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The Split tree for option pricing

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To my parents.

Abstract

In this dissertation convergence of binomial trees for option pricing is investigated. The focus is on American and European put and call options. For that purpose variations of the binomial tree model are reviewed.

In the first part of the thesis we examine the convergence behavior of the already known trees from the literature (CRR, RB, Tian and CP) for the European options. The CRR and the RB tree suffer from irregular convergence, so our first aim is to find a way to get the smooth convergence. We first show what causes these oscillations. That will also help us to improve the rate of convergence. As a result we introduce the Tian and the CP tree and we prove that the order of convergence for these trees is $O\left(\frac{1}{n}\right)$.

Afterwards we introduce the Split tree and explain its properties. We prove the convergence of it and we derive an explicit first order error formula. In our setting, the splitting time $t_k = k\Delta t$ is not fixed, i.e. it can be any time between 0 and the maturity time T. This is the main difference compared to the model from the literature. Namely, we show that the good properties of the CRR tree when $S_0 = K$ can be preserved even without this condition (which is mainly the case). We achieved the convergence of $O\left(n^{-\frac{3}{2}}\right)$ and we typically get better results if we split our tree later.

In addition, we examine the behavior of the split tree when applied to American options. This will require some modifications - depending on the initial stock price and the type of option - to take care for the early exercise feature of American options.

Abstract (Deutsch)

In dieser Dissertation wird die Konvergenz von Binomialbäumen zur Optionsbewertung untersucht. Der Schwerpunkt liegt auf Amerikanischen und Europäischen Put und Call Optionen. Zu diesem Zweck werden Variationen des Binomialbaummodells besprochen.

Im ersten Teil dieser Arbeit untersuchen wir das Konvergenzverhalten von Binomialbäumen für Europäische Optionen aus der einschlägigen Literatur (CRR, RB, Tian und CP). Da die CRR und RB Bäume Oszillationen im Konvergenzverhalten aufweisen, ist unser erstes Ziel das Auffinden von Möglichkeiten, um glatte Konvergenz zu erreichen. Dazu untersuchen wir zunächst die Ursachen für diese Oszillationen, was uns letztlich zu einer Verbesserung der Konvergenzrate führt. In der Folge stellen wir die Tian und CP Bäume vor und beweisen, dass die Konvergenzordnung für diese Bäume $O(\frac{1}{n})$ ist.

Im Anschluss führen wir die sogenannten Split-Bäume ein und erläutern ihre Eigenschaften. Wir zeigen, dass diese Bäume konvergieren und leiten explizite Fehlerformeln erster Ordnung her. In unserem Setting ist die Trennungszeit $t_k = k\Delta t$ nicht fixiert, d.h. sie kann frei zwischen dem Startzeitpunkt 0 und der Laufzeit T gewählt werden. Dies ist der wesentliche Unterschied im Vergleich zur existierenden Literatur. Wir zeigen, dass die positiven Konvergenzeigenschaften der CRR Bäume im Fall $S_0 = K$ (was in der Praxis typischerweise nicht der Fall ist) erhalten bleiben, selbst wenn diese Bedingung nicht erfüllt ist. Wir erzielen eine Konvergenzordnung von $O(n^{-\frac{3}{2}})$ und zeigen auf, dass ein später Trennungszeitpunkt für unsere Binomialbäume bessere Resultate liefert.

In einem weiteren Kapitel untersuchen wir das Verhalten des Split-Baums bei der Bewertung amerikanischer Optionen. Hierzu erweist es sich als zweckmäßig - in Abhängigkeit vom Startwert des Aktienpreises und des Typs der Option-, das Verfahren so zu modifizieren, dass der Möglichkeit der vorzeitigen Ausübung bei amerikanischen Optionen Rechnung getragen wird.

Contents

Chapter 1

Introduction

In the last decades derivatives have become increasingly important in the world of finance. Trading call options started in the Chicago Board Options Exchange in 1973 (CBOE). Options had also been traded earlier but CBOE created an orderly market with well - defined contracts. Trading put options started in 1977 and today many other exchanges throughout the world trade options. The first completely satisfactory equilibrium option pricing model has been introduced by F. Black and M. Scholes [5]. R.C. Merton [32] extended their model in several important ways in the same year. They made a basis for many subsequent academic studies. Their studies have shown that option pricing theory is relevant to almost every area of finance. Unfortunately, Black - Scholes and Merton articles are quite advanced. Luckily, W.F. Sharpe derived the same results using only elementary mathematics.

The binomial approach to option pricing grew out of a discussion between M. Rubinstein and W.F. Sharpe at a conference in Ein Borek, Israel. They suggested the following principle: if an economy with three securities can only attain two future states, one such security will be redundant. In other words, each single security can be replicated by the other two, a fact later known as market completeness. This observation lead them to a two-state model, but it should be verified that the economic properties of the Black-Scholes diffusion approach are preserved. This was the birth of the binomial option pricing.

Applying binomial trees is a useful and very popular technique for pricing an option, since it is easy to implement. It was introduced by J.C. Cox, S.A. Ross and M. Rubinstein in [9] and R.J. Redleman and B.J. Bartter in [40] independently. A binomial tree can be identified with a diagram that represents all different possible paths that might be followed by a stock price over the life of the option. One of the advantages of the binomial tree is that it can be adapted to options with different types of payoff structure, including American options. Since there is no closed form solution for the price of American options, this is an advantage of the binomial trees.

The binomial approach is based on the concept of the weak convergence. We construct the discrete - time model $S^{(n)}$ so that convergence in distribution to the continuous process S is ensured. This means that the expectations calculated in the binomial tree can be used as approximations of the option prices in the continuous models, at least for bounded option payoffs (and with further considerations for more general payoffs). Unfortunately, the models defined by J.C. Cox, S.A. Ross and M. Rubinstein (short CRR) and by R.J. Redleman and B.J. Bartter (short RB) have a slow convergence which also is often highly irregular. This motivated other scientists to modify binomial trees suitably to obtain an improved convergence.

Developments of the tree methods can be categorized into three groups. The first group is "Modifications of conditions on parameters". Namely, CRR and RB specified the moment-matching conditions so that the moments of the binomial tree converge to those of the continuous process when the number of steps used for the discretization goes to infinity (in the CRR tree the first moment is matched exactly and the second only asymptotically, while for RB both are matched exactly). The construction of the new trees differ mainly in the discretization process used, the number of moments to be matched and the way it is done as well as wheter to impose symmetry on the tree or not. Some important trees from this category are suggested by R. Jarrow and A. Rudd in [16]. They discretized the log risk-neutral stock process and imposed symmetry on the movement probabilities. Tian in [46] suggested to match the first three raw moments exactly in spot space. More about trees from this category can be found in [20], p.425.

The second group is "Design of trees with special features". The main focus of this thesis is on this category. Tian in [47] introduced a tilt parameter into the model. The idea is to tilt a tree so that a node in the tree coincides with the strike price at the maturity of the option. As a result, we get a smooth convergence which allows us to additionally use extrapolation methods. Chang and Palmer in [7] (short CP) proposed a similar idea. They constructed the tree so that the strike is at the geometric average of two nodes, i.e. the midpoint of two nodes in log-space. Leisen and Reimer in [31] first specified the movement probabilities in the bond measure and the stock measure and then from there they determined the movement sizes. In addition, the tree was constructed such that the strike is at the center of the tree. Joshi [19] proved that Leisen and Reimer's tree has a second order convergence. Joshi in [18] introduced the Split tree. The idea of this tree is to have the tree centered around the strike value in log scale. To achieve this, a time depended drift is introduced for the first k

steps of the tree and afterwards, when we are already in desired position, there is no drift, i.e. we continue with a CRR tree for the rest of the steps. In this paper k is fixed to be $k = \lfloor \frac{n}{2} \rfloor$. The main novel result of this thesis is to prove the convergence of the Split tree and to see what is happening if we choose another k.

The third group is the "Introduction of acceleration techniques". Hull and White in [15] applied the "Control Variate" technique to the binomial trees. The idea is to use European options as a control to American options, so the American option price is adjusted by the error we get by pricing European options. Broadie and Detemple in [6] introduced the so called "Smoothing" technique. The idea is to replace the continuation value one time step before the maturity with the Black - Scholes price. The motivation is that the Black - Scholes price has a continuous derivative, while the continuation value does not. As a result, the price converges with much fewer oscillations and more smoothly. In the same paper they also used Richardson extrapolation. This helps with removing the error.

Outline of the thesis

Chapter 2 contains an introduction to binomial trees. We gave definitions of some basic type of options and the weak convergence is introduced. Afterwards some properties of the binomial trees are explained and we introduced CRR and RB tree.

In Chapter 3 we discussed more about properties of the CRR tree. We prove that this model has a good performance if we have $S_0 = K$. Afterwards we state and prove the Main theorem. This theorem (Theorem 6) gives the order of convergence of the binomial trees for the digital and European call options, but also it gives the error term. These results motivated the introduction of the Tian and the CP tree. These two models achieve a convergence of $O\left(\frac{1}{n}\right)$

Chapter 4 is the central part of our work. First, we introduce the Split tree and explain its properties. Afterwards, we prove the convergence of it and we found an explicit first order error formula (helpful discussions on this topic with G. Leduc have been very much appreciated). In our setting, the splitting time $t_k = k\Delta t$ is not fixed, i.e. it can be any time between 0 and the maturity time T. This is the main difference compared to the model suggested by Joshi. Namely, we show that the good properties of the CRR tree when $S_0 = K$ can be preserved even without this condition (which is mainly the case). We achieved the convergence of $O\left(n^{-\frac{3}{2}}\right)$ and we typically get better results if we split our tree later.

The last chapter is reserved for American options. We state the result of Amin and

Khanna and we perform a numerical analysis. Different situations are observed and we compare all introduced models. Some discussion about drift in general is presented here, which might help for further research for American options.

Chapter 2

Introduction to binomial trees

Derivatives have become very important in the world of finance over the last decades. In many stock exchanges all over the world options are traded actively. Consequently, there is a market request for mathematical models to price them which makes pricing options one of the most important problems in applied mathematics. In this chapter we present some types of options, the fundamental Black - Scholes formula for pricing European call and put options, and theoretical and practical aspects of binomial trees as numerical method for option pricing.

2.1 Option types

A derivative is a financial instrument whose value depends on the values of other financial instruments (or derives from), its underlying assets. An option is a simple financial derivative. For example, a stock option is a derivative whose value depends on the price of a stock. There exist also options on equity, foreign currencies, bonds, goods as oil, gold, etc. There exist two basic types of so-called vanilla options.

Definition 1. *A **call option** is a contract which gives its holder the right (but not the obligation) to **buy** a certain fixed amount of the underlying asset for an already agreed price on or before a certain date. A **put option** is the contract which gives its holder the right (but not the obligation) to **sell** a certain fixed amount of the underlying asset for an already agreed price on or before a certain date.*

An option is not binding, i.e. the holder is not obligated to buy or sell, which gives rise to the term "option". There exist three important groups of options:

- European options,
- American options,

- Exotic options (barrier options, Asian options, Bermudan options, etc).

In this thesis the focus will be on European and American options. The name is not connected with the geographical position of the holder, but on the way the option can be used. European options can be exercised only on a certain date, the exercise date (or maturity time). American options can be exercised at any time between now and the maturity time. We denote the maturity time by T. If T is $+\infty$ the option is called perpetual.

European call option

Let us assume that a holder wants to buy a European call on one share of a stock. This means that the holder has the right to buy this share at time $t = T$ for the strike price K, which is known at the beginning of the contract, i.e. at time $t = 0$. Let $S(T)$ be the share price at maturity. If $S(T) > K$, the holder of the option buys the share for the price K and he can sell it immediately at the market for the price $S(T)$. His gain will be $S(T) - K$. If $S(T) < K$, the rational holder will decide not to exercise the option. As a consequence, there is no gain from holding the option. Combining these two cases, we get that the final payment of

$$V(S) = (S(T) - K)^+ = \max\{S(T) - K, 0\}.$$

This is also presented in the Figure 2.1.

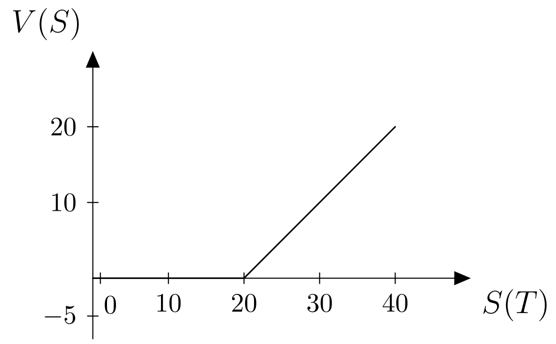


Figure 2.1: Payoff of a European call option with strike price $K = 20$

Example 1. *Let us say we have the situation where an investor buys a European call option with a strike price of* \$60 *to purchase 100 shares of some company. The current stock price is* \$58*, the expiration date of the option is one year and the price of an option to purchase one share is* \$5*. Since we have* 100 *shares, the initial investment is* \$500*. If the stock price, after one year, is less than* \$60*, the investor will choose not to exercise, since there is no point in buying a share* *for* \$60 *which value is less than* \$60*. In this case the investor will lose the initial investment of* \$500*. If the stock price is above* \$60*, the investor will decide to exercise the option. Let us assume that the option price is* \$70*. By exercising the option, the investor will buy* 100 *shares for* \$60 *per share. If she sells the shares immediately, she will gain* \$10 *per share, i.e.* \$1000*, ignoring transaction costs. After taking the initial costs into account, the investor is left with a net profit of* \$500*. Nevertheless, it is possible to make a loss. Assume that the value of the company's option at the maturity is* \$63*. After exercising the option, the investor would gain* $100 \times (\$63 - \$60) = \$300$ *and taking initial costs into account, she is left with* $-\$200$*. However, without exercising she would lose* \$500*, which is a bigger loss than* \$200*. We can conclude that call option should always be exercised at the expiration date if the stock price is above the strike price.*

European put option

The holder of a European put has the right to sell one share of a stock at maturity time T for the strike price K. Now we have that in case when $K > S(T)$ the holder will decide to sell the share for price K. On the other hand, if $K < S(T)$ the holder will decide not to exercise the option. Combining these two cases, we get that the final payment is

$$V(S) = (K - S(T))^+ = \max\{K - S(T), 0\}.$$

There is also a case for the put option where the loss can occur, similar to the call option. Payoff of the European put option is also presented in Figure 2.2.

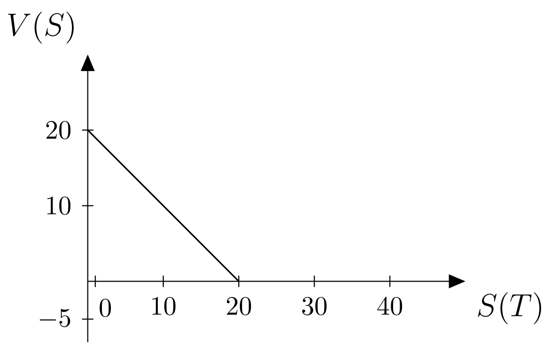


Figure 2.2: Payoff of a European put option with strike price $K = 20$

American options

There exist also American put and call options. As we have seen, European calls and puts can be exercised only at the maturity time T. American puts and calls can be exercised at any time between 0 and time T. A closed-form solution of American options is hardly ever available, therefore the usage of the numerical methods is of great importance. In Chapter 6 we will see how to apply binomial trees to calculate prices of American options numerically.

Exotic options

Put and call options are also known as plain vanilla options. An exotic option is an option which differs from a common American or European option in terms of the underlying asset or when and how the investor receives a certain payoff. We will present Asian and Barrier options as an example of exotic options.

Asian options

An Asian option is a path dependent exotic option. The payoff of an Asian option depends on the average price of the underlying asset over a certain period of time. An Asian option can protect the investor from short term market "manipulations", which can occur close to maturity. We have Asian call options with payoff $(\overline{S}-K)^+$ and Asian put option with payoff $(K-\overline{S})^+$, where $\overline{S}$ is an average price over time period $[0, T]$. Average in continuous time is obtained by:

$$\overline{S} = \frac{1}{T}\int_0^T S(t)dt.$$

For the discrete case, with sample points $t_1, t_2, ..., t_n$, we consider arithmetic and geometric average:

$$\overline{S} = \frac{1}{n}\sum_{i=1}^{n} S(t_i)$$
$$\overline{S} = \left(\prod_{i=1}^{n} S(t_i)\right)^{\frac{1}{n}}.$$

Barrier options

Another example of path dependent options are barrier options. A barrier option is an option where the payoff depends on whether the path of the underlying asset has reached some pre-specified barrier B during the life time of the contract. There are two main types of barrier options: knock-out and knock-in barrier options.
Knock-out barrier options have the following payoffs:

$$(S(T)-K)^+ \cdot \mathbb{1}\left\{\max_{t\in[0,T]} S(t) < B\right\} \quad \text{up-and-out barrier call}$$
$$(K-S(T))^+ \cdot \mathbb{1}\left\{\max_{t\in[0,T]} S(t) < B\right\} \quad \text{up-and-out barrier put}$$
$$(S(T)-K)^+ \cdot \mathbb{1}\left\{\min_{t\in[0,T]} S(t) > B\right\} \quad \text{down-and-out barrier call}$$
$$(K-S(T))^+ \cdot \mathbb{1}\left\{\min_{t\in[0,T]} S(t) > B\right\} \quad \text{down-and-out barrier put}$$

Knock-in barrier options have the following payoffs:

$$(S(T)-K)^+ \cdot \mathbb{1}\left\{\max_{t\in[0,T]} S(t) \geq B\right\} \quad \text{up-and-in barrier call}$$
$$(K-S(T))^+ \cdot \mathbb{1}\left\{\max_{t\in[0,T]} S(t) \geq B\right\} \quad \text{up-and-in barrier put}$$
$$(S(T)-K)^+ \cdot \mathbb{1}\left\{\min_{t\in[0,T]} S(t) \leq B\right\} \quad \text{down-and-in barrier call}$$
$$(K-S(T))^+ \cdot \mathbb{1}\left\{\min_{t\in[0,T]} S(t) \leq B\right\} \quad \text{down-and-in barrier put}$$

There exist more types of options. For more details we refer to [41].

2.2 The stock price model

After introducing the main types of options our aim is to find a model which represents the dynamics of the underlying asset. In the 1970s the famous formulae for pricing European puts and calls were introduced. They were introduced by Black and Scholes [5] and by Merton [32], and are known as Black-Scholes formulae (short BS). Merton and Scholes received the Nobel Memorial prize in Economics Sciences in 1997 for this work. We next present the one-dimensional BS model.

In the Black-Scholes model, the dynamics of the stock prices are described as

$$S(t) = s_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}, s_0 = S(0),$$

where μ is the drift, σ is the volatility and W is a one-dimensional Brownian motion with respect to the probability measure P. This dynamic can also be represented as an SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t).$$

The arbitrage free price of a derivative can be calculated, under some suitable conditions, as an expectation of the discounted payoff under the equivalent martingale measure

$$V = \mathbb{E}(e^{-rT}g(S(T))) \tag{2.2.1}$$

where g is a payoff function, r is the risk-free interest rate and T is the maturity time. Moreover, using Girsanov's Theorem our SDE can be rewritten as

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t),$$

where $\widetilde{W}$ is a one-dimensional Brownian motion with respect to the risk-neutral probability measure Q. The corresponding solution is given by

$$S(t) = S(0)e^{(r-\frac{1}{2}\sigma^2)t+\sigma\widetilde{W}(t)}.$$

More details and conditions specified for the payoff function of the option such that the results we stated above hold can be found in, e.g. [24]. The next result is the famous result for pricing European put and call options. It was introduced by Fischer Black, Myron Scholes and Robert Merton in 1973 and it is perhaps the world's most well-known model for option pricing. Now we will state the Black-Scholes formula (from now on BS formula).

Corollary 1. *Consider a market model consisting of one stock and a bond with constant market coefficients and maturity time $T > 0$, $t \in [0, T]$. Then, the price of a European call option with the strike price $K > 0$ is given by*

$$C_C(t) = S(t)\Phi(\mathfrak{d}_1(t)) - Ke^{-r(T-t)}\Phi(\mathfrak{d}_2(t)),$$

where

$$\mathfrak{d}_1(t) = \frac{\ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{2.2.2}$$

$$\mathfrak{d}_2(t) = \frac{\ln\left(\frac{S(t)}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = \mathfrak{d}_1(t) - \sigma\sqrt{T-t}, \tag{2.2.3}$$

where Φ is the distribution function of the standard normal distribution. The price of the European put is given by

$$C_P(t) = Ke^{-r(T-t)}\Phi(-\mathfrak{d}_2(t)) - S(t)\Phi(-\mathfrak{d}_1(t)).$$

The proof can be found in [24], Corollary 3.9, page 88.

2.3 Binomial trees

2.3.1 Option valuation and replication

In this section we introduce the concept of binomial trees. Namely, we first show how one-step binomial trees behave and afterwards we extend our theory to n-step binomial trees. A binomial tree can be represented as a diagram which shows us different possible paths that might be followed by the stock price over the life of the option.

Let us assume we have a stock whose price is $S(t)$ and a risk-free bond $B(t)$. For the sake of notation we will use S_t and B_t respectively, and since we are considering only a one-step binomial tree, $t \in \{0, T\}$. The risk-free bond, for simplicity, will be set to $B_0 = 1$ and is assumed to grow linearly in time $[0, T]$ with interest rate r. The stock price, on the other hand, has the initial value $S_0 = s_0$ with the possibility to increase to $S_0 u$ with probability p or to decrease to $S_0 d$ with probability $1 - p$, where we require the coefficients u, d to be $u > d$. The principle of modern financial mathematics will help us choosing the factors u and d. Namely, we want to have an arbitrage - free market. What does an arbitrage - free market mean? For that purpose, let us first give a definition of arbitrage.

Definition 2. *A self financing and admissible pair (ϕ, c) consisting of a trading strategy ϕ and a consumption process c is called an arbitrage opportunity if the corresponding wealth process satisfies $X(0) = 0$ and for $X(T)$ we have*

$$X(T) \geq 0, \mathbb{P}(X(T) > 0) > 0.$$

A typical example of arbitrage is any kind of free lottery, i.e. any possibility to win something where we have no initial costs. Since we will assume that our market is an arbitrage-free market, u and d have to satisfy

$$d < 1 + rT < u. \tag{2.3.1}$$

If relation (2.3.1) does not hold, we can generate money without investing our own funds by either selling the stock short (if we have $u \leq 1 + rT$) or financing a stock purchase by a credit (in the case $d \geq 1 + rT$). From now on, we will assume that relation (2.3.1) holds.

Let us assume that we want to price a European call option with strike price K using a one-step binomial tree. This means that the payoff is

$$V = (S_T - K)^+.$$

Then, the price of this option at time $t = 0$ using the relation (2.2.1) is

$$\mathbb{E}\left(\frac{1}{1+rT}(S_T - K)^+\right) = \frac{1}{1+rT}(us_0 - K)q,$$

where $ds_0 < K < us_0$. To justify this, we will introduce the replication strategy.

Corollary 2. *In the arbitrage-free market we have: if two cash flows are identical at some time in the future, then they have the same value today.*

This means that for our example of a European call we have to find a replication strategy (φ_0, φ_1), where φ_0 is the amount invested in the money market account at time $t = 0$ and φ_1 is the number of shares which have to be held at time $t = 0$ so that the investor, at time T, has exactly the same amount of money as if he has a call option. Due to the two possible final payments $us - K$ and 0 we have that φ_0 and φ_1 have to satisfy

$$\begin{aligned} \varphi_0(1+rT) + \varphi_1 us &= us - K \\ \varphi_0(1+rT) + \varphi_1 ds &= 0. \end{aligned}$$

Solving this system we find that the unique replication strategy is

$$\varphi_0 = -\frac{d(us-K)}{(u-d)(1+rT)}, \varphi_1 = \frac{us-K}{(u-d)s}$$

which leads us to the price of the call option via replication

$$\begin{aligned} s_{call} &= \varphi_0 + \varphi_1 s \\ &= -\frac{d(us-K)}{(u-d)(1+rt)} + \frac{us-K}{u-d} \\ &= \frac{1}{1+rT}\frac{1+rT-d}{u-d}(us-K) \\ &=: \frac{1}{1+rT}q(us-K). \end{aligned}$$

This brings us to the next result.

Theorem 1. ***(Risk-neutral valuation and replication)*** *In the one - period binomial tree, where $u > 1 + rT > d$, the unique arbitrage-free price s_H of the option $H = g(S(T))$ in the arbitrage - free model is given by*

$$
\begin{aligned}
s_H &= \frac{1}{1+rT}[qg(us) + (1-q)g(ds)] \\
&= \mathbb{E}_Q\left(\frac{1}{1+rT}H\right),
\end{aligned}
$$

where the expected value $\mathbb{E}_Q$ is given with respect to the probability measure Q which is determined by

$$
Q\left(\frac{S_T}{s_0} = u\right) = q = \frac{1+rT-d}{u-d},
$$

and $0 < q < 1$. *Moreover,*

$$
\mathbb{E}_Q\left(\frac{S_T}{s_0}\right) = 1 + rT = \frac{B_T}{B_0}.
$$

The proof can be found in [23] on page 17.

2.3.2 General binomial trees

In this subsection, we examine the general n-period binomial model. The n-period binomial model was first introduced by Cox, Ross and Rubinstein in 1979 in [9] and by Rendleman and Barrter in [40]. Basically, it is not very different to the one-period binomial model, i.e. we repeat the one-step binomial tree over and over. Let T be the maturity time. Then, we divide the time interval $[0, T]$ into n equally sized subintervals, i.e. let

$$
\Delta t = \frac{T}{n}, t_i = i\frac{T}{n} = i\Delta t, i = 0, 1, ..., n.
$$

Then, the risk-free security and the stock, with the fixed interest rate, are given by

$$
\begin{aligned}
B_{t_i}^{(n)} &= (1 + r\Delta t)^i, \\
S_{t_i}^{(n)} &= s_0 u^{X_i} d^{i-X_i}, i = 0, 1, ..., n,
\end{aligned}
$$

where the number of up movements X_i is binomially distributed with parameters i and p, i.e. $X_i \sim B(i, p)$, where p is the probability. Then the growth of the stock price is given by

$$
S_{i+1}^{(n)} = \begin{cases} uS_i^{(n)} \text{ with probability } p \\ dS_i^{(n)} \text{ with probability } 1-p, \end{cases}
$$

for $i = 0, 1, ..., n-1$. We again assume we have an arbitrage-free market, i.e.

$$d < 1 + r\Delta t < u.$$

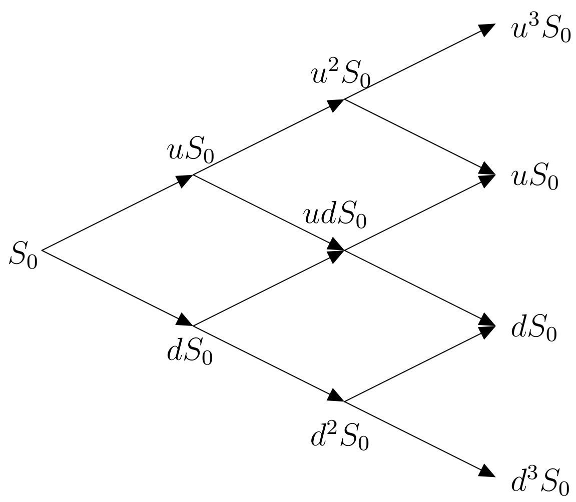


Figure 2.3: One-dimensional binomial tree with 3 steps

Assume we have a one-dimensional Black-Scholes model with stock price dynamics under the risk-neutral measure given by

$$dS(t) = S(t)(rdt + \sigma dW_t), S(0) = s_0 > 0,$$

where $\sigma > 0$ is volatility parameter, r is the risk-free interest rate and W is a one dimensional Brownian motion with respect to the risk-neutral measure Q. The time horizon $T > 0$ is fixed.

