New Aspects of Inflation Modeling

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Abstract

Inflation modeling is a very important tool for conducting an efficient monetary policy. This doctoral thesis reviewed inflation models, in particular the Phillips curve models of inflation dynamics. We focused on a well known and widely used model, the so-called three equation model which is a system of equations consisting of a new Keynesian Phillips curve (NKPC), an investment and saving (IS) curve and an interest rate rule.

We gave a detailed derivation of these equations. The interest rate rule used in this model is normally determined by using a Lagrangian method to solve an optimal control problem constrained by a standard discrete time NKPC which describes the inflation dynamics and an IS curve that represents the output gaps dynamics. In contrast to the real world, this method assumes that the policy makers intervene continuously. This means that the costs resulting from the change in the interest rates are ignored. We showed also that there are approximation errors made, when one log-linearizes non linear equations, by doing the derivation of the standard discrete time NKPC.

We agreed with other researchers as mentioned in this thesis, that errors which result from ignoring such log-linear approximation errors and the costs of altering interest rates by determining interest rate rule, can lead to a suboptimal interest rate rule and hence to non-optimal paths of output gaps and inflation rate.

To overcome such a problem, we proposed a stochastic optimal impulse control method. We formulated the problem as a stochastic optimal impulse control problem by considering the costs of change in interest rates and the approximation error terms . In order to formulate this problem, we first transform the standard discrete time NKPC and the IS curve into their high-frequency versions and hence into their continuous time versions where error terms are described by a zero mean Gaussian white noise with a finite and constant variance. After formulating this problem, we use the quasi-variational inequality approach to solve analytically a special case of the central bank problem, where an inflation rate is supposed to be on target and a central bank has to optimally control output gap dynamics. This method gives an optimal control band in which output gap process has to be maintained and an optimal control strategy, which includes the optimal size of intervention and optimal intervention time, that can be used to keep the process into the optimal control band.

Finally, using a numerical example, we examined the impact of some model parameters on optimal control strategy. The results show that an increase in the output

gap volatility as well as in the fixed and proportional costs of the change in interest rate lead to an increase in the width of the optimal control band. In this case, the optimal intervention requires the central bank to wait longer before undertaking another control action.

Zusammenfassung

Die Modellierung von Inflation ist ein wichtiges Hilfsmittel um eine Geldpolitik effizient durchzuführen. Im Rahmen dieser Dissertation wurde eine Übersicht über mehrere Inflationsmodelle gegeben, insbesondere über die Modelle der Phillips-Kurve für die Inflationsdynamik. Hierbei haben wir den Schwerpunkt auf die so gennante Drei-Gleichung -Modell gelegt, eins der bekanntesten und am weitesten benutztes makroökonomisches Modell. Dieser Makroökonomische Ansatz umfasst die Neukeynesianische Phillips-Kurve (NKPC), die Investitions-und Sparkurve und eine Zinssatzregel. Zunächst wurde eine detaillierte Herleitung diseser Drei Gleichungen gemacht. Die in diesem Ansatz benutzte Zinssatzregel, ist normalerweise eine Lösung von einem Optimalsteuerungsproblem, die durch eine inflationsdynamik beschreibende NKPC und Investitions-und Sparkurve eingeschränkt und mit Hilfe einer Lagrange-Methode bestimmt wird. Im Gegensatz zu der Realwelt geht diese Methode davon aus, dass die Entscheidungsträger durchgehend eingreifen. Somit bleiben die Kosten, die aus Veränderung des Zinssatzes resultieren, unberücksichtigt. Außerdem wurde gezeigt, dass Näherungsfehler entstehen, wenn man eine Loglinearisierung von nicht linearen Gleichungen während der Herleitung der diskreten Zeit Standard NKPC macht.

Im Rahmen dieser Arbeit wurden Forschungsergebnisse aus in dieser Arbeit zitierten Quellen bestätigt, dass Fehler, die aus log-linear Näherungsfehlern und den Veränderungskosten des Zinssatzes durch die Festlegung der Zinssatzregel resultieren, dazu führen können, dass eine suboptimale Zinssatzregel und somit nichtoptimale Pfad von Output-Lücke und Inflationsrate entsteht.

Um dieses Problem zu überwinden, wurde eine stochastische optimale Impuls-Control-Methode vorgeschlagen. Das Problem wurde als stochastisch optimalen Impuls-Control-Problem formuliert, wobei die Veränderungskosten des Zinssatzes und die Näherungsfehler in das Modell miteinbezogen wurden.

Um das Problem zu formulieren wurde zuerst eine Transformation der diskreten Zeit Standard NKPC und der Investitions-und Sparkurve zu ihrer "high-frequency" Versionen und damit zu ihrer stetige Zeit Versionen gemacht, wobei die Fehlerterme durch Gaußsches weißes Rauschen mit einer Mittelwert Null und einer endlichen und konstanten Varianz beschrieben werden.
Nach der Formulierung dieses Problems, wurde die Quasi-Variationsungleichung Ansatz verwendet, um analytisch ein Sonderproblem der Zentralbank zu lösen. Bei dem Problem wird davon ausgegangen, dass eine Inflationsrate im Plan liegt und die Zentralbank den Output-Lücke-Prozess optimal kontrollieren muss.

Dieses Verfahren ergibt eine optimales Kontrollband, in dem ein Output-Lücke Prozess erhalten werden muss, und eine optimale Regelsstraegie, die die Optimalen Eingriffsgrösse und -zeit umfasst, die dazu dient den Prozess in dem optimalen Kontrollband zu halten.

Im letzten Teil wurde durch ein numerisches Beispiel die Auswirkung von solchen Modellparametern auf optimalen Steuerungsstrategie untersucht. Die Ergebnisse zeigen, dass eine Erhöhung sowohl in der output gap volatilität als auch in den fixen und variablen Kosten in Abhängigkeit von der Zinssatzänderung zu einer erhöhten Breite des optimalen Kontrollband führen. In diesem Zusammenhang sollte die Zentralbank länger warten um einen Steuerungsseingriff vorzunehmen.

Chapter 1

Introduction

Motivation. Unstable and high inflation rate is regarded as a considerable economic, social and political problem because it results in arbitrary redistribution of wealth favoring a group of society (e.g. debtors) and hurting another (e.g. creditors and fixed income earners). This makes people feel insecure and can affect the public morality. It additionally causes capital flight, which can lead to a disappearance of wealth, since many people can decide to invest their money in foreign assets. As a consequence, falling investment and savings can inhibit economic growth. For this reason, there was a broad consensus around the world that maintenance of low inflation rate should be a primary goal of the central banks (see also [73] P.2). This mandate has been extended for most central banks after the global financial crisis in 2008 and effectively promoted the goals of low unemployment, maximum economic growth and low inflation reinforced by an increased adoption of flexible inflation targeting. But quite often, these goals conflict. This leads to an optimal control problem whose solution is an optimal policy that can be applied to achieve these three objectives. In order to formulate such a problem mathematically, one needs a good mathematical description (or model) of the dynamics of variables that must be controlled. This is one of the main reasons why a great variety of macroeconomic models has been developed that represent the dynamics of the key macroeconomic variables such as inflation, output gap (or unemployment) and interest rates. The most widely used model in modern monetary macroeconomics is the 3-equation new keynesian model (see for example [32]). This is a dynamic system which involves a new keynesian Phillips curve that relates inflation to output gap, an IS equation that links the evolution of the output gap to the nominal interest rate and an interest rate rule.
This dissertation will focus on this class of models of inflation dynamics.

Contribution. In this dissertation we contribute to the existing literature on inflation modeling and controlling as follows:

First, we do a detailed derivation of the existing discrete time New Keynesian Phillips curve model of inflation dynamics,
we transform it into its corresponding high-frequency version (i.e. the length period between two consecutive observed data is reduced) and hence into a continuous time New Keynesian Phillips curve model by considering rational expectation and approximation errors (which are made by doing Log-linearization of some equations during the derivation of the model) in form of two dimensional stochastic differential equations. Furthermore, the obtained system of two stochastic differential equations will be solved analytically.
Second, we introduce an alternative framework for controlling inflation and output gaps by means of interest rates. Indeed, the interest rate rule in the Standard discrete time New Keynesian Phillips Curve model is usually determined by solving an optimal control problem via Lagrangian methods. It is assumed that the control must be applied every time instant and that the control computed in this manner would lead to an optimal trajectory of inflation and output gap over time. However, in the real world, the central banks alter interest rates discontinuously, due to the costs related to the manipulation of interest rates. As it has been illustrated by Sami et al. 2013 (see [66]), and King et al.2005 (see [82]), the interest rate rule determined by using the method above can lead to the sub-optimal trajectory of output gap and inflation over time. To overcome such problems, this dissertation proposes a stochastic impulse control method. This method gives an optimal band and optimal control strategy (which consists of the sequence of optimal intervention times and the corresponding actions) that can be applied to keep output gap and inflation process into an optimal band. To this end, we formulate the central bank problem where inflation and output have to be maintained near their targets by means of interest rate as a stochastic impulse control problem constrained by the two obtained stochastic differential equations which describe inflation and output gap process. Moreover, we will solve analytically the special case, where the central bank has to control only the output gap process through interest rates by assuming that inflation is on target, using quasi-variational inequality method. Finally, we examine the influence of some model parameters to the optimal band.

1.1 Outline of the Dissertation

The remainder of this dissertation proceeds as follows:
The next chapter introduces some mathematical tools, concepts and stochastic control theory that will be used in subsequent chapters. In chapter 3 we will first briefly discuss the meaning and measurement of inflation.
Section 2 of this chapter presents a wide view of inflation modeling. Finally, we discuss the relationship between inflation and other three key macroeconomic variables,

namely output gap, unemployment and interest rates, via the Phillips curve models of inflation dynamics. In this section a detailed derivation of the discrete time New Keynesian Phillips Curve model of inflation and its continuous time version will be given.
The fourth chapter will deal with the problem of controlling inflation and the output gap process. In section one of this chapter, we rigorously formulate the problem as a stochastic optimal impulse control problem where the central bank has to find optimal strategies in order to keep inflation and output gap process inside an optimal control band by means of interest rates. In section two, we will apply the quasi-variational inequality method and solve analytically the problem where under the assumption that inflation is on target, the central bank has to maintain the output gap process inside an optimal interval using interest rates.
Finally, the last section of this chapter will present a numerical example to illustrate the effects of some model parameters to the optimal impulse strategies.

Chapter 2

Some Mathematical Preliminaries

In this chapter we introduce some mathematical tools, concepts and stochastic control theory that will be used throughout this work. For the reason of keeping this chapter a reasonable size, many proofs are omitted and we will give references where the interested reader can find them. Basic references used for this chapter are [113], [39], [114], [25], [72],[87] and [110]

2.1 Stochastic Processes

In real world, we usually observe most variables and especially economic variables whose values change randomly over time. This phenomenon is usually called a stochastic (random) process and it is formally defined as follows:

Definition 2.1.1. ***A stochastic process*** *is a family of random variables* $\{X_t, t \in \mathcal{I}\}$ *defined on the same probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, where* Ω *is a collection of outcomes* ω*,* $\mathcal{F}$ *stands for a* σ*-algebra of subsets of* Ω *and* $\mathbb{P}$ *is a probability measure.*
If $\mathcal{I} = \mathbb{N}$, the process is said to be a **discrete time stochastic process** and it is expressed by X_t.
If $\mathcal{I} = [0, \infty)$, the process is called a **continuous time stochastic process** and it is denoted by X(t). In this dissertation t and X_t represent the time and the position or "state "of the process at time t respectively.
For every fixed $\omega \in \Omega$, the mapping $t \mapsto X_t(\omega)$ is called the **sample path** or the **sample realization** or the **trajectory** of the process.

An important example of a stochastic process that will be used in this dissertation is the Brownian motion. To define this process, let us start with a definition of a filtered probability space.

Definition 2.1.2. *A quadruple $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F}$ is an increasing collection of sub-σ-algebra $\{\mathcal{F}_t\}_{t\geq 0}$ with $\mathcal{F}_t \subset \mathcal{F}$ and $\mathcal{F}_s \subset \mathcal{F}_t, \quad \forall\ 0 \leq s < t < \infty$, is called a **filtered probability space** or a **stochastic basis** and $\mathbb{F}$ is a filtration.*
We say that a filtration $\mathbb{F}$ is **complete** if the following conditions are satisfied:
(i) $\mathcal{F}_0$ contains all subsets of $\mathbb{P}$-null sets of $\mathcal{F}$ and
(ii) the filtration $\mathbb{F}$ is right-continuous, i.e $\mathcal{F}_{t^+} := \bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t, \quad \forall t \geq 0$.
A filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is **complete** if the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the Filtration $\mathbb{F}$ are complete [1].
Intuitively, one may see the filtration $\mathbb{F}$ as a means of modeling the complete flow of information over time assuming that no information is lost and $\mathcal{F}_t$ as the set of informations available at time t.
A filtration $\mathcal{F}_t^X$ **generated** by the process $X = \{X_t, t \geq 0\}$, which is also called **natural filtration** or (**canonical**) of X, is $\mathcal{F}_t^X = \sigma\left(X_s, o \leq s \leq t\right), t \in [0, \infty)$, the smallest σ-algebra under which X_s is measurable for all $0 \leq s \leq t$. One can think $\mathcal{F}_t^X$ as the set of all informations which can be extracted from the observation of the paths of X between 0 and t.

