \mathbb{P}\Lambda^2 V$ is the plane spanned by $C(\mathcal{E})$, [8]. The section σ is a line containing the point $\langle a\rangle$ and intersecting $C(\mathcal{E})$ in two points or a double point contained in $P(\mathcal{E})$, see 3.4. Hence, $\langle a\rangle \in P(\mathcal{E})$. #



Now I am prepared to prove the main result of this paragraph. Let $\mathcal{E}$ again be an instanton of NC-type with monad

$$0 \rightarrow m\mathcal{O}(-1) \xrightarrow{S} (m+1)\mathcal{E}_a \xrightarrow{T^\vee \circ J} m\mathcal{O}(1) \rightarrow 0$$

It is already proved in 4.30 that if the matrix S has entries in a two-dimensional subspace $H_2 \subset H^0\mathcal{E}_a(1)$ only, then the instanton $\mathcal{E}$ is a special 't Hooft bundle. The next proposition shows that this is in fact an equivalence

Proposition 5.4 *Let $\mathcal{E}$ be an instanton bundle of NC-type being the cohomology of the monad*

$$0 \rightarrow m\mathcal{O}(-1) \xrightarrow{S} (m+1)\mathcal{E}_a \xrightarrow{T^\vee \circ J} m\mathcal{O}(1) \rightarrow 0$$

then the following is equivalent:

(1) There exists a two-dimensional subspace $H_2 \subset H^0\mathcal{E}_a(1)$, such that all entries of S are in H_2.

(2) $\mathcal{E}$ is a special 't Hooft bundle. (i.e. $h^O\mathcal{E}(1) = 2$)

Proof:

1 $\Longrightarrow$ 2 This is the result of proposition 4.30.

2 $\Longrightarrow$ 1 One shows first that there exists a two dimensional subspace $H_2 \subset H^0\mathcal{E}_a(1)$ such that $H^0\mathcal{K}(1)$ has a basis of elements $(\alpha_1, \cdots, \alpha_{m+1}) \otimes s$ where $s \in H_2$ and $\alpha_i \in k$. I proceed now lifting the first display sequence of the NC-type monad to the corresponding sequence

$$0 \rightarrow \mathcal{K}'(1) \rightarrow (2m+1)\Omega^1(1) \rightarrow 4m\mathcal{O} \rightarrow 0$$

of the Beilinson-II-monad for the instanton $\mathcal{E}$.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (m+1)\mathcal{O}(-1) & \to & (m+1)\mathcal{O}(-1) \oplus m\Omega^1(1) & \to & m\Omega^1(1) & \to & 0 \\
 & & \downarrow & & \downarrow \begin{pmatrix} aI_{m+1} & 0 \\ 0 & I_m \end{pmatrix} & & \downarrow & & \\
0 & \to & \mathcal{K}' & \stackrel{R}{\to} & (m+1)\Omega^1(1) \oplus m\Omega^1(1) & \stackrel{B'}{\to} & 4m\mathcal{O} & \to & 0 \\
 & & \downarrow & & \searrow \quad \swarrow & & \downarrow & & \\
0 & \to & \mathcal{K} & \to & (m+1)\mathcal{E}_a & \stackrel{T^\vee \circ J}{\to} & m\mathcal{O} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The morphism

$$(m+1)\mathcal{E}_a \stackrel{T^\vee \circ J}{\to} m\mathcal{O}(1)$$

can be lifted to the morphism

$$(2m+1)\Omega^1(1) \stackrel{B'}{\to} 4m\mathcal{O}$$

because the obstruction groups vanish. Following proposition 2.1 in [8] there exits a base change such that the matrix B' is of normal form in the meaning of 5.6. Therefore $H^0\mathcal{K}(1)$ has a basis generated by the matrix R, see chapter 5.6 page 127. The entries of R, ξ, η, μ, span the plane $P(\mathcal{E})$, see [8] 3.1. In the previous lemma 5.2 it was verified that $\langle a \rangle \in P(\mathcal{E})$. Therefore the images $\bar{\xi}, \bar{\eta}$ and $\bar{\mu}$ of ξ, η and μ under the projection through $\langle a \rangle$, $P(\mathcal{E}) \to P(\mathcal{E})/\langle a \rangle$, span a two dimensional subspace of $H^0\mathcal{E}_a(1)$. The image of the basis of $H^0\mathcal{K}'(1)$, generated by R, is a basis of $H^0\mathcal{K}(1)$ because the sequence

$$0 \quad \to \quad (m+1)H^0\mathcal{O}(1) \quad \stackrel{A}{\to} \quad H^0\mathcal{K}'(1) \quad \to \quad H^0\mathcal{K}(1) \quad \to \quad 0$$

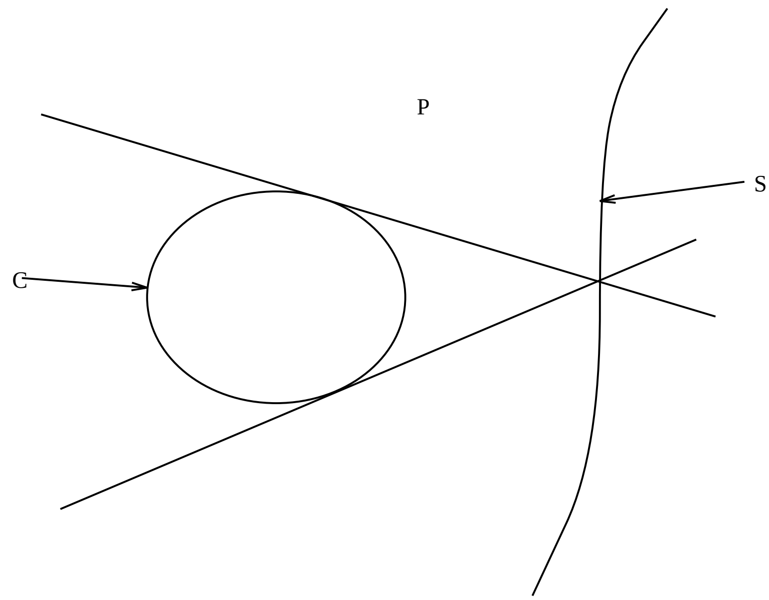
is exact. Hence S has entries in a two dimensional subspace.

#

5.3 Complements on Poncelet pairs and poles

Poncelet situations are studied for more than hundred years beginning with Poncelet (Traité des propriétés projectives des figures, 1822) and continuing with Darboux [10], 1917. Böhmer and G. Trautmann related the classical results to special 't Hooft bundles, see [8]. I use for the rest of this paper properties of Poncelet Pairs to classify special 't Hooft bundles of NC-type. Therefore I recall here the definition and basic features of Poncelet pairs.

Definition 5.5 *Let C be a conic in $\mathbb{P}_2$, $\mathbb{P}L \subset \mathbb{P}H^0\mathcal{O}_C(2m+2)$ a pencil of divisors on C of degree $2m+2$. A curve S of degree $2m-1$ in $\mathbb{P}_2$ is called Poncelet related to C with respect to $\mathbb{P}L$ if and only if for any two points of any of the divisors the tangents to C intersect on S. I shall also say (C, S) is a* **Poncelet pair** *with respect to L. The curve S is named the* **Poncelet curve**.

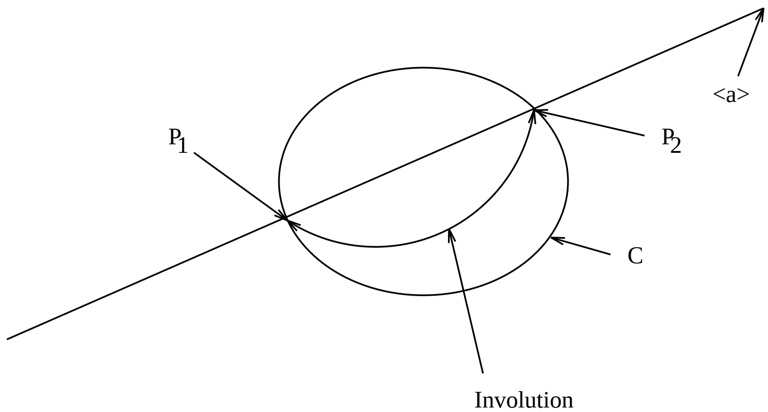


I want to study now those Poncelet pairs for which the curve S splits into a line ℓ and a curve S' (i.e.$S = S' \cup \ell$) Therefore it is necessary to introduce the notion of an involution and a pole here.

Definition 5.6 *Let C be a smooth conic on a projective plane $P := \mathbb{P}_2 \simeq \mathbb{P}(W)$ and $\langle a \rangle$ a point on P not contained in the conic C. This point $\langle a \rangle$ defines an* **involution** *$\iota_a : C \to C$ as follows. All lines in P through $\langle a \rangle$ form a line $\mathbb{P}_1^\vee \subset \mathbb{P}_2^\vee = \mathbb{P}(W^\vee)$ in the dual plane of all lines in P. Each line intersects the conic in multiplicity 2 (generically two points). This gives a covering*

$$\pi_a : \quad C \quad \overset{2:1}{\to} \quad \mathbb{P}_1$$

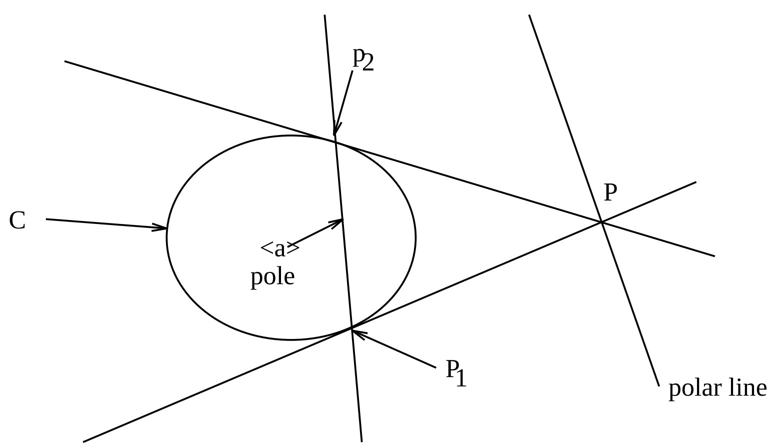
The involution $\iota_a \neq id$ is the covering transformation not equal to the identity. This is visualized in the picture below.