The focus of this section is on risk-neutral binomial approximations to the log-normal stock price process. In the theory of option pricing a fundamental result states that in a complete market the option price is given as the discounted expected payoff under the risk-neutral measure Q

$$E_Q\left(e^{-rT}g(S(t), t \in [0,T])\right), \tag{2.3.2}$$

where g is a payoff function of the path-dependent financial instrument with maturity T and underlying S. Here we are interested mostly in European and American put and call options. We will consider American options in greater detail in Chapter 5. Usually, this expectation cannot be calculated explicitly so we have to apply numerical methods to approximate the desired quantity. The most famous numerical methods are Monte Carlo simulations, numerical methods for PDEs and the lattice approach. The topic of this thesis is the lattice approach, more precisely binomial trees.

What is a lattice approach? The aim of the lattice approach is to construct a discrete-time process $S^{(n)}$ and we want that this process converges weakly to our continuous-time process S. On the next couple of pages we will explain how to do it.

Let us put now the binomial tree in a more formal shape. If $n \in \mathbb{N}$ is the number of periods in the discrete model, then in binomial approximation of the stock price we have two possible scenarios per period

$$\mathcal{E}^{(n)} := \{\omega : \{1,2,...,n\} \to \{1,-1\}\}$$

endowed with the product σ-field

$$\mathcal{F}^{(n)} := \bigotimes_{k=1}^{n} \mathcal{P}\left(\{1,-1\}\right) := \sigma\left(Z_k^{(n)} | k = 1,2,...,n\right).$$

Here $\mathcal{P}\left(.\right)$ is a power set and $Z_k^{(n)} : \mathcal{E}^{(n)} \to \{1,-1\}$ is the coordinate mapping with $Z_k^{(n)}\left(\omega\right) = \omega_k$.

Now we can define a binomial process on $\left(\mathcal{E}^{(n)}, \mathcal{F}^{(n)}\right)$ with the same starting point as the initial value of the continuous time process s_0

$$S^{(n)}(k+1) = S^{(n)}(k)e^{\alpha(n)\Delta t + \beta\sqrt{\Delta t}Z_{k+1}^{(n)}},\ k = 0,1,...,n-1 \tag{2.3.3}$$

where $\beta > 0$ is a constant , $\alpha(n)$ is a deterministic function of n and $\Delta t = \frac{T}{n}$ is the grid size of the discrete time model.
From here we have that

$$u = u(n) := e^{\alpha(n)\Delta t + \beta\sqrt{\Delta t}}$$

and

$$d = d(n) := e^{\alpha(n)\Delta t - \beta\sqrt{\Delta t}} \tag{2.3.4}$$

Theorem 2. ***(Risk-neutral valuation and replication)*** *Each option g in an n-period binomial model can be replicated by an investment strategy in the stock and the bond. The initial costs of this strategy determine the option price and are given by*

$$c_0 = E_{Q^{(n)}}(e^{-rT} g),$$

where the measure $Q^{(n)}$ is the product measure of the one-period transition measures $Q_i^{(n)}$ which are determined by

$$Q_i^{(n)} \left(\frac{S_{i+1}^{(n)}}{S_i^{(n)}} = u \right) = q = \frac{e^{r\Delta t} - d}{u - d},$$

and for which we have

$$S_0^{(n)} = E_{Q^{(n)}}(e^{-rT} S_n^{(n)}).$$

From the last equation, we can see that the expected relative return of the stock and the bond, under the measure $Q^{(n)}$, coincide. This motivates calling $Q^{(n)}$ the risk-neutral measure. The risk-neutral probability q gives us the market view on the likelihood that the favorable one-period return u is attained and it can be different from the physical probability p. More details can be found in, e.g [23], [25].

Let us now generalize this approach. For that purpose we will give two algorithms: one for European options and later one for American options.

The algorithm

Consider first a path-independent European option, i.e. an option with payoff $g(S_T)$, which can only be exercised at maturity. We want to approximate the price $E_Q(g(S))$ by using $E_n(g(S^{(n)}))$. The next algorithm explains how to do it:

1. Tree initialization

   - Calculate the possible values of the stock at maturity

     $$S_{k+1}^{(n)} = S_k^{(n)} e^{\alpha(n)\Delta t + \beta\sqrt{\Delta t} Z_{k+1}^{(n)}}, \;\; k = 0, 1, ..., n-1$$

   - Calculate the option value at maturity

     $$V^{(n)}(T, S_n^{(n)}) = g(S_n^{(n)})$$

     for all possible values of $S_n^{(n)}$.

2. Backward induction

- For $i = n-1, ..., 0$ do

$$V^{(n)}(i\Delta t, S_i^{(n)}) = e^{-r\Delta t}\Big[qV^{(n)}((i+1)\Delta t, uS_i^{(n)}) \\ + (1-q)V^{(n)}((i+1)\Delta t, dS_i^{(n)})\Big]$$

3. Return $E_n(g(S_n^{(n)})) = V^n(0, s)$ as the discrete time approximation for the option price, $S_0^{(n)} = s$.

The option price is equal to its expected payoff in a risk-neutral world discounted at the risk-free interest rate. If we add more steps to the binomial tree, the risk-neutral valuation principle will still hold.

2.4 Trees and continuous-time models

As we have seen, a binomial approach leads to a modeling framework for option pricing which is easy to compute. Naturally, we are now interested if our model is in any reasonable way related to the Black-Scholes stock price model for which the stock price $\{S_t, t \in [0, T]\}$ is assumed to follow a geometric Brownian motion, i.e.

$$S_t = s \cdot \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right),$$

where W_t is one-dimensional Brownian motion and $\sigma > 0$ a given constant which describes the volatility of the stock price movements. More precisely, as the period length Δt tends to zero

- do we have weak convergence of the stock price paths in the sequence of increasing binomial models to the given geometric Brownian motion?
- does the sequence of binomial option prices $\left(E_{Q^{(n)}}\left(e^{-rT}g\right)\right)_n$ converge to the corresponding option price in the Black-Scholes model?

The answer to these questions is related to the concept of weak convergence of the corresponding stochastic process. The Central Limit Theorem and Donsker's theorem are here very helpful.

2.4.1 Weak convergence

Let us first understand what we mean by weak convergence.

Definition 3. *Let M be a metric space with the Borel σ-field $\mathcal{B}(M)$, i.e. the smallest σ field containing all open subsets of M, and let P_n, $n \in \mathbb{N}$, and P be probability measures on $(M, \mathcal{B}(M))$. The sequence $\{P_n\}$ is said to converge weakly to P, denoted by*

$$P_n \stackrel{w}{\Longrightarrow} P,$$

if for every bounded, continuous function $f : M \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \int_M f(x) dP_n(x) = \int_M f(x) dP(x).$$

Moreover, if X_n and X are random variables, $n \in \mathbb{N}$, with state space M defined on probability space $(\Omega_n, \mathcal{F}_n, P_n)$ and $(\Omega, \mathcal{F}, P)$ respectively, then the sequence $\{X_n\}$ is said to converge weakly to X, denoted by

$$X_n \stackrel{w}{\Longrightarrow} X,$$

if for every bounded, continuous function $f : M \to \mathbb{R}$ we have

$$\lim_{n\to\infty} E_{P_n}\left(f(X_n)\right) = E_P(f(X)),$$

i.e. if the corresponding probability distributions converge weakly.

Weak convergence of random variables is the same as weak convergence of their distributions. The range of all the random elements has to be the same, while the underlying probability spaces may be different.

One of the main results of weak convergence is Donsker's Theorem which can be seen as a process version of the Central Limit Theorem. Here we will give a special case of it

Theorem 3. ***Donsker's theorem (special case)*** *For given stock price parameters r (drift) and σ (volatility) the price process of the binomial tree converges (in distribution) towards the price process in the Black-Scholes model if the first two moments of the relative log-returns of both models coincide, i.e. if we have*

$$\begin{aligned} E\left(\ln\left(\frac{S(\Delta t)}{S(0)}\right)\right) &= E^{(n)}\left(\ln\left(\frac{S^{(n)}(1)}{S^{(n)}(0)}\right)\right) \\ E\left(\ln\left(\frac{S(\Delta t)}{S(0)}\right)^2\right) &= E^{(n)}\left(\ln\left(\frac{S^{(n)}(1)}{S^{(n)}(0)}\right)^2\right). \end{aligned}$$

By Donsker's theorem, to achieve weak convergence, we need to have the first two moments of the log-increments of the discrete and the continuous model matched. For practical purposes, it is very important that weak convergence also holds if the moment matching conditions are only valid asymptotically (see [33]). To achieve that, we will choose the sequance $(\alpha(n))_n$, the constant $\beta > 0$ and the sequence of probability measures $\left(P^{(n)}\right)_n$ on $\left(\mathcal{E}^{(n)}, \mathcal{F}^{(n)}\right)_n$ such that the following conditions are satisfied:

1. $\forall n \in \mathbb{N}$, the random variables $Z_k^{(n)}, k = 1, 2, ..., n$, are independent and identically distributed.

2. The first two moments of the one-period log-returns of S are asymptotically matched, i.e.

$$\begin{aligned}
\mu(n) :&= \frac{1}{\Delta t} E_{P^{(n)}} \left(\ln \left(\frac{S^{(n)}(k+1)}{S^{(n)}(k)} \right) \middle| S^{(n)}(k) \right) \\
&= \frac{1}{\Delta t} E_{P^{(n)}} \left(\ln \left(\frac{S^{(n)}(k) e^{\alpha(n)\Delta t + \beta\sqrt{\Delta t} Z_{k+1}^{(n)}}}{S^{(n)}(k)} \right) \middle| S^{(n)}(k) \right) \\
&= \alpha(n) + \beta \sqrt{\frac{1}{\Delta t}} E_{P^{(n)}} \left(Z_{k+1}^{(n)} \right) \\
&= \alpha(n) + \beta \sqrt{\frac{1}{\Delta t}} (2p_n - 1)
\end{aligned} \tag{2.4.1}$$

$$\begin{aligned}
\sigma^2(n) :&= \frac{1}{\Delta t} Var_{P^{(n)}} \left(\ln \left(\frac{S^{(n)}(k+1)}{S^{(n)}(k)} \right) \middle| S^{(n)}(k) \right) \\
&= \frac{1}{\Delta t} Var_{P^{(n)}} \left(\ln \left(\frac{S^{(n)}(k) e^{\alpha(n)\Delta t + \beta\sqrt{\Delta t} Z_{k+1}^{(n)}}}{S^{(n)}(k)} \right) \middle| S^{(n)}(k) \right) \\
&= \frac{1}{\Delta t} Var_{P^{(n)}} \left(\alpha(n)\Delta t + \beta\sqrt{\Delta t} Z_{k+1}^{(n)} \right) \\
&= \beta^2 Var_{P^{(n)}} \left(Z_k^{(n)} \right) \\
&= 4\beta^2 p_n (p_n - 1)
\end{aligned} \tag{2.4.2}$$

are such that as $n \to \infty$,

$$\mu(n) \to r - \frac{1}{2}\sigma^2 \tag{2.4.3}$$

and

$$\sigma^2(n) \to \sigma^2. \tag{2.4.4}$$

Under these conditions we will have that the linear interpolation of

$$S^{(n)}(k+1) = S^{(n)}(k) e^{\alpha(n)\Delta t + \beta\sqrt{\Delta t} Z_{k+1}^{(n)}}, \; k = 0, 1, ..., n-1$$

converges weakly to the stock price process. The drift parameteres $(\alpha(n))_n$ are allowed to be non-constant in n. The schemes with $\alpha(n) \equiv \alpha$ constant are referred to as conventional schemes. We will impose following condition:

Assumption: *The sequence* $(\alpha(n))_n$ is assumed to be bounded, i.e. it is assumed to be of order $O(1)$.

The next theorem tells us the conditions which should be satisfied so that we have (2.4.3) and (2.4.4) (see [25] and [33], Proposition 2, page13).

Theorem 4. ***(Theorem on drift invariance)*** *Let $S^{(n)}$ be a risk-neutral binomial scheme. Then the moment matching conditions (2.4.3) and (2.4.4) are satisfied if and only if $\beta = \sigma$. In particular, convergence to the first two moments of the one-period log-returns in the continuous time model is of order*

$$\left|\mu(n) - \left(r - \frac{1}{2}\sigma^2\right)\right| = O\left(\frac{1}{n}\right) \text{ and } \left|\sigma^2(n) - \sigma^2\right| = O\left(\frac{1}{n}\right). \tag{2.4.5}$$

Proof. By Taylor expansion of the exponential function and (2.3.4) we have

$$\begin{aligned} u(n) &= 1 + \beta\left(\frac{T}{n}\right)^{\frac{1}{2}} + \left(\alpha(n) + \frac{1}{2}\beta^2\right)\frac{T}{n} + \left(\frac{1}{6}\beta^3 + \alpha(n)\beta\right)\left(\frac{T}{n}\right)^{\frac{3}{2}} + \\ &\quad \left(\frac{1}{2}\alpha^2(n) + \frac{1}{2}\beta^2\alpha(n) + \frac{1}{24}\beta^4\right)\left(\frac{T}{n}\right)^2 + o\left(\frac{1}{n^2}\right) \end{aligned}$$

$$\begin{aligned} d(n) &= 1 - \beta\left(\frac{T}{n}\right)^{\frac{1}{2}} + \left(\alpha(n) + \frac{1}{2}\beta^2\right)\frac{T}{n} - \left(\frac{1}{6}\beta^3 + \alpha(n)\beta\right)\left(\frac{T}{n}\right)^{\frac{3}{2}} + \\ &\quad \left(\frac{1}{2}\alpha^2(n) + \frac{1}{2}\beta^2\alpha(n) + \frac{1}{24}\beta^4\right)\left(\frac{T}{n}\right)^2 + o\left(\frac{1}{n^2}\right) \end{aligned}$$

which implies

$$\begin{aligned} q(n) &= \frac{e^{r\Delta t} - d(n)}{u(n) - d(n)} \\ &= \frac{1}{2} + c_1(n)\left(\frac{T}{n}\right)^{\frac{1}{2}} + c_2(n)\left(\frac{T}{n}\right)^{\frac{3}{2}} + o\left(\frac{1}{n^{\frac{3}{2}}}\right) \end{aligned} \tag{2.4.6}$$

with

$$c_1(n) = \frac{1}{2\beta}\left(r - \alpha(n) - \frac{1}{2}\beta^2\right)$$

$$c_2(n) = \frac{1}{2\beta}\left(\frac{1}{2}\left(\alpha(n) - r\right)^2 + \frac{1}{6}\beta^2\left(\alpha(n) - r\right) + \frac{1}{24}\beta^4\right)$$

where we used a Taylor expansion for $e^{r\Delta t}$, too. We should notice that $c_1(n)$ and $c_2(n)$ are of order $O(1)$. Now consider

$$\mu(n) \overset{(2.4.1)}{=} \alpha(n) + \beta \left(\frac{n}{T}\right)^{\frac{1}{2}} (2q(n) - 1) = r - \frac{1}{2}\beta^2 + 2\beta c_2(n)\frac{T}{n} + o\left(\frac{1}{n}\right)$$

and

$$\sigma^2(n) \overset{(2.4.2)}{=} 4\beta^2 q(n)(1 - q(n)) = \beta^2 - 4\beta^2 c_1^2(n)\frac{T}{n} + o\left(\frac{1}{n}\right).$$

The moment matching conditions (2.4.2) and (2.4.3) are satisfied if and only if $\beta = \sigma$. From those equations we can also conclude that

$$\left|\mu(n) - \left(r - \frac{1}{2}\sigma^2\right)\right| = O\left(\frac{1}{n}\right) \text{ and } \left|\sigma^2(n) - \sigma^2\right| = O\left(\frac{1}{n}\right)$$

which proves the theorem. □

We have only a condition on β, i.e. the weak convergence is ensured independently of the choice of the sequence $(\alpha(n))_n$. This leaves space for suitable good choices of $\alpha(n)$ to improve the convergence behavior. These choices will play a central role in the next chapter where we will concentrate more on the choice of the drift parameter and explore more properties of it.

2.4.2 CRR and RB trees

In Theorem 4 we have seen some conditions which a binomial tree should satisfy such that we have week convergence to the Black-Scholes stock price process. This means, if we have a bounded payoff, the parameters of the binomial tree must be chosen such that option prices converge to the continuous-time limit, i.e. discrete distribution of the asset price (represented by the binomial tree) must converge to the continuous-time limit of the lognormal distribution. As we have shown in the previous section, this means that we must have (approximate) equality of the first two moments of continuous and discrete model. There are many possibilities to satisfy the moment matching conditions since we have two equations and three parameters: p, u and d. In this chapter we will focus on two of them. The first suggestion was given by Cox, Ross and Rubinsten (CRR from now on) in [9] and they suggested the following parameters:

$$u = e^{\sigma\sqrt{\Delta t}}, d = \frac{1}{u}, p = \frac{1}{2}\left(1 + \left(r - \frac{1}{2}\sigma^2\right)\frac{1}{\sigma}\sqrt{\Delta t}\right).$$

Since p is a probability of an up-movement, it is only well defined if

$$0 \leq \frac{1}{2}\left(1 + \left(r - \frac{1}{2}\sigma^2\right)\frac{1}{\sigma}\sqrt{\Delta t}\right) \leq 1 \iff n > \frac{(r - \frac{1}{2}\sigma^2)^2}{\sigma^2}T.$$

Here we have $\alpha \equiv 0$ and $\beta = \sigma$. For these parameters we have that the first moment of the log increments is exactly matched but the second one is matched asymptotically:

$$\mu(n) = r - \frac{1}{2}\sigma^2,$$

$$\sigma(n)^2 = \sigma^2 - \left(r - \frac{1}{2}\sigma^2\right)^2 \Delta t.$$

Another property of this model is, since $\alpha \equiv 0$, there is no drift and the tree is symmetric in the log-scale around the initial value $\log(S_0)$.
The main drawback of this model is that the convergence is not smooth. We will later show that the order of convergence is $O\left(\frac{1}{\sqrt{n}}\right)$, but since it is not smooth we cannot apply an extrapolation method to speed it up. Also oscillations between even and odd n can be observed. This is known as the even-odd effect. But if we consider only even n (or odd n) then these oscillations are not present. In the Figure 2.4 the even-odd problem is shown. More details about the convergence of the CRR tree are presented in Section 3.1.

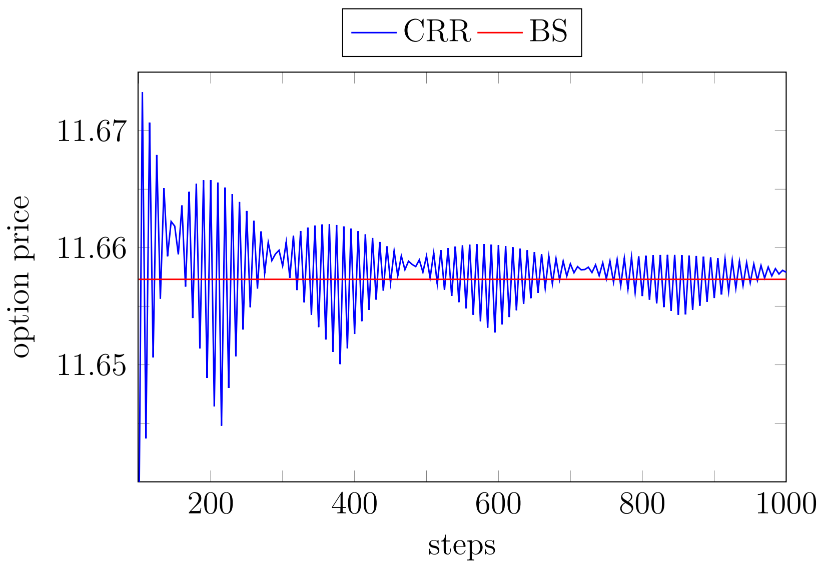


Figure 2.4: European call option price obtained using CRR model: $S_0 = 95$, $K = 100$, $\sigma = 0.25$, $r = 0.1$, $T = 1$, Black-Scholes = 11.6573.

The second suggestion was given by Rendleman and Bartter [40](RB from now on):

$$p = \frac{1}{2}, u = e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}}, d = e^{(r-\frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}},$$

and with these parameters we match both moments exactly. Here we have $\alpha = r - \frac{1}{2}\sigma^2$, which means this tree is tilted with regard to initial value, but the probability p is fixed which is an advantage in the implementation of the model since it requires less operation counts because of the symmetry in the probabilities. This speeds up the algorithm. The RB model unfortunately also suffers from irregular convergence which can be seen in Figure 2.5.

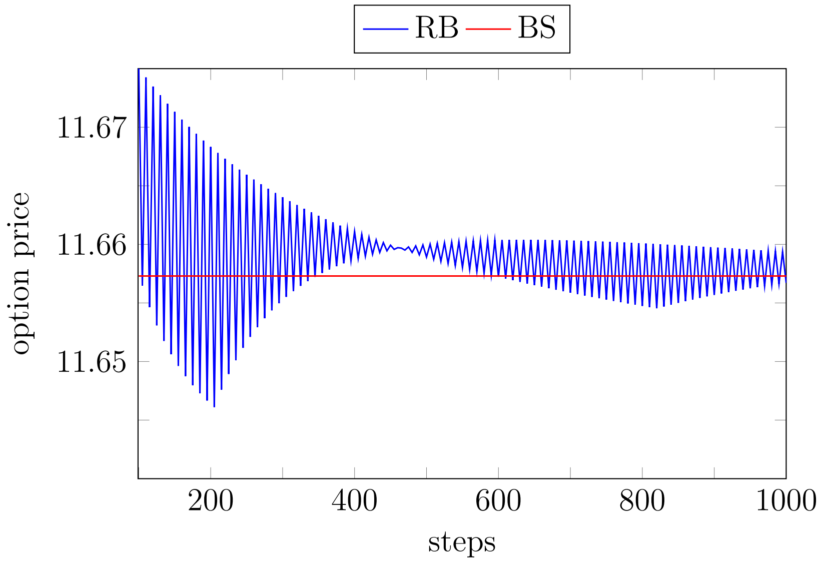


Figure 2.5: European call option price obtained using RB model: $S_0 = 95$, $K = 100$, $\sigma = 0.25$, $r = 0.1$, $T = 1$, Black-Scholes = 11.6573.

A comparison between these two models is given in Bock [1], where the author has shown that the absolute error for both models has a very similar convergence pattern, but the CRR model delivers better results for the relative error. In this thesis the focus will be on the models derived from the CRR model.

Chapter 3

Advanced and modified trees

In the previous chapter we have seen the basic structure of the binomial trees. Now the natural question which arises is: can we have better results using suitably modified binomial trees? Answering this question will be the main topic of this chapter.

3.1 Why and when is a CRR tree good?

Let us first look in great detail at the convergence behavior of the CRR tree. Diener and Diener showed in [10] that the convergence rate for CRR trees for European calls is $O\left(\frac{1}{n}\right)$. We want to see now under which circumstances we get the best behavior of the CRR tree. These results will motivate us to introduce modified binomial trees. We will first focus on results given by Diener and Diener and afterwards we will expand our research. Their result in our notation is:

Theorem 5. *In the n-period CRR binomial model, if $S_0 = 1$, the binomial price $C(n)$ at $t = 0$ of the European call with strike price K and maturity $T = 1$ satisfies*

$$C(n) = C_{BS} + \frac{e^{-\frac{d_1^2}{2}}}{24\sigma\sqrt{2\pi}} \frac{A - 12\sigma^2(\Delta_n^2 - 1)}{n} + O(\frac{1}{n\sqrt{n}}),$$

where

$$\Delta_n = 1 - 2\left\{\frac{\ln(\frac{S_0}{K}) + n\ln d}{\ln(\frac{u}{d})}\right\},$$

$$A = -\sigma^2(6 + \mathfrak{d}_1^2 + \mathfrak{d}_2^2) + 4(\mathfrak{d}_1^2 - \mathfrak{d}_2^2)r - 12r^2,$$

$\{x\}$ is the fractional part of x, C_{BS} is the price obtained by the Black - Scholes formula and $\mathfrak{d}_1$ and $\mathfrak{d}_2$ are the coefficients in the Black - Scholes formula, i.e. as in (2.2.3).

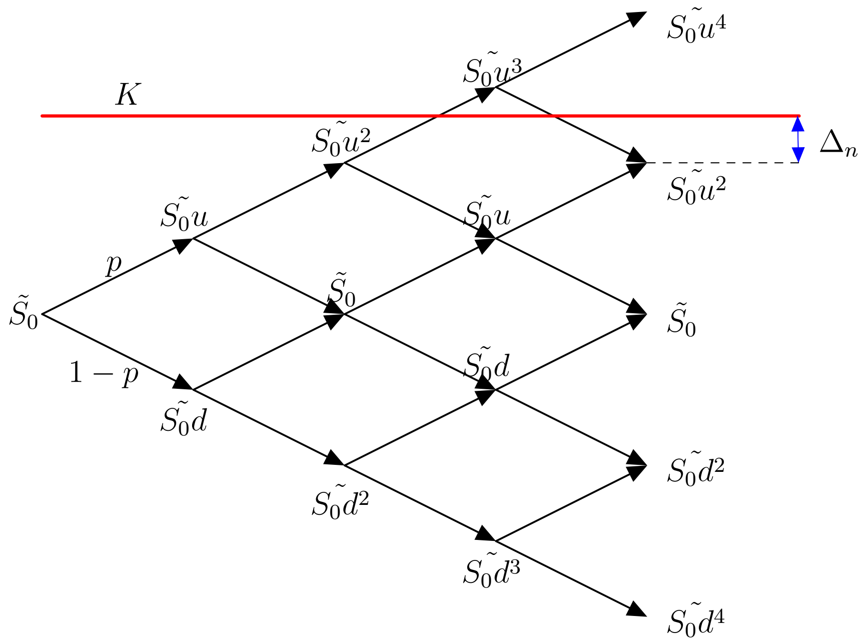


Figure 3.1: Binomial tree with Δ_n, log-scale

As we can see, the coefficient of $\frac{1}{n}$ in the error depends on the quantity Δ_n. Let us investigate this coefficient.