Definition 2.1.3. *Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, a stochastic process $B = (B(t))_{t\geq 0}$ starting at 0 with mean value μ and variance σ^2 on this space is called a $\mathbb{P}$-**Brownian motion** (or a $\mathbb{P}$-Wiener process) if:*
i) B(0)= 0 $\mathbb{P}$-a.s [2] that is $\mathbb{P}\left(B(0) = 0\right) = 1$
ii) B has independent increments, i.e., $B(t) - B(s)$ is independent of

$$B(t^{'}) - B(s^{'}) \quad \forall 0 \leq s^{'} \leq t^{'} \leq s \leq t < \infty.$$

iii) B has stationary increments, i.e. the distribution of $B(t+u) - B(t)$ only depends on u, $\quad \forall u \geq 0$.
iv) B has a.s. continuous sample paths.
$(B(t))_{t\geq 0}$ is called a normalized (or standard) Brownian motion if $\mu = 0$ and $\sigma^2 = 1$.

We call B with $B = (B_1, \ldots, B_d) = (B_1(t), \ldots, B_d(t))_{t\geq 0}$ a d-dimensional Brownian motion, $d \in \mathbb{N}$, if its components $B_i, i = 1, \ldots, d$, $d \in \mathbb{N}$, are independent Brownian motions.

As we will see in subsection 2.2.1, a stochastic optimal impulse control problem

[1] *A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is **complete** if any subset of any $\mathbb{P}$-null set A is also in $\mathcal{F}$.* It is always possible to make any probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space by adding to a σ-algebra $\mathcal{F}$ a $\mathbb{P}$ -null set $N \subset \mathcal{N}$, where

$$\mathcal{N} := \{N : N \subseteq A \ for\ some\ \ A \subset \mathcal{F} \ with\ \mathbb{P}(A) = 0\}\ (see[39]p.14).$$

[2] a.s., is an abbreviation for an event that almost surely happens.

formulation requires the existence of a unique solution of the stochastic differential equations which represent the processes to be controlled. For this reason, we first define what we call a unique (strong) solution. Then we present two results that will be used in order to check if the system of stochastic differential equations describing inflation and output gap processes has a strong unique solution.

Definition 2.1.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $B = \left\{B(t), \mathcal{F}_t^B; 0 \leq t < \infty\right\}$ be a m-dimensional Brownian motion on it. In addition, assume that this space is rich enough to accommodate a random vector $\xi \in \mathbb{R}^d$ which is independent of $\mathcal{F}_\infty^B$ and whose distribution is $F_\xi(A) = \mathbb{P}\left(\xi \in A\right)$; $A \in \mathcal{B}\left(\mathbb{R}^d\right)$.*
when one considers the left-continuous filtration $\{\mathcal{G}_t\}_{t\geq 0} := \sigma\left(\xi, B(s); 0 \leq s \leq t\right)$ for $0 \leq t < \infty$, and creates the augmented filtration [3]
(2.0) *$\{\mathcal{F}_t\}_{t\geq 0} := \sigma\left(\mathcal{G}_t \cup \mathcal{N}\right), 0 \leq t < \infty;\quad \mathcal{F}_\infty := \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$,*
where $\mathcal{N} := \{N \subseteq \Omega; \exists G \in \mathcal{G}_\infty \text{ with } N \subseteq G \text{ and } \mathbb{P}\left(G\right) = 0\}$,
one gets an m-dimensional Brownian motion $\{B(t), \mathcal{F}_t, 0 \leq t < \infty\}$ as it has been proven in [78] .
*Considering all these above, a **strong solution** X of the stochastic differential equation*

$$dX(t) = b(X(t), t)dt + \sigma(X(t), t)dB(t), t \geq 0, \tag{2.1}$$

with
$b : \mathbb{R}^d \times [0, \infty) \longrightarrow \mathbb{R}^d$,
$\sigma : \mathbb{R}^d \times [0, \infty) \longrightarrow \mathbb{R}^{d\times m}$ measurable, on $(\Omega, \mathbb{F}, \mathbb{P})$ with respect to the fixed m-dimensional Brownian motion B and the initial condition ξ (which is independent of B) over this probability space is a stochastic process $(X(t), t \geq 0)$ which obeys the following properties:
a) X is adapted [4] to the augmented filtration $\{\mathcal{F}_t\}_{t\geq 0}$ represented by (2.0) above;
b) X is a continuous process;
c) $\mathbb{P}\left(X(0) = \xi\right) = 1$;
d) $\int_0^t ||b\left(X(s), s\right)|| + ||\sigma\left(X(s), s\right)||^2 ds$ is finite almost surely and
e) $X(t) = X(0) + \int_0^t b\left(X(s), s\right) ds + \int_0^t \sigma\left(X(s), s\right) dB(s);\ \forall t \geq 0$, holds almost surely.

Assume that, whenever $(\Omega, \mathbb{F}, \mathbb{P})$ is a probability space equipped with a Brownian motion B and an independent random variable ξ, any two strong solutions X, Y of the stochastic differential equation (2.1) with initial conditions ξ satisfy

$$\mathbb{P}\left(\forall t \geq 0, X(t) = Y(t)\right) = 1.$$

[3] This filtration is right-continuous and hence complete (for the proof see [78] P.90).
[4] Adapted process is defined as follows:
*Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$ be a filtered probability space. A stochastic process $\{X(t), t \geq 0\}$ on Ω is **adapted** to the filtration $\{\mathcal{F}_t, t \geq 0\}$ if for every $t \geq 0$ the random variable $X(t)$ is $\mathcal{F}_t$ measurable* (In this case the processe $X(t)$ is also called "**non-anticipating process**"). One can say that an **adapted process** is a process whose value at any time t is revealed by the information $\mathcal{F}_t$.

Then one says that the strong uniqueness holds for (2.1)

Theorem 2.1.1 (Existence and uniqueness theorem for stochastic differential equations).
Let $T > 0$ and $b(.,.) : [0,T]\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\sigma(.,.) : [0,T]\times\mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m}$ be measurable functions for which there exist constants C and D such that

$$\textbf{a)}\ \|b(t,x)\| + \|\sigma(t,x)\| \leq C(1 - \|x\|);$$

$$\textbf{b)}\ \|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq D(\|x - y\|);$$

$\forall x, y \in \mathbb{R}^n$ *and* $\forall t \in [0,T]$ *(where* $\|\sigma\|^2 = \sum \|\sigma_{ij}\|^2$*) and let*

c) *ξ be a random variable which is independent of the σ-algebra $\mathcal{F}_\infty^{(m)}$ generated by $B_s(.)$, $s \geq 0$ with finite second moment(i.e $E\left[\|\xi\|^2\right] < +\infty$).*

Then the stochastic differential equation

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t, \tag{2.2}$$

for $t \in [0,T]$ and $X_0 = \xi$; possesses a unique continuous strong solution $X_t(\omega)$ with the property that $X_t(\omega)$ is adapted to filtration $\mathcal{F}_t$ (as in (2.0)) generated by ξ and $B_s(.)$; $s \leq t$ and $E\left[\int_0^T \|X_t\|^2 dt\right] < \infty$.

Proof of the theorem 2.1.1 see [114] P.69-71

The main technical tool in stochastic calculus is Itô's Formula which we state below.

Itô's Formula: (Multi-Dimensional Itô's formula).[5] *Let u(t,z) be a continuous function in $[0,\infty) \times \mathbb{R}^m$ with continuous partial derivatives $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial z^i}$, $\frac{\partial u}{\partial z^i z^j}$. Further, let $X(t)$ be a m-dimenional process presented by a stochastic differential equation:*

$$dX(t) = \mu(t)dt + \sigma(t)dB(t) \tag{2.3}$$

where $\mu_1, \ldots \mu_m$ and $\sigma = (\sigma_{ij})$, $(1 \leq i \leq m)$, $(1 \leq j \leq n)$, belong to $L_w^1[0,T]$ and $L_w^2[0,T]$[6] *respectively. Then $u(t, X(t))$ has a stochastic differential representation:*

[5] Itô's formula and its proof can be found for example in[114] , [108] and [98].

[6] Recall that $L_w^p[\alpha,\beta]$ is the set consisting of classes of all separable f_t - adapted measurable processes $f(t)$ with $(1 \leq p \leq \infty)$ satisfying: $\mathbb{P}\left\{\int_\alpha^\beta \|f(t)\|^p dt < \infty\right\} = 1, 1 \leq p < \infty$ and $\mathbb{P}\left\{ess\ sup_{\alpha<t<\beta}\|f(t)\| < \infty\right\} = 1, p = \infty$.

$$du(t,X(t)) = \left[\frac{\partial u}{\partial t}(t,X(t)) + \sum_{i=1}^{m}\frac{\partial u}{\partial X^i}(t,X(t))\mu_i(t)\right.$$
$$\left. + \frac{1}{2}\sum_{l=1}^{n}\sum_{i,j=1}^{m}\frac{\partial u}{\partial X^i X^j}(t,X(t))\sigma_{il}(t)\sigma_{jl}(t)\right]dt + \sum_{l=1}^{n}\sum_{i=1}^{m}\frac{\partial u}{\partial X^i}(t,X(t))\sigma_{il}(t)dw_l(t)$$

The next theorem is useful when one needs to interchange the limit and expectation. It gives conditions that guarantee the validity of this interchange. Before presenting it, let us define the following terms.

Definition 2.1.5. *Let $X : \Omega \longrightarrow \mathbb{R}$ be a random variable.*
Define

$$X^+(\omega) = \begin{cases} X(\omega) & \text{if } X(\omega) \geq 0 \\ 0 & \text{if } X(\omega) < 0 \end{cases}, \quad X^-(\omega) = \begin{cases} 0 & \text{if } X(\omega) > 0 \\ -X(\omega) & \text{if } X(\omega) \leq 0 \end{cases}$$

Then X^+ and X^- are non-negative, $X(\omega) = X^+(\omega) - X^-(\omega)$ for all ω, and $|X(\omega)| = X^+(\omega) + X^-(\omega)$.
*If at least one of $\int_\Omega X^+(\omega)d\mathbb{P}(\omega) < \infty$ or $\int_\Omega X^-(\omega)d\mathbb{P}(\omega) < \infty$ holds, X is said to be (Lebesgue)-**integrable** with respect to the probability measure $\mathbb{P}$ and we define $\int_\Omega X(\omega)d\mathbb{P}(\omega) := \int_\Omega X^+(\omega)d\mathbb{P}(\omega) - \int_\Omega X^-(\omega)d\mathbb{P}(\omega)$.*
*X is called **summable** if and only if $\int_\Omega |X(\omega)|d\mathbb{P}(\omega) < \infty$.*

*If X is integrable then the number $E(X) = \int_\Omega X(\omega)d\mathbb{P}(\omega)$ is called the **expectation** of X with respect to the probability measure $\mathbb{P}$.*

Theorem 2.1.2 (Dominated convergence theorem (DCT)). *Let x_n be a sequence of Borel functions converging to x almost surely. Assume that there exists a μ-summable function $y \geq |x_n|$ almost surely.*
Then $\lim_{n\to\infty} E(x_n) = E(\lim_{n\to\infty} x_n) = E(x)$. Proof see [121].

Another important concept that will be used in this work, is the conditional expectation (CE). In the following, we start with the construction of the space on which it is defined and thereafter we give a formal definition of the CE.

Definition 2.1.6. *For a random variable $X : \Omega \longrightarrow \mathbb{R}^d$ and $1 \leq P < \infty$ we define the L^P-norm of X, $||X||_P$, by $||X||_P = ||X||_{L^P(\Omega,\mathcal{F},\mathbb{P})} = \left(\int_\Omega |X(\omega)|^P d\mathbb{P}(\omega)\right)^{\frac{1}{P}}$. If $P = \infty$, $||X||_\infty = ||X||_{L^\infty(\Omega,\mathcal{F},\mathbb{P})} = sup\{|X(\omega)|; \omega \in \Omega\}$.*
The corresponding spaces $L^P(\Omega,\mathcal{F},\mathbb{P}) = \left\{X : \Omega \longrightarrow \mathbb{R}^d; ||X||_P < \infty\right\}$ are called L^P- spaces.