Definition 5.7 (pole) *Let L be a linear system on a conic $C \subset P$ where P is the projective plane. I say that the linear system L has pole if there exists a point $\langle a \rangle \notin C$ such that each divisor of L is invariant under the involution $\iota_a : C \to C$, defined as above, ($\iota_a(D)$= D for all $D \in L$).*

Remark 5.8 A conic C is a quadric, here smooth, in a plane . Hence this conic induces a quadratic form q of full rank on P such that $C = V(q)$. The line $\ell_a := \{x \in P | q(a, x) = 0\}$ is **the polar line** with respect to q and the pole $\langle a \rangle$.

Remark 5.9 I want to recall here a fact from classical geometry. The polar line is the locus where two tangents to C in opposite points P_1 and P_2 with respect to $\langle a \rangle$ intersect. This is a consequence following from the fact that all three points $\langle a \rangle$, P_1 and P_2 lie a the line ℓ. Thus in $\mathbb{P}(W^\vee)$ one has the dual incidence relations; i.e.: the lines ℓ_a, ℓ_{p_1} and ℓ_{p_2} associated with $\langle a \rangle$, P_1 and P_2 intersected in the point representing the line ℓ. This is also true for the polar lines in $\mathbb{P}(W)$. The picture below visualizes the situation.



I can prove now:

Lemma 5.10 *Let (C,S) be a Poncelet pair with respect to L and let L be a basepoint free pencil of* $\deg = 2m+2$ *with a pole $\langle a\rangle \notin C$. Then the Poncelet curve S splits into a line ℓ and a curve S' of degree $2m$ such that $S = S' \cup \ell$. Moreover ℓ is the polar line of the pole $\langle a\rangle$ with respect to the conic C. All tangents on opposite points with respect to the involution ι_a, see definition 5.7, intersect on ℓ and The pencil L defines a linear system L' on ℓ with $2deg(L') = deg(L)$.*

Conversely every such pair (ℓ, L') determines a basepoint free pencil L with a pole on C with $\deg(L) = 2\deg(L')$ *such that ℓ is a component of the Poncelet curve.*

Proof: The tangents on two opposite points on C with respect to the involution ι_a intersect on the polar line, see definition 5.9. This is clear by the remark 5.22. Obviously ℓ is a component of S because it intersects S in infinitely many points.

Conversely the polar line ℓ determines the pole $\langle a\rangle$. Hence, every point on the polar line determines a line through the pole $\langle a\rangle$. This line intersects the conic C in two opposite points. In the way the pair (ℓ, L') defines a pencil of $\deg(L) = 2\deg(L')$ on C. Again ℓ must be a component of the Poncelet curve S with respect to L. #

Remark 5.11 The "Satz" 6.4 in [21] shows that the existence of the pencil is necessary. The splitting $S = S' \cup \ell$ is not enough. He shows:
Let (C,S) be a Poncelet pair with respect to L and let L be a pencil of $\deg = 2m+2$ with a pole $\langle a\rangle \notin C$. If less than $m+1$ tangents intersect on ℓ then L is not basepoint free.

There is one more remark to be made on the uniqueness of poles for a linear system. First, I present a sufficient and necessary condition for the existence of linear system of deg 4 on a conic with 3 poles.

Proposition 5.12 *Let $L \subset |H^0\mathcal{O}_C(4)|$ be a pencil on C then the following are equivalent:*

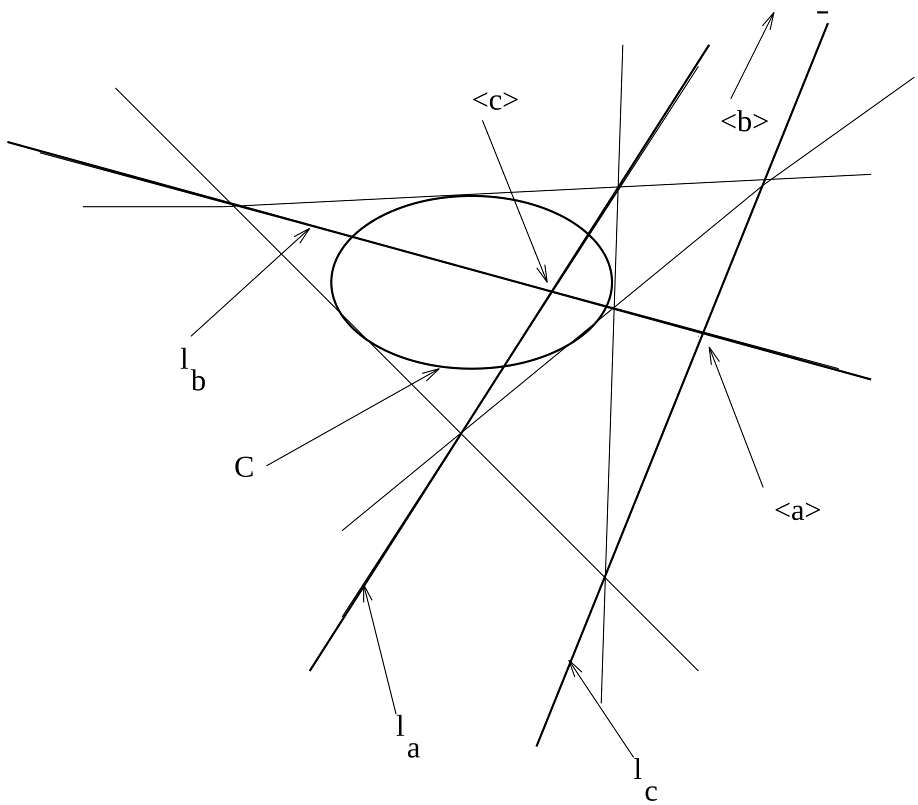
(1) L has 3 poles: $\langle a\rangle, \langle b\rangle$ and $\langle c\rangle$

(2) $C = \ell_a \cup \ell_b \cup \ell_c$ where ℓ_a, ℓ_b and ℓ_c are the polar lines attached with the poles a, b and c. The involutions ι_a, ι_b and ι_c form the Kleinian Group. (hence one point determines a whole divisor in L.

(3) There exist the following incidence relations:

$$\ell_a \cap \ell_b = c$$
$$\ell_b \cap \ell_b = a$$
$$\ell_a \cap \ell_c = b$$

For the degree=4 of the linear system is small, the situation can be visualized in a nice picture:



In fact those examples exist. The conic C is the image of $\mathbb{P}_1$ under the Veronese embedding:

$$\begin{array}{ccc} \mathbb{P}_1 & \rightarrow & \mathbb{P}_2 \\ (s,t) & \mapsto & (s^2, st, t^2) \end{array}$$

Let $L \subset H^0\mathcal{O}_C(4)$ be a pencil generated by $(s^4 - t^4, s^2t^2)$ and s, t are the coordinate functions on $\mathbb{P}_1$. In [28] the equation is the Poncelet curve S is computed directly. Given coordinates e_0, e_1, e_2 for the plane P and the Conic $C = V(e_1e_2 - e_0^2)$ then the Poncelet curve $S = V(e_1(e_0 + ie_2)(e_0 - ie_2))$.

5.3.1 Further remarks on Poncelet pairs

The remarks I mention here are partially known since the days of classical algebraic geometry. They are directly connected with the results in this thesis. These results are collected and in detail written in [21], and treat reducibility questions of Poncelet curves.

The general situation is again given by a smooth conic C in $\mathbb{P}_2$ a pencil $L \subset H^0\mathcal{O}_C(n+1))$ on C and a curve S of degree n such that (C, S) is a Poncelet pair with respect to L, see definition 5.5.

Remark 5.13 It is obvious that for base point free pencils a pole is possible if and only if $n+1$ is even. In addition to the result for basepoint free pencils on page 114, is in [21] a description of Poncelet curves of even degree which contain a line l.

Proposition 5.14 ([21]6.3) *Let $S = S' \cup l$ be the Poncelet curve for a smooth conic $C \subset \mathbb{P}_2$ and a pencil L of odd degree. Then L has basepoints.*

The behaviour of those Poncelet curves which contain a conic was already known to Darboux:

Proposition 5.15 ([10]pp 625, [21],pp 25) *Let (C, S) be a Poncelet pair with respect to a pencil L of degree n, where $C \subset \mathbb{P}_2$ is a smooth conic and S is the Poncelet curve of degree $n-1$. If there exits for a divisor $D \in L$ an ordering of all points in D, $(p_1, \cdots, p_n)$, such that $\forall i \in Z_n$ the tangents to C in the points p_i and p_{i+1} intersect on a conic C' and moreover, if $C' \subset S$, then S splits into $\frac{n-1}{2}$ conics if n is odd and into $\frac{n-1}{2}$ conics and a line if n is even.*

In the case of $n = 5$ this splitting describes Pascal's theorem in one direction and a case of the famous Cayley-Bacharach theorem in the other direction. [12]pp 673.