Let $l := l(n)$ be an integer value such that

$$S^{(n)}(l-1) := S_0 u^{l-1}(n) d^{n-l+1}(n) < K \leq S_0 u^l(n) d^{n-l}(n) := S^{(n)}(l),$$

i.e. $S^{(n)}(l-1)$ and $S^{(n)}(l)$ are the terminal nodes adjacent to the strike value K. The non constant factor Δ_n can be rewritten also for the general type of trees, i.e. using $u(n)$ and $d(n)$ from (2.3.4) we get (with, of course $\beta = \sigma$)

$$\Delta_n := 1 - 2\{-a(n)\} := 1 - 2\left\{-\frac{1}{2}n + \frac{\ln\left(\frac{s_0}{K}\right) + \alpha(n)T}{2\sigma\sqrt{T}}\sqrt{n}\right\}. \tag{3.1.1}$$

Chang and Palmer have shown in [7] that Δ_n can also be represented as

$$\Delta_n = 1 - 2\frac{\ln\left(\frac{S^{(n)}(l)}{K}\right)}{\ln\left(\frac{S^{(n)}(l)}{S^{(n)}(l-1)}\right)}. \tag{3.1.2}$$

and thus, it measures the distance between the strike value and its adjacent terminal values in log-scale, which is illustrated in Figure 3.1. Now we can conclude that Δ_n is strictly increasing on $\big(S^{(n)}(l-1), S^{(n)}(l)\big]$ with

$$\Delta_n = \begin{cases} 1 \text{ for } K = S^{(n)}(l) \\ 0 \text{ for } K = \sqrt{S^{(n)}(l-1)S^{(n)}(l)} \end{cases} , \tag{3.1.3}$$

and it converges to -1 as K tends to $S^{(n)}(l-1)$. This is the reason why we have oscillations in the CRR binomial call price around the Black - Scholes price. Controlling this might help us getting smooth convergence.

Let $a(n)$ be, as in (3.1.3), the quantity

$$a(n) := \frac{\ln\left(\frac{K}{S_0}\right) - n\ln d}{\ln u - \ln d}. \tag{3.1.4}$$

Then we have

$$[x] := k(n) := [a(n)] + 1 = a(n) + 1 - \{a(n)\}, \tag{3.1.5}$$

where $[x]$ denotes the integer part of x and $\{x\} := x - [x]$. Then, the fractional part $\{a(n)\}$ is always bounded between 0 and 1.

Now we assume the following: u and d have a converging expansion of type

$$u = u(n) = \frac{1}{2} + \sigma\frac{1}{\sqrt{n}} + u_2\frac{1}{n} + ...$$
$$d = d(n) = \frac{1}{2} - \sigma\frac{1}{\sqrt{n}} + d_2\frac{1}{n} + ...$$

For example, parameters u and d from the CRR model satisfy this converging expansion. Then we have the following result (see [10])

Corollary 3. *The integer $k(n)$ defined by (3.1.5) has the following asymptotic expansion with bounded coefficients:*

$$k(n) = \frac{1}{2}n + a_{-1}\sqrt{n} + a_0 + 1 - \{a(n)\} + O\left(\frac{1}{\sqrt{n}}\right) \tag{3.1.6}$$

where

$$a_{-1} = \frac{1}{4\sigma}\left(2\ln\left(\frac{K}{S_0}\right) - (u_2 + d_2) + \sigma^2\right),$$
$$a_0 = 2\left(2\ln\left(\frac{K}{S_0}\right) - (u_2 + d_2) - \sigma^2\right)(u_2 - d_2))$$

This means, the function $a(n)$ has an asymptotic expansion of type $a(n) = \frac{1}{2}n + a_{-1}\sqrt{n} + O\left(\frac{1}{\sqrt{n}}\right)$ in the model under consideration here. The first term does not give any contribution to k for even n (its fractional part is zero), and for odd n it brings a contribution of $\frac{1}{2}$ to k. This explains the oscillations of order $\frac{1}{n}$ between even and odd values of n in the price $C(n)$. Further, for the same parity of n, when $a(n)$ is far away from an integer value, its fractional part k changes continuously, but it will have a discontinuity every time $a(n)$ crosses an integer. On the other hand, in the case of the call at the money, i.e. $K = S_0$, one has $a(n) = \frac{n}{2}$ (for the CRR tree, i.e. when $u = \frac{1}{d}$) and thus the price $C(n)$ oscillates but the asymptotic expansion with bounded coefficient is much easier to compute as k is simply equal to 0 for even n and $\frac{1}{2}$ for odd n. Thus, there is no scallop here. In Figure 3.2 we can see that the CRR tree is indeed smooth for $K = S_0$. We will take this up again in a more general setting when introducing the split tree in Chapter 4.

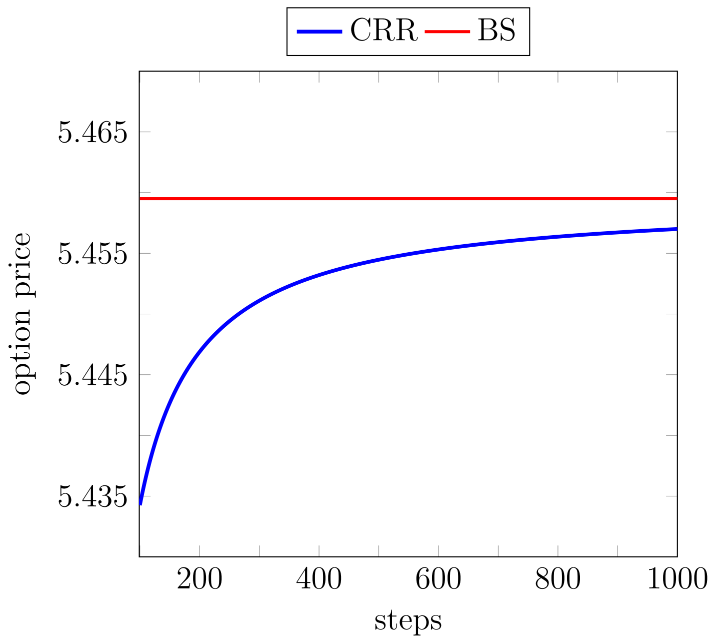


Figure 3.2: European put option obtained using CRR model: $S_0 = 100$, $K = 100$, $\sigma = 0.25$, $r = 0.1$, $T = 1$, Black-Scholes =5.4595

Now combining all these results we end up with

Corollary 4. *The CRR tree admits a smooth convergence for $S_0 = K$ and the order of it for this case is $O\left(\frac{1}{n}\right)$.*

We have seen that the CRR model behaves better compared to other models in case of $S_0 = K$, i.e. there is no scallop. But, usually we have initial values given and we cannot choose them freely. Can we maybe create a similar behavior of the CRR tree if we plays it suitably around K? Answering this question will be the main idea of the next few sections.

From now on, we will say that

$$u = e^{\alpha\sigma^2\Delta t + \sigma\sqrt{\Delta t}}, d = e^{\alpha\sigma^2\Delta t - \sigma\sqrt{\Delta t}}$$

where $\alpha = \alpha(n)$ is in general a bounded sequence depending of n. In [7] they consider all possible choices of u and d such that $\frac{u}{d} = e^{2\sigma\sqrt{\Delta t}}$ and $n\ln(ud)$ is bounded. If we look at our statement again, we will see that these two statements are equivalent. This is a more general approach, since for $\alpha = 0$ we get the CRR model, for $\alpha = \frac{r}{\sigma^2} - \frac{1}{2}$ we get the RB model. The reason why we consider this more general case is that it allows us to see how our tree behaves for different values of α, also known as a drift.

3.2 Advanced trees

In [7] Chang and Palmer concentrated on getting smooth convergence rather than faster convergence, since once we have smooth convergence we can apply extrapolation methods and improve our rate of convergence. The next theorem is the main theorem from [7] and it tells us that the rate of convergence of digital and European options is $O\left(\frac{1}{n}\right)$ but it gives us motivation how to improve the results. It is a slight generalization of Diener and Diener [10], but it has a simpler proof.

Theorem 6 (Main Theorem - Chang and Palmer)**.** *For the n-period binomial model, where*

$$u = e^{\alpha\sigma^2\Delta t + \sigma\sqrt{\Delta t}}, d = e^{\alpha\sigma^2\Delta t - \sigma\sqrt{\Delta t}}, \tag{3.2.1}$$

with α an arbitrary bounded function of n, if the initial stock price is S_0 and the strike price is K with maturity time T. Then

1. *the price of a digital call option satisfies*

$$C_d(n) = e^{-rT}\Phi(d_2) + \frac{e^{-rT}e^{\frac{-d_2^2}{2}}}{\sqrt{2\pi}}\left[\frac{\Delta_n}{\sqrt{n}} - \frac{d_2\Delta_n^2}{2n} + \frac{B_n}{n}\right] + O\left(\frac{1}{n}\right)$$

and

2. *the price of a European call options satisfies*

$$C(n) = C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}}}{24\sigma\sqrt{2\pi T}} \frac{A_n - 12\sigma^2 T(\Delta_n^2 - 1)}{n} + O(\tfrac{1}{n}),$$

where

$$B_n = \frac{d_1^3 + d_1 d_2^2 + 2d_2 - 4d_1}{24} + \frac{(2 - d_1 d_2 - d_1^2)\sqrt{T}}{6\sigma}(r - \alpha\sigma^2) + \frac{T d_1}{2\sigma^2}(r - \alpha\sigma^2)^2,$$

$$A_n = -\sigma^2 T(6 + d_1^2 + d_2^2) + 4T(d_1^2 - d_2^2)(r - \alpha\sigma^2) - 12T^2(r - \alpha\sigma^2)^2.$$

The price of a European call option with maturity T is given by (from [37]):

$$C(n) = S_0 \sum_{k=j}^{n} \binom{n}{k} \hat{p}^k \hat{q}^{n-k} - K e^{-rT} \sum_{k=j}^{n} \binom{n}{k} p^k q^{n-k},$$

where $0 < p < 1$, $p = \frac{e^{r\Delta t} - d}{u - d}$, $q = 1 - p$, $\hat{p} = pue^{-r\Delta t}$, $\hat{q} = 1 - \hat{p}$ and

$$j = [\gamma] \text{ where } \gamma = \frac{\ln\left(\frac{K}{S_0}\right) - n \ln d}{\ln\left(\frac{u}{d}\right)}.$$

The price of a digital call option is given by:

$$C_d(n) = e^{-rT} \sum_{k=j}^{n} \binom{n}{k} p^k q^{n-k}$$

and its Black-Scholes price is given by $e^{-rT}\Phi(d_2)$, while for the European call it is given by $S\Phi(d_1) - Ke^{-rT}\Phi(d_2)$.
The following lemma, an extension of a result of Uspensky from [48], is our fundamental tool for proving the Main theorem, Theorem 6.

Lemma 1. *Provided that $p \to \frac{1}{2}$ as $n \to \infty$ and $0 \leq j = j_n \leq n + 1$ for n sufficiently large,*

$$\sum_{k=j}^{n} \binom{n}{k} p^k q^{n-k} = \frac{1}{\sqrt{2\pi}} \int_{\xi_1}^{\xi_2} e^{-\frac{u^2}{2}} du + \frac{q - p}{6\sqrt{2\pi npq}} \left((1 - \xi_2^2) e^{-\frac{\xi_2^2}{2}} - (1 - \xi_1^2) e^{-\frac{\xi_1^2}{2}} \right)$$
$$+ \frac{1}{12n\sqrt{2\pi}} \left(\xi_2 e^{-\frac{\xi_2^2}{2}} (\xi_2^2 - 1) - \xi_1 e^{-\frac{\xi_1^2}{2}} (\xi_1^2 - 1) \right) + o\left(\frac{1}{n}\right),$$

where $\xi_1 = \frac{j - np - \frac{1}{2}}{\sqrt{npq}}$ and $\xi_2 = \frac{nq + \frac{1}{2}}{\sqrt{npq}}$.

Proof of the Main theorem. Assume that n is large enough such that we have $0 < p < 1$ and $0 \leq \gamma \leq n+1$, which implies that $0 \leq [\gamma] \leq n+1$.
Proof of part 1: Let us first expand probability p in powers of $\Delta t^{\frac{1}{2}}$ up to third order. Then by definition of u and d and using the Taylor expansion

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{1}{2} + \alpha \Delta t^{\frac{1}{2}} + \beta \Delta t^{\frac{3}{2}} + o\left(\Delta t^{\frac{3}{2}}\right) \tag{3.2.2}$$

where

$$\alpha = \frac{r - (\lambda + \frac{1}{2})\sigma^2}{2\sigma}, \quad \beta = \frac{\sigma^4(4\lambda + 1) - 4\sigma^2 r + 12(r - \lambda\sigma^2)^2}{48\sigma}.$$

Next, we compute

$$\xi_1 = \frac{j - np - \frac{1}{2}}{\sqrt{npq}} = -\frac{1}{2\sqrt{npq}}(-2j + 2np + 1). \tag{3.2.3}$$

Since $j = \gamma + \{-\gamma\}$, $-2\gamma + n + 2n\alpha\Delta t^{\frac{1}{2}} = d_2\sqrt{n}$ and using (3.2.2):

$$\begin{aligned} -2j + 2np + 1 &= -2j + 1 + n + 2n\alpha\Delta t^{\frac{1}{2}} + 2n\beta\Delta^{\frac{3}{2}} + o(\Delta t^{\frac{1}{2}}) \\ &= \Delta_n + d_2\sqrt{n} + 2\beta T\Delta t^{\frac{1}{2}} + o(\Delta t^{\frac{1}{2}}). \end{aligned} \tag{3.2.4}$$

Also, because $pq = p(1-p) = \frac{1}{4} - \alpha^2\Delta t^{\frac{3}{2}}$ and using the binomial series theorem

$$\frac{1}{2\sqrt{pq}} = 1 + 2\alpha^2\Delta t + O(\Delta t^{\frac{3}{2}}). \tag{3.2.5}$$

Hence, using (3.2.3), (3.2.4) and (3.2.5)

$$-\xi_1 = d_2 + \frac{\Delta_n}{\sqrt{n}} + \frac{\delta}{n} + o\left(\frac{1}{n}\right), \text{with } \delta = 2T(\alpha^2 d_2 + \beta\sqrt{T}). \tag{3.2.6}$$

Next, we examine the terms in Lemma 1 one by one. Let

$$I := \int_{\xi_1}^{\xi_2} e^{-\frac{u^2}{2}} du = \int_{\xi_1}^{\infty} e^{-\frac{u^2}{2}} du - \int_{\xi_2}^{\infty} e^{-\frac{u^2}{2}} du =: I_1 - I_2.$$

First we estimate I_1. Using $f(x) := \int\limits_{d_2}^{x} e^{-\frac{u^2}{2}} du$, which is an even function, we get

$$I_1 = \int\limits_{-\infty}^{-\xi_1} e^{-\frac{u^2}{2}} du = \int\limits_{-\infty}^{d_2} e^{-\frac{u^2}{2}} du + \int\limits_{d_2}^{-\xi_1} e^{-\frac{u^2}{2}} du = \Phi(d_2)\sqrt{2\pi} + f(-\xi_1).$$

For some η between $-\xi_1$ and d_2, by a Taylor expansion we have

$$f(-\xi_1) = e^{-\frac{d_2^2}{2}}(-\xi_1 - d_2) - \frac{d_2 e^{-\frac{d_2^2}{2}}}{2}(-\xi_1 - d_2)^2 + \frac{f'''(\eta)}{3!}(-\xi_1 - d_2)^3,$$

where $f'''(\eta) = -e^{-\frac{\eta^2}{2}} + \eta^2 e^{-\frac{\eta^2}{2}}$ is bounded. Then it follows from (3.2.6) that

$$f(-\xi_1) = e^{-\frac{d_2^2}{2}} \left[\frac{\Delta_n}{\sqrt{n}} + \frac{\delta}{n} - \frac{d_2 \Delta_n^2}{2n} \right] + o\left(\frac{1}{n}\right),$$

so

$$I_1 = \Phi(d_2)\sqrt{2\pi} + e^{-\frac{d_2^2}{2}} \left[\frac{\Delta_n}{\sqrt{n}} + \frac{\delta}{n} - \frac{d_2 \Delta_n^2}{2n} \right] + o\left(\frac{1}{n}\right). \tag{3.2.7}$$

Now we are going to estimate I_2. Since $p \to \frac{1}{2}$ and $q \to \frac{1}{2}$ as $n \to \infty$, we have $\frac{\xi_2}{\sqrt{n}} = \frac{nq+\frac{1}{2}}{n\sqrt{pq}} \to 1$ as $n \to \infty$, which implies that we can find n_0 such that $\xi_2 \geq 2$ for $n \geq n_0$. Then, when $n \geq n_0$,

$$|I_2| \leq \int\limits_{\xi_2}^{\infty} e^{-u} du = e^{-\xi_2} = e^{-\frac{nq+\frac{1}{2}}{\sqrt{npq}}} = o\left(\frac{1}{n}\right). \tag{3.2.8}$$

From (3.2.5) and (3.2.2)

$$\begin{aligned} \frac{1}{2\sqrt{pq}} &= 1 + \frac{2\alpha^2 T}{n} + O\left(\frac{1}{n^{\frac{3}{2}}}\right) \\ p - q &= \frac{2\alpha\sqrt{T}}{\sqrt{n}} + O\left(\frac{1}{n^{\frac{3}{2}}}\right) \end{aligned}$$

so that we have

$$\frac{q-p}{\sqrt{npq}} = -\frac{4\alpha\sqrt{T}}{n} + O\left(\frac{1}{n^2}\right).$$

Next, note that $(1-\xi_2^2)e^{-\frac{\xi_2^2}{2}} - (1-\xi_1^2)e^{-\frac{\xi_1^2}{2}} \to -(1-d_2^2)e^{-\frac{d_2^2}{2}}$ as $n \to \infty$. Hence

$$\frac{q-p}{6\sqrt{2\pi npq}}\left((1-\xi_2^2)e^{-\frac{\xi_2^2}{2}} - (1-\xi_1^2)e^{-\frac{\xi_1^2}{2}}\right) = \frac{-4\alpha\sqrt{T}}{6n\sqrt{2\pi}}\left(-(1-d_2^2)e^{-\frac{d_2^2}{2}}\right) + o\left(\frac{1}{n}\right). \tag{3.2.9}$$

Using $-\xi_1 \to d_2$ and $\xi_2 \to \infty$ as $n \to \infty$, we get

$$\frac{1}{12n\sqrt{2\pi}}(\xi_2 e^{-\frac{\xi_2^2}{2}}(\xi_2^2-1) - \xi_1 e^{-\frac{\xi_1^2}{2}}(\xi_1^2-1)) = \frac{d_2 e^{-\frac{d_2^2}{2}}(d_2^2-1)}{12\sqrt{2\pi}}\frac{1}{n} + o\left(\frac{1}{n}\right). \tag{3.2.10}$$

Putting (3.2.7) - (3.2.10) in Lemma 1, we obtain

$$\begin{aligned} e^{rT}C_d(n) &= \sum_{k=j}^{n}\binom{n}{k}p^k q^{n-k} \\ &= \Phi(d_2) + \frac{e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}}\left\{\frac{\Delta_n}{\sqrt{n}} - \frac{d_2\Delta_n^2}{2n}\right. \\ &\left. + \left[\delta + \left(\frac{2\alpha\sqrt{T}}{3} - \frac{d_2}{12}\right)(1-d_2^2)\right]\frac{1}{n}\right\} + o\left(\frac{1}{n}\right). \end{aligned} \tag{3.2.11}$$

Now multiplying the last term by e^{-rT} and replacing α and δ by their definitions, we prove the first part of the main theorem.

Proof of part 2: Now we will use Lemma 1 with p, q, ξ_1, ξ_2 replaced by their hatted versions $\hat{p}$, $\hat{q}$, $\hat{\xi}_1$, $\hat{\xi}_2$ and obtain

$$\begin{aligned} \sum_{k=j}^{n}\binom{n}{k}\hat{p}^k\hat{q}^{n-k} &= \frac{1}{\sqrt{2\pi}}\int_{\hat{\xi}_1}^{\hat{\xi}_2} e^{-\frac{u^2}{2}}du + \frac{\hat{q}-\hat{p}}{6\sqrt{2\pi n\hat{p}\hat{q}}}\left((1-\hat{\xi}_2^2)e^{-\frac{\hat{\xi}_2^2}{2}} - (1-\hat{\xi}_1^2)e^{-\frac{\hat{\xi}_1^2}{2}}\right) \\ &+ \frac{1}{12n\sqrt{2\pi}}\left(\hat{\xi}_2 e^{-\frac{\hat{\xi}_2^2}{2}}(\hat{\xi}_2^2-1) - \hat{\xi}_1 e^{-\frac{\hat{\xi}_1^2}{2}}(\hat{\xi}_1^2-1)\right) + o\left(\frac{1}{n}\right), \end{aligned}$$

where

$$\hat{\xi}_1 = \frac{j - n\hat{p} - \frac{1}{2}}{\sqrt{n\hat{p}\hat{q}}} \text{ and } \hat{\xi}_2 = \frac{n\hat{q}}{+}\frac{12}{\sqrt{n\hat{p}\hat{q}}}.$$

Our next step is to estimate $\hat{p}$. Using formula (3.2.1) and another Taylor expansion, we arrive at

$$\hat{p} = \frac{u - ude^{-r\Delta t}}{\sqrt{n\hat{p}\hat{q}}} = \frac{1}{2} + \hat{\alpha}\Delta t^{\frac{1}{2}} + \hat{\beta}\Delta t^{\frac{3}{2}} + o\left(\Delta t^{\frac{3}{2}}\right)$$

where

$$\hat{\alpha} = \frac{r - (\lambda - \frac{1}{2})\sigma^2}{2\sigma}, \quad \beta = \frac{\sigma^4(4\lambda - 1) - 4\sigma^2 r - 12(r - \lambda\sigma^2)^2}{48\sigma}. \tag{3.2.12}$$

Now by replacing p, q, α and β in the derivation of (3.2.6) by the same quantities with hats on and using the fact that $-2\gamma + n + 2n\hat{\alpha}\Delta t^{\frac{1}{2}} = d_1\sqrt{n}$, we get

$$-\hat{\xi} = d_1 + \frac{\Delta_n}{\sqrt{n}} + \frac{\hat{\delta}}{n} + o\left(\frac{1}{n}\right), \text{ where } \hat{\delta} = 2T(\hat{\alpha}^2 d_1 + \hat{\beta}\sqrt{T}).$$

Proceeding as we did to get (3.2.11) and using the hatted version of p, q, ξ_1, ξ_2, α, β, δ and using d_1 instead of d_2, we obtain the relation

$$\sum_{k=j}^{n} \binom{n}{k} \hat{p}^k \hat{q}^{n-k} = \Phi(d_1) + \frac{e^{-\frac{d^1}{2}}}{\sqrt{2\pi}} \times \left\{ \frac{\Delta_n}{\sqrt{n}} + \left[\hat{\delta} - \frac{d_1\Delta_n^2}{2} + \left(\frac{2\hat{\alpha}\sqrt{T}}{3} - \frac{d_1}{12} \right)(1 - d_1^2) \right] \frac{1}{n} \right\} + o\left(\frac{1}{n}\right). \tag{3.2.13}$$

Now if we multiply (3.2.13) with S_0, (3.2.11) with Ke^{-rT}, then subtracting and using $S_0 e^{-\frac{d_1^2}{2}} = Ke^{-rT}e^{-\frac{d_2^2}{2}}$, we get

$$C(n) = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2) + \frac{S_0 e^{-\frac{d_2^2}{2}} C_n}{\sqrt{2\pi}} \frac{1}{n} + o\left(\frac{1}{n}\right),$$

where C_n is

$$C_n = \hat{\delta} - \delta - \frac{\sigma\Delta_n^2\sqrt{T}}{2} + \frac{(1 - d_1^2)\sqrt{T}}{12}(8 - (\hat{\alpha} - \alpha) - \sigma) - \frac{\sigma\sqrt{T}(d_1 + d_2)}{12}(8\alpha - d_2).$$

Simplifying the last equation we get the second part of the Main theorem.

□

In the Main theorem we have again the coefficient Δ_n. Until now we just proved the theorem, but our next step is to use it and see how we can improve our results by controlling this coefficient.

3.2.1 The Tian tree

Convergence in the CRR and RB trees is almost always nonsmooth because the position of K oscilates between the two adjacent stock prices so that Δ_n oscillates between 1 and -1. To overcome this problem, Tian [47] suggested that we take a new drift which will move our tree such that the adjacent node is placed exactly at the point K, i.e. $\Delta_n = 1$. This will be done in the following way: We determine the node closest to the strike price K by solving the following equation:

$$K = S_0 u(n)^a d(n)^{n-a}.$$

This leads us to:

$$a(n) = \frac{\ln\left(\frac{K}{S_0}\right) - n\ln\left(d(n)\right)}{\ln(u(n)) - \ln(d(n))}.$$

The right hand side of the last equation is usually not an integer, i.e. $l_\alpha(n) - 1 < a(n) < l_\alpha(n)$, $l_\alpha(n) \in \mathbb{N}$. To ensure that the terminal node is placed exactly on the strike value K, for the given sequence $(l_\alpha(n))_n$ we define sequence $(\tilde{\alpha}(n))_n$ such that

$$\tilde{\alpha}(n) := \frac{\ln\left(\frac{K}{S_0}\right) - (2l_\alpha(n) - n)\sigma\sqrt{\Delta t}}{T}. \tag{3.2.14}$$

For any number of periods n, the Tian model $S^{(n)}_{\tilde{\alpha}(n)}$ is defined as the process (2.4.1) with $\beta = \sigma$ and with the new drift $\tilde{\alpha}(n)$. The new parameter is not constant, it depends on the number of periods. The process $S^{(n)}_{\tilde{\alpha}(n)}$ is defined such that the corresponding equation $K = S_0 u(n)^a d(n)^{n-a}$ is solved by

$$a_{\tilde{\alpha}(n)}(n) = \frac{1}{2}n + \frac{\ln\left(\frac{K}{S_0}\right) - \tilde{\alpha}(n)T}{2\sigma\sqrt{T}}\sqrt{n} = l_\alpha(n), \tag{3.2.15}$$

where the last equation follows from (3.2.14). It follows now that the quantity $a_\alpha(n)$ obtained for the Tian model is integer-valued. Hence,

$$\frac{\ln\left(\frac{S^{(n)}_{\tilde{\alpha}(n)}(l)}{K}\right)}{\ln\left(\frac{S^{(n)}_{\tilde{\alpha}(n)}(l)}{S^{(n)}_{\tilde{\alpha}(n)}(l-1)}\right)} = 0$$

i.e. by (3.1.3)

$$S^{(n)}_{\tilde{\alpha}(n)} = K, \forall n \in \mathbb{N}.$$

This means that for any number of periods n, the terminal distribution of the Tian model assigns probability mass to the point K. As a consequence, we get $\Delta_n = 1$, i.e. it does not depend on n.