Definition 2.1.7. *let X be an element of $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$. The* ***conditional expectation*** $E(X|\mathcal{G})$ *of X given $\mathcal{G}$ is a random variable which satisfies the following conditions:*
i) $E(X|\mathcal{G})$ is measurable with respect to $\mathcal{G}$, and
ii) for any $A \subset \mathcal{G}$ we have $E\{E(X|\mathcal{G})1_A\} = E\{X1_A\}$ where 1_A is the indicator function defined by

$$1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

The following are the properties of conditional expectation[7]:
1. If y is $\mathcal{G}$-measurable, then $E(YX|\mathcal{G}) = YE(X|\mathcal{G})$, and
2. If X and $\mathcal{G}$ are independent, then $E(X|\mathcal{G}) = E(X)$.

Second-order Linear Ordinary Differential Equations

For solving stochastic control problems one typically reduces the search for the value function (i.e. the optimal utility as a function of the initial time and position of controlled process) to the solution of an ordinary or a partial differential equation. This part provides the results that will help us to solve a second order differential equation presented in chapter 4.

Definition 2.1.8. *Equations of the form*

$$y^{''}(x) + P(x)y^{'}(x) + Q(x)y(x) = R(x), \tag{2.4}$$

where P, Q and R stand for continuous functions on an open interval I are called the ***Second -Order Linear Ordinary Differential equations****. If $R(x) = 0$ for all x, then the equation (2.4) becomes*

$$y^{''}(x) + P(x)y^{'}(x) + Q(x)y(x) = 0, \tag{2.5}$$

and we call it the ***second order linear homogeneous ODE****. If $R(x) \neq 0$ for some x, equation (2.5) is* ***inhomogeneous****. A solution y_1 to such equation is a function that*
satisfies this equation. A ***particular solution*** *y_p is a solution with no arbitrary constant. A* ***general solution*** *y is a solution with arbitrary constants from which every particular solution can be determined by appropriate choice of coefficients. A homogeneous solution is the general solution to a linear homogeneous ODE.* ***Homogeneous solution*** *of inhomogeneous equation $y_h$is the general solution of the corresponding homogeneous ODE.*

[7]See [125].

The following theorem gives sufficient conditions for such equations to have a unique solution.

Theorem 2.1.3. Existence and uniqueness (see also [25]): *consider an initial value problem* $y''(x) + P(x)y'(x) + Q(x)y(x) = R(x), \quad y(x(0)) = y(0)$ $y'(x(0)) = y'(0)$ *where* P*,* Q *and* R *are continuous functions on an open interval* (a, b) *containing the point* $x = x(0)$*. Then there exists a unique solution* $y = \phi(x)$ *of this problem, and the solution exists throughout the interval* (a, b).

How can we form the general solution of equation 2.4 and 2.5? The answer is given by the proposition 2.1.1 and 2.1.2 below.

Proposition 2.1.1: *The general solution of equation (2.5) is of the form*

$$C_1y_1 + C_2y_2$$

, where y_1 *and* y_2 *are two linearly independent solutions of equation (2.5).*
The proof of this proposition can be found for example in [139]P.125.

Proposition 2.1.2: *every solution of inhomogeneous equation can be written in the form*

$y(x) = \underbrace{C_1y_1(x) + C_2y_2(x)}_{:=y_h(x)} + y_p$, *where* $y_h(x)$ *is homogeneous (or general) solution of equation (2.5) and* $y_p(x)$ *is some specific (or particular) solution of inhomogeneous equation.* (Proof see [25]).

How can we verify that two solutions are linearly independent?
In order to answer this question, we need to define first the Wronskian determinant which is a very useful tool for :
1. checking if the two solutions are linearly independent,
2. finding a second solution if we know one solution,
3. determining a particular solution of inhomogeneous equation.

Definition 2.1.9. *A* 2×2 *determinant for two differentiable functions* y_1 *and* y_2 *of the form*

$W[y_1, y_2](x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x)$ is called the **Wronskian determinant**.

Proposition 2.1.3. *Let $y_1(x)$ and $y_2(x)$ be the solutions of equation (2.5). Then $y_1(x)$ and $y_2(x)$ are linearly independent if and only if $W[y_1, y_2](x) \neq 0$ for all $x \in I$.*
(The proof can be found in [25])

Remark 2.1.1. If we know one solution of the equation (2.5), say $y_1(x)$, then

$$y_2(x) = y_1(x) \int^x \frac{\exp\left(-\int^t P(u)du\right)}{(y_1(t))^2} dt. \tag{2.6}$$

Proof of the remark 2.1.1. Let us consider

$\frac{W[y_1,y_2](x)}{(y_1(x))^2} = \frac{y_1(x)y_2'(x) - y_2(x)y_1'(x)}{(y_1(x))^2} = \left(\frac{y_2(x)}{y_1(x)}\right)'.$

Applying the integral on both sides and the Abel's theorem [8] we have

$\frac{y_2(x)}{y_1(x)} = \int^x \frac{W[y_1,y_2](x)}{(y_1(x))^2} dx = \int^x \frac{\exp\left(-\int^t P(u)du\right)}{(y_1(t))^2} dt + C.$
The constant C can be dropped because it only adds to $y_2(x)$ a multiple of $y_1(x)$, and thus our final formula for $y_2(x)$ will be

$y_2(x) = y_1(x) \int^x \frac{\exp\left(-\int^t P(u)du\right)}{(y_1(t))^2} dt.$ □

One can determine a particular solution of an inhomogeneous ODE as follows.

Theorem 2.1.4 (see also [25]). *If the functions $y_1(x)$ and $y_2(x)$ are linear independent solutions of equation (2.5), then a particular solution of inhomogeneous equation can be given by*

$y_p(x) = -y_1(x) \int \frac{y_2(x)R(x)}{W[y_1,y_2](x)} dx + y_2(x) \int \frac{y_1(x)R(x)}{W[y_1,y_2](x)} dx.$

Depending on the form of some functions $R(x)$ in equation (2.5), the forms of particular solution of inhomogeneous equation are known.

[8] The Abel's states that:
If $y_1(x)$ and $y_2(x)$ are two solutions of the equation (2.5), then

$W[y_1, y_2](x) = D \exp\left(-\int P(u)du\right),$
where D is a certain constant that is independent on x, but dependent on y_1 and y_2. Further, $W[y_1, y_2](x) = 0$ for all $x \in I$, only if $D = 0$. (Proof see [25]).

For example, if $R(x) = a_n x^n + \ldots + a_0$, then the form of y_p is $a_n x^n + \ldots + a_0$ (see also [25] p.175).

2.2 Stochastic Optimal Control Theory

Stochastic optimal control theory (SOCT) is a mathematical description of how to find an optimal control among all possible ones for the purpose of optimizing the expected discounted cost or reward function either over a given finite or an infinite interval of time subject to the constraint equations in the form of stochastic differential or difference equations. SOCT has played an important role in solving the range of problems which arise in a wide variety of disciplines including different fields of engineering, financial mathematics and economics. These problems can be classified into two main groups depending on the kind of control which is used [9]:
• Classical stochastic optimal control problem, where the control has to be applied at every time instant and
• Stochastic optimal stopping problem, where at each time point the control is either to stop or continue. In this case, there is no further control action after stopping the system. A key example of stochastic optimal stopping problem is the American option pricing problem.
In real world situation, one often deals with a mixture of the above mentioned problems where the decision-maker can influence continuously the dynamic behavior of the system and furthermore has the opportunity to stop it at an optimal stopping time.
Another significant intermediate case is:
Stochastic optimal impulse control problems in which the control is applied only at distinct stopping times. In finance, impulse control is applied to the problems where the fixed and proportional transaction costs are taken into account in contrast to the classical stochastic optimal control where transaction costs are neglected. The other difference between these two stochastic controls is that in stochastic optimal impulse control the effect of the control is to shift the process without affecting either the drift or the volatility as it is done in the classical stochastic optimal control. In the absence of fixed transaction costs (when one considers only proportional transaction costs), stochastic optimal impulse control problems can be reduced to singular stochastic control problems (see [140]).
Some other important stochastic optimal control problems known in literature and their applications can be found for instance in [120] and [74].

Motivated by Korn (1999) (see [87]), Long et al. 2012 (see [2]) , Verhangen et al. 1999 (see [147]) and Caicedo et al. 2014 (see [30]), we choose the stochastic impulse control problem in infinite horizon from others to be studied in this section and we

[9]See [34].

will apply it to the central bank problem of controlling inflation and output gap dynamics in chapter 4.

2.2.1 Stochastic optimal impulse control problem

In the following, we start with the problem formulation and then we will introduce the quasi-variational inequality approach[10] to solve this type of problem.

Problem formulation

Assume that in the absence of intervention, the uncontrolled process $X(t)$ is given as the solution of one-dimensional stochastic differential equation of the form:

$$dX(t) = b\left(X(t)\right)dt + \sigma\left(X(t)\right)dB(t), \quad X(0) = \xi, \tag{2.7}$$

where $B\left(X(t)\right) : \mathbb{R} \longrightarrow \mathbb{R}$ and $\sigma\left(X(t)\right) : \mathbb{R} \longrightarrow \mathbb{R}$ are functions satisfying the usual conditions for the existence and uniqueness solution of the equation (2.7)[11] for every initial condition $\xi \in \mathbb{R}$ on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$,
$B(t)$ stands for one-dimensional Brownian motion defined on the complete filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ and ξ denotes a real valued random variable independent of $B(t)$ with a finite second moment (i.e., $E\left(|\xi|^2\right) < \infty$).
At chosen intervention times τ_i the decision-maker can shift the process $X(t)$ to another value $X(\tau_i) = X(\tau_i^-) - \Delta X_i$, where $X(\tau_i^-)$ represents the state of the process before control is applied and $\Delta X_i \in \mathbb{R}$ denotes the control action at time τ_i which is also chosen by the decision-maker. After an action, the process follows its original dynamics until the decision-maker decides to shift it again.
Now the decision-maker faces the problem of how to select an impulse control strategy S from a set of admissible impulse control strategies Z that solve the following problem:

$$\min_{\{(\tau_i,\Delta X_i), i\in\mathbb{N}\}\in\mathcal{A}} E_x^S\left(\int_0^\infty e^{-\rho t} f\left(X(t)\right)dt + \sum_{i=1}^{\infty} e^{-\rho\tau_i}\left(K + k|\Delta X_i|\right)1_{\{\tau_i<\infty\}}\right), \tag{2.8}$$

where $f : \mathbb{R} \longrightarrow [0, \infty)$ is a continuous function which represents the running cost,
$K \in (0, \infty)$ denotes the fix cost per intervention,
$k \in (0, \infty)$ stands for the proportional cost per intervention,

[10] This is one of the two known standard approaches to solve the stochastic optimal impulse control problem. The second method is called iterative method where one uses iterative approach to find a solution from a sequence of optimal stopping which will converge to the solution of impulse control (see[110]).

[11] These conditions are the linear growth and the global Lipschitz conditions presented in the theorem 2.1.1.

$E_x^S(.)$ is the expectation when the process $X(t)$ starts with initial value x and the strategy $S = \{(\tau_i, \Delta X_i), i \in \mathbb{N}\}$ is selected by the decision-maker,
ρ is a contant discount rate and hence $e^{-\rho t}$ is a discount factor, and
impulse control strategy and a set of admissible impulse control strategies are defined as follows:

Definition 2.2.1. *An* ***impulse control strategy*** $S = \{(\tau_i, \Delta X_i), i \in \mathbb{N}\}$ *is a sequence of intervention times* τ_i *and control actions* ΔX_i *that obeys the following conditions:*
i) $0 \leq \tau_i \leq \tau_{i+1}$ *a.s* $\forall i \in \mathbb{N}$
ii) τ_i *is a stopping time* [12] *with respect to the filtration* $\mathcal{F}_t := \sigma\{X(s^-), s \leq t\}, t \geq 0$
iii) ΔX_i *is measurable with respect to* $\mathcal{F}_{\tau_i}$
iV) $X(\tau_i) = X(\tau_i^-) - \Delta X_i$
An impulse control strategy will be called ***admissible*** *if the following conditions are fulfilled:*
V) $\mathbb{P}(\lim_{i\to\infty} \tau_i \leq T) = 0 \ \forall \ T \geq 0.$
Vi) $E_x^S\left(\int_0^\infty e^{-\rho t} f(X(t))\, dt\right) < \infty$
Vii) $\lim_{T\to\infty} E_x^S\left(e^{-\rho T} X(T)\right) = 0$

Definition 2.2.2. *The* ***value function*** $V : [0,\infty) \longrightarrow \mathbb{R}$ *associated with the problem which is described in (2.8) is defined by*

$$V(x) := \inf_{S\in\mathcal{A}} E_x^S\left(\int_0^\infty e^{-\rho t} f(X(t))\, dt + \sum_{i=1}^{\infty} e^{-\rho\tau_i}(K + k|\Delta X_i|)\, 1_{\{\tau_i<\infty\}}\right). \quad (2.9)$$

Quasi-Variational Inequalities (QVI)

A quasi-variational inequalities approach consists of constructing the value function $V(x)$ as a solution to the system of inequalities (commonly referred to as quasi-variational inequalities) whose heuristic derivations will be given below.
In order to find the optimal impulse control strategy and its corresponding value function $V(x)$, it is necessary to define first the following operator:

Definition 2.2.3. *Assume that* $\mathcal{H}$ *is a space of all measurable functions* $\phi : \mathbb{R} \longrightarrow \mathbb{R}$. *We define the intervention operator (or minimum cost operator)* $M : \mathcal{H} \longrightarrow \mathcal{H}$ *by*
$M\phi(x) := \inf_{\Delta x\in\mathbb{R}} [\phi(x - \Delta x) + K + k|\Delta x|]$.