Furthermore a complete classification of Poncelet curves up to degree 4 is contained in [21].

5.4 Main theorems and the moduli for special 't Hooft bundles of NC-type

In this section the main results on special 't Hooft bundles of NC-type and a description of their moduli are presented. The question of the normal form of monads for special 't Hooft bundles of NC-type is treated in a section 5.6.

In the next two theorem the intersection of the set of NC-type instantons and the special 't Hooft bundles in the moduli space $MI(c_2)$ is characterized. I recall now theorem 5.4 where the special 't Hooft bundles are described among the instantons of NC-type by:

Theorem 5.16 *Let $\mathcal{E}$ be an instanton bundle of NC-type with $c_2(\mathcal{E}) = 2m+1$. Then the following conditions are equivalent:*

(1) $h^0\mathcal{E}(1) = 2$.(i.e. $\mathcal{E}$ is a special 't Hooft bundle)

(2) $\mathcal{E}$ has a monad of NC-type:

$$0 \rightarrow m\mathcal{O}(-1) \stackrel{S}{\rightarrow} (m+1)\mathcal{E}_a \stackrel{T^\vee \circ J}{\rightarrow} m\mathcal{O}(1) \rightarrow 0$$

and in addition $S \in Mat(m, m+1; H_2)$ where $H_2 \subset H^0\mathcal{E}_a(1)$ is a 2-dimensional subspace.

Proof:

1 $\Longrightarrow$ **2** This is proven in lemma 5.4.

2 $\Longrightarrow$ **1** This is verified in proposition 4.30.

#

In theorem 5.17 the NC-type instantons are classified among the special 't Hooft bundles:

Theorem 5.17 *Let $\mathcal{E}$ be a special 't Hooft bundle with $c_2(\mathcal{E})$ odd*

(1) $\mathcal{E}$ is of NC-type:

(2) The basepoint free linear system $g^1_{2m+2}(\mathcal{E})$, see 5.1, has a pole, which is not contained in the conic $C(\mathcal{E})$

(3) The Poncelet curve $S(\mathcal{E})$ splits into a line and a curve, i.e.: $S(\mathcal{E}) = \ell \cup S'$ and there is a basepoint free pencil L' on ℓ of $deg(L') = m+1$ defined by the intersection points of the tangents of $m+1$ pairs of points of every divisor D of $g^1_{2m+2}(\mathcal{E})$.

Proof:

2 $\Longleftrightarrow$ **3** This is verified in lemma 5.10.

1 $\Longrightarrow$ **2** This is the content of lemma 5.3(1).

2 $\Longrightarrow$ **1** The proof of of this statement uses the normal form for monads of NC-type and is thus postponed, see proposition 5.32.

#

As a direct consequence of the theorem 5.17 one can describe now the moduli space of special 't Hooft bundles of NC-type. The moduli space for special 't Hooft bundles in general was studied in [8] and [18].

In the general case the moduli problem for special 't Hooft bundles with fixed second Chern class $c_2(\mathcal{E}) = n$ is bijective to the set of all pairs (P, L) where $P \subset \mathbb{P}\Lambda^2 V$ is a plane with $P \cap G(2,4) = C (C$ is a regular conic) and $L \subset H^0\mathcal{O}_C(n+1)$ a basepoint free linear system. For special 't Hooft bundles of NC-type we have:

Theorem 5.18 *Let $\mathbb{P}_3 = \mathbb{P}(V)$ and let U open in $(\mathbb{P}\Lambda^2 V \setminus G(2,4)) \times G(2,5) \times G(2,2m+1)$ be the set of all triples $(\langle a\rangle, P, L)$ such that $\langle a\rangle \in \mathbb{P}\Lambda^2 V \setminus G(2,4)$, and P is a plane in $\mathbb{P}\Lambda^2 V$ containing $\langle a\rangle$ and $L \subset |H^0\mathcal{O}_C(\frac{2m+2}{2})|$ is a basepoint free pencil where C is the conic $C := P \cap G(2,4)$. Then there exists a finite map, generically 1 to 1, from U to the moduli space of all instantons of NC-type with $h^0\mathcal{E}(1) = 2$ and $c_2(\mathcal{E}) = 2m+1$. i.e.:*

$$\begin{array}{ccc} U & \to & MI(2m+1) \\ (\langle a\rangle, P, L) & \mapsto & [\mathcal{E}] \end{array}$$

Remark 5.19 (1) The reconstruction of the instanton $\mathcal{E}$ from a triple $(\langle a\rangle, P, L)$ is done in section 5.6.

(2) One can not expect bijectivity in the theorem 5.18 because lemma 5.12 that there exist pencils with more than one pole.

It is also possible to compute the dimension of the family of special 't Hooft bundles which have a monad

$$0 \to m\mathcal{O}(-1) \to (m+1)\mathcal{E}_a \to m\mathcal{O}(1) \to 0$$

directly from the geometry as in theorem 5.18. There is a different way computing the number which is already derived in lemma 4.34. According to theorem 5.18 it remains to compute the dimension of the space $\mathbb{P}\Lambda^2 V \setminus G(2,4) \times G(2, \Lambda^2 V/\langle a\rangle) \times G(2, m+1)$:

(1) $\mathbb{P}\Lambda^2 V \setminus G(2,4)$ is 5 dimensional.

(2) $G(2, \Lambda^2 V/\langle a\rangle)$ is 6 dimensional.

(3) $G(2, m+2)$ has dimension $2m$.

Hence the dimension is $c_2 + 10 = 2m + 11$.

Remark 5.20 The choice of the subspace $H_2 \subset H^0\mathcal{E}_a(1)$ is equivalent to the choice of the pole $\langle a\rangle$ and a plane P containing the pole $\langle a\rangle$.

5.5 Linear systems on a conic having a pole

In this section I study properties of linear systems on a smooth conic in $P := \mathbb{P}_2 \simeq \mathbb{P}(W)$ which shall be applied to the normal form question of monads for special 't Hooft bundles of NC-type in the sequel. Actually I shall parameterize all pencil on a smooth conic $C \in P$, which allow a pole with respect to the conic. The precise definitions of the involution and of a pole are already given in chapter 4 , see 5.7 and 5.6. The consideration in the chapter are explicit calculations in coordinates which reflects the more geometric reasoning in chapter 5.3 "Complements on Poncelet pairs and poles".

Remark 5.21 Let P be the projective plane $\mathbb{P}(W)$. I choose coordinates $z_0, z_1, z_2 \in W^\vee$ such that $C := \{z_0z_2 - z_1^2 = 0\}$ then the following is true:

(1) $z_0z_2 - z_1^2 = \frac{1}{2}(z_0, z_1, z_2)\begin{pmatrix} & & 1 \\ & -2 & \\ 1 & & \end{pmatrix}\begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix}$

(2) Therefore the conic defines after this choice of coordinates a duality pairing between W and $W^\vee$.

$$\begin{array}{ccc} W & \to & W^\vee \\ \begin{pmatrix} e_0 \\ e_1 \\ e_2 \end{pmatrix} & \mapsto & \begin{pmatrix} z_2 \\ -2z_1 \\ z_0 \end{pmatrix}. \end{array}$$

e_0, e_1, e_2 is a basis of W and z_0, z_1, z_2 the basis of $W^\vee$. This induces moreover a pairing between $\mathbb{P}(W)$ and $\mathbb{P}(W^\vee)$

(3) This pairing is non degenerate because C is a smooth conic in $\mathbb{P}_2$.

(4) If the pole in $\mathbb{P}(W)$ has coordinates (a_0, a_1, a_2) then the polar line $\ell_a := \{x \in \mathbb{P}(W) | q(a,x) = 0\}$ has coordinates $(a_2, -2a_1, a_0)$ in $\mathbb{P}(W^\vee)$, i.e.: it is the line $\{a_2z_0 - 2a_1z_1 + a_0z_2 = 0\} \subset \mathbb{P}(W)$. The polar line is the image of the line $\mathbb{P}_1^\vee$ in 5.6 under the duality pairing between $\mathbb{P}(W)$ and $\mathbb{P}(W^\vee)$.

Lemma 5.22 *Choose coordinates s, t of $\mathbb{P}_1$, then the Veronese embedding is:*

$$\begin{array}{ccc} \mathbb{P}_1 & \to & C \subset \mathbb{P}_2 \\ (t, s) & \mapsto & (t^2, ts, s^2) \end{array}$$

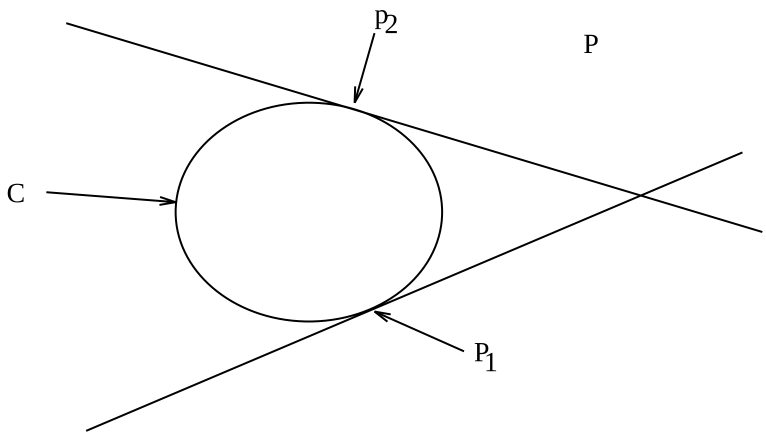
The intersection point of the two tangent lines on C in the points with coordinates $\{t_i^2, t_is_i, s_i^2\}$ $i = 1, 2$ has coordinates $(t_1t_2, \frac{t_2s_1+t_1s_2}{2}, s_1s_2)$.