It is left to show that the model suggested by Tian converges to the Black - Scholes value, i.e. we have to ensure weak convergence. Since we have $\beta = \sigma$, it is left to show that the sequence of drift parameters $(\widetilde{\alpha}(n))_n)$ is bounded. In that case, the moment matching conditions are satisfied for the risk-neutral transition probabilities and weak convergence follows by the Main theorem. Comparing to the CRR model, in Tian's model the mass points are moved by a small distance. This can be seen by writing the new drift $(\widetilde{\alpha}(n))_n)$ in terms of the original one. By (3.2.14) and (3.2.15):

$$\widetilde{\alpha}(n) = \frac{2\sigma}{\sqrt{T}\sqrt{n}} \left(a_\alpha(n) - l_\alpha(n)\right) + \alpha,$$

and this implies, since $-1 \leq a_\alpha(n) - l_\alpha(n) \leq 0$,

$$-\frac{2\sigma}{\sqrt{T}\sqrt{n}} + \alpha \leq \widetilde{\alpha}(n) < \alpha.$$

Observe that the new drift $\widetilde{\alpha}(n)$ in Tian's model differs from the original drift by

$$\widetilde{\alpha}(n) = \alpha + o(1). \tag{3.2.16}$$

Moreover, the new drift satisfies Assumption 1, i.e. $\widetilde{\alpha}(n) = O(1)$, and we can formulate:

Proposition 1. *The sequence of processes $\left(S^{(n)}_{\widetilde{\alpha}(n)}\right)_{n\in\mathbb{N}}$ defined by Tian's model with linear interpolation and an appropriate time-scaling converges weakly to the stock price process S.*

Since Δ_n is found to be independent of n, compared to other methods with constant drift α, Tian's model shows a more improved convergence behavior of the discretization error in the approximation of the terminal stock price. Let us now see how the Main theorem changes with Δ_n obtained by Tian's model.

Proposition 2. *Let $K \in \mathbb{R}$. The binomial model $S^{(n)}_{Tian}$ suggested by Tian admits the following representation of the discretization error:*

- *The discretization error of the digital call option is*

$$C_d(n) = e^{-rT}\Phi(d_2) + \frac{e^{-rT}e^{\frac{-d_2^2}{2}}}{\sqrt{2\pi}} \left[\frac{1}{\sqrt{n}} - \frac{d_2}{2n} + \frac{B_n}{n}\right] + O\left(\frac{1}{n}\right)$$

and

- *The discretization error of the price of a European call options satisfies*

$$C(n) = C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}}}{24\sigma\sqrt{2\pi T}} \frac{A_n}{n} + O\left(\frac{1}{n}\right),$$

where

$B_n = \frac{d_1^3 + d_1 d_2^2 + 2d_2 - 4d_1}{24} + \frac{(2 - d_1 d_2 - d_1^2)\sqrt{T}}{6\sigma}(r - \alpha\sigma^2) + \frac{T d_1}{2\sigma^2}(r - \alpha\sigma^2)^2,$

$A_n = -\sigma^2 T(6 + d_1^2 + d_2^2) + 4T(d_1^2 - d_2^2)(r - \alpha\sigma^2) - 12T^2(r - \alpha\sigma^2)^2.$

In Tian's model, the discretization error converges smoothly along the upper bound given by $\exp\left(\frac{-\frac{1}{2}d_2^2(x)}{\sqrt{2\pi n}}\right)$. The convergence for digital options is not faster, but for European call options we get convergence order $O\left(\frac{1}{n}\right)$. By smooth convergence we imply that the coefficient of the leading error term is constant and the oscillations of the higher order terms are negligible. As a result, the Berry Esséen bound remains tight and extrapolation methods can be applied.

As we already know, the CRR model is not smooth, but we have just shown that Tian's model is. The model suggested by Tian shows smaller pricing errors as more time steps are used, which is not the case in the CRR model. The rate of convergence is not improved, but this is not a drawback since extrapolation methods can be used to improve the rate of convergence.

We applied Tian's model to the same example as we did with the CRR and RB models, where irregular convergence and the even-odd problem was visible. In Figure 3.3 the even-odd problem is still noticeable, but the scallop effect is gone. To avoid the even-odd problem, in practice we usually use only even or odd step numbers. If we now look at Figure 3.4, we can notice the almost smooth convergence and that the even-odd oscillations have disappeared.

3.2.2 The Chang and Palmer tree

Similar to Tian's approach, Chang and Palmer [7] (CP from now on) suggested that we move our tree such that the strike value is placed on the geometric average of two adjacent nodes, i.e. to have $\Delta_n = 0$. The original drift α in this model is replaced by some sequence of drift parameters $(\underline{\alpha}(n))_{n\in\mathbb{N}}$ such that $K = \sqrt{S^{(n)}(l(n))S^{(n)}(l(n)-1)}$. As a consequence, the discretization error will exhibit a higher order of convergence. This model is also known as "the centered binomial model". Let us investigate this in more detail: Assume $S_{\alpha}^{(n)}$ is the binomial

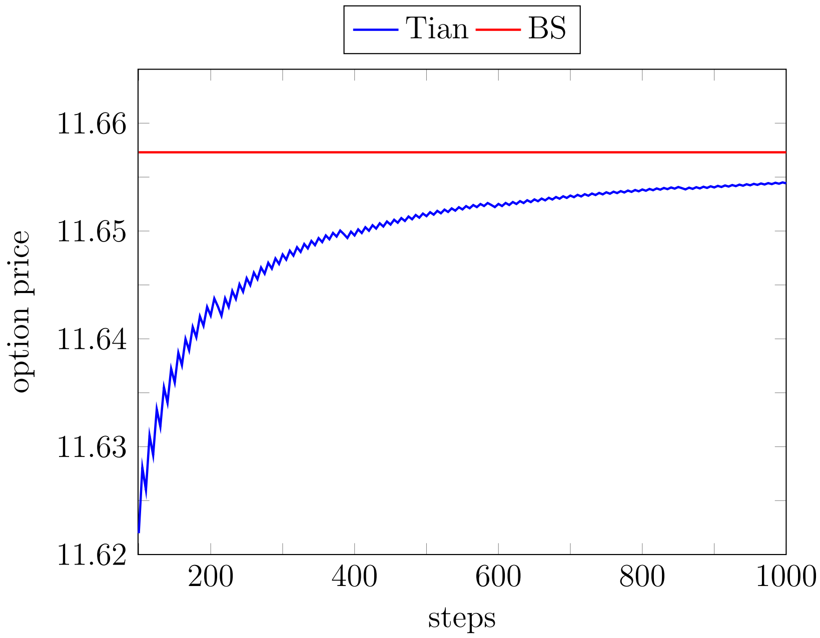


Figure 3.3: European call option price obtained using Tian model: $S_0 = 95$, $K = 100$, $\sigma = 0.25$, $r = 0.1$, $T = 1$, Black-Scholes =11.6573.

process (2.4.1) with $\beta = \sigma$ and $\alpha(n) = \alpha$ constant in n. As in Tian's model, $l_\alpha(n)$ denotes the corresponding integers such that $K \in \left(S_n^{(n)}(l_\alpha(n)-1), S_n^{(n)}(l_\alpha(n))\right]$. Then, the sequence for the new drift parameter will be given by

$$\underline{\alpha}(n) = \frac{\ln\left(\frac{K}{S_0}\right) - (2l_\alpha(n) - n - 1)\sigma\sqrt{\delta t}}{T}. \tag{3.2.17}$$

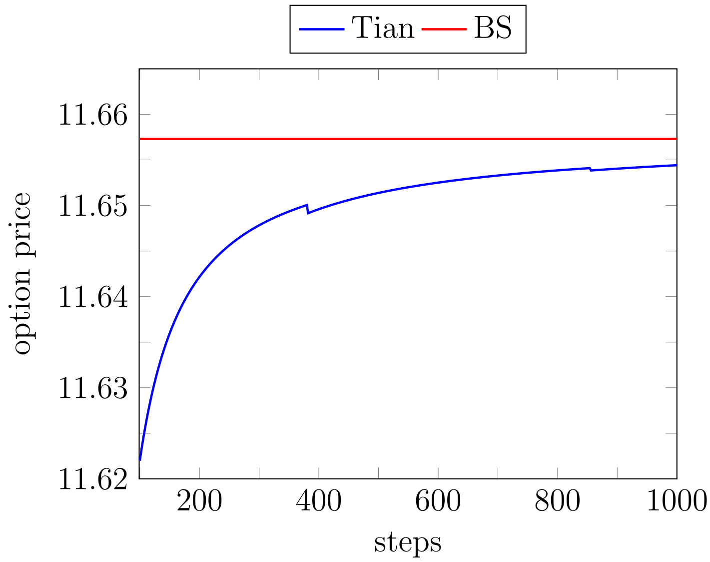
By (3.2.17) the CP model is defined such that the equation $s_0 u(n)^a d(n)^{n-a} = K$ is solved by

$$a_{\underline{\alpha}(n)}(n) := \frac{1}{2}n + \frac{\ln\left(\frac{K}{S_0}\right) - \underline{\alpha}(n)T}{2\sigma\sqrt{T}}\sqrt{n} = l_\alpha(n) - \frac{1}{2}.$$

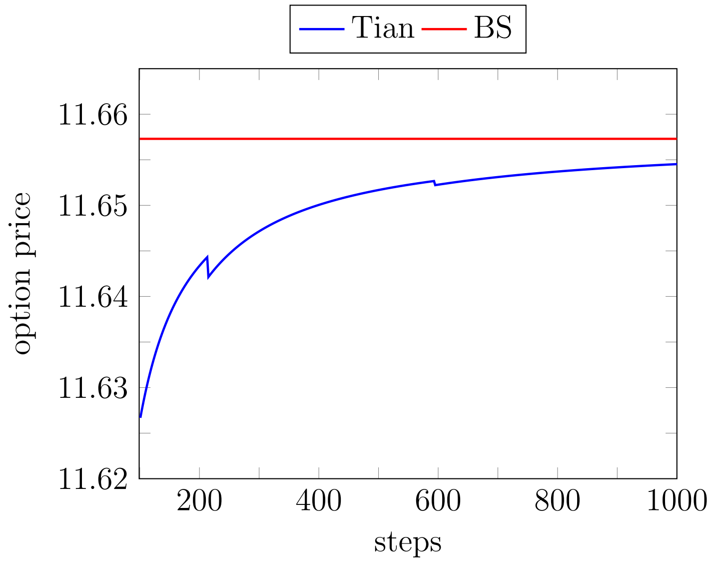
which leads us to

$$\frac{\ln\left(\frac{S^{(n)}_{\underline{\alpha}(n)}(l)}{K}\right)}{\ln\left(\frac{S^{(n)}_{\underline{\alpha}(n)}(l)}{S^{(n)}_{\underline{\alpha}(n)}(l-1)}\right)} = \frac{1}{2}$$

i.e. by (3.1.3)



(a) Tian with even number of steps



(b) Tian with odd number of steps

Figure 3.4: European call option price obtained using Tian model: $S_0 = 95$, $K = 100$, $\sigma = 0.25$, $r = 0.1$, $T = 1$, Black-Scholes =11.6573.

$$K = \sqrt{S^{(n)}_{\underline{\alpha}(n)}(l_\alpha(n)-1)S^{(n)}_{\underline{\alpha}(n)}(l_\alpha(n))}, \forall n \in \mathbb{N}. \tag{3.2.18}$$

As we can see from (3.2.18), for any number of periods n, the terminal distribution of the corresponding CP model is such that the point K is at the geometric average of two neighboring mass points. Now we can conclude, using (3.1.3), that in this model $\Delta_n = 0$ always and this will improve our order of convergence.

Our next aimis to show that we want to show $\underline{\alpha}(n) = \alpha + o(1)$. Since the probability mass is moved only by a small distance, this is a direct consequence. Similar to the case in Tian's model, we can conclude

$$\underline{\alpha}(n) = \frac{2\sigma}{\sqrt{T}\sqrt{n}}\left(a_\alpha(n) - l_\alpha(n) + \frac{1}{2}\right) + \alpha, \tag{3.2.19}$$

where $a_\alpha(n)$ is solution of the equation $S_0 u(n)^a d(n)^{n-a} = K$ in the original model. From (3.2.19)

$$\alpha - \frac{\sigma}{\sqrt{T}\sqrt{n}} \leq \underline{\alpha}(n) < \alpha + \frac{\sigma}{\sqrt{T}\sqrt{n}},$$

which shows our assertion.

We now can conclude these results in the following position.

Proposition 3. *The sequence of processes* $\left(S^{(n)}_{\underline{\alpha}(n)}\right)_{n\in\mathbb{N}}$ *defined by the Chang and Palmer model with linear interpolation and an appropriate time-scaling converges weakly to the stock price process* S.

Then, we can formulate the Main theorem as follows.

Theorem 7 (Main Theorem - Chang and Palmer)**.** *For the n-period binomial model, where*

$$u = e^{\alpha\sigma^2\Delta t + \sigma\sqrt{\Delta t}},\ d = e^{\alpha\sigma^2\Delta t - \sigma\sqrt{\Delta t}},$$

with α *an arbitrary bounded function of* n, *if the initial stock price is* S_0 *and the strike price is* K *with maturity time* T, *we have that*

1. *the price of a digital call option satisfies*

$$C_d(n) = e^{-rT}\Phi(d_2) + \frac{e^{-rT}e^{\frac{-d_2^2}{2}}}{\sqrt{2\pi}}\frac{B_n}{n} + O\left(\frac{1}{n}\right)$$

   and

2. *the price of a European call option satisfies*

$$C(n) = C_{BS} + \frac{S_0 e^{-\frac{d_1^2}{2}}}{24\sigma\sqrt{2\pi T}} \frac{A_n + 12\sigma^2 T}{n} + O(\tfrac{1}{n}),$$

where

$$B_n = \frac{d_1^3 + d_1 d_2^2 + 2d_2 - 4d_1}{24} + \frac{(2 - d_1 d_2 - d_1^2)\sqrt{T}}{6\sigma}(r - \alpha\sigma^2) + \frac{T d_1}{2\sigma^2}(r - \alpha\sigma^2)^2,$$

$$A_n = -\sigma^2 T(6 + d_1^2 + d_2^2) + 4T(d_1^2 - d_2^2)(r - \alpha\sigma^2) - 12T^2(r - \alpha\sigma^2)^2.$$

This theorem tells us that the order of convergence is improved from $\frac{1}{\sqrt{n}}$ to $\frac{1}{n}$ for both digital and European call options, i.e. we have

$$\limsup_{n\to\infty} \left\{ \sup_{K\in\mathbb{R}} \left| Q^{(n)}\left(S_{CP}^{(n)} \geq K\right) - \Phi(d_2(K)) \right| \sqrt{n} \right\} = 0$$

and

$$\limsup_{n\to\infty} \left\{ \sup_{K\in\mathbb{R}} \left| Q^{(n)}\left(S_{CP}^{(n)} \geq K\right) - \Phi(d_2(K)) \right| n \right\} > 0,$$

which means if the binomial process is defined according to the CP model, the Berry-Esséen bound ceases to be tight and, in addition, the leading discretization error term converges monotonically.

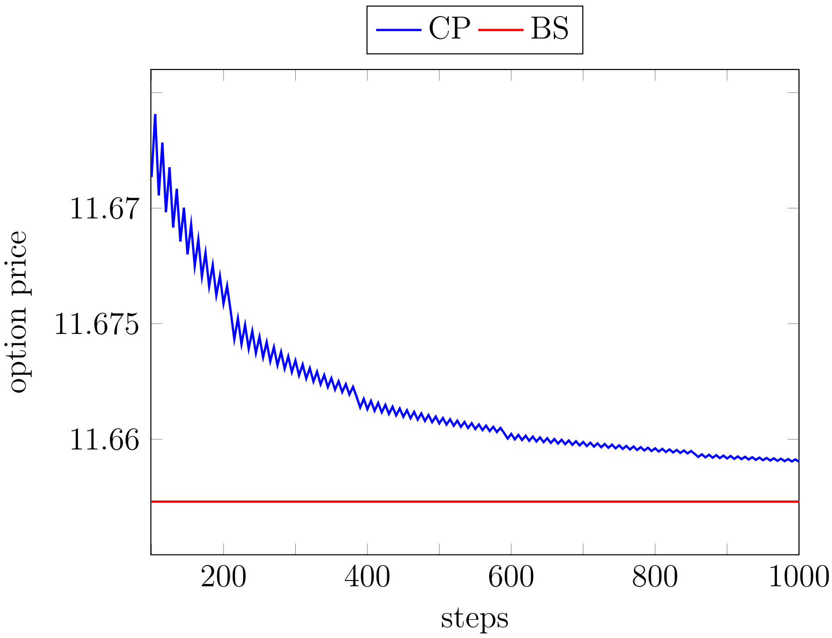
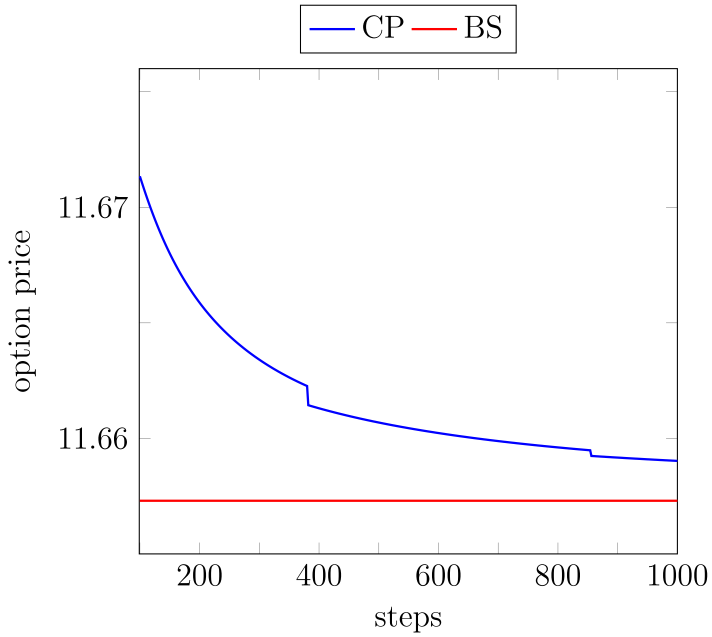
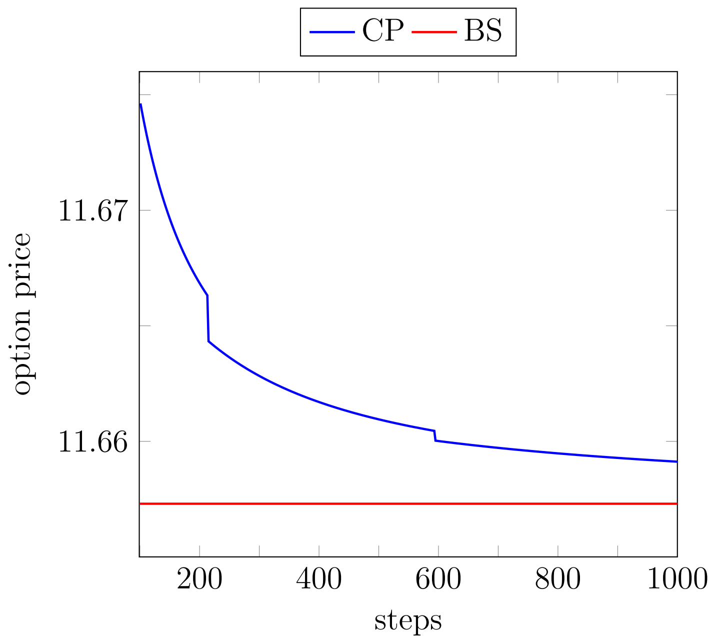


Figure 3.5: European call option prices obtained using CP model: $S_0 = 95$, $K = 100$, $\sigma = 0.25$, $r = 0.1$, $T = 1$, Black-Scholes =11.6573.



(a) CP with even number of steps



(b) CP with odd number of steps

Figure 3.6: European call option prices obtained using CP model: $S_0 = 95$, $K = 100$, $\sigma = 0.25$, $r = 0.1$, $T = 1$, Black-Scholes =11.6573.

Let us now see how the previous example works with the CP model. As mentioned before, also in Figure 3.5 the even-odd problem appears. On the other hand, in Figure 3.6 oscillations are not visible. From now on, we will use only even or odd step numbers.

Next we show CRR, Tian and CP model together in Figure 3.7. It is visible that only CRR model is oscillating. CP and Tian have almost the same behavior, but CP model is slightly faster.

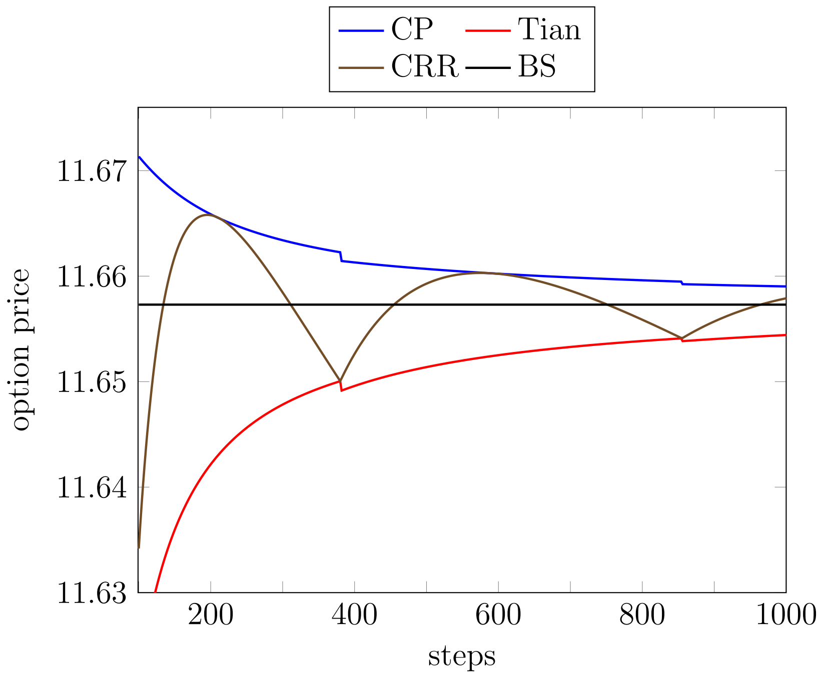


Figure 3.7: European call: $S_0 = 95$, $K = 100$, $\sigma = 0.25$, $r = 0.1$, $T = 1$, Black-Scholes =11.6573.

Tillting the tree so that the neighboring node hits the strike value, or the geometric average of the two neighboring nodes, in CP and Tian's model is done in the last step and it does not have to be that this adjacent node is in the middle. In the next chapter, we present a different model which has the same idea of tilting the tree so that we get better convergence, but carries it out in a slightly different way.

Chapter 4

The split tree

We have seen that Tian [47] had a good idea to shift a tree a little bit so that the strike price ends up on the closest node which leads to smoother convergence. Chang and Palmer[1] [7] followed this idea, but they shifted the tree so that the strike price ends up on the geometrical average of the neighboring nodes which leads to faster convergence. But for both models we get a well positioned strike price at the end of the tree. Would we have better results if the strike price is on a node somewhere in the middle of the tree and then continue with the plain CRR method, since we have shown that the CRR model works well for the case $S_0 = K$? This gives us an idea to combine the Tian/CP tree with the CRR tree. In [18], Joshi[2] suggested tjos idea and introduced the so-called Split tree. Combining two different models into one will be the main topic of this chapter. First we introduce the Split tree and afterwards we prove convergence of it. For this purpose we follow the work of G. Leduc[3] from [27]. Results are shown numerically and there we also discussed results for different splitting times.

In the Split tree we want our tree to be centered around the strike value in log scale. For that purpose, we will use a time dependent drift. As we have seen, in the CRR tree in log-scale there is no drift, i.e. $\alpha = 0$, which means that it is symmetric around its initial value. The main drawback of this tree is its slowness and non-smooth convergence, i.e. $O(\frac{1}{\sqrt{n}})$, hence we cannot improve it using extrapolation methods. But, if $S_0 = K$ we have shown that we have smooth convergence. This has motivated us to introduce the Split tree. The idea behind this tree is that we combine two trees. Let us say we have n steps. In the first $k = \lfloor \frac{n}{2} \rfloor$ steps we want to have a drift which will get the center of the tree at the same level as a strike, and after that we do not have any drift, i.e. we continue

[1] K. Palmer was really helpful with explaining the Main theorem personally

[2] M. Joshi was really helpful with explaining properties of this model personally.

[3] G. Leduc participated directly in proving convergence of the Split tree and getting the explicit error formula.

with the CRR tree

$$\alpha_1 = \frac{n \ln(\frac{K}{S_0})}{kT} \tag{4.0.1}$$

which leads us to

$$u_1 = e^{\frac{\ln\left(\frac{K}{S_0}\right)}{k} + \sigma\sqrt{\Delta T}} \tag{4.0.2}$$

and

$$d_1 = e^{\frac{\ln\left(\frac{K}{S_0}\right)}{k} - \sigma\sqrt{\Delta T}} \tag{4.0.3}$$

for the first half of the tree, and

$$\alpha_2 = 0, u_2 = e^{\sigma\sqrt{\Delta t}}, d_2 = \frac{1}{u}. \tag{4.0.4}$$

for the second half of the tree. This model was first introduced in [18]. First of all, we can conclude that Assumption 1 is satisfied - for both drifts. Let us now say we have Δ_{n_1} for the first half of the steps, and Δ_{n_2} for the rest (the non-constant factor defined in (3.1.2)). The tree in the first half can be considered as Tian tree (if we put $\frac{n}{2}$ in Tian's drift, we get the same drift as in the Split tree), so we can conclude that $\Delta_{n_1} = 1$. After that we always have that the strike value is on the node (actually in every second step), but the tree is symmetric around the strike value in the log space so there are no oscillations which means that $\Delta_{n_2} = 1$.

To avoid arbitrage opportunities, we have to have

$$d_1 < e^{r\Delta t} < u_1.$$

Let us investigate this property in more details. Before that we assume that $k \in (0, n)$, i.e. we allow to split the tree at any time. Here we excluded 0 and n since, if $k = 0$, it means we already have $S_0 = K$ and if $k = n$ then we do tilting until the end, i.e. there is no splitting. The above no arbitrage condition leads to the relation

$$\begin{aligned} \frac{1}{k}\ln\left(\frac{K}{S_0}\right) - \sigma\sqrt{\Delta t} < r\Delta t < \frac{1}{k}\ln\left(\frac{K}{S_0}\right) - \sigma\sqrt{\Delta t} \\ \iff \ln\left(\frac{K}{S_0}\right) < k(r\Delta t + \sigma\sqrt{\Delta t}) < \ln\left(\frac{K}{S_0}\right) + 2\sigma\sqrt{\Delta t}. \end{aligned}$$

The expression $r\Delta t + \sigma\sqrt{\Delta t}$ is always positive and it will not cause any problems, but $r\Delta t - \sigma\sqrt{\Delta t}$ might be negative. Therefore we will consider two cases:

1. $$k > \frac{\ln\left(\frac{K}{S_0}\right)}{r\Delta t + \sigma\sqrt{\Delta t}}$$

which yields a condition on k only reasonable for $K > S_0$, since $r\Delta t + \sigma\sqrt{\Delta t} > 0$.

2. If $r\Delta t - \sigma\sqrt{\Delta t} < 0$ then

$$k(r\Delta t - \sigma\sqrt{\Delta t} < 0) < \ln\left(\frac{K}{S_0}\right),$$

which only yields a reasonable condition on k for $K < S_0$. In this case we have

$$k > \frac{\ln\left(\frac{K}{S_0}\right)}{r\Delta t - \sigma\sqrt{\Delta t}}.$$

In our model, we will assume $r\Delta t - \sigma\sqrt{\Delta t} < 0$ since we are interested in the case when $n \to \infty$. This yields that we do not have an upper bound for k. This conclusion is also expected, since if we do not split at all we will end up with Tian tree. Another interesting case is when $S_0 = K$. If we plug this into u_1 and d_1, we end up with a CRR model, i.e. there is no splitting needed since we are already in a desired position.