Heuristic derivation of the QVI for the problem (2.8)

[12] A stopping time is defined as follows.
Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$ *be a filtered probability space. A random variable* $\tau : \Omega \longrightarrow [0,\infty]$ *is called a* ***stopping time*** *if* $\{\omega : \tau(\omega) \leq t\} \subset \mathcal{F}_t \quad \forall t \geq 0.$

Now we are interested in M applied to the value function V. In this case, the value function $MV(x)$ will represent the value of strategy which consists of taking the best immediate action when starting in x and then following an optimal strategy. However, due to the fact that the fixed cost $K > 0$, there are states where an immediate action is not optimal at all, and therefore we have $V(x) < MV(x)$. $V(x)$ and $MV(x)$ should be equal at the first time after starting in x when it is optimal to do an action. Then we have:

$$V(x) \leq MV(x) \ for \ all \ x \in \mathbb{R}. \tag{2.10}$$

In the following we will use the minimum cost operator M to formulate a dynamic programming principle that will serve us to derive the QVI heuristically.
Let $I \in \mathbb{R}$ be a continuation region such that $\tau^* = \inf\{t \geq 0 : X(t) \notin I\}$ is optimal for all $x \in \mathbb{R}$ where $\tau^* < \infty$ almost surely.
Fix $x \in \mathbb{R}$ and let τ be an arbitrary $\{\mathcal{F}_t\}$-stopping time with $\tau < \infty$ almost surely and consider the following strategy:
Before time τ there is no intervention. At time τ we intervene and apply the optimal impulse value at this time and then we proceed optimally afterward.
According to the minimum cost operator M, we know that starting from the state $X(\tau^-)$, the cost of our strategy is given by $MV(X(\tau^-))$. Also, because no action is done over $[0,\tau]$ the cost of our strategy on this interval is just $E_x^S\left[\int_0^\tau e^{-\rho u} f(X(u))\,du\right]$.
Thus , because τ is arbitrary, we have

$$V(x) \leq E_x^S\left[\int_0^\tau e^{-\rho u} f(X(u))\,du + e^{-\rho u} MV\left(X(u^-)\right)\right],$$

where the right hand side is the total of our strategy.
Now assume that we use the same strategy above with τ replaced by τ^* which is the first time at which it is optimal to apply the control. Then according to our strategy we are always acting optimally. Thus considering X^* to be the corresponding controlled process, we have

$$MV\left(X^*\left(\tau^{*-}\right)\right) = V\left(X^*\left(\tau^{*-}\right)\right). \tag{2.11}$$

and

$$V(x) = E_x^S\left[\int_0^{\tau^*} e^{-\rho u} f(X^*(u))\,du + e^{-\rho\tau^*} V\left(X^*(\tau^{*-})\right)\right]. \tag{2.12}$$

Combining (2.11) and (2.12) yield the following dynamic programming principle:

$$V(x) = \inf E_x^S\left[\int_0^\tau e^{-\rho u} f(X(u))\,du + e^{-\rho\tau} MV\left(X(\tau^-)\right)\right]. \tag{2.13}$$

Now let us consider the optimal stopping time τ^* for which infimum in (2.13) will be attained and we additionally suppose that $V(x)$ is sufficiently smooth to apply Itô's formula.

Under these assumptions, the combination of (2.13) and (2.11), and the application of Itô's formula to $e^{-\rho u}V\left(X(u)\right)$, we can rewrite $e^{-\rho u}V\left(X(u^{-})\right)$, yield

$$\begin{aligned}
V(x) &= \inf E_x^S\left[\int_0^{\tau} e^{-\rho u} f\left(X(u)\right) du + e^{-\rho\tau} MV\left(X(\tau^{-})\right)\right] \\
&= E_x^S\Bigg[V(x) + \int_0^{\tau^*} e^{-\rho u}\Bigg(f\left(X(u)\right) - \rho V\left(X(u)\right) + V'\left(X(u)\right) b\left(X(u)\right) + \\
&\frac{1}{2}\sigma^2\left(X(u)\right) V''\left(X(u)\right)\Bigg) du\Bigg] + E_x^S\left[\int_0^{\tau^*} e^{-\rho u} V'\left(X(u)\right)\sigma\left(X(u)\right) dB(u)\right] \\
&\leq E_x^S\Bigg[V(x) + \int_0^{t} e^{-\rho u}\Bigg(f\left(X(u)\right) - \rho V\left(X(u)\right) + V'\left(X(u)\right) b\left(X(u)\right) + \\
&\frac{1}{2}\sigma^2\left(X(u)\right) V''\left(X(u)\right)\Bigg) du\Bigg] + E_x^S\left[\int_0^{t} e^{-\rho u} V'\left(X(u)\right)\sigma\left(X(u)\right) dB(u)\right],
\end{aligned} \tag{2.14}$$

for a fixed but otherwise arbitrary $t > 0$.
Assuming that the expectation of the stochastic integral vanishes, substracting $V(x)$ on both sides of (2.14) and multiplying both sides by $\frac{1}{t}$ we get

$$\begin{aligned}
0 \leq \frac{1}{t} E_x^S\Bigg[\int_0^{t} e^{-\rho u}\Big(f\left(X(u)\right) - \rho V\left(X(u)\right) + V'\left(X(u)\right) b\left(X(u)\right) \\
+ \frac{1}{2}\sigma^2\left(X(u)\right) V''\left(X(u)\right) du\Bigg].
\end{aligned}$$

Applying the mean value theorem for integrals, letting t converge to zero and assuming that this limit can be interchanged with the expectation, we have

$$\begin{aligned}
0 &\leq \frac{1}{t} E_x^S\Bigg[\int_0^{t} e^{-\rho u}\Big(f\left(X(u)\right) - \rho V\left(X(u)\right) + V'\left(X(u)\right) b\left(X(u)\right) \\
&+ \frac{1}{2}\sigma^2\left(X(u)\right) V''\left(X(u)\right) du\Bigg] \\
&= E_x^S\left[f\left(X(0)\right) - \rho V\left(X(0)\right) + V'\left(X(0)\right) b\left(X(0)\right) + \frac{1}{2}\sigma^2\left(X(0)\right) V''\left(X(0)\right)\right] \\
&= f(x) - \rho V(x) + V'(x) b(x) + \frac{1}{2}\sigma^2(x) V''(x).
\end{aligned}$$

That is [13]:

$$\mathcal{L}V(x) + f(x) \geq 0 \quad for\ all\ x\ \in \mathbb{R}, \tag{2.15}$$

where $\mathcal{L}V(x) = \frac{1}{2}\sigma^2(x)V_{xx}(x) + b(x)V_x(x) - \rho V(x)$.
If the optimal stopping time τ^* would be equal to zero, then it is optimal to intervene immediately and so

$$V(x) = MV(x)\ for\ any\ x\ such\ that\ \tau^* = 0. \tag{2.16}$$

If $\tau^* > 0$, then with the equality that we have in equation (2.14) for τ^* and by following the same calculations used to produce (2.15) we arrive at the following equation:

$$\mathcal{L}V(x) + f(x) = 0,\ for\ any\ x\ such\ that\ \tau^* > 0. \tag{2.17}$$

Hence we have two inequalities given by (2.15) and (2.10) which must hold for all $x \in \mathbb{R}$ and in addition, equalities represented by (2.16) and (2.17) imply that both inequalities can not hold at the same time. This means that we must have

$$[V(x) - MV(x)]\,[\mathcal{L}V(x) + f(x)] = 0\ for\ all\ x\ \in \mathbb{R}. \tag{2.18}$$

Definition 2.2.4. *The three relations (2.10), (2.15) and (2.18) are called the **Quasi-variational inequalities(QVI)** for the problem (2.8) .*
Given a continuous solution of the QVI, the following control can be constructed:

Definition 2.2.5. *Let V be a continuous solution of the QVI. Then the following impulse control strategy is called **QVI-control**:*

(i) $(\tau_0, \Delta X_0) := (0, 0)$.

And for every $i \geq 1$,:

(ii) $\tau_i := \inf\{t \geq \tau_{i-1} : V(X(t^-)) = MV(X(t^-))\}$,

(iii) $\Delta X_i := arg\ min_{\Delta X}\left[V\left(X(\tau_i^-) - \Delta X\right) + K + k|\Delta X|\right]$.

The theorem below guarantees that given the assumptions (2.19) and (2.20), a smooth solution of the QVI coincides with the value function and the admissible control associated with it solves the problem presented in (2.8).

Theorem 2.2.1 (Verification theorem [14]): *Assume that there exists a solution $V^* \in C^2$ (or better: a sufficiently regular solution [15]) of QVI to the problem (2.8).*

[13] Derivative appearing in equation (2.15) are only supposed to exist as left hand derivatives.
[14] This is the version from [87].
[15] In the proof of this theorem (see Appendix), we will see that the C^2 assumption for the solution V^* of the QVI is only required to apply Itô's formula. But the proof can also go through even

If V^* satisfies the growth conditions

$$E_x^S \left[\int_0^\infty \left(e^{-\rho t} \sigma \left(X(t) \right) V_x^* \left(X(t) \right) \right)^2 dt \right] < \infty, \tag{2.19}$$

and

$$\lim_{T \to \infty} E_x^S \left[e^{-\rho T} V^* \left(X(T) \right) \right] = 0 \tag{2.20}$$

for every $X(t)$ corresponding to an admissible impulse control strategy

$$S = \{(\tau_i, \Delta X_i)\}_{i \in \mathbb{N}},$$

then we have $V(x) \geq V^*(x)$ for every $x \in \mathbb{R}$.
Moreover, if the QVI-control associated with V^* is admissible then it is an optimal impulse control, and for every $x \in \mathbb{R}$
$V(x) = V^*(x)$.
The proof of this theorem is provided in Appendix.

Remark 2.2.1. The solution V^* of the QVI splits the real line ($\mathbb{R}$) in two regions:
1. Intervention region IR defined as

$$IR := \{x \in \mathbb{R} : V^*(x) = MV^*(x) \text{ and } \mathcal{L}V^*(x) + f(x) > 0\}.$$

and

2. Non-intervention region (or continuation region) defined as

$$NIR := \{x \in \mathbb{R} : V^*(x) < MV^*(x) \text{ and } \mathcal{L}V^*(x) + f(x) = 0\}.$$

Therefore, the solution V^* will be in the form of optimal band, say $[a, b]$, with optimal restarting points $\alpha, \beta \in (a, b)$ at which the process X should be shifted by the controller when the process hits the boundaries. This means that the problem expressed by (2.8) can be reduced to the free boundary problem, where the problem of searching for the optimal policy boils down to search for the optimal boundaries of the continuation region and optimal restarting points inside the continuation region when the process is pushed back into the interval (a, b) after reaching the boundaries. As shown by Dixit (1991) (see [40]) and Buckley and Korn (1998) (see [72]) among others, the optimal function V^* corresponding to impulse control band strategy characterized by four parameters $-\infty < a < \alpha \leq \beta < b < \infty$ must satisfy the continuous pasting conditions (or value matching conditions) and the smooth pasting conditions which are used to determine unknown boundary and specify the value function.
The figure 2.2.1 below illustrates such control strategy.

under the weaker assumptions on V^* where some generalised versions of Itô's formula that require weaker regularity assumptions (see also [86]) can be used.