Proof:
The tangents at $p_i \in C$ $i = 1, 2$ are:

$$\{z_2(p)z_0 + 2z_1(p)z_1 + z_0(p)z_2 = 0\} \simeq$$

$$\{s_i^2z_0 + 2s_it_i(p)z_1 + t_iz_2 = 0\} \ i = 1, 2$$

Hence the intersection point of the lines is $(t_1t_2, \frac{t_2s_1+t_1s_2}{2}, s_1s_2)$.
The picture below visualizes the situation of the lemma.



#

With these introductory facts and fixing of conventions we are now able to compute explicitly the involution in terms of the fixed pole $\langle a \rangle$. For the rest of the chapter $\langle a \rangle$ has the coordinates a_0, a_1, a_2. The isomorphism $C \simeq \mathbb{P}_1$ given by the Veronese embedding reduces the number of coordinates to 2, therefore I shall work for the rest of this chapter on $\mathbb{P}_1$. I describe now the induced involution $\tilde{\iota_a}$ on $\mathbb{P}_1$.

Lemma 5.23 *Let*

$$\begin{array}{cccc} v: & \mathbb{P}_1 & \rightarrow & C \\ & (t,s) & \mapsto & (t^2, ts, s^2) \end{array}$$

be the Veronese map, then the involution $\tilde{\iota_a}$ defined by

$$\begin{array}{ccc} \mathbb{P}_1 & \xrightarrow{\tilde{\iota_a} := v \circ \iota_a \circ v^{-1}} & \mathbb{P}_1 \\ v \downarrow & \# & v \downarrow \\ C & \xrightarrow{\iota_a} & C \end{array}$$

is expressed in coordinates as:

$$\begin{array}{cccc} \tilde{\iota_a}: & \mathbb{P}_1 & \rightarrow & \mathbb{P}_1 \\ & (t_1 : s_1) & \mapsto & (t_2 : \frac{(-a_2t_1 + a_1s_1)t_2}{a_0s_1 - a_1t_1}) = (t_2 : s_2) \end{array}$$

Proof:
In 5.22 we have proven that the line connecting two opposite points (t_1^2, t_1s_1, s_1^2) and (t_2^2, t_2s_2, s_2^2) has coordinates $(s_1s_2, -(t_2s_1 + t_1s_2), t_1t_2) \in \mathbb{P}_2^\vee$. It contains the point (a_0, a_1, a_2). Therefore: we can determine s_2 as a function of t_2 and vice versa:

$$\begin{array}{rcl} s_1s_2a_0 - (t_2s_1 + t_1s_2)a_1 + a_2t_1t_2 & = & 0 \\ \Longrightarrow s_2(a_0s_1 - a_1t_1) & = & -a_2t_1t_2 + a_1t_2s_1 \\ \text{If } a_0s_1 - a_1t_1 \neq 0 \text{ and } t_2 \neq 0 \Longrightarrow s_2 & = & \dfrac{(-a_2t_1 + a_1s_1)t_2}{a_0s_1 - a_1t_1} \end{array}$$

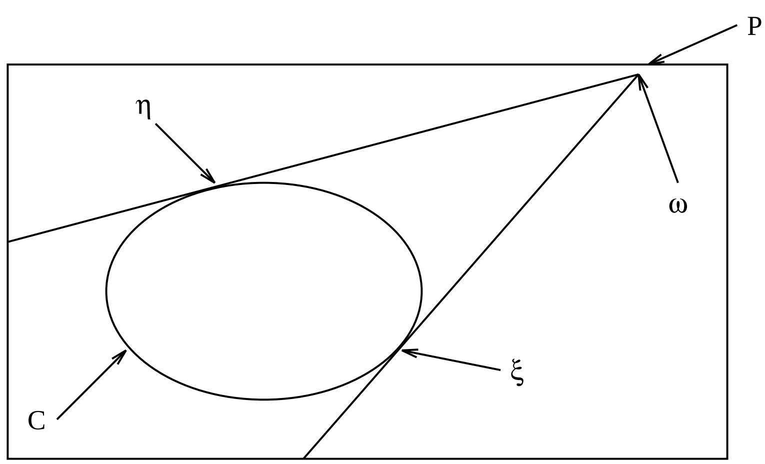
If $a_0s_1 - a_1t_1 = 0$ then $(t_2 : s_2) = (0 : 1)$.
If $t_2 = 0$ then $-a_2t_1 + a_1s_1 = 0$ because s_2 cannot be zero at the same time. #

Remark 5.24 *In particular we have:*

$$\begin{array}{rcl} \tilde{\imath}_a : I\!P_1 & \to & I\!P_1 \\ (1:0) & \mapsto & (1 : \frac{a_2}{a_1}) \\ (0:1) & \mapsto & (\frac{a_0}{a_1} : 1) \end{array}$$

5.5.1 The induced action $\tilde{\imath}_a^*$ on $H^0\mathcal{O}_C(2m+2)$

In this subsection the contravariant action $\tilde{\imath}_a^*$ of $\tilde{\imath}_a$ on the space $H^0\mathcal{O}_C(2m+2)$ is described. I construct explicitly a basis of the vector space of invariant polynomials in $H^0\mathcal{O}_C(2m+2)^{\tilde{\imath}_a^*} \simeq H^0\mathcal{O}_{I\!P_1}(m+1)$. I shall start with an arbitrary basis of $H^0\mathcal{O}_{I\!P_1}(m+1)$ and construct $m+2$ invariant elements in $H^0\mathcal{O}_C(2m+2)^{\tilde{\imath}_a^*}$ which are independent. I can visualize the geometric background in the picture below. The two points $(0:1)$ and $(1:0)$ on $I\!P_1$ are mapped under the Veronese map to two points $\eta = (0:0:1)$ respectively $\xi = (1:0:0)$ on C. For a basis of $I\!P_2$ we need another point. It is convenient to take $\omega := (0:1:0)$ as the intersection point of the two tangents at η and ξ.



Lemma 5.25 *(1) In the situation above the equation*

$$-\bar{\omega} = a_0\bar{\xi} + a_2\bar{\eta}$$

is true in $H^0\mathcal{E}_a(1) \simeq \Lambda^2V/\langle a\rangle$. $\bar{\omega}, \bar{\eta}$ and $\bar{\xi}$ denote the images of ω, η and ξ under the projection

$$\Lambda^2V \to \Lambda^2V/\langle a\rangle$$

(2)

$$\begin{aligned}\tilde{\imath}_a\xi &= (1, a_2) = \tilde{\imath}_a(1:0)\\ \tilde{\imath}_a\eta &= (a_0, 1) = \tilde{\imath}_a(0:1)\end{aligned}$$

Proof:

(1) Without any loss of generality I fix $\langle a\rangle = (a_0 : 1 : a_2)$, hence

$$a = a_0\xi + \omega + a_2\eta \ .$$

Thus $a_0\xi + \omega + a_2\eta \equiv 0 \bmod a$. Remember that $\bar{\langle a\rangle} \equiv \bar{0} \in H^0\mathcal{E}_a(1)$.

(2) This is obvious from 5.23

#

Lemma 5.26 *The contravariant action $\tilde{\imath}_a^*$ induced by $\tilde{\imath}_a$ is*

$$\begin{aligned}H^0\mathcal{O}_{\mathbb{P}_1}(m+1) &\to H^0\mathcal{O}_{\mathbb{P}_1}(m+1)\\ s &\mapsto (s - a_0 t)\\ t &\mapsto (t - a_2 s)\end{aligned}$$

where s, t is a basis of $H^0\mathcal{O}_{\mathbb{P}_1}(1)$.

Remark 5.27 In the sequel I shall use for convenience the notation

$$\begin{aligned}a &= -a_0\\ b &= a_2\end{aligned}$$

Next I construct the basis of $H^0\mathcal{O}_C(2m+2)^{\tilde{\imath}_a^*}$. I start with an arbitrary basis $\{u^\mu v^\nu\}_{\{\forall \mu+\nu=m+1\}}$ of $H^0\mathcal{O}_{\mathbb{P}_1}(m+1)$. If I multiply each element $u^\mu v^\nu_{\mu+\nu=m+1}$ by its image under $\tilde{\imath}_a^*$, then the result is invariant according to the next lemma.