We summarize the method in the form of an algorithm as follows

Algorithm for the Split tree

1. Tree Initialization
   Divide the tree into 2 subtrees:

   - For $m = 1, 2, ..., k-1$ do

     $$S^n(m+1) = S^n(m)e^{\frac{\ln K - \ln S}{k} + \sigma\sqrt{\Delta t}Z^n_{k+1}}, S_0 = s_0.$$

     For $m = k, ..., n-1$ do

     $$S^n(m+1) = S^n(m)e^{\sigma\sqrt{\Delta t}Z^n_{k+1}}.$$

   - Calculate the option value at maturity for all possible values of S^n_n

     $$V_n(n) = g\left(S^n_n\right).$$

2. Backward induction

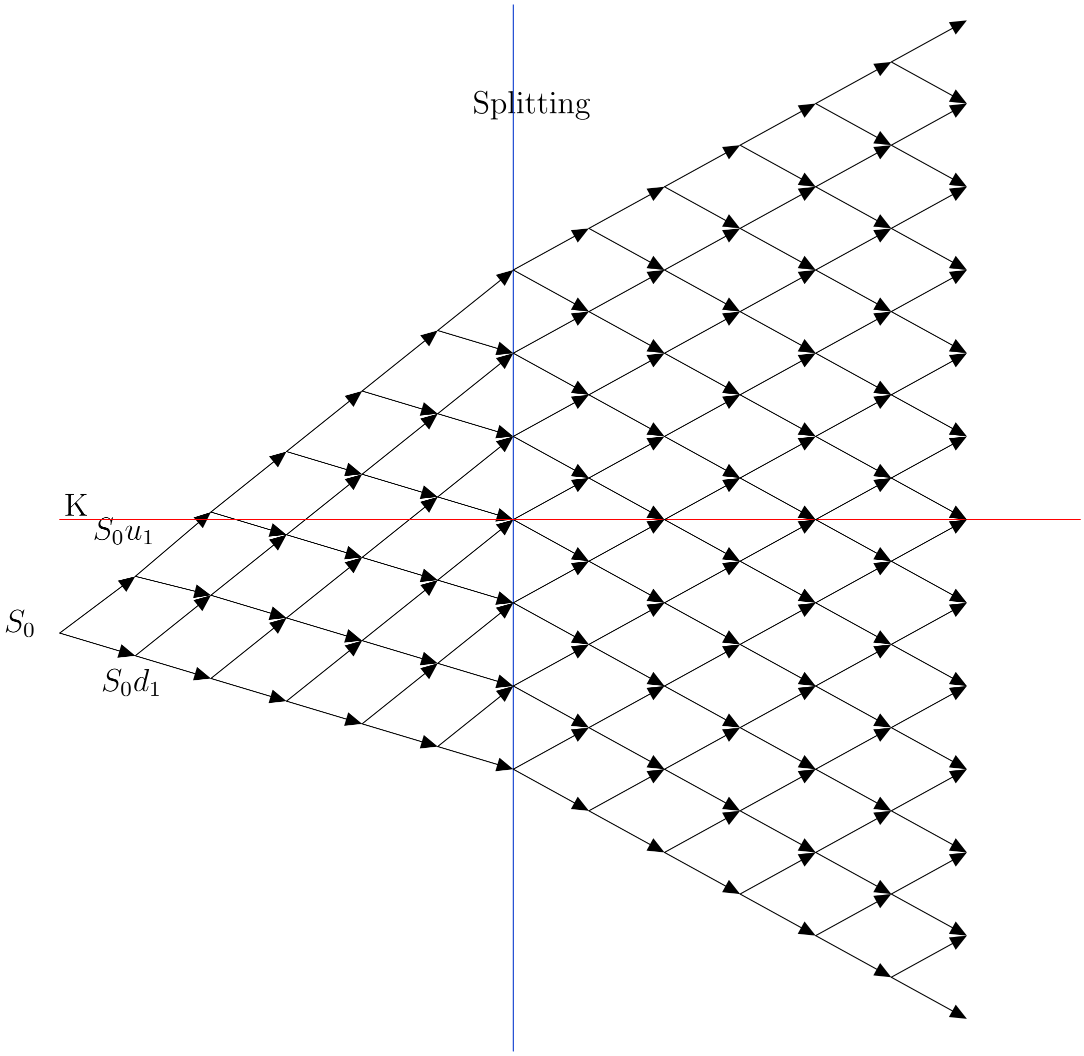


Figure 4.1: Split tree

- For $i = n-1, ..., k+1$ calculate the value of the option by

$$V_n(i\Delta t, S_i^{(n)}) = e^{-r\Delta t}\Big[p_2 V^{(n)}((i+1)\Delta t, u_2 S_i^{(n)}) \\ + (1-p_2)V_{(n)}((i+1)\Delta t, d_2 S_i^{(n)}\Big].$$

- For $i = k, ..., 1$ calculate the value of the option by

$$V_n(i\Delta t, S_i^{(n)}) = e^{-r\Delta t}\Big[p_1 V^{(n)}((i+1)\Delta t, u_1 S_i^{(n)}) \\ + (1-p_1)V_{(n)}((i+1)\Delta t, d_1 S_i^{(n)}\Big].$$

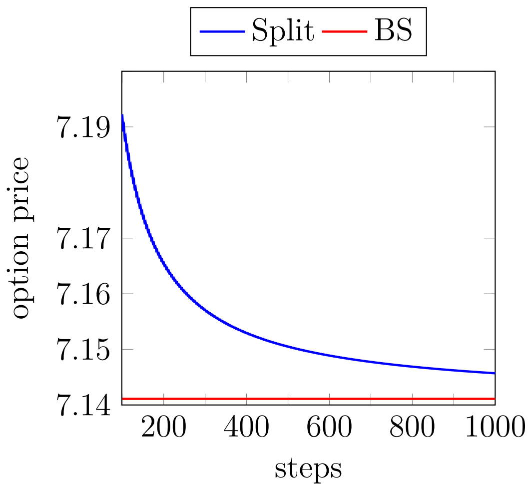
3. Return $e^{-rT}V_n(0)$.

We are now interested in the following questions:

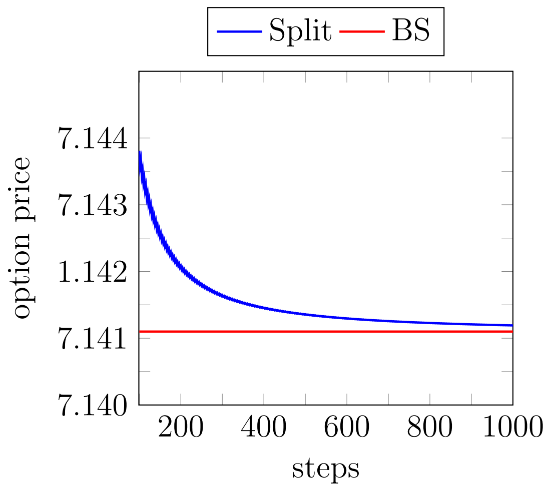
1. Is it the best way to split the tree in the middle, i.e. with respect to time?
2. Do we have weak convergence?
3. What is the order of convergence?
4. Is the Split tree really good?

4.1 Optimal time for splitting the tree - best tilting

Joshi introduced the Split tree as we have defined it in the previous section in [18]. In that paper he splits the tree after $k = \lfloor \frac{n}{2} \rfloor$ steps. We now want to investigate what will happen if we split the tree earlier or later. First, we will show numerically how does the Split tree behave for different time of splitting and then we will prove it afterwards. In Figure 4.2 we can see results for European put options where $K = 100$, $S_0 = 95$, $r = 0.1$, $\sigma = 0.25$, $n = 100 : 2 : 4000$ and the black line is the Black - Scholes value for these parameters. As we can see in Figure 4.2, we have better performance in the second picture, when we do splitting later, i.e. convergence is faster. This gives us some space for research.



(a) Splitting after 1/4 steps



(b) Splitting after 3/4 steps

Figure 4.2: European put option prices obtained using Split model: $S_0 = 95$, $K = 100$, $\sigma = 0.25$, $r = 0.1$, $T = 1$, Black-Scholes =7.1411.

n	1/4	1/2	3/4	CP	Tian
100	7.1923	7.1559	7.1438	7.1551	7.1057
200	7.1656	7.1480	7.1421	7.1496	7.1259
400	7.1530	7.1443	7.1415	7.1450	7.1333
500	7.1505	7.1436	7.1414	7.1444	7.1351
800	7.1469	7.1426	7.1412	7.1434	7.1376
1000	7.1457	7.1423	7.1412	7.1428	7.1382
2000	7.1434	7.1417	7.1411	7.1420	7.1397
4000	7.1422	7.1414	7.1411	7.1415	7.1404

Table 4.1: European put: $S_0 = 95$, $K = 100$, $\sigma = 0.25$, $r = 0.1$, $T = 1$, Black - Scholes value = 7.1411.

These results can be also seen in Table 4.1. In this table, we compared the results obtained by splitting after $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$ of the steps, from the CP model and results obtained from the Tian model. At the first glance, we can conclude that all models are smooth. Second, we can notice better performance for later splitting. Note that by splitting after $k = \frac{3}{4}n$ steps, we already have after 800 steps the best result from CP and Tian, while CP and Tian need $n = 4000$. On top of this, only for the split tree with $k = \frac{3}{4}n$, we obtain the exact result. To verify the order of convergence we used a log-log plot (Figure 4.3). We plotted the errors of the Split tree for splitting after $\frac{n}{4}$, $\frac{n}{2}$ and for $\frac{3n}{4}$, then Tian, CP and CRR. It

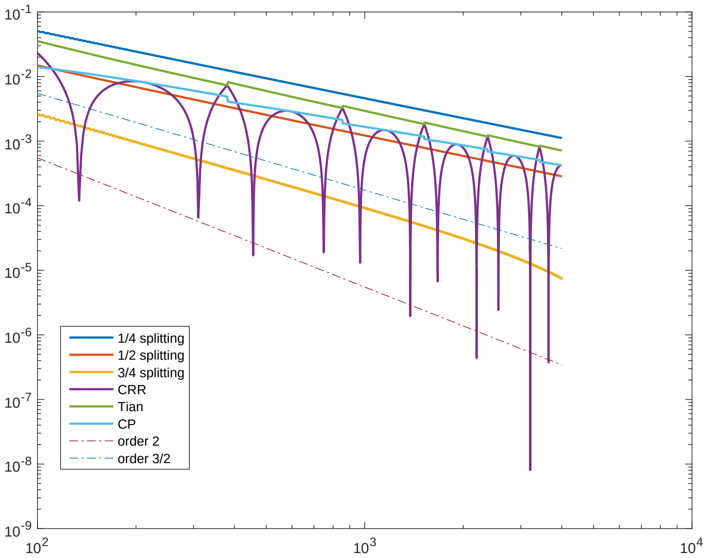


Figure 4.3: Order of convergence

is visible that splitting after $\frac{3n}{4}$ has the best performance and it is the only tree which has order of convergence between $\frac{3}{2}$ and 2. More about the comparison between the models can be found in the next section.

4.2 Convergence of the Split tree

As it is not clear at all how fast the split tree converges, the main aim of this section is to find a first-order error formula for approximation of S_t by $\{S_t^{(n)}\}_{n=1}^{\infty}$. For that purpose, we will follow the work of G. Leduc from [27]. Let us first introduce the flexible CRR tree. In this model, we define the probability p_n of jumping from the current state $S_t^{(n)}$ to $S_t^{(n)}u_n$ as

$$p_n := \frac{e^{r\Delta t} - d_n}{u_n - d_n},$$

where

$$\begin{aligned} u_n &: = \exp\left(\sigma\sqrt{\Delta t} + \alpha\sigma^2\Delta t + \mu_n \frac{2\sigma}{T}(\Delta t)^{\frac{3}{2}}\right), \\ d_n &: = \exp\left(-\sigma\sqrt{\Delta t} + \alpha\sigma^2\Delta t + \mu_n \frac{2\sigma}{T}(\Delta t)^{\frac{3}{2}}\right), \end{aligned}$$

with $|\mu_n| \leq \mathcal{L}$, for some $\mathcal{L} > 0$. As we can notice, for the case when $\alpha = 0$ and $\mu_n = 0$ we have the classic CRR scheme, and for $\alpha = \frac{r-\frac{1}{2}\sigma^2}{\sigma^2}$, $\mu_n = 0$ we end up with the RB tree.

Let us take a closer look at our Split tree. There we have

$$S^{(n)}(k+1) = S^{(n)}(k)e^{\frac{\ln K - \ln S}{k} + \sigma\sqrt{\Delta t}Z^n_{k+1}}$$

for the first part, and

$$S^{(n)}(k+1) = S^{(n)}(k)e^{\sigma\sqrt{\Delta t}Z^n_{k+1}}$$

for the rest, which leads us to

$$\begin{aligned}\alpha_1 &= \frac{\ln K - \ln S_0}{k\sigma^2\Delta t}, \mu_{n_1} = 0,\\ \alpha_2 &= 0, \mu_{n_2} = 0.\end{aligned}$$

This means that we can apply already known results for the flexible CRR tree also to our case. For that purpose, let us first check the first two moments:

$$\mu(n) := \frac{1}{\Delta t}E_{P^n}\left(\ln\left(\frac{S^n(k+1)}{S^n(k)}\right)|S^n(k)\right) = \frac{\ln K - \ln S}{k} + \sigma\sqrt{\frac{1}{\Delta t}}E_{P^n}(Z_{k+1}),$$

$$\sigma^2(n) := \frac{1}{\Delta t}Var_{P^n}\left(\ln\left(\frac{S^n(k+1)}{S^n(k)}\right)|S^n(k)\right) = \sigma^2 Var_{P^n}\left(Z^n_k\right),$$

and for the second part of the tree

$$\mu(n) := \frac{1}{\Delta t}E_{P^n}\left(\ln\left(\frac{S^n(k+1)}{S^n(k)}\right)|S^n(k)\right) = \sigma\sqrt{\frac{1}{\Delta t}}E_{P^n}(Z_{k+1}),$$

$$\sigma^2(n) := \frac{1}{\Delta t}Var_{P^n}\left(\ln\left(\frac{S^n(k+1)}{S^n(k)}\right)|S^n(k)\right) = \sigma^2 Var_{P^n}\left(Z^n_k\right).$$

In Chapter 2.4 we have seen that for convergence we need to match the first two moments at least asymptotically and to have a bounded drift, i.e. the following assumption has to be satisfied.

Assumption 1: *The sequence* $(\alpha(n))_n$ is assumed to be bounded, i.e. it is assumed to be of order $O(1)$.

Since $k \leq n$, we have that $\alpha_1 = \frac{\ln K - \ln S}{k\sigma^2\Delta t}$ and $\mu_1 = \frac{\ln K - \ln S}{k\Delta t}$ are bounded, i.e. our new $(\alpha(n))_n$ satisfies Assumption 1 and knowing our results from Chapter 2, we can conclude that as $n \to \infty$,

$$\begin{aligned}\mu(n) &\to r - \tfrac{1}{2}\sigma^2\\ \sigma^2(n) &\to \sigma^2.\end{aligned}$$

So, we have shown that the moment-matching conditions, in both parts of the split tree where we are given that $k(n)$ and $n-k(n)$ go to infinity for $n \to \infty$, of the one-period log-return are fulfilled and $\beta = \sigma$, which implies weak convergence. Now it is left to investigate the rate of convergence.

Let us assume we have n steps and let $\mathbb{N} \ni k \in [0, n]$. If Δ_1 is for the first part of the tree (Tian), and Δ_2 for the second part (CRR), then we have, since

$$\Delta_n = 1 - 2\left\{\frac{\ln\left(\frac{S_0}{K}\right) + n\ln(d)}{\ln\left(\frac{u}{d}\right)}\right\}$$

that

$$\begin{aligned}\Delta_1 &= 1 - 2\left\{\frac{\ln\left(\frac{S_0}{K}\right) + k\left(\frac{\ln\left(\frac{K}{S_0}\right)}{k} - \sigma\sqrt{\Delta t}\right)}{2\sigma\sqrt{\Delta t}}\right\}\\ &= 1 - 2\left\{-\frac{k}{2}\right\} = \begin{cases}1 & \text{for } k \text{ even}\\ 0 & \text{for } k \text{ odd}\end{cases}\end{aligned}$$

and

$$\Delta_2 = 1 - 2\left\{-\frac{k\sigma\sqrt{\Delta t}}{2\sigma\sqrt{\Delta t}}\right\} = 1 - 2\left\{-\frac{k}{2}\right\} = \begin{cases}1 & \text{for } n-k \text{ even}\\ 0 & \text{for } n-k \text{ odd.}\end{cases}$$

If n is even, then we have that both Δ_1 and Δ_2 are equal to 1. However, if n is odd, we have $\Delta_1 = 1$ and $\Delta_2 = 0$ or vice versa.

Our next aim is to investigate the rate of convergence of the Split tree and for that purpose we follow the work of [27]. In this paper, Leduc found a general first-order error formula of the flexible CRR tree for European options which motivated us to search for a general error formula for the Split tree, i.e. to find an error formula depending on splitting time.

We are now interested in payoff functions which are piecewise smooth. For that purpose we consider a payoff function g which is piecwise $\in C^{(3)}$ and

$$|g^{(l)}(x)| \leq Q(1 + x^p) \text{ for } l = 0, 1, 2, 3 \text{ and every } x \geq 0, \tag{4.2.1}$$

for some integer $p \geq 1$ and some $Q \in \mathbb{R}$. By g is piecewise $C^{(3)}$ function we mean there exists a partition $0 < K_1 < ... < K_N < \infty$ of $[0, \infty)$ and $N+1$ functions $g_0, ..., g_N \in C^{(3)}$ such that

$$g = g_0 1_{[0,K_1)} + g_1 1_{[K_1,K_2)} + ... + g_N 1_{[K_N,\infty)}.$$

This class of payoffs we will call $\mathcal{K}_p^{(3)}$. We define a norm $||\cdot||_p^{(3)}$ on $\mathcal{K}_p^{(3)}$ being equal to the smallest value of Q such that (4.2.1) holds. For any integer $m \geq 0$ we define $\mathcal{K}_p^{(m)}$ and $||\cdot||_p^{(m)}$ analogously.

Let us assume that we want to split the tree after k steps, $\mathbb{N} \ni k \in (0, n)$, which means that we now have the freedom to choose when to split the tree. Then t_k is the maturity time for the first part of the tree, and $T - t_k$ is the maturity time for the second part of the tree. Let $\mathcal{E}_{t_k}^{1,n,\alpha} g\,(x)$ and $\mathcal{E}_{T-t_k}^{2,n} g\,(x)$ be defined as the discounted expectation of a payoff function g when the spot price is x and the maturity is T. The first discounted expectation relates to the first part of our tree, i.e. here t_k is the maturity time and α tells us there is a drift included. The second discounted expectation relates to the second part of the tree, i.e. the maturity time for this part of the tree is $T - t_k$ and we do not have any drift. We are now interested in finding an explicit first order error formula for

$$\mathcal{E}_{t_k}^{1,n,\alpha}\left(\mathcal{E}_{T-t_k}^{2,n} g\right)(x).$$

Notation For every $t, x \geq 0$ and a polynomially bounded function g we have

$$\begin{aligned} \mathcal{E}_t g(x) &:= e^{-rt} E_x(g(S_t)), \\ \mathcal{E}_t^{(n)} g(x) &:= e^{-rt} E_x(g(S_t^{(n)})), \end{aligned}$$

which tells us that $\mathcal{E}_t$ and $\mathcal{E}_t^{(n)}$ are discounted expectations. Since they are semi-group operators and independent, we have $\mathcal{E}_{t+s} g = \mathcal{E}_t \mathcal{E}_s g$, $\mathcal{E}_{t+s}^{(n)} g = \mathcal{E}_t^{(n)} \mathcal{E}_s^{(n)} g$ and $\mathcal{E}_t^{(n)} \mathcal{E}_s g = \mathcal{E}_s \mathcal{E}_t^{(n)} g$.

If $g \in \mathcal{K}_p^{(3)}$, then g can be split into a linear combination of a function which is differentiable and in $\mathcal{K}_p^{(3)}$ plus digital options and call options, i.e.

$$g(x) = h(x) + \sum_{l=1}^{N} \Delta g(K_l) 1_{[K_l,\infty)}(x) + \sum_{l=1}^{N} \Delta g'(K_l) \max(x - K_l, 0), \qquad (4.2.2)$$

where h is also in $C^{(1)}$ and belongs to $\mathcal{K}_p^{(3)}$. From [7] and [10] the error formulae for digital and call options are known, now the idea is to find the error formulae for the $C^{(1)}$ part of g, i.e. for h. For simplicity, we will focus on continuous payoff functions g. For that purpose we will again use our Main theorem.

If $S_0 = x$ is given, then for the European option with payoff g, maturity T obtained by pricing with a flexible CRR scheme, an error is expressed by

$$\begin{aligned} Err_T^n(g)(x) &:= e^{-rT}E_x(g(g(S)) - e^{-rT}E_x(g(g(S_T^{(n)})) \\ &:= \mathcal{E}_t g(x) - \mathcal{E}_t^{(n)} g(x) \end{aligned}$$

Let us recall once again the main theorem, but this time we will write it up with the new notation so that it is easier to follow the work of [27].

Theorem 8 (Main Theorem - Chang and Palmer)**.** *For the n-period binomial model, where*

$$u = e^{\sigma\sqrt{\Delta t}+\alpha\sigma^2\Delta t},\ d = e^{-\sigma\sqrt{\Delta t}+\alpha\sigma^2\Delta t},$$

with α an arbitrary bounded function of n. For every $x > 0$, the error of a European call option $Err_T^n(C_K)(x)$ satisfies

$$\mathcal{E}^{2,n}_{T-t_k} h\left(x\right) = \mathcal{E}_{T-t_k} h\left(x\right) + \frac{\Lambda^n_{T-t_k}(x)\left(\frac{n}{n-k}\right)}{n} + O\left(n^{-3/2}\right)$$

where

$$\begin{aligned} \Lambda^n_{T-t_k}(x) &= \frac{xe^{-\frac{\mathfrak{d}_1^2}{2}}}{24\sigma\sqrt{2\pi\left(T-t_k\right)}}\left(A - 12\sigma^2\left(T-t_k\right)\left(\Delta^2_{n-k}-1\right)\right), \\ A &= -\sigma^2\left(T-t_k\right)\left(6+\mathfrak{d}_1^2+\mathfrak{d}_2^2\right) - 12\left(T-t_k\right)^2\left(r\right)^2 + 4\left(T-t_k\right)\left(\mathfrak{d}_1^2-\mathfrak{d}_2^2\right)\left(r\right), \\ \Delta_{n-k} &= 1 - 2\left\{\frac{\ln\left(x/K\right) + n\ln\left(d_2\right)}{\ln\left(u_2/d_2\right)}\right\}, \\ \mathfrak{d}_1 &= \frac{\left(\ln\left(\frac{x}{\mathsf{K}}\right)+\left(r+\frac{1}{2}\sigma^2\right)\left(T-t_k\right)\right)}{\sigma\sqrt{\left(T-t_k\right)}} \\ \mathfrak{d}_2 &= \frac{\left(\ln\left(\frac{x}{\mathsf{K}}\right)+\left(r-\frac{1}{2}\sigma^2\right)\left(T-t_k\right)\right)}{\sigma\sqrt{\left(T-t_k\right)}} = \mathfrak{d}_1 - \sigma\sqrt{\left(T-t_k\right)}. \end{aligned}$$

Adding our discussion about Δ, we get

1. Case: $\Delta_1 = 1$ and $\Delta_2 = 1$
   We have $\Lambda^n_{T-t_m}(x) = \frac{xAe^{-\frac{\mathfrak{d}_1^2}{2}}}{24\sigma\sqrt{2\pi(T-t_m)}}$ in both parts
2. Case: $\Delta_1 = 1$ and $\Delta_2 = 0$
   We have $\Lambda^n_{T-t_m}(x) = \frac{xAe^{-\frac{\mathfrak{d}_1^2}{2}}}{24\sigma\sqrt{2\pi(T-t_m)}}$ for the first part, and for the second $\Lambda^n_{T-t_m}(x) = \frac{x(A+12\sigma^2 T)e^{-\frac{\mathfrak{d}_1^2}{2}}}{24\sigma\sqrt{2\pi(T-t_m)}}$, which is identical to $\Lambda^n_{T-t_m}(x)$ with A replaced by $\widetilde{A} = A + 12\sigma^2 T$ so it has identical convergence behavior.

This result will help us to understand better the rate of convergence. Another important result from [27] is given as follows.

Theorem 9 (General first-order error formula)**.** *Let $\{S^{(n)}\}$ be a flexible CRR scheme and let $p \geq 1$. For every continuous g in $\mathcal{K}_p^{(3)}$, if $0 < K_1 < ... < K_N < \infty$ defines a partition of $[0, \infty)$ for which g is $C^{(1)}$ on the corresponding closed subintervals, then for every $x > 0$*

$$Err^n_T(g)(x) = \frac{\Upsilon_T(g,x) + \sum_{l=1}^{N} \Delta g'(K_l)\Lambda^n_T(K_l, x)}{n} + O\left(n^{-\frac{3}{2}}\right), \tag{4.2.3}$$

where

$$\begin{aligned}
\Upsilon_T(g,x) &= \left(\frac{1}{2}\hat{\Delta}_2 - \frac{1}{3}\hat{\Delta}_3 + \frac{1}{4}\hat{\Delta}_4\right) e^{-rT} E_x\left(S_T^2 h''(S_T)\right) \\
&+ \frac{1}{24}\frac{4\hat{\Delta}_3 - 5\hat{\Delta}_4}{\sigma\sqrt{T}} e^{-rT} E_x\left(S_T^2 h''(S_T)\eta_T\left(\frac{S_T}{x}\right)\right) \\
&+ \frac{1}{24}\frac{\hat{\Delta}_4}{T\sigma^2} e^{-rT} E_x\left(S_T^2 h''(S_T)\left(\eta_T^2\left(\frac{S_T}{x}\right) - 1\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
\eta_T(z) &= \frac{\ln(z) - (r - \frac{1}{2}\sigma^2 T)}{\sigma\sqrt{T}}, \\
\hat{\Delta}_2 &= -\sigma^4 T^2\alpha + \alpha^2\sigma^4 T^2 + r^2 T^2 + rT^2\sigma^2 + \frac{5}{12}\sigma^4 T^2 - 2T^2 r\sigma^2\alpha, \\
\hat{\Delta}_3 &= 2rT^2\sigma^2 - 2\sigma^4 T^2\alpha + 2\sigma^4 T^2, \\
\hat{\Delta}_4 &= 2\sigma^4 T^2.
\end{aligned}$$

Since the proof of this theorem is rather technical, we will focus only on the main things which we need for our problem. A detailed proof can be found in [27].

Let us first state some important properties of the flexible CRR scheme. The next lemma, also from [27], tells us that the values obtained by the flexible CRR scheme are indeed close to the real values.