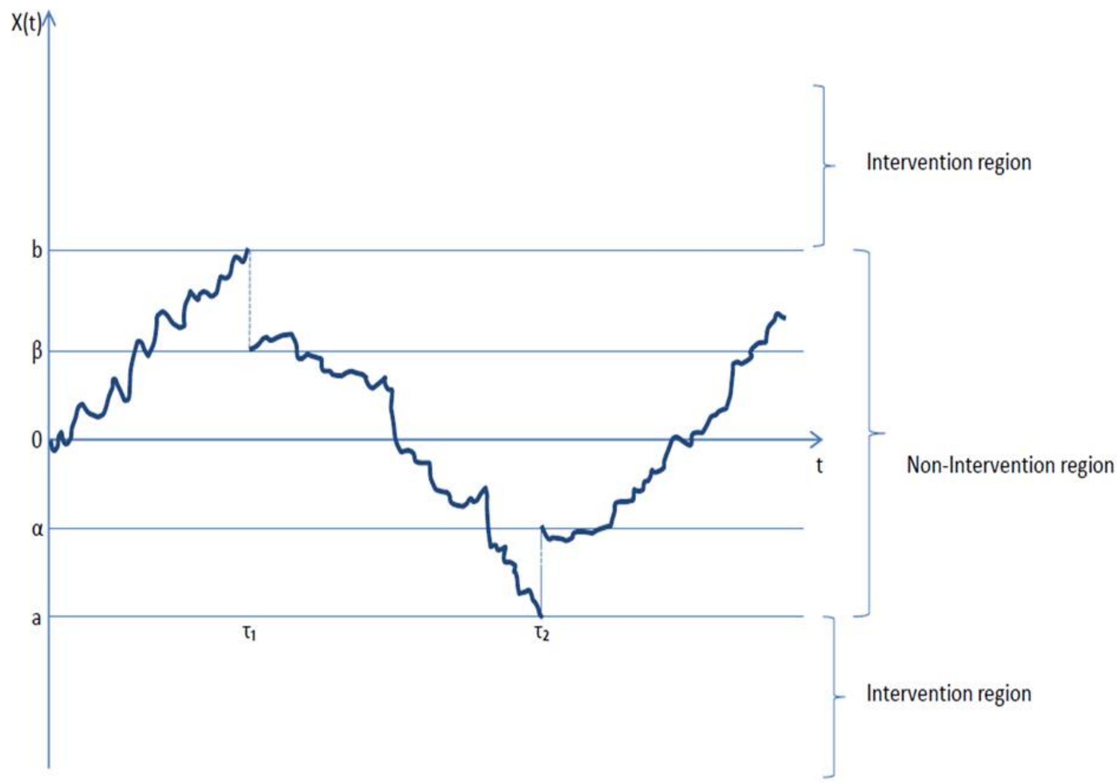


Figure 2.2.1: A path of a controlled process and the impulse control band. The figure shows that the controller intervenes immediately with an intervention size $\Delta X_i = a - \alpha$ (resp. $\Delta X_i = b - \beta$) each time the process hits the boundary a (resp. b) in order to push it upwards to α (resp. downwards to β).

In this case of free boundary problem, the definition 2.2.1 will be modified as follows: $ii^{*})$ τ_i is a stopping time with respect to the filtration $\mathcal{F}_t := \sigma\left\{X(s^-), s \le t\right\}, t \ge 0$, with $\tau_i \le \inf\left\{t \ge \tau_{i-1} | X(t) \notin (a,b)\right\}$,

$$iV^{*}) \quad X(\tau_i) = X(\tau_i^-) - \Delta X_i = \alpha 1_{\left\{X(\tau_i^-)=a\right\}} + \beta 1_{\left\{X(\tau_i^-)=b\right\}}$$

.

Further modifications will be done for:

• The operator M (where it is only minimised over such values of ΔX such that condition iV^* above is fulfilled), and

• The definition of the QVI (where the condition $V(x) = MV(x)$ for $x \in (-\infty, a] \cup [b, \infty)$ in (2.10) have to be added and $\mathcal{L}V(x) + f(x) = 0, \quad \forall x \in (a,b)$).

Definition 2.2.5 can be adapted according to the modifications done above. With the modification of QVI-control the verification theorem remains valid. but in this case V^* is only required to be continuous on $[a,b]$ and to be a C^2-function on (a,b).

Remark 2.2.2 (see also [87]) The results obtained above, in particular the verification theorem hold also for:
• Multi-dimensional case where the process X is a vector process with dynamics represented by

$dX_i(t) = b_i\left(X(t)\right)dt + \sum_{j=1}^{k} \sigma_{ij}\left(X(t)\right)dB_j(t),\ i = 1, \ldots, n$, and the operator $\mathcal{L}$ in QVI will be expressed by

$\mathcal{L}V(x) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{k} \sigma_{im}(x)\sigma_{jm}(x)V_{x_i x_k}(x) + \sum_{i=1}^{n} b_i(x)V_{x_i}(x) - \rho V(x).$

Chapter 3

Inflation Modeling

Inflation attracts a lot of attention from both professional economists and media and general public due to the fact that there are many economic, social and political costs of a high and unstable inflation rate. As a consequence, inflation controlling has become a primary goal of central banks, and various types of inflation linked products (for example inflation linked bonds and derivatives) have been created to hedge the effects of inflation. These lead to a need of developing for sophisticated inflation models which can be used by inflation linked market participants to price inflation linked products and Monetary policy makers in order to analyze and hence to control inflation dynamics.

In this chapter we will first briefly discuss two main aspects of inflation, namely, the meaning and the measurement of inflation.
Furthermore, a review of the literature on inflation modeling will be presented before discussing the Phillips curve models of the inflation process. Finally, we will give a detailed derivation of the discrete time new Keynesian Phillips curve and we will also transform it into its corresponding high frequency, and hence continuous time versions for the purpose of being able to apply stochastic impulse control techniques for solving the central bank's problem presented in Chapter 4.

3.1 Inflation and Deflation

There are numerous accepted definitions of inflation in economic literature (see also [35]). For instance, as documented by Dwivedi(2010) (see [45]), some economists like Coulborn (1963) define inflation as a situation of too much money chasing too few goods. According to the group led by Friedman (1970), inflation is a phenomenon of continuously rising prices resulting from excess money supply.
Modern economists like Johnson (1970) define inflation as a sustained rise in prices.

In the opinion of Ackley (1978), inflation means a situation of persistent and appreciate rise in general level or average of prices. Samuelson (1955) regards inflation as a state of rise in general level of prices. However, economists generally seem to agree that inflation refers to a persistent and considerable increase in the general level of prices which causes a decline in the purchasing power per unit of money. This definition is criticized, saying that a considerable rate of increasing in price level, which means the rate higher than the desirable rate, is not clear because it varies from country to country and from time to time. For example, the European Central bank (ECB) considers an inflation rate below but close to 2% over the medium term to be a desirable inflation rate (see [54]). According to the South African Reserve Bank (SARB), an inflation rate between 3 and 6% was a desirable rate in 2002 and 2003, while in 2004 and 2005, the desirable inflation rate was set to the range of 3 to 5% (see [90]).
The term **deflation** refers to a situation in which the inflation rate is below 0. This is very rarely observed in practice. It can occur, for example, when there is a lack of aggregate demand in the economy, forcing suppliers to reduce prices in order to attract customers.
Deflation is also dangerous because, for instance, it can cause a falling in profitability and ultimately increased unemployment [16]. In contrast to the deflation, the state of shrinking but still positive inflation rate is called the **disinflation**.

Remark 3.1.1. *Any increase in price level in excess of a desirable rate can not always be taken to be inflationary. The following factors must be considered and the price rise has to be adjusted accordingly, while deciding on whether the rate of increase in price level in excess of a desirable rate is really inflationary, especially when one has to formulate anti-inflationary policies (see [45]).*
1. Increase in price level due to change in the composition of GDP in which the low-price farm products are replaced by the high-price industrial goods,
2. Increase in price level caused by the qualitative change in the products across the board,
3. Increase in price level due to the change in price indexing system, and
4. Recovery in price after a period of recession.

Measurement of Inflation

A formal definition of inflation says that a (simple) inflation rate π_t is the rate of change in the price index per year expressed in percentages (see [61]) i.e.

$$\pi_t = \frac{P_t - P_{t_0}}{P_{t_0}},$$

[16] More details about the causes and consequences of the deflation can be found in [27].

where P_t and P_{t_o} are the consumer price indexes at the time point t and the time point (or the base period) t_0 respectively.
Therefore, measuring inflation requires the determination of price indexes. The most widely used price indexes to monitor inflation are the consumer price indexes (CPIs)[17]. These indexes are mostly calculated using a fixed-weighted Laspeyres formula which is presented by the following equation: [18]

$$I_t = \frac{\sum_{i=1}^{n} P_{i,t} W_{i,0}}{\sum_{i=1}^{n} P_{i,0} W_{i,0}} \times 100,$$

where I_t is the price index at time point t,

$P_{i,t}$ is the price of the i^{th} item in the basket of goods and services at the time t,
$P_{i,0}$ denotes the price of the i^{th} item in the basket of goods and services at the base period,
$W_{i,0}$ represents the quantity of an item i in basket of good and prices at the base period,
n is the number of items in basket of goods and prices being priced.

3.2 Literature Review on Inflation Models

Approaches for modeling inflation are generally classified into two types (see [17] P.18) :
Macroeconomic based models which are analytical tools designed to describe and quantify the impact of macroeconomic variables like the level of exchange rates, nominal interest rates, output gap, unemployment and money supply in order to describe the fundamentals that may have an impact on the inflation level. These models are widely used for example by central banks to generate inflation forecasts.
Another class of models are option pricing based models. These models do not consider any fundamental impact but rather take the dynamic of inflation for granted and aim at providing option prices based on the assumed dynamics. These models play an important role when it comes to the pricing of complex inflation-indexed products and to determining hedging solutions for them.

[17] See also [88].
[18] See also [129] and [69].

3.2.1 Some macroeconomic based approaches used in inflation modeling literature

Over the years, a considerable number of macroeconomic based models are developed in order to forecast and to model inflation. Among them, Phillips curves are popular models used by many central banks in describing and forecasting inflation (see [112] P.1 and [9] P.112).

For examples: Barkubu (2005) (see [58]) utilized hybrid new Keynesian Phillips curve to study the relationship between inflation and the marginal cost both for United States and Euro area over the period 1975-2003 and they found out that the hybrid new Keynesian Phillips curve approach fits the data for the United States and the Euro area.
Kapur (2012) (see [77]) employed expectations-augmented Phillips curve to model and to forecast inflation in India and discovered that demand factors (like unemployment gap or output gap) and supply factors (like imported inflation or exchange rate movements) are the main factors of inflation in India.
Ivo Krznar (2011) (see [89]) analyzed the domestic inflation rate in Croatia using different versions of Phillips curve but as a result from his study, the hybrid new Keynesian version is better than the others to explain the dynamics of the domestic inflation rate.
The empirical study done by Hasan (2012) (see [63]) stressed that Phillips curve is a good tool for inflation modeling in Bangladesh.
Even though Phillips curve is also some times used in sub-saharan Africa,
Durevall (2012) (see [43]) claimed that the Phillips curve[19] approach is inappropriate for describing inflation in sub-saharan african countries, whose economy is mostly are agricultural based. They argued that there is a weak or even no relationship between unemployment, aggregate demand and wage increases due to the extensive self- and underemployment, large informal markets and a low degree of labour-market organization that some times characterize these countries.
Like Durevall, most researchers proposed models based on quantity theory of money to describe the dynamic of inflation in developing countries in general, and in East African Community (EAC) in particular (see [76] P.5). Error Correction Model (ECM) and P^* model are two examples of these models. ECM has been employed for instance by Alain (1999) ([42]) to analyze the dynamic of inflation in Kenya.
Samuel and Ussif (2001) (see [95]) applied ECM to estimate Tanzania's inflation rates and Emilio (2001) (see [133]) used ECM model to study dynamics of inflation in Madagascar in the period 1971-2000.
P^* model is used by many researchers for the purpose of analyzing inflation in both developed and developing countries. Among them, Katrin (1998) (see [153]) studied

[19]The Phillips Curve argues that an increase in aggregate demand leads to higher employment, which in turn exerts pressure on wages and then to general price level.

inflation in the European area using P^* model and found that it is an adequate model to explain inflation on the national level and for the whole European area. Also Abdul et al. 2005(see [124]) applied this model to determine the leading indicator of inflation in Pakistan.
In the literature, there are other approaches like linear time series, in particular Auto-regressive Integrated Moving Average model (ARIMA) and Vector Autoregressive model, which are utilized for modeling and forecasting inflation rates.
These models have been widely documented in the literature, see e.g. Junttila (2001) (see [75]) and Samuel et al. 2011 (see [4]) who employed ARIMA to study inflation process in Finland and Ghana respectively.
Caesar (2006) (see [92]) applied VAR model in forecasting Swiss inflation and Gichondo and Kimenyi (2012) (see [6]) used VAR model to study inflation process in Rwanda.
In addition to these mentioned approaches for inflation modeling and forecasting, nonlinear models have been developed. The most common nonlinear models in literature are Markov-Switching Autoregressive model (MSAR), Smooth Transition Autoregressive model (STAR) and Threshold Autoregressive model (TAR).
These models may be superior to linear models to explain the behavior of inflation processes for the reason that there exists a high degree of nonlinearity and the presence of jumps in correspondence with some crucial dates and historical episodes (see [7]).
These aspects of inflation comportment have been reported for many countries. For instance, empirical study done by Shyh-Wei Chen (see [33] P.55) for eleven OECD countries[20] gave the result that inflation rates in these nations are nonlinear series.