Lemma 5.28 *The $\tilde{\imath}_a^*$-invariant basis of $H^0\mathcal{O}_{\mathbb{P}_1}(2m+2)$ is*

$$\{s^\mu(s+at)^\mu t^\nu(t+bs)^\nu\}_{\{(\mu,\nu)|\mu+\nu=m+1\}}$$

Proof: Certainly a polynomial $p\tilde{\imath}_a^*(p)$ is invariant under the involution $\tilde{\imath}_a^*$. The set of all such monomials consist of $2m+2$ independent ones. #

Remark 5.29 It is also useful to normalize as follows:

$$\begin{aligned} \tilde{\imath}_a \xi &= (1, a_2) = \tilde{\imath}_a(1:0) \\ \tilde{\imath}_a \eta &= (a_0, 1) = \tilde{\imath}_a(0:1) \end{aligned}$$

This gives us a basis :

$$\{s^\mu (s+at)^\mu t^\nu (s+bt)^\nu\}_{\{(\mu,\nu)|\mu+\nu=m+1\}}$$

The next lemma is not difficult but very technical and it is easier for the reader to see the ideas in the case of low second Chern classes $c_2(\mathcal{E}) = 2m+1$. I present these matrices after the lemma. The next lemma describes explicitly the embedding in terms of matrices with respect to the two bases chosen above the embedding

$$e \;:\; H^0\mathcal{O}_{\mathbb{P}_1}(m+1) \hookrightarrow H^0\mathcal{O}_C(2m+2)$$

Lemma 5.30 *The embedding*

$$\begin{aligned} e: \quad H^0\mathcal{O}_{\mathbb{P}_1}(m+1) &\rightarrow H^0\mathcal{O}_C(2m+2) \\ f &\mapsto f\tilde{\imath}_a^* f \end{aligned}$$

is represented by the matrix :

$$A_m := (a_{m+1-\nu-k,\nu})_{\substack{\nu=0,\dots,m+1\\ k=0,\dots\nu}} = \left(\sum_{j=0}^{k} \binom{m+1-\nu}{k-j} \binom{\nu}{j} b^{(m+1-\nu)-k+j} a^j \right)_{\substack{\nu=0,\dots,m+1\\ k=0,\dots\nu}}$$

if we choose the basis $\{s^\mu (s+at)^\mu t^\nu (s+bt)^\nu\}_{\{\mu,\nu|\mu+\nu=m+1\}}$ *for* $H^0\mathcal{O}_{\mathbb{P}_1}(m+1)$ *and* $\{s^\mu (s+at)^\mu t^\nu (s+bt)^\nu\}_{\{\mu,\nu|\mu+\nu=2m+2\}}$ *for* $H^0\mathcal{O}_C(2m+2)$.

Proof: It is enough to compute the coefficients of the polynomials $s^\mu t^\nu (s+at)^\mu (t+bs)^\nu$.

$$\begin{aligned} s^\mu t^\nu (s+at)^\mu (t+bs)^\nu &= \left(\sum_{i=1}^{\mu} \binom{\mu}{i} s^{\mu-i}(at)^i\right) \left(\sum_{j=1}^{\nu} \binom{\nu}{j} (bs)^{\nu-j} t^j\right) \\ \text{“Cauchy summation''} \Rightarrow \quad &= \sum_{k=1}^{\mu+\nu} \sum_{i+j=k} \binom{\mu}{i} \binom{\nu}{j} t^{i+j} s^{\mu-\nu-i-j} a^j b^{\mu-i} \\ \mu+\nu = m+1 \Rightarrow \quad &= \sum_{k=1}^{m+1} t^k s^{m+1-k} \sum_{i+j=k} \binom{\mu}{i} \binom{\nu}{j} a^j b^{\mu-i} \\ &= \sum_{k=1}^{m+1} t^k s^{m+1-k} \underbrace{\sum_{j=0}^{k} \binom{m+1-\nu}{k-j} \binom{\nu}{j} a^j b^{(m+1-\nu)-k+j}}_{:= a_{m+1-\nu-k,\nu} \text{ the coefficient for } t^k s^{m+1-k} = s^\mu t^\nu} \end{aligned}$$

#

What do these matrices look like? I present now A_m for the values -1,0,1 and 2:

$$\begin{pmatrix} 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & a & 0 \\ 0 & b & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2a & a^2 & 0 & 0 \\ 0 & b & 2ab+1 & a & 0 \\ 0 & 0 & b^2 & 2b & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3a & 3a^2 & a^3 & 0 & 0 & 0 \\ 0 & b & 2ab+1 & 2a+a^2b & a^2 & 0 & 0 \\ 0 & 0 & b^2 & 2b+ab^2 & 2ab+1 & a & 0 \\ 0 & 0 & 0 & b^3 & 3b^2 & 3b & 1 \end{pmatrix}$$

There is one more technical detail to mention about these matrices. It is possible to compute the coefficients of A_{m+1} from the coefficients of A_m inductively.

Lemma 5.31 *If the matrices A_m are defined as in 4.31 then the following is true:*

(1) $a^{m+1}_{(m+1)-k-\nu,\nu} = a a^m_{m-k-\nu+1,\nu} + a^m_{m-k-\nu,\nu}$

(2) $a^{m+1}_{(m+1)-k-\nu,\nu} = a^m_{m-k-\nu+2,\nu-1} + b a^m_{m-k-\nu-1,\nu-1}$

The upper index indicates to which matrix A_m the coefficients belong.

Proof:
We already know the meaning of the matrices A_{m+1}. The image of a basis of monomials $s^{m+1}, s^m t, \ldots, s t^m, t^{m+1}$ of $H^0\mathcal{O}_{\mathbb{P}_1}(m+1)$ under A_{m+1} is our basis of invariant polynomials. The rows are in our convention the image vectors of a monomial under the embedding e as before. Hence it is sufficient for the proof to calculate the coefficients of $p(s+at)$ $p \in H^0\mathcal{O}_{\mathbb{P}_1}(2m+2)$ as $p(s+at) = ps + pat$:

$$\begin{aligned}
& s^{m+1-\nu}t^\nu(s+at)^{m+1-\nu}(t+bs)^\nu \\
= \quad & s^{m+1-\nu}t^\nu(s+at)^{m+-\nu}(t+bs)^\nu(s+at) \\
\text{by 4.31} \quad = \quad & s^{m+1-\nu}t^\nu\left(\sum_{k=0}^{m} t^k s^{m-k}\left(\sum_{j=0}^{k}\binom{m-\nu}{k-j}\binom{\nu}{j}a^j b^{(m-\nu)-k+j}\right)\right)(s+at) \\
= \quad & s^{m+1-\nu}t^\nu\left(\sum_{k=0}^{m} t^k s^{m+1-k}\left(\sum_{j=0}^{k}\binom{m-\nu}{k-j}\binom{\nu}{j}a^j b^{(m-\nu)-k+j}\right)\right. \\
& \left. + \sum_{k=0}^{m} t^{k+1}s^{(m+1)-k-1}a\left(\sum_{j=0}^{k}\binom{m-\nu}{k-j}\binom{\nu}{j}a^j b^{(m-\nu)-k+j}\right)\right)
\end{aligned}$$

$$= s^{m+1-\nu}t^{\nu}\left(\sum_{k=0}^{m} t^k s^{m+1-k} a^m_{m-k-\nu,\nu} \right.$$
$$\left. + \sum_{k=1}^{m+1} t^k s^{(m+1)-k} a \left(\sum_{j=0}^{k-1} \binom{m-\nu}{k-j-1}\binom{\nu}{j} a^j b^{(m-\nu-1)-k+j}\right)\right)$$
$$= s^{m+1-\nu}t^{\nu}\left(\sum_{k=0}^{m} t^k s^{m+1-k} a^m_{m-k-\nu,\nu} + \sum_{k=1}^{m+1} t^k s^{(m+1)-k} a a^m_{m-k-\nu-1,\nu}\right)$$

This proves (1). The proof of case (2) is completely analogous. #

Example: Let us look at A_1 and A_2:

$$A_1 = \begin{pmatrix} 1 & 2a & a^2 & 0 & 0 \\ 0 & b & 2ab+1 & a & 0 \\ 0 & 0 & b^2 & 2b & 1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & 3a & 3a^2 & a^3 & 0 & 0 & 0 \\ 0 & b & 2ab+1 & 2a+a^2b & a^2 & 0 & 0 \\ 0 & 0 & b^2 & 2b+ab^2 & 2ab+1 & a & 0 \\ 0 & 0 & 0 & b^3 & 3b^2 & 3b & 1 \end{pmatrix}$$

I compute for instance with the formula (1) in lemma 5.31 the entry $a^2_{4,3}$ of the matrix A_2 from the entries of the matrix A_1. $a^2_{4,3} = aa^1_{3,3} + a^1_{4,3}$ i.e.: $2b + ab^2 = ab^2 + 2b$.

5.6 A normal form for NC-type-monads

This chapter has several purposes which I am going to outline now. The guideline of this chapter will be the reconstruction of the special 't Hooft bundle from its pencils $g^1_{2m+2}(\mathcal{E})$ or $g^1_{m+1}(\mathcal{E})$ respectively. In the previous chapter 5.4 the preparatory considerations on $\tilde{\iota}^*_a$-invariant linear systems on $\mathbb{P}_1$ are made. It turns out that a pencil admitting a pole defines a NC-type monad. At the same time this pencil defines a Beilinson-II-monad according to the constructions described in [8]. In this section a map between both monad displays will be given.