Lemma 2 (Properties of $\{S^{(n)}\}_{n=1}^{\infty}$). *Every flexible CRR scheme $\{S^{(n)}\}_{n=1}^{\infty}$ satisfies the following properties:*

P1 (Berry-Esseen property) *There exists a constant Q such that for every $t \in \frac{T}{n}\mathbb{N}$, in the interval $\frac{T}{2} \leq t \leq T$,*

$$\sup_z |F_{S_t^{(n)}}(z) - F_{S_t}(z)| \leq Q n^{-\frac{1}{2}}.$$

where $F_{S_t^{(n)}}$ and F_{S_t} are cumulative distribution functions of $S_t^{(n)}$ and S_t, with $S_0^{(n)} = S_0$.

P2 (Local estimate of the distance to 1) *For integers $k \geq 0$ and I the identity function,*

$$\delta_k^{(n)} := \mathcal{E}_{\Delta t}^n(|I-1|^k)(1) = O\left(n^{-\frac{k}{2}}\right).$$

P3 (Local error of the difference from 1) *For integers $k = 2, 3, 4$, there exists $\hat{\Delta}_k$ such that*

$$\begin{aligned}
\Delta_k^{(n)} &:= Err_{\Delta t}^n((I-1)^k)(1) \\
&= e^{-r\Delta t} E_1((S_{\Delta t} - 1)^k - (S_{\Delta t}^{(n)} - 1)^k) \\
&= \frac{\hat{\Delta}_k}{n^2} + O\left(n^{-\frac{5}{2}}\right).
\end{aligned}$$

P4 (Local and global estimates for log and power functions) *We have*

$$\mathcal{E}_{\Delta t}^n(|\ln I|)(1) = O\left(n^{-\frac{1}{2}}\right).$$

Furthermore, for every fixed real number γ,

$$\mathcal{E}_{\Delta t}^n(I^\gamma) = \mathcal{E}_{\Delta t}(I^\gamma) + O\left(n^{-2}\right)$$

and consequently

$$\begin{aligned}
\mathcal{E}_{\Delta t}^n(I^\gamma - 1) &= O\left(n^{-\frac{1}{2}}\right), \\
\max_{j=0,\ldots,n} |\mathcal{E}_{j\Delta t}^n(I^\gamma)(x) - \mathcal{E}_{j\Delta t}(I^\gamma)(x)| &= x^\gamma O\left(n^{-1}\right), \\
\max_{j=0,\ldots,n} |\mathcal{E}_{j\Delta t}^n(I^\gamma)(x)| &= x^\gamma O(1).
\end{aligned}$$

P5 (Remainder related local estimate) *For any integer β and any integer $N > 0$,*

$$\mathcal{E}^{n}_{\Delta t}\left(\left|\int_{1}^{I} u^{\beta}(I-u)^{N} du\right|\right)(1) = O\left(n^{-\frac{N+1}{2}}\right).$$

Next we want to introduce Error localization and Error localization expansion formula. We want to have a step-by-step approach of the error corresponding to the steps in the binomial approach. It is already known that

$$\mathcal{E}^{(n)}_{\Delta t} g(x) = g\left(xu(n)\right) p(n) + g\left(xd(n)\right)\left(1-p(n)\right)$$

is a discounted expectation in a single - time step tree, while $\mathcal{E}^{(n)}_{t_j} g(x)$ is for a j-time-step tree. This leads us to

$$\mathcal{E}^{(n)}_{t_{j+1}} g = \mathcal{E}^{(n)}_{\Delta t} g\left(\mathcal{E}^{(n)}_{t_j} g\right)$$

i.e,

$$\mathcal{E}^{(n)}_{t_{j+1}} g = \underbrace{\mathcal{E}^{(n)}_{\Delta t} g\left(\mathcal{E}^{(n)}_{\Delta t} g\left(\ldots\left(\mathcal{E}^{(n)}_{\Delta t} g\right)\right)\right)}_{j+1 \text{ times}}. \tag{4.2.4}$$

Because of the semigroup and commutation properties of $\mathcal{E}^{(n)}$, we have

$$\mathcal{E}^{(n)}_{t_j+t_k} g = \mathcal{E}^{(n)}_{t_j}\mathcal{E}^{(n)}_{t_k} g = \mathcal{E}^{(n)}_{t_k}\mathcal{E}^{(n)}_{t_j} g,$$

which can be extended to

$$\mathcal{E}^{(n)}_{t_j}\mathcal{E}_{t_k} g = \mathcal{E}_{t_k}\mathcal{E}^{(n)}_{t_j} g.$$

This tells us that we can evaluate an option in 2 steps: let T be a maturity time and t_j is some time before maturity. Then, one can first value the option with the maturity of $T - t_j$, which depends on the stock price, and then consider it as an option with maturity t_j to get the value at time 0. We can use an even more general approach, similar like in (4.2.4): we first evaluate an option with maturity Δt, then use this as a payoff of another option with maturity Δt and continue this way until we reach the full maturity T. So, our first step is to calculate $\mathcal{E}^{(n)}_{\Delta t} g$. Since the goal is to estimate $\mathcal{E}_T$ using $\mathcal{E}^{(n)}_T$, using this procedure we get

$$\mathcal{E}^{(n)}_{\Delta t} g = \mathcal{E}_{\Delta t} g - (\mathcal{E}_{\Delta t} g - \mathcal{E}^{(n)}_{\Delta t} g) = \mathcal{E}_{\Delta t} g - Err^n_{\Delta t} g.$$

Then in the second step

$$\mathcal{E}^{(n)}_{2\Delta t} g = \mathcal{E}^{(n)}_{\Delta t}(\mathcal{E}^{(n)}_{\Delta t} g) = \mathcal{E}^{(n)}_{\Delta t}(\mathcal{E}_{\Delta t} g - Err^n_{\Delta t} g) = \mathcal{E}^{(n)}_{\Delta t}(\mathcal{E}_{\Delta t} g) - \mathcal{E}^{(n)}_{\Delta t}(Err^n_{\Delta t} g),$$

and since

$$\mathcal{E}^{(n)}_{\Delta t}\mathcal{E}_{\Delta t} g = \mathcal{E}_{\Delta t}\mathcal{E}_{\Delta t} g - (\mathcal{E}_{\Delta t}\mathcal{E}_{\Delta t} g - \mathcal{E}^{(n)}_{\Delta t}\mathcal{E}_{\Delta t} g) = \mathcal{E}_{2\Delta t} g - Err^n_{\Delta t}\mathcal{E}_{\Delta t} g,$$

we get

$$\mathcal{E}^{(n)}_{2\Delta t} g = \mathcal{E}_{2\Delta t} g - \mathcal{E}^{(n)}_{\Delta t}(Err^n_{\Delta t} g) - Err^n_{\Delta t}\mathcal{E}_{\Delta t} g.$$

If we continue this procedure, we end up with

$$\mathcal{E}^{(n)}_{t_m} g = \mathcal{E}_{t_m} g - \sum_{j=0}^{m-1} \mathcal{E}^{(n)}_{t_m - t_{j+1}}(Err^n_{\Delta t}\mathcal{E}_{t_j} g). \tag{4.2.5}$$

This shows that if we can calculate the local errors with sufficient accuracy, then we can estimate the global error from these local errors. Equation (4.2.5) is equivalent to

$$Err^n_{t_m} g = \sum_{j=0}^{m-1} \mathcal{E}^{(n)}_{t_m - t_{j+1}}(Err^n_{\Delta t}\mathcal{E}_{t_j} g), \tag{4.2.6}$$

i.e. evaluating a European option with maturity t_m, we will decompose the error (with respect to the Black - Scholes model) into the sum of the discounted expected values, with respect to $\mathcal{E}^{(n)}_{t_m - t_{j+1}}$, of the single step errors $Err^n_{\Delta t}\mathcal{E}_{t_j} g$. Since

$$\mathcal{E}^{(n)}_{t_m - t_{j+1}} = \mathcal{E}_{t_m - t_{j+1}} - Err^n_{t_m - t_{j+1}},$$

replacing the term $\mathcal{E}^{(n)}_{t_m - t_{j+1}}$ by $\mathcal{E}_{t_m - t_{j+1}}$ on the right-hand side of (4.2.5), results in additional error terms - the compound error is then

$$-Err^n_{t_m - t_{j+1}}(Err^n_{\Delta t}\mathcal{E}_{t_j} g).$$

This leads us to the following theorem.

Theorem 10 (Error localization formula). *Let $n, m \geq 1$ be integers and let g be a polynomially bounded function. Then*

$$Err^n_{t_m} g = \sum_{j=0}^{m-1} \mathcal{E}^{(n)}_{t_m - t_{j+1}} (Err^n_{\Delta t} \mathcal{E}_{t_j} g) - \sum_{j=0}^{m-1} Err^n_{t_m - t_{j+1}} (Err^n_{\Delta t} \mathcal{E}_{t_j} g). \tag{4.2.7}$$

Theorem 10 tells us that the error $Err^n_{m\Delta t} h$ can be decomposed into two terms - the main term of the error, denoted by $MErr^n_{m\Delta t} h$, which is the sum of the local errors, and the compound error term, denoted by $CErr^n_{m\Delta t} h$, which is the sum of the errors of the local errors, and both of them should be analyzed separately i.e.

$$MErr^n_{m\Delta t} h(x) := \sum_{j=0}^{m-1} \mathcal{E}^{(n)}_{m\Delta t - (j+1)\Delta t} (Err^n_{\Delta t} \mathcal{E}_{j\Delta t} h),$$

$$CErr^n_{m\Delta t} h(x) := \sum_{j=0}^{m-1} Err^n_{m\Delta t - (j+1)\Delta t} (Err^n_{\Delta t} \mathcal{E}_{j\Delta t} h).$$

Analyzing the local errors using Taylor expansion, where the summation $\sum_{k=2}^{N}$ is understood to vanish in the case $N < 2$, we get

Lemma 3 (Local error expansion formula). *For every integer $N \geq 0$, $p \geq 1$, $x \geq 0$ and $h \in C^{(n)} \cap \mathcal{K}^{(N+1)}_p$,*

$$Err^n_{\Delta t} h(x) = \sum_{k=2}^{N} \frac{\Delta^{(n)}_k x^k h^{(k)}(x)}{k!} + \mathcal{R}^{n,N}_{\Delta t} (I^{N+1} h^{(n+1)})(x), \tag{4.2.8}$$

where, for every function Ψ,

$$\mathcal{R}^{n,N}_{\Delta t}(\Psi)(x) := \frac{1}{N!} Err^n_{\Delta t} \left(\int_1^I \frac{\Psi(xu)(I-u)^N}{u^{N+1}} du \right)(1).$$

Proofs can be found in [27]. Combining the error localization formula with the local error expansion formula, we end up with

Proposition 4 (Error localization expansion formula). *Let $m\Delta t$ be the $m-th$ time step, $M \geq 0$ be an integer and assume that $h \in C^{(M)} \cap \mathcal{K}^{(M+1)}_p$. Then, for every integer $N \geq 0$ and every $x > 0$*

$$Err^n_{m\Delta t} h(x) = MErr^n_{t_m} h(x) - CErr^n_{t_m} h(x)$$

where

$$
\begin{aligned}
MErr^n_{t_m}h(x) = {} & m\sum_{k=2}^{N}\frac{\Delta_k^{(n)}}{k!}x^k\frac{\partial^k}{\partial x^k}\mathcal{E}_{t_{m-1}}(h)(x) \\
& + m\mathcal{R}^{n,N}_{\Delta t}\left(I^{N+1}\frac{\partial^{N+1}}{\partial x^{N+1}}\mathcal{E}_{t_{m-1}}h\right)(x),
\end{aligned} \tag{4.2.9}
$$

$$
\begin{aligned}
CErr^n_{t_m}h(x) = {} & \sum_{j=1}^{m-1}\sum_{k=2}^{N}\frac{\Delta_k^{(n)}}{k!}Err^n_{t_m-t_{j+1}}\left(I^k\frac{\partial^k}{\partial x^k}\mathcal{E}_{t_j}h\right)(x) \\
& + \sum_{j=1}^{m-1}\mathcal{R}^{n,N}_{\frac{T}{n}}\left(Err^n_{t_m-t_{j+1}}\left(I^{N+1}\frac{\partial^{N+1}}{\partial x^{N+1}}\mathcal{E}_{t_j}h\right)\right)(x) \\
& + \sum_{k=2}^{N}\frac{\Delta_k^{(n)}}{k!}Err^n_{t_{m-1}}\left(I^k\frac{\partial^k}{\partial x^k}h\right)(x) \\
& + \mathcal{R}^{n,M}_{\frac{T}{n}}\left(Err^n_{t_{m-1}}\left(I^{M+1}\frac{\partial^{M+1}}{\partial x^{M+1}}h\right)\right)(x).
\end{aligned} \tag{4.2.10}
$$

The proof can be found in [27].

Application to our case

We have seen that we can calculate the error after every step - which is good for our case since we are splitting our tree, which can happen after any step, but we still do not know the optimal splitting time. In Proposition 4 we will use this formula with $N = 4$, because if $h \in C^{(4)} \cap \mathcal{K}_p^{(5)}$ then each of the remainders in fomulae (4.2.9) and (4.2.10) are of order $n^{-\frac{5}{2}}$, which makes them in total of order $n^{-\frac{3}{2}}$ and thus negligible. Our first aim is to show that A, B, C, D and E are negligible, with $h_k(x) = \mathcal{E}_{T-t_k}g(x) - \frac{\Lambda^n_{T-t_k}(x)\frac{n}{n-k}}{n}$,where

$$
\begin{aligned}
MErr^n_{t_k}h(x) = {} & k\sum_{m=2}^{N}\frac{\Delta_m^{(n)}}{m!}x^m\frac{\partial^m}{\partial x^m}\mathcal{E}_{t_{k-1}}(h)(x) \\
& \underbrace{+ k\mathcal{R}^{n,N}_{\frac{T}{n}}\left(I^{N+1}\frac{\partial^{N+1}}{\partial x^{N+1}}\mathcal{E}_{t_{k-1}}g\right)(x),}_{\text{A}}
\end{aligned}
$$

and

$$
\begin{aligned}
CErr^n_{t_k} h(x) = & \underbrace{\sum_{j=1}^{k-1} \sum_{m=2}^{N} \frac{\Delta_m^{(n)}}{m!} Err^n_{t_k - t_{j+1}} \left(I^m \frac{\partial^m}{\partial x^m} \mathcal{E}_{t_j} h \right)(x)}_{\text{B}} \\
& \underbrace{+ \sum_{j=1}^{k-1} \mathcal{R}^{n,N}_{\frac{T}{n}} \left(Err^n_{t_k - t_{j+1}} \left(I^{N+1} \frac{\partial^{N+1}}{\partial x^{N+1}} \mathcal{E}_{t_j} h \right) \right)(x)}_{\text{C}} \\
& \underbrace{+ \sum_{m=2}^{N} \frac{\Delta_m^{(n)}}{m!} Err^n_{t_{k-1}} \left(I^m \frac{\partial^m}{\partial x^m} h \right)(x)}_{\text{D}} \\
& \underbrace{+ \mathcal{R}^{n,M}_{\frac{T}{n}} \left(Err^n_{t_{k-1}} \left(I^{M+1} \frac{\partial^{M+1}}{\partial x^{M+1}} g \right) \right)(x)}_{\text{E}} .
\end{aligned}
$$

To prove that all these terms are negligible, we consider each of them separately. But first we have to show that the function $h_k(x) = \mathcal{E}_{T-t_k} g(x) - \frac{\Lambda^n_{T-t_k}(x) \frac{n}{n-k}}{n}$ is bounded. We will do this in 2 steps, i.e. first we will show that $h_k^{(1)}(x) = \mathcal{E}_{T-t_k} g(x)$ is bounded, and afterwards the same for $h_k^{(2)}(x) = \frac{\Lambda^n_{T-t_k}(x) \frac{n}{n-k}}{n}$, where $h_k(x) = h_k^{(1)}(x) + h_k^{(2)}(x)$. From the definitions of the put and the call options we know they are bounded, so it is their expected value. On the other hand, in the case of $h_2(x)$ we have to consider the two values of $\Lambda^n_{T-t_k}(x)$, check comments after Theorem 8. But, if we look carefully, it is enough to show the boundness only for the one case since the difference is only a constant. For $\widetilde{T} = T - t_k$

$$
\Lambda^n_{\widetilde{T}}(x) = \frac{x e^{-\frac{\mathfrak{d}_1^2}{2}}}{24\sigma\sqrt{2\pi\,(T - t_k)}} \left(-\sigma^2 \widetilde{T}(6 + \mathfrak{d}_1^2 + \mathfrak{d}_2^2) - 12\widetilde{T}^2 r^2 + 4\widetilde{T}(\mathfrak{d}_1^2 - \mathfrak{d}_2^2 \right)
$$

which is a collection of terms of the form $C_1 x e^{-\frac{\mathfrak{d}_1^2}{2}} + \mathfrak{d}_1^2 C_2 x e^{-\frac{\mathfrak{d}_1^2}{2}} + \mathfrak{d}_1 C_3 x e^{-\frac{\mathfrak{d}_1^2}{2}}$. Let us consider a function $x e^{-\frac{\mathfrak{d}_1^2}{2}}$ as $x \exp\left(-\left(\alpha \ln(x) + \beta\right)^2\right)$, where α and β are some constants. Then it is easy to show that the function is bounded using elementary calculus.

$$
\frac{d}{dx}\left(x \exp\left(-\left(\alpha \ln(x) + \beta\right)^2\right)\right) = -\exp\left(-\left(\beta + \alpha \ln x\right)^2\right)\left(2\left(\ln x\right)\alpha^2 + 2\beta\alpha - 1\right),
$$

so this function has an extremum (maximum) at $x = \exp\left(\frac{1-2\alpha\beta}{2\alpha^2}\right)$. We also have

$$\lim_{x\to\pm\infty} x\exp\left(-\left(\alpha\ln(x)+\beta\right)^2\right)=0, \lim_{x\to 0} x\exp\left(-\left(\alpha\ln(x)+\beta\right)^2\right)=0,$$

which means our function is bounded.
Now, for $j,m=0,1,2,...,F,\ F\in\mathbb{N}$

Term A
We will make the following assumption, which is typically satisfied in our applications:
Condition $\mathcal{A}$: *We have that there exists a constant Q, which does not depend on j, or k, or m, such that*

$$\|x^j h_k^{(k)}(x)\|_\infty < Q$$

where Q is bounded (definition of the norm).

Then, using Condition $\mathcal{A}$, for $N=4$ and if

$$h^*(x)=I^{N+1}\frac{\partial^{N+1}}{\partial x^{N+1}}\mathcal{E}_{t_{k-1}}h_k(x), \tag{4.2.11}$$

by Lemma 6.3 in [27], we have

$$h^*(x)=\mathcal{E}_{t_{k-1}}\left(I^{N+1}\frac{\partial^{N+1}}{\partial x^{N+1}}h_k\right)(x).$$

Now $I^{N+1}\frac{\partial^{N+1}}{\partial x^{N+1}}h_k$ is bounded. Therefore, $h^*(x)$ is bounded, i.e.

$$\begin{aligned}\|\mathcal{R}^{n,N}_{\frac{T}{n}}(h^*)\|_\infty &\le QO\left(n^{-\frac{5}{2}}\right)\\ m\|\mathcal{R}^{n,N}_{\frac{T}{n}}(h^*)\|_\infty &\le QO\left(n^{-\frac{3}{2}}\right).\end{aligned}$$

Term B
Now we consider $\displaystyle\sum_{j=1}^{k-1}\sum_{m=2}^{4}\frac{\Delta_m^{(n)}}{n^2}Err^n_{t_k-t_{j+1}}\left(I^m\frac{\partial^m}{\partial x^m}\mathcal{E}_{t_j}h\right)(x)$. Since

$$\Delta_m^{(n)}\overset{P3}{=}\frac{\hat{\Delta}_m}{n^2}+O(n^{-\frac{5}{2}}),$$

and with $h_m^{**}(x)=I^m\frac{\partial^m}{\partial x^m}\mathcal{E}_{t_j}h$ satisfying condition $\mathcal{A}$, we have

$$\|Err^n_{t_k-t_{j+1}} h^{**}_m\| = O(n^{-1}).$$

Then, (using Remark 1 bellow) we get

$$\mathbf{B} \le n \sum_{m=2}^{4} \frac{\hat{\Delta}_m}{n^2}\frac{1}{n} = O(n^{-2}).$$

Remark 1. *Suppose g satisfies condition $\mathcal{A}$. Then, for every $1 \le k \le n$, we have that terms* ***A****,* ***C****,* ***E*** *are of order $\mathcal{O}(n^{-\frac{3}{2}})$. Moreover,*

$$\begin{aligned}\sum_{j=1}^{k-1}\sum_{m=2}^{4} \frac{\Delta_m^{(n)}}{n^2} Err^n_{t_k-t_{j+1}} \left(I^m \frac{\partial^m}{\partial x^m} \mathcal{E}_{t_j} h \right)(x) &\le 2Q \sum_{j=1}^{k-1}\sum_{m=2}^{4} \frac{\hat{\Delta}_m}{n^2} \\ &\le 2Qn \sum_{m=2}^{4} \frac{\hat{\Delta}_m}{n^2} = O(n^{-1})\end{aligned}$$

Term C Since $I^{N+1}\frac{\partial^{N+1}}{\partial x^{N+1}}\mathcal{E}_{t_j} h$ is as (4.2.11) in case **A**, combining methods used in **B** and then **A** again, we can conclude that this term is also negligible.

Term D Since the function $h^{**}_m(x)$ satisfies term $\mathcal{A}$, we can use similar arguments as for **B**.

Term E Since the function $\mathcal{R}^{n,M}_{\frac{T}{n}}$ is similar to the function in **A**, we will use the same arguments.

This tells us that only the first term of (4.2.9) has to be considered, all others are $\mathcal{O}(n^{-\frac{3}{2}})$, hence negligible.

We have shown

$$\begin{aligned} Err^n_T(g)(x) &= e^{-rT} E_x(g(S_T)) - e^{-rT} E_x(g(S^{(n)}_T)) \\ &= \mathcal{E}_T g(x) - \mathcal{E}^{(n)}_T g(x). \end{aligned} \tag{4.2.12}$$

Chang and Palmer in [7] have shown

$$\mathcal{E}^n_T g\,(x) = \mathcal{E}_T g\,(x) + \frac{\Lambda^n_T(K,x)}{n} + O\left(n^{-3/2}\right). \tag{4.2.13}$$

The idea is to use this expression to get the error term for the second part of our tree. There we have that the time to maturity is $T - t_k$, i.e. $n - k$ time steps, which leads us to the expression of the error term

$$\mathcal{E}^{2,n}_{T-t_k} g(x) = \mathcal{E}_{T-t_k} g(x) + \left(\frac{n}{n-k}\right) \frac{\Lambda^n_{T-t_k}(x)}{n} + O\left(n^{-3/2}\right). \tag{4.2.14}$$

Proposition 4, i.e. the *Error localization expansion formula*, leads us to

$$Err^n_{\frac{T}{n}} \mathcal{E}_{t_{k-1}} h(x) = MErr^n_{\frac{T}{n}} \mathcal{E}_{t_{k-1}} h(x) - CErr^n_{\frac{T}{n}} \mathcal{E}_{t_{k-1}} h(x), \tag{4.2.15}$$

which we have shown is equal to

$$Err^n_{\frac{T}{n}} \mathcal{E}_{t_{k-1}} h(x) = k \sum_{m=2}^{N} \frac{\Delta^{(n)}_m}{m!} x^m \frac{\partial^m}{\partial x^m} \mathcal{E}_{t_{k-1}}(h)(x).$$

where we use that

$$h(x) = \mathcal{E}_{T-t_k} g(x) + \frac{\Lambda^n_T(K,x)\left(\frac{n}{n-k}\right)}{n} \tag{4.2.16}$$

and, since $\mathcal{E}_{\frac{T}{n}} \mathcal{E}_{t_{k-1}} = \mathcal{E}_{t_k}$, we have

$$\begin{aligned}
\mathcal{E}^{1,n,\lambda}_{t_k}\left(\mathcal{E}^{2,n}_{T-t_k} g\right)(x) &\overset{(4.2.14)}{=} \mathcal{E}^{1,n,\lambda}_{t_k}\left(\mathcal{E}_{T-t_k} g + \frac{\left(\frac{n}{n-k}\right)\Lambda^n_{T-t_k}}{n}\right)(x) + O\left(n^{-\frac{3}{2}}\right) \\
&\overset{(4.2.15)}{=} \mathcal{E}_{t_k}\left(\mathcal{E}_{T-t_k} g + \frac{\left(\frac{n}{n-k}\right)\Lambda^n_{T-t_k}}{n}\right)(x) \\
&\quad - k \sum_{m=2}^{N} \frac{\Delta^{(n)}_m}{m!} x^k \frac{\partial^m}{\partial x^m} \mathcal{E}_{t_{k-1}}\left(\mathcal{E}_{T-t_k} g + \frac{\left(\frac{n}{n-k}\right)\Lambda^n_{T-t_k}}{n}\right)(x) \\
&\quad + O\left(n^{-\frac{3}{2}}\right),
\end{aligned}$$

i.e

$$\begin{aligned}
\mathcal{E}^{1,n,\lambda}_{t_k}\left(\mathcal{E}^{2,n}_{T-t_k} g\right)(x) &= \mathcal{E}_T(g)(x) + \frac{1}{n}\mathcal{E}_{t_k}\left(\Lambda^n_{T-t_k}\left(\frac{n}{n-k}\right)\right)(x) \\
&\quad - k \sum_{m=2}^{N} \frac{\Delta^{(n)}_m}{m!} x^m \frac{\partial^m}{\partial x^m} \mathcal{E}_{t_{n-1}}(g)(x) \\
&\quad - \frac{k}{n} \sum_{m=2}^{N} \frac{\Delta^{(n)}_m}{m!} x^k \frac{\partial^m}{\partial x^m} \mathcal{E}_{t_{k-1}}\left(\Lambda^n_{T-t_k}\left(\frac{n}{n-k}\right)\right)(x) \\
&\quad + O\left(n^{-\frac{3}{2}}\right).
\end{aligned} \tag{4.2.17}$$

In the next step, we will discuss these 4 terms in more details. To make notations easier to follow, we will put $\tilde{T} = T - t_k$.