3.2.2 Some option pricing based approaches used in inflation modeling literature

The most known model among this model group is the Jarrow Yildrim's 2003 (JY) model (see [152]) . In this model, under the risk neutral measure assumption, the real and the nominal rates follow one- factor Gaussian process and the evolution of the inflation index $I(t)$ (or CPI at time t) is described by the following equation preserving the macroeconomic concept of Fisher [21] (see [107]).

$$\frac{dI(t)}{I(t)} = (R_n(t) - R_r(t))\,dt + \sigma_I dW_I(t),$$

[20]OECD is an Organization for Economic Co-operation and Development. The eleven countries considered are: Australia, Austria, Belgium, Denmark, Greece, New Zealand, Norway, Portigal, Spain, Sweden and Switzerland.

[21]Fisher equation states that the nominal interest rate is the sum of the real interest rate and the expected inflation (see also [88]).

where $R_n(t)$ and $R_r(t)$ are nominal and real instantaneous short rates respectively, W_I is the standard Brownian motion and σ_I is a positive constant.
Korn and Kruse's 2004 paper (see [88] P. 351-367) provided another modeling framework for the evolution of consumer price indexes that is related in some aspect to the Jarrow and Yildirim's 2003 model. In this paper, a consumer price index is modeled as a geometric Brownian motion with a drift which is equal to the difference of the nominal and the real interest rate. These authors also gave other possible approaches for inflation modeling which are mainly based on the interest rate modeling. In this approach, the instantaneous inflation rate is modeled as a stochastic process similarly to the short rate approaches for interest rate modeling (one example of these approaches is the Hull-White-Model). Additionally, it has been shown that under some assumptions, the macroeconomic concept of Fisher can be reflected by such models(we refer the interested reader to [88] and [17] for details on these approaches used to model inflation process).

3.3 Relationships between Key Macroeconomic Variables: Phillips Curve Models

The question "What is the connection between inflation and unemployment?" is not new in macroeconomic field. Thomas M. Humphrey in his paper (see[68]) cited economists who investigated this relationship before the birth of the Phillips curve. For example in 1752 and 1802, inflation-unemployment trade-off was an essential component of the monetary doctrines of David Hume and Henry Thornton respectively. In 1926, Irving Fisher introduced the first statistical study of the correlation between inflation and unemployment, and he found a causality relationship that runs from inflation to unemployment.
Despite many early efforts, the curve describing the relationship between inflation and unemployment was born as an empirical regularity documented by the New Zealand-born economist Alban William Phillips in 1958. In that year A. W. Phillips published a paper in which he fitted a statistical equation

$$w = g(U) = -a + bU^{-c} \tag{3.1}$$

where, $g^{'} < 0$, $g^{''} > 0$
or

$$\ln(w + a) = \ln(b) - c\ln(U) \tag{3.2}$$

(where $w + a,\ b,\ U \in \mathbb{R}^{+}$ and $a \neq 0$, b and $c \neq 0$ are the real parameters) to the scatter of annual observations on rates of growth of nominal wages (w) and unemployment rates (U) for United Kingdom between 1861 and 1957.
According to his findings, there exists a stable inverse relationship between the rate

of growth of nominal wages and unemployment rate, and a high non-linearity between the two (see [68],[16] and [111]). Graphing the above equation in the standard form with w on the ordinate and U on the abscissa gives the Phillips curve that looks like the following figure.

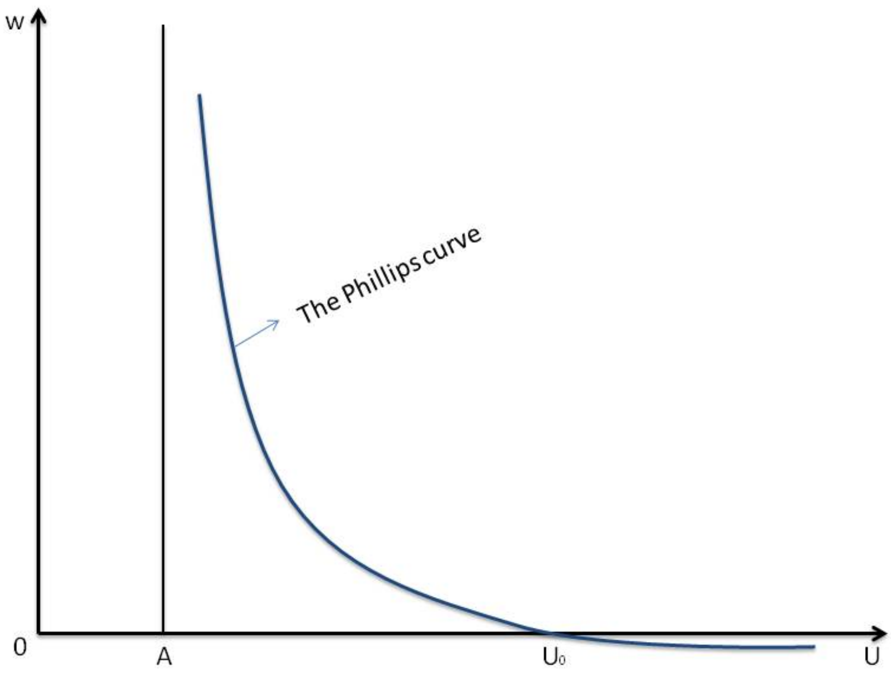


Figure 3.1: Original Phillips curve

The point $U_0 = (\frac{b}{a})^{\frac{1}{c}}$ on the graph is an intersection of the Phillips curve and the abscissa line. This means that when the rate of growth of nominal wages is stable (i.e., w=0), the Phillips curve intercepts the abscissa line. And the asymptote (the vertical line which intercepts the abscissa line in point A) means that even if the rate of growth of nominal wages were to be increased continuously, there is a threshold below which an unemployment could not fall.
Following the work of Phillips, many other macro-economists have been interested in investigating if there exists similar relationship in other countries. Most industrialized countries appeared to have a stable Phillips curve over some periods, especially the 1960s. Like others, Samuelson and Solow estimated the Phillips curve for the USA in 1960 and came to the conclusion that there is a similar relationship. They pointed out additionally that the Phillips curve represents not only a relationship between the rate of growth of nominal wages and unemployment rate, but also the relationship between inflation and unemployment (see [111] P.20).

Since then, the trade-off between inflation and unemployment presented by the Phillips curve was accepted among macro-economists and policymakers and by the end of 1960s it had become an important part of the Keynesian approach because it was consistent with the Keynesian thinking (see [67] and [10]). According to Keynes, it can not be possible to enter into a stagflation period (i.e., a period of simultaneous occurrence of high inflation and high unemployment). Based on this understanding, policymakers in 1960s could manage aggregate demand to lower or increase unemployment at the cost of higher or lower inflation. However, there were several problems with this view, among them two will be cited in the following (see [36]):

(i) One of the problems is that the original Phillips curve was essentially a statistical relationship and Phillips gave only a few theoretical explanations of his findings.

(ii) The second problem was a conflict between the traditional presumption in economic theory which said that, in long run, the real magnitudes in the economy are determined by real rather than nominal forces and the view that lower unemployment could be permanently achieved by accepting higher inflation.

In addition to these two problems, the stagflation that happened in USA in 1970s contributed to raise doubts about the legitimacy of the original Phillips curve. Consequently, this gave many other macro-economists the opportunity to criticize and to reformulate the original Phillips curve.
Criticisms, interpretations and extensions of the Phillips curve which have been done by different groups of macro-economists will be reviewed below.

Monetarist View of Phillips Curve

As we have seen before, the original Phillips curve stands for a stable trade-off between rate of growth of nominal wages and unemployment rate. This means that the nominal variable affects the real variable. If this relationship is understood in this fashion, it seems to be assumed that there is no difference between the changes in current nominal wages and the changes in expected future real wages, taking into consideration the forward looking nature of wage contracts (see [59]). Milton Friedman, the founder of monetarism, was not convinced of the assumption above. According to him this assumption is valid if it is supposed that"the price expectations are sticky in the sense that people do not expect the price level to vary and workers do not resist a decrease in their real wages caused by a high inflation ". These two assumptions are far away from the reality. Milton Friedman and Edmund Phelps agreed with the neoclassical labor market theory which states that labor markets determine real wages and employment through the interaction of labor demand and supply.

According to this theory, neither labor demand nor labor supply are affected by inflation. This implies that inflation does not also affect unemployment. Briefly Friedman and Phelps criticized Phillips to not distinguish nominal wages and real wages and not to take into account the effect of expected inflation in the fixation of the wages.
Based on the idea that the workers in wage negotiations are interested in the increase in real wages but not nominal wages, Friedman (1968) and Phelps (1968) modified the original Phillips curve - illustrated in equation 3.1- by substituting the rate of growth of nominal wages by the rate of growth of real wages and this yielded the following equation:

$$w = g(U) + \Pi^e, \tag{3.3}$$

where Π^e is the expected inflation rate. This equation is called "expectations augmented Phillips curve "in literature. According to equation (3.3), expected inflation rate determines the position of the modified Phillips curve, but does not change its slope. As it will be illustrated in the figure 2, there is a Phillips curve for each expected inflation rate and the Phillips curve will shift either to the right if expected inflation increases or to the left if it decreases. As a result, there is an inverse relationship between inflation and unemployment only in short run as it will be clarified (in the following). Friedman and Phelps concluded that expectations are formed adaptively, meaning that workers based their expectations of future inflation on recent past inflation. In this view, an increase in current inflation rate may surprise workers because they consider expected inflation rate to be equal to the inflation rate in the last period. This unexpected rise in inflation rate reduces the real wage of workers, but this occurs for a short time since people would learn about the altered policy stance as time goes by and they would adapt to the situation and would revise their expectations. This decrease in real wage of workers causes an increase in the labor demand by firms which leads to the rise in employment rate and thus to the decline in unemployment rate. Consequently, the rise in inflation makes unemployment rate go down in short run, just as predicted by the original Phillips curve but here the transmission runs from aggregate demand via unexpected inflation to unemployment rate, while in the original Phillips curve it runs from aggregate demand via unemployment rate to nominal wages and inflation (see [59]).
In the long run, expectations augmented Phillips curve implies a disappearance of negative correlation between inflation and unemployment. In long run the curve becomes a straight vertical line that intersects the X-axis at the steady state unemployment rate U_0 (This is a Natural Rate of Unemployment **NRU** according to Friedman and Phelps) which means that changes in inflation rate do not affect unemployment rate.
These behaviors of expectations-augmented Phillips curve can be illustrated as in the figure 3.2 which shows what happens in short and long term when policymakers adopt either an expansionary or contractionary policy.

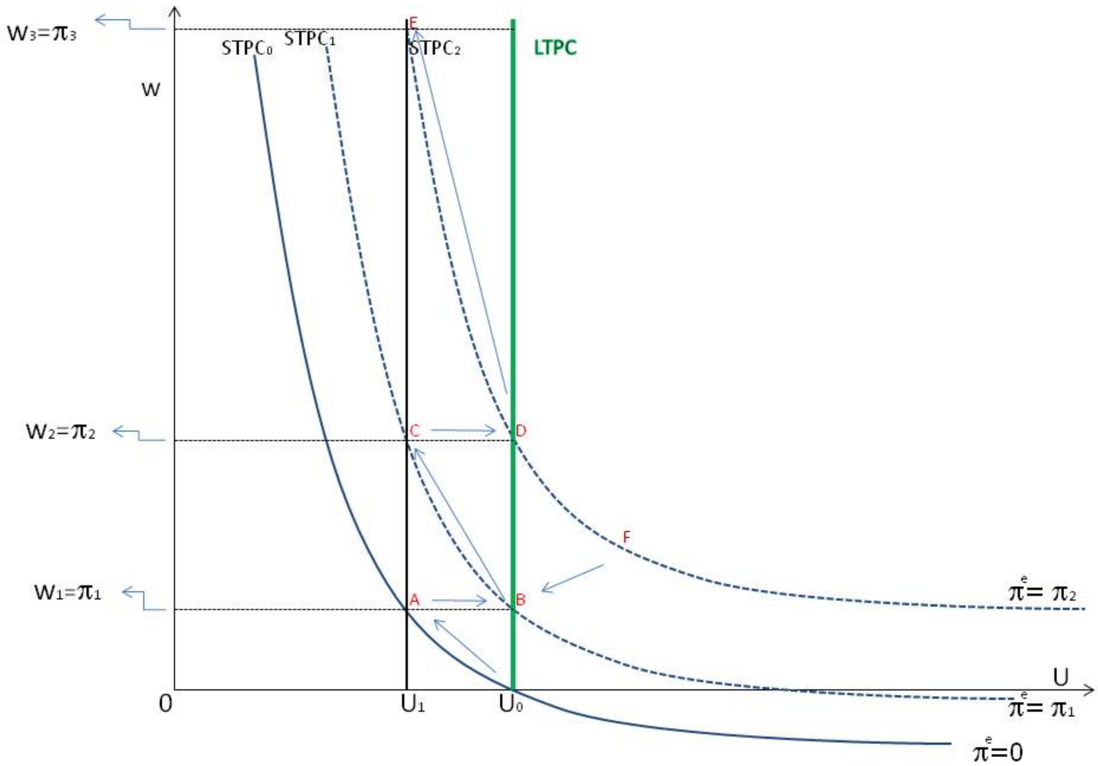


Figure 3.2: Short and Long Term Phillips Curve.