In a first step an identification is chosen for both pencils $g^1_{2m+2}(\mathcal{E})$ and $g^1_{m+1}(\mathcal{E})$ with appropriate subspaces of $(2m+2)\Omega^1(2)$ respectively $(m+1)\mathcal{E}_a(1)$. This choice is suggested by the normal form of the Beilinson-II-monad for general special 't Hooft bundles presented in [8]. I proceed then with the normal form for the NC-type-monad for special 't Hooft bundles of NC-type. Next I compare both monad displays. This means that a map between both monad displays will be given. Finally it is proven that the instanton bundles reconstructed from both pencils $g^1_{2m+2}(\mathcal{E})$ on the conic C and $g^1_{m+1}(\mathcal{E})$ on the polar line are isomorphic

Hence in chapter 5.6 the following proposition will be verified:

Proposition 5.32 *Let (L, C) a Poncelet pair where L is a basepoint free pencil of degree $2m+2$ on the conic C admitting a pole. Then (L, C) defines a Beilinson-II-monad and in addition a NC-type-monad and the cohomology bundles of both monads are isomorphic.*

I recall now W. Böhmer's and G. Trautmann's result. For every special 't Hooft bundle there exists a normal form for the Beilinson-II-monad of special 't Hooft bundles

$$0 \quad \to \quad (2m+1)\mathcal{O}(-1) \quad \stackrel{M}{\to} \quad (2m+1)\Omega^1_{\mathbb{P}_3}(1) \quad \stackrel{B}{\to} \quad 4m\mathcal{O} \quad \to \quad 0$$

as follows, [8]:
The matrix $B \in Mat(4m, 2m+1, V)$ has the normal form, see [8].3:

$$B := \underbrace{\left.\begin{pmatrix} x & y & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ x' & y' & x & y & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & x' & y' & x & y & \cdots & 0 & 0 \\ \vdots & & & & & & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & x & y \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & x' & y' \end{pmatrix}\right\}}_{4m} {\scriptstyle 2m+1}$$

where x, x', y, y' is a basis of V. The matrix $M \in Mat(2m+1, \bigwedge^2 V)$ is symmetric and a product of a persymmetric matrix P and R i.e.: $M = P \circ R$, where

$$R := \underbrace{\left.\begin{pmatrix} \xi & 0 & \cdots & 0 \\ \omega & \xi & & 0 \\ \eta & \omega & & \vdots \\ 0 & \eta & & 0 \\ 0 & 0 & \ddots & \xi \\ \vdots & \vdots & & \omega \\ 0 & 0 & \cdots & \eta \end{pmatrix}\right\}}_{2m+1} {\scriptstyle 2m+3}$$

With $\xi = x \wedge x', \omega = x \wedge y' - x' \wedge y$ and $\eta = y \wedge y'$.

Remark 5.33 Let $\mathcal{K}$ be the kernel bundle in the display of the Beilinson-II-monad and

$$0 \quad \to \quad \mathcal{K} \quad \to \quad (2m+2)\Omega^1(1) \quad \to \quad 4m\mathcal{O} \quad \to \quad 0$$

the first display sequence, then $Im(R) := Image(R)$ is isomorphic to $H^0\mathcal{K}(1) \subset (2m+2)H^0\Omega^1(2)$ Thus every vector $t \in Im(R)$ is of the form, see [8]3.1:

$$t = (\sigma_1, \cdots, \sigma_{2m+3}),\ \sigma_i = s^{2m-1-i}t^{i-1}(s^2\xi + 2st\omega + t^2\eta),\ s, t \in H^0\mathcal{O}_{\mathbb{P}_1}(1)(*)$$

Next a vector space W_a is constructed which will play for NC-type monads the role of $Im(R)$ for Beilinson-II-monads. I choose now:

$$s_1 := \bar{\xi}, s_2 := \bar{\eta}$$

where $\bar{\xi}$, $\bar{\eta}$ and $\bar{\omega}$ are the images of ξ, ω, η under the projection $\mathbb{P} \bigwedge^2 V \to \mathbb{P} \bigwedge^2 V/\langle a \rangle$ and $\langle a \rangle$ has the coordinates $(a_0 : 1 : a_2)$ in the plane $P(\mathcal{E})$, which is spanned by ξ, ω, η as on page 122 hence $-\bar{\omega} = a_0 s_1 + a_2 s_2$. Therefore the σ_i project to the s_i: $(*)$

$$s_i = s^{m-i}(s - a_0 t)^{m-i} t^i (t - a_2 s)^i \left(s(s - a_0 t) s_1 + t(t - a_2 s) s_2 \right) .$$

These s_i define now a $m+2$-dimensional subspace W_a of $(m+1)H^0\mathcal{E}_a(1) \simeq \bigwedge^2 V/\langle a \rangle$.

It is clear now that $s(s - a_0 t)s_1 + t(t - a_2 s)s_2$ parameterizes the polar line, see 5.8 in $\mathbb{P} \bigwedge^2 V/\langle a \rangle$ which is the image of $C(\mathcal{E})$ through the pole, $\mathbb{P} \bigwedge^2 V \to \mathbb{P} \bigwedge^2 V/\langle a \rangle$. Hence the two spaces of sections $Im(R)$ and W_a have the same dependency quadric $Q \subset \mathbb{P}_3$,see page 55 and 5.3.

Therefore

$$(m+1)\mathcal{E}_a^{\vee} \quad \to \quad W_a \otimes \mathcal{O} \quad \to \quad \mathcal{O}_Q(2m+2, 0) \quad \to \quad 0$$

follows from the evaluation sequence for in the case of Beilinson-II-monads,

$$0 \quad \to \quad \mathcal{K}^{\vee}(1) \quad \to \quad W \otimes \mathcal{O} \quad \to \quad \mathcal{O}_Q(2m+2, 0) \quad \to \quad 0 \ .$$

$\mathcal{K}_a$ is now the kernel defined as follows

$$0 \quad \to \quad \mathcal{K}_a^{\vee}(1) \quad \to \quad W_a \otimes \mathcal{O} \quad \to \quad \mathcal{O}_Q(2m+2, 0) \quad \to \quad 0 \ .$$

$$0 \quad \to \quad \mathcal{K}^{\vee}(1) \quad \to \quad Im(R)^{\vee} \otimes \mathcal{O} \quad \to \quad \mathcal{O}_Q(2m+2, 0) \quad \to \quad 0 \ .$$

Lemma 5.34 *Let $\mathcal{K}_a$ above then the isomorphism*

$$H^0\mathcal{O}_C(2m+2)^{\iota_a^*} \simeq H^0\mathcal{K}_a(1)$$

is valid.

Proof: Dualizing and taking global sections of the sequence

$$0 \quad \to \quad \mathcal{K}_a^{\vee}(1) \quad \to \quad W_a \otimes \mathcal{O} \quad \to \quad \mathcal{O}_Q(2m+2, 0) \quad \to \quad 0$$

yields

$$0 \quad \to \quad H^0\mathcal{O}_C(2m+2)^{\iota_a^*} \otimes \mathcal{O} \quad \to \quad \mathcal{K}_a(1) \quad \to \quad \mathcal{O}_Q(2, -2m) \quad \to \quad 0$$

Thus the lemma is true, because $W_a \simeq H^0\mathcal{O}_C(2m+2)^{\iota_a^*}$. and $H^0Q(2, -2m) \simeq H^0\mathcal{O}_{\mathbb{P}_1}(2) \otimes H^0\mathcal{O}_{\mathbb{P}_1}(-2m)$ by the Künneth formula. #

Now I can prove a normal form for monads in analogy to [8]. This normal form is called a special monad in compliance with [8]p.333

Definition 5.35 (special monad) *Let $(M, S^\vee \circ J)$ denote the monad*

$$0 \;\rightarrow\; m\mathcal{O}(-1) \;\stackrel{M}{\rightarrow}\; (m+1)\mathcal{E}_a \;\stackrel{S^\vee\circ J}{\rightarrow}\; m\mathcal{O}(1) \;\rightarrow\; 0$$

Then $(M, S^\vee \circ J)$ is special if and only if $S^\vee \circ J$ is of shape

$$S^\vee \circ J := \underbrace{\left.\left(\begin{array}{cccccc} s_2^\vee \circ J & s_1^\vee \circ J & 0 & 0 & \cdots & 0 \\ 0 & s_2^\vee \circ J & s_1^\vee \circ J & 0 & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & s_2^\vee \circ J & s_1^\vee \circ J \end{array}\right)\right\} m}_{m+1}$$

where s_1, s_2 are two independent sections in $H^0\mathcal{E}(1)$.

The next proposition gives the desired normal form for monads.

Proposition 5.36 *Let $\mathcal{E} \in MI(n)$ be an instanton bundle of NC-type with the property $h^0\mathcal{E}(1) = 2$. Then $\mathcal{E}$ has a special monad.*

Proof:
Let

$$0 \;\rightarrow\; m\mathcal{O}(-1) \;\rightarrow\; (m+1)\mathcal{E}_a \;\rightarrow\; m\mathcal{O}(1) \;\rightarrow\; 0$$

be a NC-type monad defining $\mathcal{E}$. Let $\mathcal{K}$ be the kernel bundle in the second display sequence

$$0 \;\rightarrow\; m\mathcal{O}(-1) \;\rightarrow\; \mathcal{K} \;\rightarrow\; \mathcal{E} \;\rightarrow\; 0$$

and Q the dependency locus of two independent sections in $H^0\mathcal{E}(1)$ and we have the evaluation sequence

$$0 \;\rightarrow\; k^{m+2} \otimes \mathcal{O} \;\rightarrow\; \mathcal{K}(1) \;\rightarrow\; \mathcal{O}_Q(2, -2m) \;\rightarrow\; 0$$

As in [8]3.2 I choose coordinates x, x', y, y' and $\xi = x \wedge x', \omega = x \wedge y' - x' \wedge y$ and $\eta = y \wedge y'$ such that Q is the regulus defined by

$$s^2\xi + 2st\omega + t^2\eta = 0$$

If I take now $\bar{\xi} = s_1$, $\bar{\eta} = s_2$ and $\bar{\omega} = a_0 s_1 + a_2 s_2$ then Q is parameterized by

$$V(s_1 s(s - a_0 t) + s_2 t(t - a_2 s)) = \ell_a \;,$$

the polar line. $\mathcal{K}_0$ is now defined to be the kernel of $S^\vee \circ J$ as in the definition of a special monad. Hence we obtain the evaluation sequence below:

$$0 \;\rightarrow\; k^{m+2} \otimes \mathcal{O} \;\rightarrow\; \mathcal{K}_0(1) \;\rightarrow\; \mathcal{O}_Q(2, -2m) \;\rightarrow\; 0 \;(*)$$