1.

$$\begin{aligned}
\mathcal{E}_T(g)(x) &= x\Phi(\mathfrak{d}_1) - Ke^{-r\tilde{T}}\Phi(\mathfrak{d}_2) \\
\mathfrak{d}_1 &= \frac{\left(\ln\left(\frac{x}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)\tilde{T}\right)}{\sigma\sqrt{\tilde{T}}} \\
\mathfrak{d}_2 &= \frac{\left(\ln\left(\frac{x}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)\tilde{T}\right)}{\sigma\sqrt{\tilde{T}}} = \mathfrak{d}_1 - \sigma\sqrt{\tilde{T}}
\end{aligned}$$

2. The second term of (4.2.17) is

$$\begin{aligned}
\frac{1}{n}\mathcal{E}_{t_k}&\left(\frac{n}{n-k}\left(\frac{xe^{-\frac{\mathfrak{d}_1^2}{2}}}{24\sigma\sqrt{2\pi\tilde{T}}}\left(-\sigma^2\tilde{T}\left(6+\mathfrak{d}_1^2+\mathfrak{d}_2^2\right)-12\tilde{T}^2r^2+4\tilde{T}\left(\mathfrak{d}_1^2-\mathfrak{d}_2^2\right)r\right)\right)\right)(x) \\
&= \frac{1}{n-k}\frac{-\sigma^2\tilde{T}}{24\sigma\sqrt{2\pi\tilde{T}}}\mathcal{E}_{t_k}\left(xe^{-\frac{\mathfrak{d}_1^2}{2}}(6+\mathfrak{d}_1^2+\mathfrak{d}_2^2)\right) \\
&\quad - \frac{1}{n-k}\frac{12\tilde{T}^2r^2}{24\sigma\sqrt{2\pi\tilde{T}}}\mathcal{E}_{t_k}\left(xe^{-\frac{\mathfrak{d}_1^2}{2}}\right) \\
&\quad + \frac{1}{n-k}\frac{4r\tilde{T}}{24\sigma\sqrt{2\pi\tilde{T}}}\mathcal{E}_{t_k}\left(xe^{-\frac{\mathfrak{d}_1^2}{2}}\left(\mathfrak{d}_1^2-\mathfrak{d}_2^2\right)\right).
\end{aligned}$$

Let us now first calculate these expectations. To do that, we will use Gaussian integrals and we will replace x with S_{t_k}, where $S_{t_k} = e^{(r-\frac{1}{2}\sigma^2)t_k+\sigma W_{t_k}}$, W_{t_k} - Brownian motion. Then

$$\begin{aligned}
\mathfrak{d}_1 = \mathfrak{d}_1(S_{t_k}) &= \frac{\ln\left(\frac{S_{t_k}}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)\tilde{T}}{\sigma\sqrt{\tilde{T}}} \\
&= \frac{(r-\frac{1}{2}\sigma^2)t_k+\sigma W_{t_k}-\ln(K)+rT-rt_k+\frac{1}{2}\sigma^2T-\frac{1}{2}\sigma^2t_k}{\sigma\sqrt{\tilde{T}}} \\
&= \frac{\sigma W_{t_k}-\sigma^2t_k-\left(r+\frac{1}{2}\sigma^2\right)T-\ln(K)}{\sigma\sqrt{T-t}} \\
&= \frac{W_{t_k}-\sigma t_k}{\sqrt{\tilde{T}}} - \frac{\ln K-\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{\tilde{T}}} \\
&= \frac{W_{t_k}-\sigma t_k}{\sqrt{\tilde{T}}} - \frac{C}{\sqrt{\tilde{T}}}
\end{aligned}$$

where

$$C = \frac{\ln K - \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma}.$$

On the other hand, we get

$$\mathfrak{d}_2 = \frac{W_{t_k} - \sigma T}{\sqrt{\tilde{T}}} - \frac{C}{\sqrt{\tilde{T}}}.$$

Then

$$\mathfrak{d}_1^2(S_{t_k}) = \frac{W_{t_k}^2}{\tilde{T}} - \frac{W_{t_k}\,(2\sigma t_k + 2C)}{\tilde{T}} + \widetilde{C}_1,$$
$$\mathfrak{d}_2^2(S_{t_k}) = \frac{W_{t_k}^2}{\tilde{T}} - \frac{W_{t_k}\,(2\sigma T + 2C)}{\tilde{T}} + \widetilde{C}_2,$$

with

$$\widetilde{C}_1 = \frac{(C + \sigma t_k)^2}{\tilde{T}}$$

and

$$\widetilde{C}_2 = \frac{(C + \sigma T)^2}{\tilde{T}}.$$

Now

$$\begin{aligned}
E\left(xe^{-\frac{\mathfrak{d}_1^2}{2}}\right) &= E\left(e^{(r-\frac{1}{2}\sigma^2)t_k+\sigma W_{t_k}} e^{-\frac{1}{2}\left(\frac{W_{t_k}^2}{\tilde{T}} - \frac{W_{t_k}(2\sigma t_k+2C)}{\tilde{T}} + \widetilde{C}_1\right)}\right) \\
&= E\left(e^{-\frac{1}{2}\frac{W_{t_k}^2}{\tilde{T}} + \frac{1}{2}\frac{W_{t_k}(2\sigma t_k+2C+2\sigma\tilde{T})}{\tilde{T}} - \frac{1}{2}\widetilde{C}_1 + (r-\frac{1}{2}\sigma^2)t_k}\right) \\
&= \frac{1}{\sqrt{2\pi t_k}} \int_{-\infty}^{+\infty} \underbrace{e^{-\frac{1}{2\tilde{T}}y^2 + \frac{\sigma T+C}{\tilde{T}}y - \frac{1}{2}\widetilde{C}_1 + (r-\frac{1}{2}\sigma^2)t_k} e^{\frac{-y^2}{2}}}_{\mathrm{I}} dy \qquad (4.2.18) \\
&= \frac{1}{\sqrt{2\pi t_k}} \int_{-\infty}^{+\infty} e^{-Ay^2+By-D} dy \\
&= \frac{1}{\sqrt{2t_k A}} e^{\frac{B^2}{4A}-D} \\
&= e^{\widehat{g}(t_k)},
\end{aligned}$$

where the last integral is a Gaussian integral $\int\limits_{-\infty}^{+\infty} e^{-Ay^2+By-D}dy = \sqrt{\frac{\pi}{A}}e^{\frac{B^2}{4A}-D}$ and

$$A = \frac{1}{2\tilde{T}} + \frac{1}{2},$$

$$B = \frac{\sigma T + C}{\tilde{T}},$$

$$D = \frac{1}{2}\widetilde{C}_1 - (r - \frac{1}{2}\sigma^2)t_k.$$

The second expectation is

$$\begin{aligned}
E\left(x\mathfrak{d}_1^2 e^{-\frac{\mathfrak{d}_1^2}{2}}\right) &= E\left(\left(\frac{W_{t_k}^2}{\tilde{T}} - \frac{W_{t_k}\left(2\sigma t_k + 2C\right)}{\tilde{T}} + \widetilde{C}_1\right) x e^{-\frac{\mathfrak{d}_1^2}{2}}\right) \\
&= \frac{1}{\tilde{T}\sqrt{2\pi t_k}} \int\limits_{-\infty}^{+\infty} y^2 I dy \\
&- \frac{2C + 2\sigma t_k}{\tilde{T}\sqrt{2\pi t_k}} \int\limits_{-\infty}^{+\infty} y I dy \\
&+ \frac{\widetilde{C}_1}{\sqrt{2\pi t_k}} \int\limits_{-\infty}^{+\infty} I dy \\
&= \frac{1}{\tilde{T}\sqrt{2t_k}} \frac{(2A + B^2)}{4A^{\frac{5}{2}}} e^{\frac{B^2}{4A}} e^{-D} \\
&- \frac{1}{\tilde{T}\sqrt{2t_k}} \frac{2\sigma T + 2C}{\tilde{T}} \frac{B}{2A^{\frac{3}{2}}} e^{\frac{B^2}{4A}} e^{-D} \\
&+ \frac{\tilde{C}_1}{\sqrt{2t_k}} \sqrt{\frac{1}{A}} e^{\frac{B^2}{A}} e^{-D} \\
&= g_1(t_k),
\end{aligned} \tag{4.2.19}$$

where we this time used also next Gaussian integrals

$$\int_{-\infty}^{+\infty} y^2 e^{-Ay^2+By} dy = \frac{\sqrt{\pi}(2A+B^2)}{4A^{\frac{5}{2}}} e^{\frac{B^2}{4A}},$$

$$\int_{-\infty}^{+\infty} y e^{-Ay^2+By} dy = \frac{\sqrt{\pi}B}{A^{\frac{3}{2}}} e^{\frac{B^2}{4A}}.$$

We get analogous results for $E\left(x\mathfrak{d}_2^2 e^{-\frac{\mathfrak{d}_1^2}{2}}\right)$, just instead of $\widetilde{C}_1$ we will have $\widetilde{C}_2$, so we will call it $g_2(t_k)$ which shows us that non of the three expected values depend on x and that makes our fourth step really easy.

3. The next, third, term of (4.2.17) is

$$k\sum_{m=2}^{N} \frac{\Delta_m^{(n)}}{m!} x^m \frac{\partial^m}{\partial x^m} \mathcal{E}_{t_{n-1}}(g)(x)$$

where

$$\Delta_m^{(n)} = \frac{\hat{\Delta}_m}{n^2} + O\left(n^{-\frac{5}{2}}\right), \text{ for } m = 2,3,4,$$

and

$$\hat{\Delta}_2 = -\sigma^4 T^2 \alpha + \alpha^2 \sigma^4 T^2 + r^2 T^2 + r T^2 \sigma^2 + \frac{5}{12}\sigma^4 T^2 - 2T^2 r \sigma^2 \alpha,$$
$$\hat{\Delta}_3 = 2rT^2\sigma^2 - 2\sigma^4 T^2 \alpha + 2\sigma^4 T^2,$$
$$\hat{\Delta}_4 = 2\sigma^4 T^2.$$

So, using the relation P3, we have

$$k\sum_{m=2}^{N} \frac{\Delta_m^{(n)}}{m!} x^m \frac{\partial^m}{\partial x^m} \mathcal{E}_{t_{n-1}}(g)(x) = \frac{k}{n^2}\sum_{m=2}^{N} \frac{\hat{\Delta}_m}{m!} x^m \frac{\partial^m}{\partial x^m} \mathcal{E}_{t_{n-1}}(g)(x) + O\left(n^{-\frac{3}{2}}\right).$$

And we have that

$$\frac{\partial^m}{\partial x^m} \mathcal{E}_{t_{n-1}}(g)(x) = \frac{\partial^m}{\partial x^m} \mathcal{E}_{T}(g)(x) + O\left(n^{-1}\right)$$

which gives us

$$k\sum_{m=2}^{N} \frac{\Delta_m^{(n)}}{m!} x^m \frac{\partial^m}{\partial x^m} \mathcal{E}_{t_{n-1}}(g)(x) = \frac{1}{n}\left(\frac{k}{n}\sum_{m=2}^{N} \frac{\Delta_m}{m!} x^m \frac{\partial^m}{\partial x^m} \mathcal{E}_{T}(g)(x)\right) + O\left(n^{-\frac{3}{2}}\right).$$

Let us now write the derivatives of $\mathcal{E}_T(h)(x)$. But before we start, we will first write some basic properties which we will use. We have

$$\frac{\partial \mathfrak{d}_1}{\partial x} = \frac{\partial \mathfrak{d}_2}{\partial x} = \frac{1}{x\sigma\sqrt{\tilde{T}}},$$

$$\frac{\partial \Phi(\mathfrak{d}_1)}{\partial \mathfrak{d}_1} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\mathfrak{d}_1^2},$$

$$\frac{\partial \Phi(\mathfrak{d}_2)}{\partial \mathfrak{d}_2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\mathfrak{d}_2^2},$$

$$\frac{\partial \Phi(\mathfrak{d}_1)}{\partial \mathfrak{d}_1} = \frac{K}{x} e^{-r\tilde{T}} \frac{\partial \Phi(\mathfrak{d}_2)}{\partial \mathfrak{d}_2}.$$

Then

$$\frac{\partial}{\partial x}\mathcal{E}_T(g)(x) = \Phi(\mathfrak{d}_1) + x\frac{\partial \Phi(\mathfrak{d}_1)}{\partial \mathfrak{d}_1}\frac{\partial \mathfrak{d}_1}{\partial x} - Ke^{-r\tilde{T}}\frac{\partial \Phi(\mathfrak{d}_2)}{\partial \mathfrak{d}_2}\frac{\partial \mathfrak{d}_1}{\partial x} = \Phi(\mathfrak{d}_1),$$

$$\frac{\partial^2}{\partial x^2}\mathcal{E}_T(g)(x) = \frac{\partial \Phi(\mathfrak{d}_1)}{\partial \mathfrak{d}_1}\frac{\partial \mathfrak{d}_1}{\partial x} = \frac{1}{x\sigma\sqrt{\tilde{T}}}\frac{\partial \Phi(\mathfrak{d}_1)}{\partial \mathfrak{d}_1},$$

$$\begin{aligned}\frac{\partial^3}{\partial x^3}\mathcal{E}_T(g)(x) &= \frac{-1}{x^2\sigma\sqrt{\tilde{T}}}\frac{\partial}{\partial \mathfrak{d}_1}\Phi(\mathfrak{d}_1) + \frac{\partial^2}{\partial \mathfrak{d}_1^2}\Phi(\mathfrak{d}_1)\frac{\partial}{\partial x}\mathfrak{d}_1\frac{1}{x\sigma\sqrt{\tilde{T}}} \\ &= \frac{\partial}{\partial \mathfrak{d}_1}\Phi(\mathfrak{d}_1)\frac{1}{x^2\sigma\sqrt{\tilde{T}}}\left(\frac{\mathfrak{d}_1}{x\sigma\sqrt{\tilde{T}}} - 1\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial^4}{\partial x^4}\mathcal{E}_T(g)(x) = -\frac{\partial}{\partial \mathfrak{d}_1}\Phi(\mathfrak{d}_1)\frac{1}{x^2\sigma\sqrt{\tilde{T}}}&\left[\frac{2}{x}\left(\frac{\mathfrak{d}_1}{x\sigma\sqrt{\tilde{T}}} - 1\right) + \mathfrak{d}_1\left(\frac{\mathfrak{d}_1}{x\sigma\sqrt{\tilde{T}}} - 1\right)\right. \\ &\left. + \frac{1}{x^2\sigma^2\tilde{T}}\left(\mathfrak{d}_1\sigma\sqrt{\tilde{T}} - 1\right)\right].\end{aligned}$$

4. As shown in the second step, non of the expected values depend on x, thus we get that the fourth term is equal to 0.

Now, putting everything together we end up with

Theorem 11. *For the n-period Split tree binomial model, where $t_k = k\Delta t$ is the splitting time and $x > 0$ arbitrarily, we have that the explicit first order error formula of $\mathcal{E}_{t_k}^{1,n,\mu}\left(\mathcal{E}_{T-t_k}^{2,n} g\right)(x)$ is given by*

$$
\begin{aligned}
\mathcal{E}_{t_k}^{1,n,\lambda}\left(\mathcal{E}_{T-t_k}^{2,n} g\right)(x) = x\Phi(\mathfrak{d}_1) - Ke^{-r\tilde{T}}\Phi(\mathfrak{d}_2)& \\
+ \frac{1}{n-k}\frac{-\sigma^2\tilde{T}}{24\sigma\sqrt{2\pi\tilde{T}}}e^{-rt_k}\left(6e^{\hat{g}(t_k)} + g_1(t_k) + g_2(t_k)\right)& \\
- \frac{1}{n-k}\frac{12\tilde{T}^2 r^2}{24\sigma\sqrt{2\pi\tilde{T}}}e^{-rt_k}e^{\hat{g}(t_k)}& \\
+ \frac{1}{n-k}\frac{4r\tilde{T}}{24\sigma\sqrt{2\pi\tilde{T}}}e^{-rt_k}\left(g_1(t_k) - g_2(t_k)\right)& \\
+ \frac{k}{n^2}\sum_{m=2}^{N}\frac{\hat{\Delta}_m}{m!}x^m\Bigg\{\Phi(\mathfrak{d}_1) + \frac{1}{x\sigma\sqrt{\tilde{\tilde{T}}}}\frac{\partial}{\partial x}\Phi(\mathfrak{d}_1)& \\
\frac{\partial}{\partial \mathfrak{d}_1}\Phi(\mathfrak{d}_1)\frac{1}{x^2\sigma\sqrt{\tilde{\tilde{T}}}}\left(\frac{\mathfrak{d}_1}{x\sigma\sqrt{\tilde{\tilde{T}}}} - 1\right)& \\
+ -\frac{\partial}{\partial \mathfrak{d}_1}\Phi(\mathfrak{d}_1)\frac{1}{x^2\sigma\sqrt{\tilde{\tilde{T}}}}\left[\frac{2}{x}\left(\frac{\mathfrak{d}_1}{x\sigma\sqrt{\tilde{\tilde{T}}}} - 1\right) + \mathfrak{d}_1\left(\frac{\mathfrak{d}_1}{x\sigma\sqrt{\tilde{\tilde{T}}}} - 1\right)\right.& \\
\left.\left. + \frac{1}{x^2\sigma^2\tilde{\tilde{T}}}\left(\mathfrak{d}_1\sigma\sqrt{\tilde{\tilde{T}}} - 1\right)\right]\right\} + O\left(n^{-\frac{3}{2}}\right).&
\end{aligned}
$$

where $\hat{g}(t_k)$, $g_1(t_k)$ and $g_2(t_k)$ are defined as in (4.2.18) and (4.2.19).

Theorem 11 is the main result of this thesis. We have shown in Section 3.1 that the CRR tree has a good performance for the case $S_0 = K$, i.e. it has smooth convergence and the order of convergence is $O(n^{-1})$. With Theorem 11 we have shown that these good properties of the CRR tree are preserved under splitting regardless of the position of S_0 and K.

Chapter 5

American options

In this chapter, our focus will be on American options. To be more precise, we put our focus on American put options since American calls on a stock without dividends are known to be equivalent to its European counterpart. In the previous chapters we have shown convergence of European type options and we investigated the rate of convergence. Unfortunately, something like that is not possible for American options since the American option valuation tast involves the additional problem of choosing the optimal exercise time. But, Amin and Khana in [2] proved that, under some general set of conditions, American type option values obtained by discrete time models converge to their respective continuous time values. For the European case we have seen that there exists a formula, the Black-Scholes formula, which gives us the value of European put and call options in closed form. Unfortunately, something like that does not exist for American options.

American options can be exercised at any time before maturity. The optimal exercise time is unknown and it is usually represented by a random stopping time τ^*. Because of that, a closed-form solution usually does not exist and the binomial model is of great importance for this case as it provides an easy modification to introduce the possibility to choose the exercise time in an optimal way.

The value of an American option is given by

$$\sup_{\tau \in \mathcal{T}_{0,T}} E_Q(e^{-r\tau} g(S(\tau)))$$

where g is a payoff function and $\mathcal{T}_{0,T}$ is the set of all stopping times in $[0, T]$ with respect to the natural filtration of W with values in $[0, T]$. As in a binomial model we only have finitely many states, there exists an optimal stopping time τ_n^* for every n such that the approximation in binomial model is given by

$$E_n(e^{-r\tau_n^*} g(S^{(n)}(\tau_n^*))).$$

The main difference in the algorithm for American options compared to the algorithm for European options is that in the backward induction, in each node of the tree, the exercise value has to be compared to the value obtained by holding the option at least until the next time period and then exercise it optimally afterwards.

1. Tree initialization
   - Calculate the possible values of stock at maturity
   $$S^{(n)}(k+1) = S^{(n)}(k)e^{\alpha(n)\Delta t+\beta\sqrt{\Delta t}Z^{(n)}_{k+1}},\ k = 0, 1, ..., n-1.$$
   - Calculate the option value at maturity
   $$V^{(n)}(T, S^{(n)}(n)) = g(S^{(n)}(n)).$$
2. Backward induction
   - For $i = n-1, ..., 0$ do
   $$\begin{aligned}\tilde{V}^{(n)}(i\Delta t, S^{(n)}(i)) = e^{-r\Delta t}\big[&pV^{(n)}((i+1)\Delta t, uS^{(n)}(i))\\ &+ (1-p)V^{(n)}((i+1)\Delta t, dS^{(n)}(i))\big]\end{aligned}$$
   and set
   $$V^{(n)}(i\cdot\Delta t, S^{(n)}(i)) = \max\{\tilde{V}^{(n)}(i\cdot\Delta t, S^{(n)}(i)), g(S^{(n)}(i))\}.$$
   - Return $V^n(0)$.

If the payoff function is continuous and uniformly integrable, and the limit diffusion satisfies linear growth and Lipschitz conditions, it is showed in [2] that American option prices converge to their continuous-time values. As the modification of the algorithm compared to the European option case is very simple, binomial trees are the typical choice for the numerical method when prices of American options have to be computed.

Since τ^* is in $[0,T]$ and $S^{(n)}$ is tight, we have that $(S^{(n)}, \tau^*)$ is tight in $C[0,T] \times [0,T]$. Then there exists a subsequence which converges to the weak limit. Let us say (S,τ) is that limit. The distribution of τ may depend on the particular subsequence, but the distribution of S will not. Moreover, it is not clear whether τ is a stopping time with respect to the filtration generated by S. [2] showed that τ is "in some appropriate sense" a legitimate stopping time with respect to S in two steps:

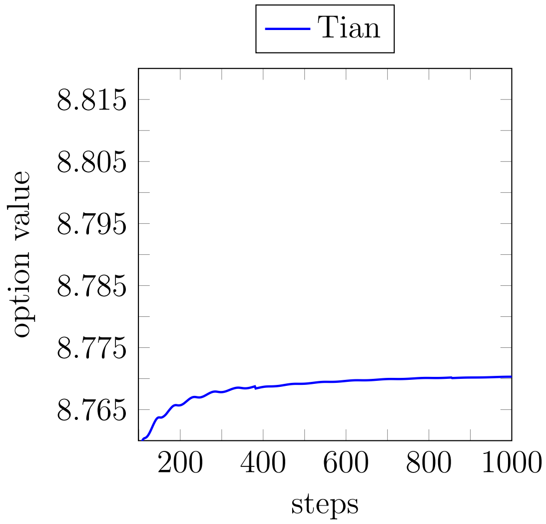
1. They showed that there exists a subsequence n_k such that $W_{n_k,\epsilon_k} \to W$. From the continuity of S and W we have that $(S_{n_k,\epsilon_k}, W_{n_k,\epsilon_k})$ is tight in $C[0,T]$.

2. Let (η, B) be the limit of a particular subsequence $(S_{n_k,\epsilon_k}, W_{n_k,\epsilon_k})$. Then they show that (η, B) is indeed a solution to

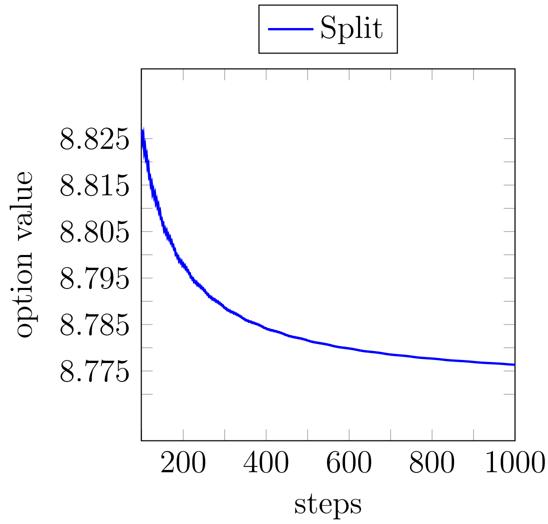
$$d\eta(t) = \mu(t, \eta(t))dt + \sigma(t, \eta(t))dB(t), 0 \leq t \leq T$$

with $\eta(0) = S(0)$ and B is a Brownian motion. This will imply that (η, B) has the same distribution as (S, W), and hence $(S_{n_k,\epsilon_k}, W_{n_k,\epsilon_k}) \to (S, W)$.

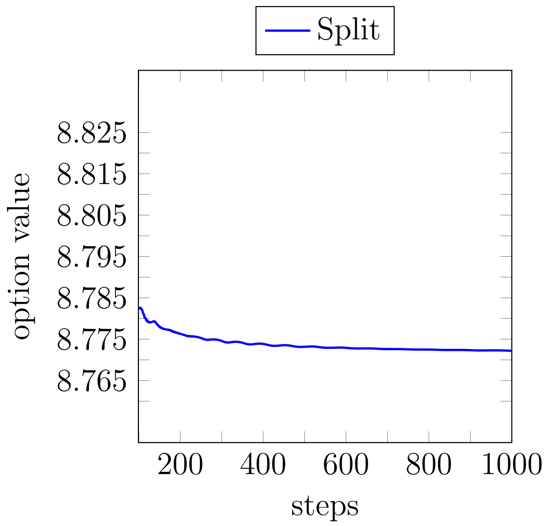
The Split tree has a comparable performance to standard methods for the American options, too. Let us discuss the behavior of American put options in more details. We will consider two cases: one when we are initially in the in-the-money situation, i.e. $K > S_0$, and another where $K < S_0$. We will start with $K > S_0$. If we are in-the-money, then we would like to stay in that position as long as it is possible, since we are already in a good position and we do not want to lose it. In this case splitting later is preferable. On the other hand, if we have $K < S_0$, then we suggest an early split to come to the area of interest, i.e. the in-the-money part as only there early exercise makes sense at all. Let us see numerically how this looks like. In this example, we use $K = 100$, $S_0 = 95$, $r = 0.1$, $\sigma = 0.25$ and $T = 1$. In Figure 5.1 some oscillations are visible at the beginning, but later we have smooth convergence. On the other hand, Tian has also oscillations. But, if we change the splitting time, we can achieve better oscillation patterns. For the American type of options, we have the opposite situation. Namely, if we split the tree after $\frac{3n}{4}$ steps, we have much more oscillations than splitting after $\frac{n}{2}$. On the other hand, we have a pretty nice and smooth convergence in the $\frac{n}{4}$ case. These results are also given in the Table 5.1. For the Tian model we have smooth convergence after 500 steps. In the case where we split the tree after $\frac{1}{4}$ steps, we have all the time smooth convergence. For the Split tree we have some oscillations between 300 and 500 steps and for the $\frac{3}{4}$ we have oscillations all the way. But, in the case of $\frac{3}{4}$ even though we have oscillations we are much closer to the real value and much earlier is visible that we are going into that direction.