In figure 3.2 above we start at point U_0 where inflation is equal to zero. Suppose that policymakers find unemployment rate at the level of U_0 too high to be tolerated and decide to stimulate aggregate demand by means of expansionary monetary policy[22] in order to reduce unemployment from U_0 to U_1. This policy will cause inflation rate to rise to π_1 and therefore real wages to decline. This fall in real wages causes firms to increase their demand for labour and consequently unemployment will decrease until it reaches U_1 (See [44] P.452). As a result the economy moves along $STPC_0$ from the point $(U_0, 0)$ to the point (U_1, π_1). But it will stay temporarily in this position for the reason that workers will feel the pinch of decrease in their real wages and will adapt their inflation expectation according to their past experiences. And as a result they will negotiate for higher monetary wage rates at the time of the renewal of the labour contract by considering their expectations so that they can resist the rise of price. This will lead to an increase in real wages again until reaching their initial level and unemployment rate U_1 will rise to U_0. Now economy is in the new equilibrium point (U_0, π_1) which corresponds to the point B on the

[22]Expansionary monetary policy is a set of actions by the monetary policymakers to increase quantity of money in circulation. They can do this for example by decreasing interest rates, by lowering reserve requirements for banks etc.

new curve $STPC_1$. This means that the Phillips curve $STPC_0$ shifts to the curve $STPC_1$.
If the policymakers try again to reduce the unemployment rate to the targeted level, they have to adopt the similar policy (here the expansionary monetary policy). In this case the consequences will be the same as the previous ones in the sense that the system will move along the curve $STPC_1$ from the point B to the point C and then from the point C to D after a period of time. This means the curve $STPC_1$ will shift towards the curve $STPC_2$ as shown in figure 3.2.
From these observations, the conclusion is that expansionary policies can not enable policymakers to reduce the natural rate of unemployment permanently. It leads only to accelerating inflation, to an upward shift in short-term Phillips curves and then to the long- run Phillips curve which is a straight vertical line joining here the points U_0, B and D. In other words there is no long-run trade-off between inflation rates and unemployment rates and the Phillips curve is a vertical line in the long run.
Point F in the figure 3.2 indicates what happen when policymakers consider inflation rate (for instance π_2) to be very high and decide to reduce it by applying an anti- inflationary monetary policy [23]. If such policies are applied, there will result a decrease in the quantity of money in circulation and consequently unemployment will rise. This situation is illustrated by the movement from the point D to the point F along $STPC_2$. This can be interpreted as a period of recession with deceleration in inflation rate (See [44] P.453).

New Classical View of Phillips Curve

Robert Lucas (the founder of the New Classical School) and his disciples criticized the way monetarists modeled expectation of inflation which have been incorporated in original Phillips curve in order to formulate expectations augmented Phillips curve. In monetarists'view, expectations are formed adaptively, meaning that economic agents anticipate inflation rates based on recent past inflation rates and they learn from their errors. According to them, the workers adjust their inflation expectations by a fraction of the error made as expressed in the following equation [24]:

$$\Pi_t^e = \Pi_{t-1}^e + \beta \left(\Pi_{t-1} - \Pi_{t-1}^e \right), \tag{3.4}$$

Where $0 < \beta < 1$ is a constant,
Π_t^e denotes current expectations of future inflation,
Π_{t-1}^e refers to previous expected inflation,
$\Pi_{t-1} - \Pi_{t-1}^e$ previous estimation error (which is the deviation between the observed

[23] Anti-inflationary monetary policy is a set of actions by the monetary policy makers to decrease the quantity of money in circulation. They can do this for example by increasing interest rates, by rising reserve requirements for banks etc.
[24] See for example [99] and [47].

and expected inflation).

New classical economists have not accepted this approach of forming expectations, since it assumes that economic agents only partially adjust their expectations by a fraction of last errors made and that they neglect additional informations which are available to them other than past values of inflation rate. According to them, expectations formed in this manner will contain systematic errors.
As alternative, Robert Lucas, Thomas Sargent and their followers believed that the only acceptable way to incorporate expectations into macroeconomic models was to adopt the rational expectations hypothesis, initially proposed by John Muth in the early 1960s. Models that incorporate rational expectations are called "forward looking "models. In such models economic agents use all accessible informations in order to make the best possible estimate of future inflation.
In the Muthian version also called version of rational expectations, agent's expectations of inflations (π_t^e) may be represented by the following equation[25]:

$$\pi_t^e = E(\pi_t|\Omega_{t-1}), \tag{3.5}$$

where π_t is the actual rate of inflation;
$E(\pi_t|\Omega_{t-1})$ is the rational expectation of the rate of inflation subject to all informations known before the time t (Ω_{t-1}).
Inflation expectations formed rationally will be correct on average, meaning that agents can make errors but rationality implies that they avoid systematic errors. If economic agents over- or underestimate inflation rates, the forecasting errors that are made are random, have a mean of zero and the lowest variance, and they are not correlated with the information set available at time when expectations are formed (See [144] P.227). More formally, the expected rate of inflation (π_t^e) is given by the equation below:

$$\pi_t^e = \pi_t + \epsilon_t, \tag{3.6}$$

where $E(\epsilon_t|\Omega_{t-1}) := E_{t-1}\epsilon_t = 0$, π_t denotes actual rate of inflation and ϵ_t is the random error term.
Substituting adaptive expectations included in expectations augmented Phillips curve illustrated in equation (3.3) by rational expectations and assuming that all markets clear continuously and instantaneously[26] via perfectly flexible price, Robert and others developed the new classical Phillips curve (see[1] and [11]).
New classical economists such as Lucas (1972, 1978) and Mindford (1983) argued that such Phillips curve is a vertical line about the natural rate of unemployment not only in the long term but also in the short term (see [143] P.351).
For new classical macro-economists, any fluctuations in unemployment rate coming from random errors in predicting inflation rates are only caused by unexpected and

[25] See [144] P. 226.
[26] Market clearing means that markets always go to where the quantity supplied equals the quantity demanded.

unannounced changes in policy by authorities. But such fluctuations will only occur in short term as economic agents learn the change in policy and correct immediately their forecasts of inflation. In this approach, the speed at which inflation rate changes is higher than that in adaptive expectations method used to predict inflation rate. The reason is that agents wait until the present becomes the past before changing their expectations because the last ones is based on the past.
The policy implication of new classical Phillips curve is that demand management policies can not affect unemployment. According to Smith (see [143]) the only policies that can affect unemployment are micro-economic, supply-side forces which increase the will and capacity to work and the will and capacity to employ. Examples of such policy recommendations are[27]: reducing replacement ratio; cutting income and profit taxes; improving labour productivity; lowering employment taxes; removing employment protection legislation and so on.

Closed Economy New Keynesian Phillips Curve(CENKPC)

The assumption of new classical models that all markets (including labour market) clear continuously and instantaneously and monetary policy ineffectiveness on real variables like employment and output as one of their implications have not been accepted by macro-economists who maintained Keynesian beliefs. According to Hicks (1974), new keynesians, in contrast to new classical macro-economists, argue that there exist markets which are characterized by rigid prices[28], predominantly the labor market and a large section of the goods markets. These disagreements stimulated incentive of many researchers to build models with coherent micro-foundations in order to explain why prices and wages do not change fast enough to always clear markets even in the presence of rational agents. In doing so, they have thought to re-establish and to justify a case for policy effectiveness where monetary policies can influence real economic variables at least in short term.
In order to explain the existence of prices and wages rigidity, several contributions of new keynesians have been done in literature. One of examples is the work that has been done by Fischer (1977) and Taylor (1979) where they initiated the nominal contract theory (see[106]). According to Fischer (1977) and Taylor (1979), price rigidity comes from the fact that a big number of firms does business on the basis of written nominal contracts for fixed term. Once an individual firm uses this kind of contract, it is impossible for the price to be adjusted to demand and supply disequilibria during the time of contract. Here the legal economic coercion is the root cause of the price rigidity. This idea is supported by Blanchard (1983). He argued that even if firms can rewrite contracts when the current ones mature, all can not

[27] See [143].

[28] Price rigidity is defined as the inability or resistance of firms to adjust instantaneously the prices of most goods and services in response to underlying cost and demand shocks. Other terms used in literature to describe price rigidity are price inertia, price stickiness, price inflexibility and nominal rigidity (see [80], [37] and [12]).

review their price at the same time and thus some inertia in price level will occur[29]. Fischer (see[48]) showed that in the presence of overlapping[30] labour contracts which put an element of stickiness into the nominal wages, monetary policy has an ability to affect real variables in short term even when the policy is fully anticipated. Fischer considered the world of overlapping labour contracts with each labour contract being made for two periods in order to explain how wage (or price) rigidity leads to non-neutrality of monetary policy in short term. Given these types of contracts, monetary authority can react (for example by supplying money) during the second period of the contract, in reaction to new information about recent economic shocks. Because in the second period the nominal wage has already been negotiated and all wage can not be adjusted at the same time, the changes caused by these reaction are not matched by one- for -one changes in expected inflation. This leads to changes in consumption and investment, and resulting in fluctuations in output and employment. All of these happen only in the short run, since in the long run, all wages adjust and the economy comes back to its natural equilibrium (see [51]).

Other popular researchers who explained the source of the price rigidity under the assumption of monopolistic competition[31] are Rotemberg (1982) and Calvo (1983). According to Rotemberg, this rigidity is due to the fact that firms face at any point in time some convex costs of adjusting their price (See [71], [79], [23], [93] and [56]). Whereas for Calvo (1983), the price rigidity explanation is based on the idea that fraction $1 - \alpha$ of firms are able to reset their price to optimize profits in response to changes in various costs while the remaining fraction α $(0 < \alpha < 1)$ of firms do not adjust their prices (See for example: [71], [1], [49] and [105]).

Based on theoretical works of Fischer (1977), Taylor (1980), Rotemberg (1982), Calvo (1983) and others, new keynesians extended new classical Phillips curve by introducing new aspects such as monopolistic competition and nominal rigidities to a standard marginal cost-based new Keynesian Phillips curve[32] (NKPC) model which describes inflation dynamics as a function of inflation expectations one period ahead and current real marginal costs (i.e., real resource costs that firms spend to produce an extra unit of their good or service) in a closed economy framework.

[29](See [106]).

[30]Overlapping(or staggered) contracts means that all contracts do not all end at the same time (see [148]; [31] and [91]).

[31]*Monopolistic competition means that each firm produces a differentiated good for which it sets the price. Because the products are not exactly the same each supplier has some ability to set the price for its own product. This assumption is important because price rigidity only make sense in a context where firms have some monopoly power to allow them to set their prices and therefore to choose when to change their price.* See [130], [154], [127] and [117].

[32]New Keynesian Phillips curve has been initially proposed by Calvo(1983) (See [94]).

This model is represented by the following equation[33]:

$$\pi_t = \beta E_t(\pi_{t+1}) + \lambda \psi_t \tag{3.7}$$

where,
π_t is inflation rate at time t,
ψ_t represents firms real marginal costs expressed as a percentage deviation around its steady state in period t,
$E_t(\pi_{t+1})$ consists in rational expectations of inflation rate at time $t+1$ formed in the current time t, that is $E(\pi_{t+1}|\Omega_t)$, where Ω_t is a set of all informations available at time t [34] .
$0 < \beta < 1$ is the subjective discount factor and $\lambda = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}$ (a positive parameter) is a slope coefficient that depends on parameter α which measures price rigidity and β.
A detailed derivation of NKPC will be done in the following subsection by applying calvo's version [35]
The equation(3.7) indicates that inflation rate will tend to rise following the rise in real marginal costs, as firms pass on higher costs in the form of higher prices, and when expectations of future inflation rate rise, due to the fact that firms raise their price today anticipating higher prices in following days. Furthermore, this equation implies that inflation is a purely forward-looking phenomenon and real marginal cost is an important driving variable for inflation dynamics. In particular, this can be emphasized by the following equation obtained when equation (3.7)[36] is solved

[33]This equation can be derived from various versions such as Taylor's (1980) fixed duration staggered wage-price contract setting, Rotemberg's (1982) adjustment cost price setting and Calvo's (1983) random time dependent price setting (see[1]).

[34]Conventionally this set contains at least current and past values of the endogenous variable π_t and the real variables (for example output gaps or other measures of real marginal costs) y_t, namely
$\rho_t = (\pi_t, \pi_{t-1}, ...; y_t, y_{t-1}, ...)$ (see [103]).