In the next step we see that there exists always a base change of k^m such that T is of special form. The evaluation sequence (*) induces an isomorphism between the extensions defined by $Ext^1(\mathcal{O}_Q(2,-2m), k^{m+2})$ and the homomorphisms $Hom(k^{m+2}, k^{m+2})$. Under this isomorphism $\mathcal{K}_0$ corresponds to the identity map and for any map ϕ $\mathcal{K}$ is the pushout of ϕ. Thus we have the diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & k^l \otimes \mathcal{O} & = & \mathcal{A} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & k^{m+2} \otimes \mathcal{O} & \longrightarrow & \mathcal{K}_0(1) & \longrightarrow & \mathcal{O}_Q(2,-2m) & \to & 0 \\
 & & \Big\downarrow \phi & & \Big\downarrow \phi' & & \| & & \\
0 & \to & k^{m+2} \otimes \mathcal{O} & \longrightarrow & \mathcal{K}(1) & \longrightarrow & \mathcal{O}_Q(2,-2m) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & k^l \otimes \mathcal{O} & = & \mathcal{B} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where A B are kernel and cokernel respectively. But since $Hom(\mathcal{K}(1), \mathcal{O}) = 0$ we must have that $l = 0$ hence ϕ is an isomorphism. Therefore $T^\vee \circ J$ can be chosen of special form. #

Remark 5.37 M is now a matrix with entries only in $H_2 \subset H^0\mathcal{E}_a(1)$, the subset which is spanned by s_1, s_2 because $M = P \circ S$ where P is a persymmetric matrix with entries in the ground field k and $S^\vee \circ J$ has entries in a two dimensional subvector space $H_2^\vee$. Therefore I have proven the converse of proposition 4.31 which I already announced.

Lemma 5.38 *Let $\mathcal{E}$ be an instanton with a special monad, then $h^0\mathcal{E}(1) = 2$*

Proof:
Let S be

$$S := \underbrace{\begin{pmatrix} s_1 & s_2 & 0 & 0 & \cdots & 0 \\ 0 & s_1 & s_2 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & s_1 & s_2 \end{pmatrix}}_{m+1} \Bigg\} m$$

Obviously the kernel of $S^\vee \circ J$ contains the span of S therefore it is of dimension $\geq m+2$ thus $h^0\mathcal{E}(1) \geq 2$ which follows from the second display sequence

$$0 \to m\mathcal{O} \to \mathcal{K}_a(1) \to \mathcal{E}(1) \to 0 .$$

#

Remark 5.39 There is also a proof for lemma 5.38 which does not refer to special monads proposition, see proposition 4.30.

Now I can formulate a comparison proposition between the resolutions of $\mathcal{O}_Q(2m+2,0)$ given in [8] and the resolution specific for special 't Hooft bundles of NC-type.

Proposition 5.40 *Let C be a smooth in $G(2,4)$ with associated quadric $Q \subset \mathbb{P}_3$, $S^\vee$ is the matrix defined below and $R^\vee$, $R^\vee$is the transpose of the matrix R and B on page 127 and the A_m are defined in proposition 5.30. then there exists a commutative diagram:*

$$\begin{array}{ccccccccc}
0 & \to & m\mathcal{O}(-2) & \xrightarrow{S\circ J} & (m+1)\mathcal{E}_a^\vee(-1) & \xrightarrow{S^\vee} & (m+2)\mathcal{O} & \longrightarrow & \mathcal{O}_Q(2m+2,0) & \to & 0 \\
 & & \downarrow A_{m-2} & & \downarrow A_{m-1} & & \downarrow A_m & & \| & & \\
0 & \to & 4m\mathcal{O}(-1) & \xrightarrow{B^\vee} & (2m+1)\Omega^2(2) & \xrightarrow{R^\vee} & (2m+3)\mathcal{O} & \longrightarrow & \mathcal{O}_Q(2m+2,0) & \to & 0
\end{array}$$

$$S^\vee := \underbrace{\left.\begin{pmatrix} s_2^\vee & s_1^\vee & 0 & 0 & \cdots & 0 \\ 0 & s_2^\vee & s_1^\vee & 0 & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & s_2^\vee & s_1^\vee \end{pmatrix}\right\} m+1}_{m+2}$$

$$J \circ S := \underbrace{\left.\begin{pmatrix} J\circ s_1 & J\circ s_2 & 0 & 0 & \cdots & 0 \\ 0 & J\circ s_1 & J\circ s_2 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & J\circ s_1 & J\circ s_2 \end{pmatrix}\right\} m}_{m+1}$$

Proof:
of the proposition
The proof consists of the 3 subsequent lemmata.

Lemma 5.41 *The dual evaluation sequence, defined as in the proposition 5.40,*

$$0 \to m\mathcal{O}(-2) \overset{J\circ S}{\to} (m+1)\mathcal{E}_a^\vee(-1) \overset{S^\vee}{\to} (m+2)\mathcal{O} \to \mathcal{O}_Q(2m+2,0) \to 0$$

is exact.

Proof:

(1) The left hand side is obviously exact by an direct computation, if one takes in to account that $s_k^\vee \circ J \circ s_i \simeq s_i \wedge s_k$ see definition 3.9

(2) The right hand side is exact, i.e. the base locus of the linear system in generated by is the quadric Q and the cokernel is $\mathcal{O}_Q(2m+2,0)$. Remember that: $\bar{\xi} = s_1$, $\bar{\omega} = (a_0 s_1 - a_2 s_2)$, $\bar{\eta} = s_2$.
I already know that a general element in the image of $S^\vee$ has the form, see page 128:

$$s_i = s^{m-i}(s-a_0t)^{m-i}t^i(t-a_2s)^i\left(s(s-a_0t)s_1 + t(t-a_2s)s_2\right) .$$

$$\begin{aligned}
&= s^{m-i}(s-a_0t)^{m-i}t^i(t-a_2s)^i(\bar{\xi}s(s-a_0t) + \bar{\eta}t(t-a_2s)) \\
&= s^{m-i}(s-a_0t)^{m-i}t^i(t-a_2s)^i(\bar{\xi}s^2 - st \underbrace{(a_0\bar{\xi} + a_2\bar{\eta})}_{\text{This is } \omega \text{ by 5.25}} + \bar{\eta}t^2) \\
&= s^{m-i}(s-a_0t)^{m-i}t^i(t-a_2s)^i \underbrace{(\bar{\xi}s^2 - st\bar{\omega} + \bar{\eta}t^2}_{\text{This is the projection of the equation of the conic in } \boldsymbol{P}_2 \text{ to the polar line}}
\end{aligned}$$

Moreover it is clear that the s_i together with C form a regulus, where the s_i define a linear system of degree $2m+2$

#

Now it remains the verify that the squares in the diagram of the proposition commute which is obvious for the right hand square. Therefore we prove lemma 5.42 and 5.44

Lemma 5.42

$$R^\vee \circ A_{m-1} = A_m \circ S^\vee$$

Proof:
The proof is now straightforward linear algebra applying 5.31. Let's compute $A_{m-1} \circ R^\vee$ first. Let $A^\nu_{m-1} := (a_{m+1-\nu-k,\nu})_{\binom{\nu=0,\dots,m+1}{k=0,\dots\nu}}$ with ν fixed the $\nu^{th.}$ row of A_m and again $\bar{\xi} = s_1^\vee \circ J$, $\bar{\omega} = (a_0 s_1^\vee \circ J - a_2 s_2^\vee \circ J)$, $\bar{\eta} = s_2^\vee \circ J$ is true. So what is the $\nu^{th.}$ row of $R^\vee \circ A_{m-1}$?

$$R^\vee \circ A_\nu = \left(\begin{array}{ccc|c|c|c|c|c|ccc} 0 & \cdots & 0 & a & \cdots & a * a^m_{m-k-\nu+1,\nu} + a^m_{m-k-\nu,\nu} & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & a^m_{m-k-\nu+1,\nu+2} + b * a^m_{m-k-\nu,\nu} & \cdots & b & 0 & \cdots & 0 \end{array}\right) \circ \begin{pmatrix} s_1^\vee \\ s_2^\vee \end{pmatrix}$$

$$= \left(\begin{array}{ccc|c|c|c|c|c|ccc} 0 & \cdots & 0 & a & \cdots & a_{(m+1)-k-\nu,\nu} & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & a_{(m+1)-k-(\nu+1),(\nu+1)} & \cdots & b & 0 & \cdots & 0 \end{array}\right) \circ \begin{pmatrix} s_1^\vee \\ s_2^\vee \end{pmatrix} = A_m \circ S^\vee$$

#

Definition 5.43 *Let K_m be the matrix inductively defined as follows:*

If K_0 is the vector $\begin{pmatrix} x' \\ -x \\ y' \\ -y \end{pmatrix}$ *and C is the matrix:* $\begin{pmatrix} x' & 0 \\ -x & 0 \\ y' & x' \\ -y & -x \end{pmatrix}$ *then*

$K_{i+1} := K_i$ "+" C where "+" is the concatenation according to the scheme:

$$\left(\begin{array}{cc|cc} K_i & & 0 & \\ \hline & y' & x' & \\ 0 & -y & -x & 0 \\ \hline & & & C \end{array}\right)$$

The commutativity of the left branch of the diagram of the proposition is due to the lemma below

Lemma 5.44

$$B^\vee \circ K_m \wedge A_{m-2} = A_{m-1} \circ J \circ S$$

Proof: It is straightforward linear algebra applying 5.31 and left to the reader. #