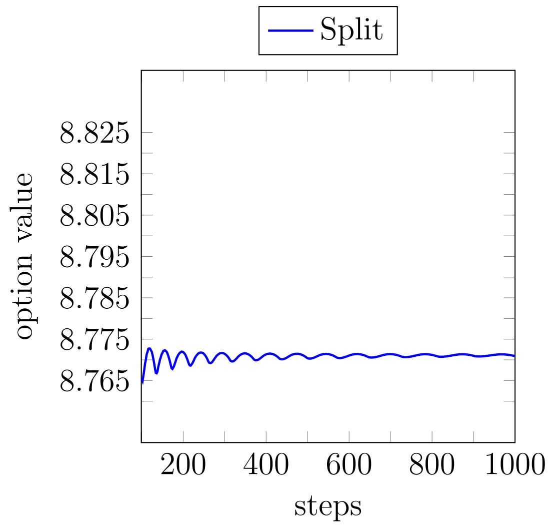
(a) Tian



(b) Splitting after 1/4



(c) Splitting after 1/2



(d) Splitting after 3/4

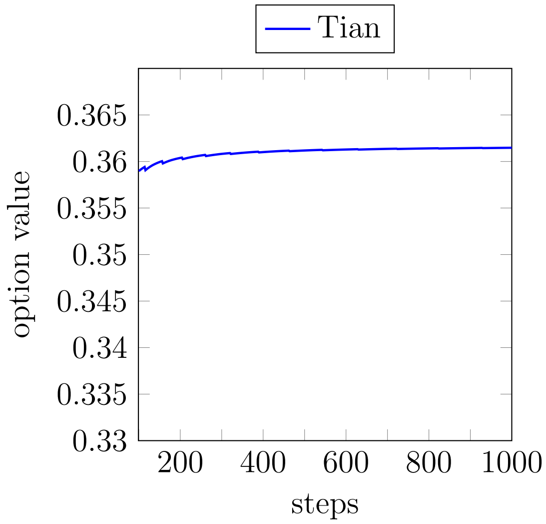
Figure 5.1: American put option: $S_0 = 95$, $K = 100$, $\sigma = 0.25$, $r = 0.1$, $T = 1$

n	Tian	1/4	1/2	3/4
40	8.7398	8.9286	8.7992	8.7785
80	8.7581	8.8429	8.7865	8.7709
100	8.7623	8.8270	8.7823	8.7654
120	8.7621	8.8169	8.7796	8.7727
140	8.7650	8.8108	8.7790	8.7683
160	8.7647	8.8051	8.7775	8.7720
180	8.7665	8.8012	8.7770	8.7692
200	8.7660	8.7981	8.7763	8.7719
280	8.7679	8.7901	8.7750	8.7709
300	8.7678	8.7887	8.7746	8.7714
380	8.7688	8.7851	8.7738	8.7700
400	8.7690	8.7842	8.7739	8.7713
420	8.7689	8.7837	8.7734	8.7712
500	8.7693	8.7816	8.7732	8.7704
1000	8.7703	8.7763	8.7722	8.7709
2000	8.7708	8.7738	8.7718	8.7712
4000	8.7711	8.7725	8.7715	8.7713

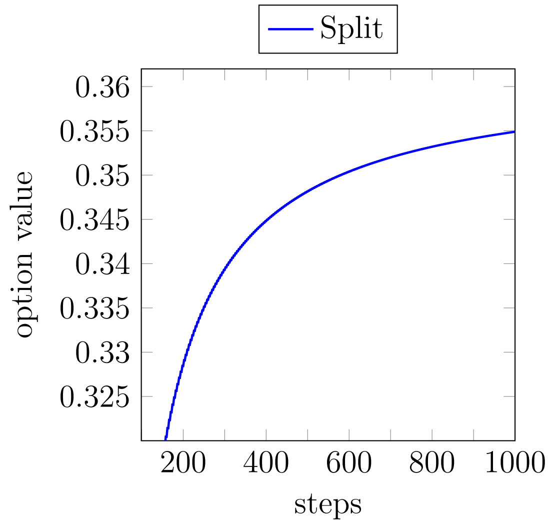
Table 5.1: American put: $S_0 = 95$, $K = 100$, $\sigma = 0.25$, $r = 0.1$, $T = 1$, BS value for European put = 7.1411.

n	Tian	1/4	1/2	3/4
40	0.3518	0.2115	0.2855	0.3155
100	0.3582	0.2971	0.3219	0.3424
200	0.3602	0.3284	0.3453	0.3518
400	0.3610	0.3284	0.3453	0.3518
420	0.3611	0.3284	0.3453	0.3518
440	0.3610	0.3284	0.3453	0.3518
500	0.3612	0.3481	0.3551	0.3577
580	0.3613	0.3481	0.3551	0.3577
600	0,3612	0.3481	0.3551	0.3577
620	0.3613	0.3481	0.3551	0.3577
800	0,3614	0.3532	0.3576	0.3592
1000	0.3615	0.3549	0.3584	0.3597
2000	0.3616	0.3583	0.3601	0.3607
4000	0.3617	0.3600	0.3609	0.3613

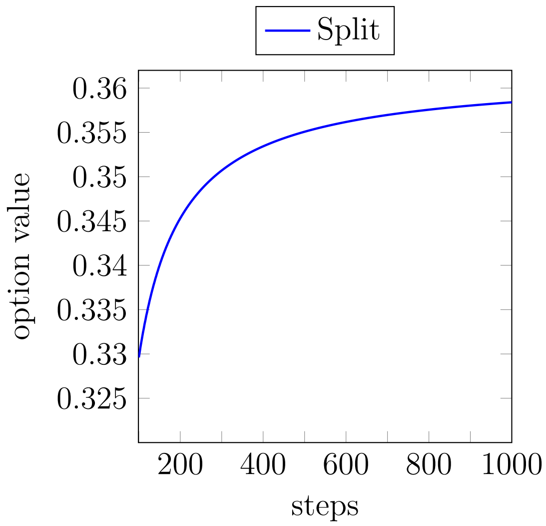
Table 5.2: American put options: $K = 40$, $S_0 = 50$, $r = 0.05$, $\sigma = 0.2$, $T = 1$, BS value for European put = 0.3436.



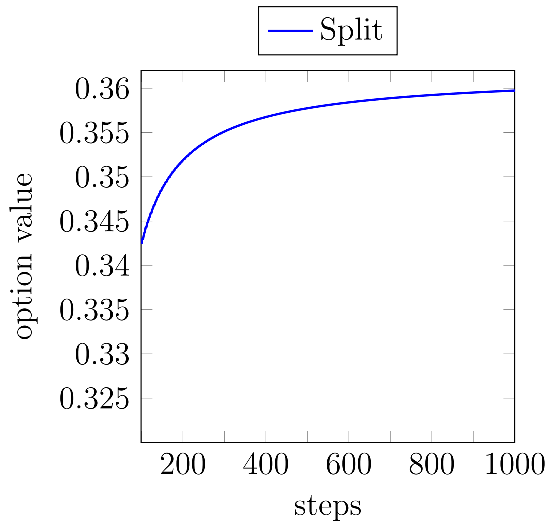
(a) Tian



(b) Splitting after 1/4



(c) Splitting after 1/2



(d) Splitting after 3/4

Figure 5.2: American put options: $K = 40$, $S_0 = 50$, $r = 0.05$, $\sigma = 0.2$, $T = 1$

Let us now consider the case where $K < S_0$. In this example we used $K = 40$, $S_0 = 50$, $r = 0.05$, $\sigma = 0.2$ and $T = 1$. We have smooth convergence for all splitting cases, but in Tian some small oscillations at the beginning are visible. Results are presented in Table 5.2 and Figure 5.2.

Now we want to compare the Split model with the CRR, Tian and CP models and we want to consider different situations. We also want to see how these models behave when the put is deep-in-the-money and deep-out-of-the-money. We will consider an example of an American put where $S_0 = 70$ $\sigma = 0.2$, $r = 0.05$, $T = 1$ are fixed and we will change K. The results can be seen in Table 5.3.

In the Figure 5.3 we have a situation where we are deep-out-of-the money. In this case we have smooth convergence for all the cases except for the CRR. In the Figure 5.4 a deep-in-the-money situation is presented and we can see much better performance if we split after $\frac{n}{2}$ and $\frac{3n}{4}$ than if we split after $\frac{n}{4}$, and some oscillations are visible for CRR and Tian.

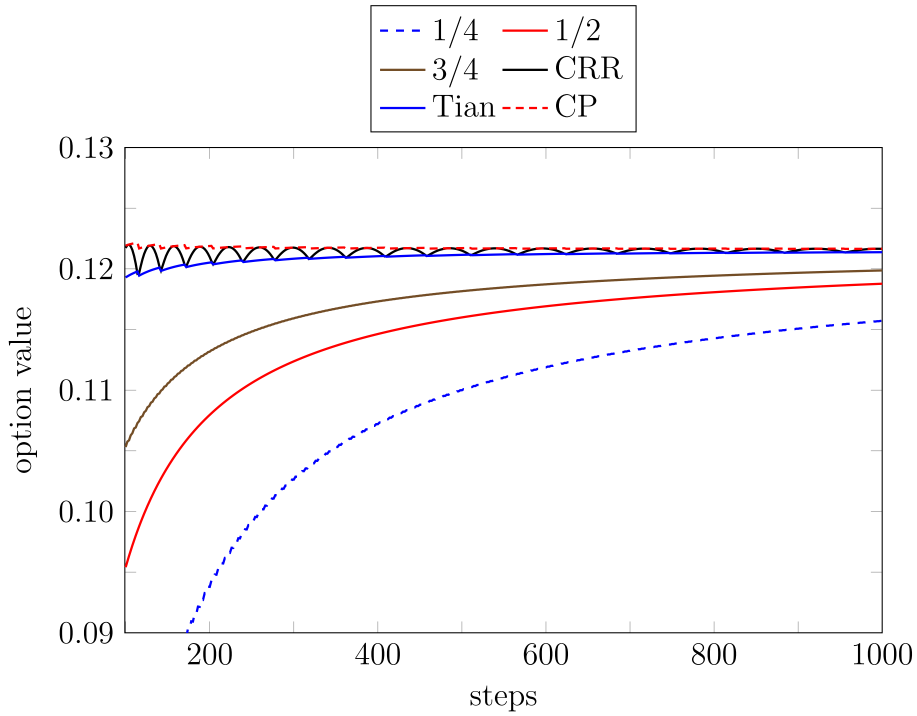


Figure 5.3: American put: $S_0 = 70$, $K = 50$ $\sigma = 0.2$, $r = 0.05$, $T = 1$

But, if we are in the deep-in-the-money situation, we would like to stay there as long as it is possible. Then moving our tree such that the strike value hits its

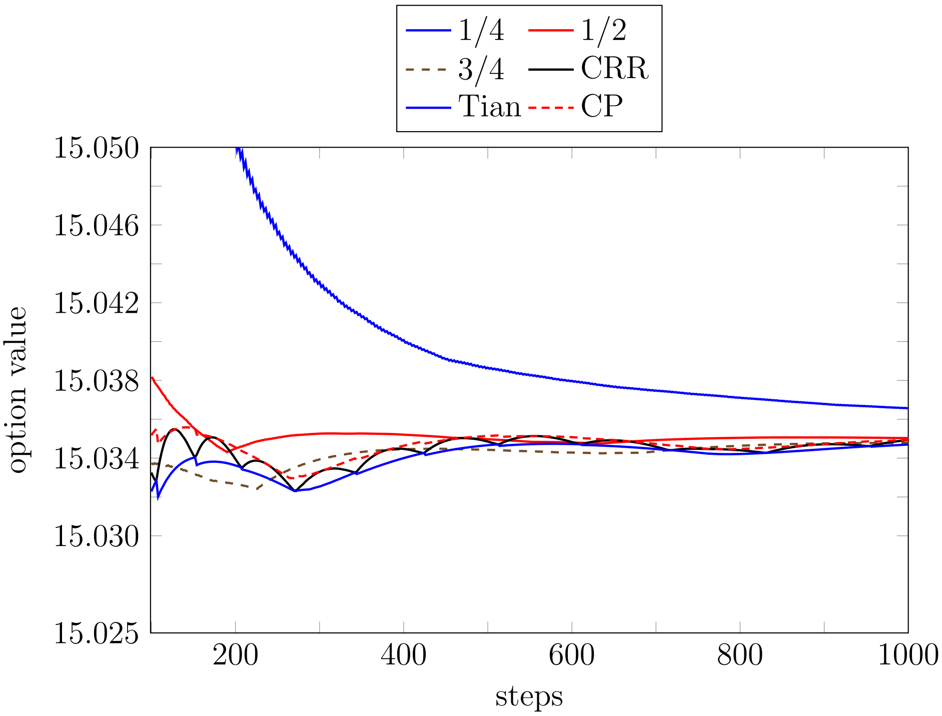


Figure 5.4: American put: $S_0 = 70$, $K = 85$ $\sigma = 0.2$, $r = 0.05$, $T = 1$

middle node does not seem to be a good solution, since half of the nodes will lose their good position. Since having a strike value at the node brings us a smoother convergence, let us try to shift the tree just as much as needed to bring the closest node to hit the strike value. In other words, we will use Tian's drift (3.2.14) but now not until the last step. Namely, we will again do splitting, but with a slightly different drift. For the first k steps we will use

$$\tilde{\alpha}_1(n) := \frac{\ln\left(\frac{K}{S_0}\right) - (2l_\alpha(n) - k)\sigma\sqrt{\Delta t}}{t_k}, \tag{5.0.1}$$

and for the rest $\tilde{\alpha}_2(n) = 0$. In Table 5.4 we present the results for the classic Split tree, CRR, Tian, CP and for the modified Split tree. There we used $K = 100$ $\sigma = 0.2$, $r = 0.05$, $T = 1$ and for the initial value we took $S_0 \in \{90, 100, 110\}$. We plot these results in Figure 5.5. The first thing which is noticeable is that we have a lot of small oscillations. Let us check why is this happening. Namely, if we plug the new drift into (3.2.15), we get

$$a_{\tilde{\alpha}_1} = \frac{1}{2}k + \frac{(2l - k)\sqrt{n}}{2\sqrt{k}}$$

and this is not an integer for every number of steps. In other words, the number of up movements in the tree is not an integer number, which means we do not end up with a strike value on the node, which leads us to an oscillation pattern. We obtain a similar behavior if we use the drift from the CP model.

Given the performance of the split tree for American options we see potential in its application. However, it seems that we need a more specified approach than in the European option case to cope with the early exercise feature of American options in a satisfying way. This is clearly an area for future research.

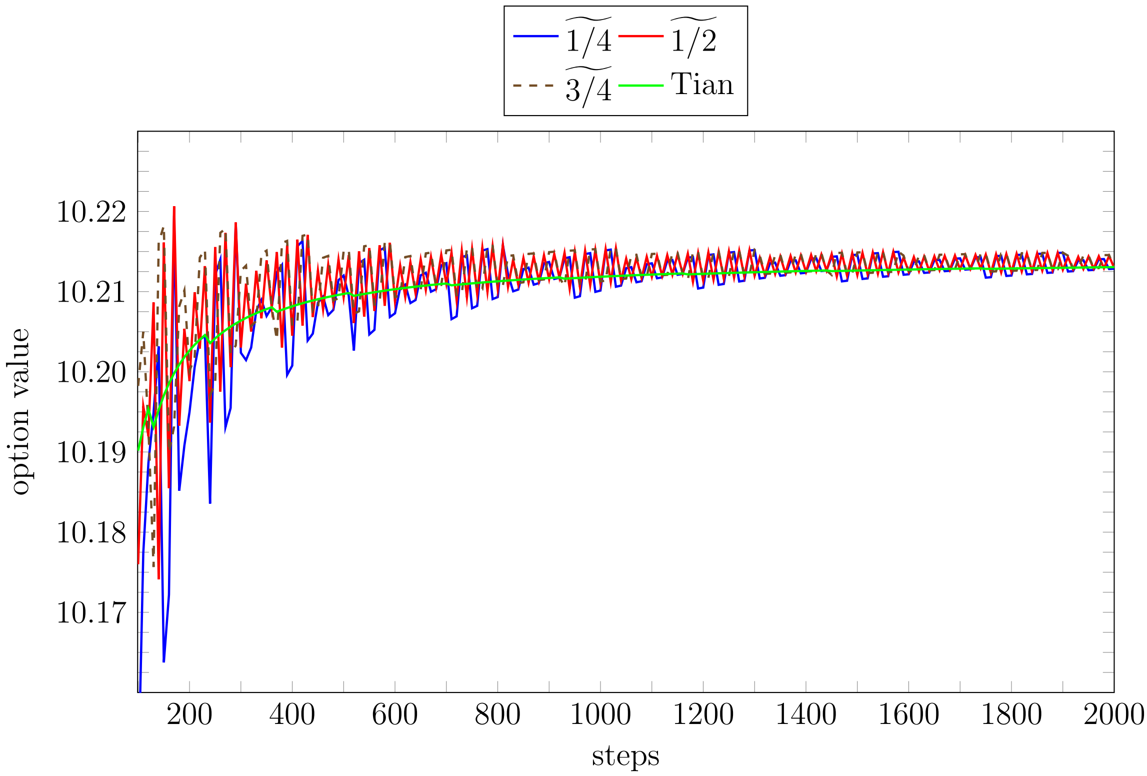


Figure 5.5: American put options: $K = 100$, $S_0 = 90$, $r = 0.05$, $\sigma = 0.2$, $T = 1$

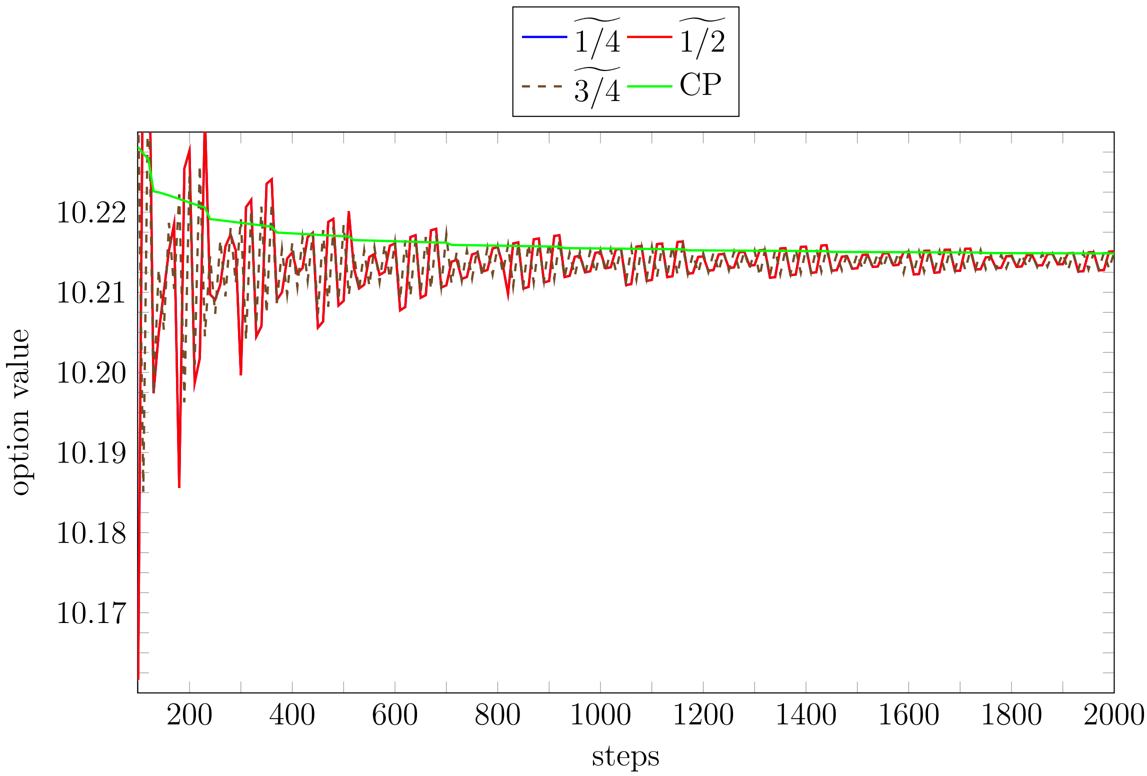


Figure 5.6: American put options: $K = 100$, $S_0 = 90$, $r = 0.05$, $\sigma = 0.2$, $T = 1$

Parameters	n	1/4	1/2	3/4	CRR	Tian	CP
	100	4.2576	4.2576	4.2576	4.2576	4.2576	4.2732
	200	4.2605	4.2605	4.2605	4.2605	4.2605	4.2683
	400	4.2619	4.2619	4.2619	4.2619	4.2619	4.2658
K = 70	500	4.2622	4.2622	4.2622	4.2622	4.2622	4.2653
BS = 3.9015	800	4.2626	4.2626	4.2626	4.2626	4.2626	4.2645
	1000	4.2627	4.2627	4.2627	4.2627	4.2627	4.2643
	100	10.7686	10.6629	10.6443	10.6433	10.6338	10.6434
	200	10.6923	10.6503	10.6420	10.6410	10.6379	10.6422
	400	10.6633	10.6450	10.6410	10.6409	10.6391	10.6413
K =80	500	10.6582	10.6440	10.6409	10.6412	10.6394	10.6411
BS = 9.3662	800	10.6511	10.6426	10.6407	10.6409	10.6398	10.6409
	1000	10.6488	10.6422	10.6406	10.6409	10.6400	10.6409
	100	15.0974	15.0382	15.0337	15.0333	15.0323	15.0352
	200	15.0507	15.0345	15.0327	15.0339	15.0336	15.0346
	400	15.0401	15.0352	15.0345	15.0345	15.0340	15.0346
K = 85	500	15.0387	15.0350	15.0344	15.0349	15.0347	15.0351
BS = 12.9238	800	15.0371	15.0350	15.0346	15.0344	15.0342	15.0345
	1000	15.0366	15.0350	15.0348	15.0349	15.0347	15.0349
	100	0.9753	1.0240	1.0439	1.0728	1.0670	1.0779
	200	1.0225	1.0475	1.0576	1.0746	1.0693	1.0747
	400	1.0468	1.0595	1.0646	1.0729	1.0706	1.0733
K = 60	500	1.0518	1.0620	1.0660	1.0729	1.0708	1.0730
BS=1.009	800	1.0592	1.0656	1.0682	1.0717	1.0712	1.0726
	1000	1.0617	1.0668	1.0689	1.0719	1.0713	1.0724
	100	0.0695	0.0954	0.1053	0.1217	0.1193	0.1219
	200	0.0938	0.1080	0.1132	0.1210	0.1205	0.1219
	400	0.1072	0.1146	0.1173	0.1214	0.1210	0.1217
K = 50	500	0.1100	0.1160	0.1182	0.1215	0.1212	0.1217
BS = 0.1168	800	0.1143	0.1181	0.1194	0.1216	0.1213	0.1217
	1000	0.1157	0.1188	0.1199	0.1216	0.1214	0.1217

Table 5.3: American put: $S_0 = 70$ $\sigma = 0.2$, $r = 0.05$, $T = 1$,

Parameters	n	1/4	1/2	3/4	CRR	Tian	CP	$\widetilde{1/4}$	$\widetilde{1/2}$	$\widetilde{3/4}$
$S_0 = 90$	100	11.6415	11.5255	11.5007	11.4986	11.4857	11.5009	11.4433	11.4785	11.4916
	200	11.5582	11.5076	11.4960	11.4945	11.4893	11.4966	11.4833	11.4875	11.4886
	400	11.5229	11.4997	11.4940	11.4940	11.4909	11.4946	11.4852	11.4892	11.4902
	500	11.5164	11.4982	11.4938	11.4922	11.4913	11.4942	11.4907	11.4912	11.4937
	800	11.5071	11.4961	11.4934	11.4936	11.4918	11.4936	11.4903	11.4914	11.4917
	1000	11.5041	11.4954	11.4932	11.4934	11.4920	11.4935	11.4907	11.4916	11.4919
$S_0 = 100$	100	6.0824	6.0824	6.0824	6.0824	6.0824	6.1045	6.0719	6.0824	6.0990
	200	6.0864	6.0864	6.0864	6.0864	6.0864	6.0975	6.0864	6.0864	6.0864
	400	6.0884	6.0884	6.0884	6.0884	6.0884	6.0940	6.0884	6.0884	6.0884
	500	6.0888	6.0888	6.0888	6.0888	6.0888	6.0933	6.0904	6.0888	6.0928
	800	6.0894	6.0894	6.0894	6.0894	6.0894	6.0922	6.0894	6.0894	6.0894
	1000	6.0896	6.0896	6.0896	6.0896	6.0896	6.0918	6.0896	6.0896	6.0896
$S_0 = 110$	100	2.9734	2.9802	2.9829	2.9879	2.9857	2.9879	2.9614	2.9711	2.9916
	200	2.8594	2.9244	2.9508	2.9969	2.9780	2.9993	2.9770	2.9799	2.9812
	400	2.9540	2.9708	2.9775	2.9885	2.9845	2.9899	2.9805	2.9827	2.9836
	500	2.9604	2.9739	2.9793	2.9884	2.9849	2.9892	2.9862	2.9843	2.9886
	800	2.9701	2.9786	2.9820	2.9877	2.9855	2.9882	2.9842	2.9849	2.9852
	1000	2.9734	2.9802	2.9829	2.9879	2.9857	2.9879	2.9850	2.9854	2.9855

Table 5.4: American put: $K = 100$ $\sigma = 0.2$, $r = 0.05$, $T = 1$,

Chapter 6

Conclusion

In this thesis we have introduced the most famous models for option pricing using binomial trees. First, we have seen the basic properties of the the CRR and the RB tree and the first problems appeared there. Afterwards we tried to fix these problems by introducing the Tian and the CP model. But the real focus of this thesis was on the Split tree model. The split tree model combines the Tian and the CRR tree. Namely, the idea of this model is to use the best from both models: we start with a drift which moves our tree so that we end up with the strike value on the middle node and afterwards we continue with the CRR tree, since we have shown that the CRR tree behaves pretty well when it starts with the strike value. Our main aim was to prove that this new model converges to the price obtained by the continuous model. We were also interested in the rate of convergence and what would be the optimal splitting time. As the result we ended up with Theorem 11. The importance of this theorem is that it tells us that we will have smooth convergence regardless of our parameters - the position of the strike and asset price will not ruin it. Also, the speed of convergence is at least as good as in the Tian and in the CP.

The foregoing chapter also showed that the Split tree has a big potential for the application on Anerican options. However, while we have smooth convergence in many cases, the convergence is not as fast as the one of the CRR, the Tian and the CP model. Possible ways to improve this can be the use of extrapolation methods or suitable modifications of the drift in the first part of the Split tree. There, our results showed an improved speed of convergence for the price of losing the smooth convergence. Thus, further research on the application of the Split tree for American options is necessary.

Appendix

Theorem 12. ***(Donsker's theorem)*** *If ξ_1, ξ_2,... are independent and identically distributet with mean 0 and variance σ^2, and if X^n is the random function defined by*

$$X_t^n(\omega) = \frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor}(\omega) + (nt - \lfloor nt \rfloor) \frac{1}{\sigma\sqrt{n}} \xi_{\lfloor nt \rfloor + 1}(\omega)$$

with $S_n = \xi_1 + \xi_2 + ... + \xi_n$, $(S_0 = 0)$. Then $X^n => W$ as $n \to \infty$, where W denotes Brownian motion (compare [4]).

Theorem 13. ***(Slutsky's theorem)*** *Let (M, d) be a metric space. Let $(X_1^n, X_2^n)_n$ be a sequence of $(M \times M)$-values random variables defined on a probability space $\left(\Omega^{(n)}, \mathcal{F}^{(n)}, P^{(n)}\right)$. Suppose that $X_1^{(n)} =>_w X_1$ for some M-valued random variable X_1. If for all $\varepsilon > 0$,*

$$P^{(n)}\left(d\left(X_1^n, X_2^n\right) > \varepsilon\right) \to 0,$$

then $X_2^{(n)} =>_w X_1$ (compare [11])

Theorem 14. ***(Berry-Esséen inequality)*** *Let $X_1, X_2, ...X_n$ be independent random variables such that $EX_j = 0$ and $E\left|X\right|^3 < \infty$ $(j = 1, ..., n)$. We write*

$$\sigma_j^2 = EX_j^2, \quad B_n = \sum_{j=1}^{n} \sigma_j^2, \quad F^{(n)}(x) = P\left(B_n^{-\frac{1}{2}} \sum_{j=1}^{n} X_j < x\right)$$

and

$$L^{(n)} = B_n^{-\frac{3}{2}} \sum_{j=1}^{n} E\left|X_j\right|^3.$$

Then,

$$\sup\left|F^{(n)}(x) - \Phi(x)\right| \leq AL^{(n)},$$

where $\Phi(x)$ denotes the standard normal distribution function and A is some positive constant (compare [36]).

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