So far I have only studied properties of the right hand monad arrow $S^\vee \circ J$ which has a convenient normal form and is only dependent on the conic $C(\mathcal{E})$ associated to the dependency quadric of two sections in $\mathcal{E}(1)$. The next step shows how a base point free pencil in $H^0\mathcal{O}(2m+2)$ or $H^0\mathcal{O}(m+1)$ determines an instanton bundle and vice versa. This is equivalent to the description of the right arrow in the monad. Furthermore a pencil on $C(\mathcal{E})$ gives isomorphic bundles if we construct them from a Beilinson-I type monad via $g^1_{2m+2}(\mathcal{E})$ or a monad of type

$$0 \rightarrow m\mathcal{O}(-1) \rightarrow (m+1)\mathcal{E} \rightarrow m\mathcal{O}(1) \rightarrow 0$$

via $g^1_{m+1}(\mathcal{E})$ iff A_m maps $g^1_{m+1}(\mathcal{E})$ on $g^1_{2m+2}(\mathcal{E})$. There now the commutative diagrams joining the pencils $g^1_{m+1}(\mathcal{E})$ and $g^1_{2m+2}(\mathcal{E})$ with $H^0\mathcal{O}(2m+2)$ and $H^0\mathcal{O}(m+1)$ respectively.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \rightarrow & \mathcal{E}^\vee & \longrightarrow & g^1_{m+1}(\mathcal{E}) \otimes \mathcal{O} & \longrightarrow & \mathcal{O}_Q(2m+2,0) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \rightarrow & \mathcal{K}^\vee(-1) & \longrightarrow & (2m+3)\mathcal{O} & \longrightarrow & \mathcal{O}_Q(2m+2,0) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & (2m+1)\mathcal{O} & = & (2m+1)\mathcal{O} & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{E}'^{\vee} & \longrightarrow & g^1_{2m+2}(\mathcal{E})\otimes\mathcal{O} & \longrightarrow & \mathcal{O}_Q(2m+2,0) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathcal{K}_a^{\vee}(-1) & \longrightarrow & (m+2)\mathcal{O} & \longrightarrow & \mathcal{O}_Q(2m+2,0) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & m\mathcal{O} & = & m\mathcal{O} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Lemma 5.45 *If $g^1_{m+1}(\mathcal{E})$ maps to $g^1_{2m+2}(\mathcal{E})$ under the surjective embedding*

$$A_m: \quad H^0\mathcal{O}(m+1) \quad \to \quad H^0\mathcal{O}(2m+2)^{\bar{\iota}_a^*}$$

then the vector bundles $\mathcal{E}'$ and $\mathcal{E}$ are isomorphic.

Proof:
This is true because the pullback is unique up to isomorphism. #

Let be left column of the last diagram

$$0 \quad \to \quad \mathcal{E}'^{\vee} \quad \to \quad \mathcal{K}_a^{\vee}(-1) \quad \stackrel{M^{\vee}}{\to} \quad m\mathcal{O} \quad \to \quad 0$$

and

$$0 \quad \to \quad m\mathcal{O}(-2) \quad \stackrel{J\circ S}{\to} \quad (m+1)\mathcal{E}_a^{\vee}(-1) \quad \to \quad \mathcal{K}_a^{\vee}(-1) \quad \to \quad 0$$

be the resolution of $\mathcal{K}_a^{\vee}(-1)$ given by 5.40

$$\begin{array}{ccccccccccc}
\to & m\mathcal{O}(-2) & \to & (m+1)\mathcal{E}_a^{\vee}(-1) & & \to & & \mathcal{K}^{\vee}(-1) & \to & (2m+3)\mathcal{O} & \to & 0 \\
 & & & & \searrow & & \nearrow & & & & & \\
 & & & & & (2m+1)\mathcal{O} & & & & & &
\end{array}$$

Then after dualizing I can tie together both exact sequences to the desired monad of NC-type monad for the instanton:

$$0 \quad \to \quad m\mathcal{O}(-1) \quad \stackrel{M}{\to} \quad (m+1)\mathcal{E}_a \quad \stackrel{S^{\vee}\circ J}{\to} \quad m\mathcal{O}(1) \quad \to \quad 0$$

Lemma 5.46 *Let M be $J \circ S^{\vee} \circ A$ and M is symmetric. Hence as in the general case of [8], A with entries in the field k, is persymmetric, i.e.: A has shape:*

$$A(g^1_{m+1}(\mathcal{E})) = \underbrace{\left.\left(\begin{array}{ccccc} a_0 & a_1 & a_2 & \cdots & a_m \\ a_1 & a_2 & & & \\ a_2 & & & & \\ \vdots & & & & \\ a_m & & & & \end{array}\right)\right\}}_{m+1} {\scriptstyle m+1}$$

Proof:
It is linear algebra analogous to [8]p.333. #

References

[1] A. Altman-S. Kleiman, Introduction to Grothendieck Duality Theory, Lecture notes in Mathematics 146, Springer Verlag, 1970.

[2] V. Ancona-G. Ottaviani, Canonical resolutions of sheaves on Schubert and Brieskorn varieties, Preprint 1989.

[3] A. Beilinson , Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebra, Funkt. Analiz. Prilozhenia 12, 68-69, 1978.

[4] W. Barth, Some properties of stable rank 2 vector bundles on $\mathbb{P}_n$, Math. Ann. 226, 125-150, 1977.

[5] W. Barth-K. Hulek, Monads and moduli of vector bundles, manuscripta math. 25, 323 - 347, 1978.

[6] C. Bănică - O. Forster, Multiple Structures on Space & Curves in Contemporary Mathematics vol. 58, Proceedings of the Lefschetz Centennial Conference 1984, part I, AMS.

[7] C. Bănică - O. Forster, Sur les structures multiples, hand written manuscript.

[8] W. Böhmer-G. Trautmann, Special instanton bundles and Poncelet curves, in Singularities, Representations of Algebras and Vector Bundles, Proceedings, Lambrecht, 1985.

[9] Chang Mei Chu, A Bound on the Order of Jumping lines, Math. Ann. 262, 511-516, 1983.

[10] G.Darboux, Principés de Géométrie Analytique, Gauthier-Villars et Cie, Éditeurs. Paris, 1917.

[11] R. Godement, théorie des faisceaux, Actualités scientifiques et industrielles 1252, Hermann. Paris, 1964.

[12] Ph.Griffith-J.Harris Principles of algebraic Geometry, Pure and applied mathematics, John Wiley & Sons. New York, 1978.

[13] Ph.Griffith-J.Harris On Cayley's explicit solution of Poncelet's porism, L'Enseignement mathématique 24, 31-40, 1978.

[14] R. Hartshorne, Algebraic Geometry, Graduate Text in Mathematics 50, Springer Verlag, 1977.

[15] R. Hartshorne, Stable vector bundles and instantons, Comm. Math. Phys. 59, 1-15, 1978.

[16] R. Hartshorne, Stable vector bundles of rank 2 on $\mathbb{P}_3$, Math. Ann. 238, 229-280, 1978.

[17] A. Hirschowitz-M.S. Narasimhan, Fibres de 't Hooft speciaux et applications, Proc. Nice Conf. 1981, Birkhäuser.

[18] H.-J. Kowalski, Lineare Algebra, 9.Auflage, de Gruyter, 1979.

[19] M. Maruyama, Moduli of stable sheaves I & II, J. Math. Kyoto Univ. 17, 91-126, 1977 & 18, 557 - 614, 1978.

[20] M. Maruayama - G. Trautmann, Limits of Instantons, Intern. Journal of Math. 3, 213-276, 1992.

[21] D.Müller, Ebene Poncelet-Kurven Singularitäten und Zerlegbarkeit, Examensarbeit, Kaiserslautern, 1995.

[22] M.S. Narasimhan - G. Trautmann, Compactification of $M_{\mathbb{P}_3}(0,2)$ and Poncelet Pairs of Conics, Intern. Journal of Math. 3, 213-276, 1992.

[23] T. Nüßler - G. Trautmann, Multiple Koszul Structures on Lines and Instanton Bundles, Int. Jour. of Math. Vol.5, No 3, 373-388, 1994.

[24] T. Nüßler - H. Schönemann, Gröbner Bases in Algebras with Zerodivisors, DFG-Forschungsschwerpunkt Komplexe Mannigfaltigkeiten Preprint Nr.178.

[25] Ch. Okonek - M. Schneider - H. Spindler, Vector bundles on complex projective spaces, Birkhäuser 1980.

[26] P. Rao, manuscript, University of Missouri, ST Louis 1995.

[27] M. Skiti, Sur une famille de fibrés instanton, to appear in Math. Z.

[28] G. Trautmann, Generalities about Instanton Bundles, manuscript Kaiserlautern 1994.

[29] J. Wehler, Moduli space and versal deformation of stable vector bundles, Revue Roum. de Math. Pure et Appl. 30, 69-78, 1985.

[30] G.-M. Greuel - G.Pfister - H. Schönemann, SINGULAR Reference Manual, Kaiserlautern University, ftp.mathematik.uni-kl.de/pub/Math/Singular

[31] D.Bayer - M.Stillman - M.Stillman, Macaulay User Manual ftp.zariki.harvard.edu/Macaulay

Fachbereich Mathematik
Universität Kaiserslautern
D- 6750 Kaiserslautern
Email:
thomas@mathematik.uni-kl.de
thomas@math.uni-sb.de

Hiermit versichere ich, daß ich die vorliegende Arbeit selbständig verfaßt und keine anderen Hilfsmittel als die angegebenen Quellen verwandt habe.