[35] Calvo's version is chosen because of its simplicity and the similar equations of NKPC emerge under other models of nominal rigidity like those of Rotemberg(1982) and Taylor (1980) as it has been demonstrated by Robert(1995) (see for example: [1] and [123]).

[36]This equation can be derived by using the law of iterated projection, which states that $E(E(Y|X,Z)|X) = E(Y|X)$ (see [62]), and iterating equation (3.16) forward in the following way:

- Iterating one step forward, we get

$$\pi_t = \beta E_t(\overbrace{\beta E_{t+1}(\pi_{t+2} + \lambda\psi_{t+1})}^{\pi_{t+1}}) + \lambda\psi_t. \tag{3.8}$$

The law of iterated projection implies that $E_t(E_{t+1}(\pi_{t+2})) = E_t(\pi_{t+2})$ so that

$$\pi_t = \beta^2 E_t(\pi_{t+2}) + \beta\lambda E_t(\psi_{t+1}) + \lambda\psi_t. \tag{3.9}$$

-Iterating one step forward and once again apply the law of iterated projection, we get

$$\pi_t = \beta^3 E_t(\pi_{t+3}) + \beta^2\lambda E_t(\psi_{t+2}) + \beta\lambda E_t(\psi_{t+1}) + \lambda\psi_t. \tag{3.10}$$

forward.

$$\pi_t = \lambda \sum_{i=0}^{\infty} \beta^i E_t(\psi_{t+i}) \tag{3.12}$$

The NKPC has become attractive on the theoretical grounds because of its features that it is derived from optimizing models under rational expectations, its policy implication that monetary policy can affect real economic variables in short term due to the nominal rigidity assumption and its simplicity. These characteristics have made the NKPC to be widely applied by many central banks in order to simulate inflation consequences of alternative monetary policies (see [94], [146], [112] and [51]).

However, despite its appeal on theoretical side, there is a growing literature on empirical problems of NKPC regarding, for example, the measurement of real marginal cost and its implications which are at odds with the findings from the study of real world data.

For one, equation (3.21) implies that shifts in either current or expected future marginal costs affect immediately current inflation. This makes inflation to be a jump variable and thus not a persistent[37] variable in contrast to the results from the data[38].

An other problem that occurs when it comes to implementing the NKPC empirically relates to the finding of a proxy for the real marginal cost, since it is not directly observable from the data. To deal with this issue, most of empirical literature uses output gap as a proxy for real marginal cost since one can find a correlation between the two variables under certain conditions, as it is expressed by the following equation [39] :

$$\psi_t = \kappa(y_t - y_t^n), \tag{3.13}$$

where κ is a parameter,
y_t is actual output and y_t^n the natural level of output(or flexible-price equilibrium output).

Given this proportionality, equation (3.7) can be transformed to the output gap - based on new Keynesian Phillips curve:

$$\pi_t = \beta E_t(\pi_{t+1}) + \tilde{\lambda}(y_t - y_t^n) \tag{3.14}$$

Continuing the process, we get $\pi_t = \lambda \lim_{k\longrightarrow\infty} \sum_{i=0}^{k} \beta^i E_t(\psi_{t+i}) + \lim_{k\longrightarrow\infty} \beta^{k+1} E_t(\pi_{t+k+1})$.
Imposing that $\lim_{t\longrightarrow\infty} |\pi_t| < \infty$ holds, we have
$\lim_{k\longrightarrow\infty} \beta^{k+1} E_t(\pi_{t+k+1}) = 0$ because $0 < \beta < 1$. then we have

$$\pi_t = \lambda \sum_{i=0}^{\infty} \beta^i E_t(\psi_{t+i}). \tag{3.11}$$

[37]Inflation persistence means the tendency of inflation to converge sluggishly towards a targeted value (see [41]).

[38] See for example [151] P.254; [41] and related literature.

[39]Derivation of relationship between real marginal cost and output gap (see Appendix A of this work).

and equation (3.12) becomes:

$$\pi_t = \tilde{\lambda} \sum_{i=0}^{\infty} \beta^i E_t(y_{t+i} - y^n_{t+i}), \tag{3.15}$$

where $\tilde{\lambda} = \lambda\kappa$.
An other proxy for marginal cost known in literature is for example the labor income share (or Real unit labor share). It has been proposed for instance by Lopez-Salido et al. 2001 (see [55]) after finding that the output gap is a poor measure of marginal cost when NKPC is estimated for USA and Euro- areas. According to these authors, under the assumption that the production function follows a Coob-Dooglas technology, firms take wages as given and there are no labour adjustment costs, the real marginal cost can be expressed by:

$$\psi_t = w_t + l_t - p_t - y_t = s_t,$$

where w_t , l_t , p_t , s_t and y_t are respectively nominal wage, employment, price level and value added output in log-deviations from steady state (see for example [15], [52] and [55]).
From equation (3.15), it is obvious that there is a positive correlation between inflation and output gap, meaning that as far as a central bank can commit to stabilizing of output gap, it can achieve price stability. But the reverse is observed in the data [40]
Furthermore, rearranging equation (3.14) in the following way:

$$\pi_t = \beta E_t\pi_{t+1} + \tilde{\lambda}(y_t - y^n_t) \Longleftrightarrow E_t\pi_{t+1} = \frac{1}{\beta}\pi_t - \frac{\tilde{\lambda}}{\beta}(y_t - y^n_t), \tag{3.16}$$

so

$$\begin{aligned} E_t\pi_{t+1} - \pi_t &= \frac{1}{\beta}\pi_t - \pi_t - \frac{\tilde{\lambda}}{\beta}(y_t - y^n_t) \\ &= \frac{1-\beta}{\beta}\pi_t - \frac{\tilde{\lambda}}{\beta}(y_t - y^n_t), \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} (y_t - y^n_t) &= \frac{\beta}{\tilde{\lambda}}[\frac{1-\beta}{\beta}\pi_t + \pi_t - E_t\pi_{t+1}] \\ &= \frac{\beta}{\tilde{\lambda}}[\pi_t - E_t\pi_{t+1}] + \frac{1-\beta}{\tilde{\lambda}}\pi_t, \end{aligned} \tag{3.18}$$

[40] See for example [55] , [1] and related literatures.

it is clear from equation (3.18) to conclude that an expected disinflation (i.e $\pi_t > E_t\pi_{t+1}$) leads to an output boom which is opposite to the outcomes from the data [41].

Moreover, due to its purely forward-looking aspect, the NKPC does not imply the hump-shaped property[42] of impulse response function [43] for inflation in response to monetary policy shock that estimated Vector Auto-regressive Models (VAR) characterize [44].

In order to avoid such unrealistic implications many authors suggest that an additional lagged inflation term to the NKPC is required [45]. In this vein, several modifications of the NKPC theoretical formulations have been developed to introduce the lagged inflation term in standard new Keynesian Phillips curve [46]. The most popular and widely used model is the modification of the basic Calvo formulation by Gali and Gertler in 1999 (see [52] P.210). According to these authors, instead of allowing all $1-\alpha$ firms (i.e. a set of firms which are allowed to adjust their prices) to set prices of their good in a rational manner, a fraction $1-\omega$ of them set their prices in forward looking manner as before, while the remaining proportion ω of $1-\alpha$ firms use a simple backward-looking rule of thumb[47] .

Based on this assumption, Gali and Gertler extended the NKPC to the standard forward- and backward-looking NKPC named as Hybrid New Keynesian Phillips

[41] See for example [101] , [57] , [1] and related literature.

[42] Hump-shaped property means the gradual rise of inflation after an expansionary monetary policy or gradual fall of inflation after a contractionary monetary policy.

[43] As it has been defined by Mankiw (See [100]), impulse response function is a dynamic path of some variables (for instance: Inflation) in response to some shock (for example: shock to monetary policy).

[44] See [83], [104], [149] and [46] for detailed discussions.

[45]See [71], [89], and [81] and related literature.

[46]Other examples of these modifications are:

• Relative contracting model developed by Jeffrey Fuhrer and George Moore in 2005 as an extension of Taylor's (1980) model. They suggested that workers negotiate their real wages with reference to the real wages that other workers earned in the past. The result was a HNKPC with $\gamma_f = \gamma_b = \frac{1}{2}$ (See [64] P.3).

• An other modification is an extension of Calvo's formulation introduced in Sbordone, and Smets and wouter's papers (See [136] and [142] respectively). They assume that all firms adjust their price at each period. But only a random proportion of firms can reset their prices rationally (i.e. in a manner consistent with profit maximization) and the remaining (i.e. firms which are not able to reoptimize their price) updates last period's price only by indexing partially to past inflation. This yields the HNHPC with $\gamma_f = \frac{\beta}{1+k\beta}$ and $\gamma_b = \frac{k}{1+k\beta}$ where k denotes the partial indexation parameter and $\beta \leq \gamma_f + \gamma_b \leq 1$ is the discount factor.

• Christiano, Eichenbaum, and Evans (See [46]) proposed an other extension of Calvo's model. According to these authors, the firms which can not reoptimize their prices updates their last period's price simply by indexing to lagged inflation while in model of Sbordone and his partners as mensioned above, these firms updates their price by indexing partially to the lagged inflation. This leads to the HNKPC with $\gamma_f = \frac{\beta}{1+\beta}$ and $\gamma_b = \frac{1}{1+\beta}$. For more details about these modifications see for example [71], [131] and [19].

[47]According to the simple backward looking rule of thumb, firms adjust prices at time t by first checking the price from the previous period and correcting it for inflation at time t-1.

curve (HNKPC) and expressed by the following equation: [48]

$$\pi_t = \gamma_f E_t \pi_{t+1} + \gamma_b \pi_{t-1} + \lambda \psi_t, \tag{3.19}$$

where

$$\lambda \equiv (1-\alpha)(1-\omega)(1-\beta\alpha)\phi^{-1},$$

$$\gamma_f \equiv \beta\alpha\phi^{-1}\ ,\ \gamma_b \equiv \omega\phi^{-1},$$

with

$$\phi \equiv \alpha + \omega\left[1-\alpha\left(1-\beta\right)\right].$$

Open Economy New Keynesian Phillips Curve (OENKPC)

In 1990s the world has experienced a decline in inflation rates in general and a significant increase in economic integration (i.e. Globalization[49]) due to agreements signed between countries in order to facilitate international trade [50] . This motivated researchers to study the impact of trade openness (T.O) on inflation and to analyze channels through which T.O can influence inflation dynamic. The results of several empirical studies suggested that globalization may affect inflation dynamics. Some examples of these studies about the role of open-economy aspects on inflation process are:
Romer's (1993)[51], where He found that an increase in openness leads to the disinflation. Other empirical studies whose results provide support for Romer's outcome have been done for example by Nasser in 2009 [52], Chen ,Imbs and Scott in 2004 [53], Gruben and MCleord in 2004 [54], Bowdler and Malik in 2005[55] and Allard in 2007 [56] among others. Furthermore, Banerjee and Botini's (2004)[57] studies show that openness affects real marginal cost and hence inflation. They found that real import prices can play a relevant role in determining inflation in UK, Italy, France and

[48] Derivation of HNKPC see for example [55]and [102].
[49] *Globalization is taken to indicate a process of increasing the connectivity and interdependence of markets and business by removing restrictions and barriers on exchange of knowledge, products and commodities across the borders and regions.*
[50] Examples of agreements can be found in [28] P.15. Important episodes of globalization are the 1992 single market reform in Europe and the formation of the Euro zone.
[51]See [128].
[52]See [134].
[53]See[70].
[54]See [60].
[55] See [24].
[56]See [3].
[57]See [15].

Australia. They argued that this result may be connected with different exchange rate dynamics across these countries. Moreover, in order to confirm this findings they considered an open economy external competitive pressures on imported substitutable final goods as one of ways through which the openness can affect the evolution of inflation (since it can vary the equilibrium price markup on marginal cost which is assumed to be constant and equal to actual markup in the case of CONKPC) [58]. Another is Balakrishnan's (2002)[59] research which also concluded that movements in real prices of imported materials caused changes in the marginal cost and thus in inflation.

Taking these considerations into account, researchers found that it is more important to build open economy aspects into the NKPC in order to improve the fit of NKPC models.

In this vein, several authors have done extensions of the CENKPC to the OENKPC [60]. These reformulations can be generally classified in two main groups based on their assumptions and the ways they introduce open economy factors into the basic closed economy new keynesian phillips curve models (See [1] P.27).

The first group modifies CENKPC by incorporating imported intermediate inputs into the production function. This group can be also decomposed into two subgroups based on their assumptions. The first part assumes that trade is only done at the level of intermediate goods [61] whereas for the second, the trade takes place at the level of the final and intermediate goods [62] .

The second group focuses on the interaction between exchange rate dynamics, price setting and inflation instead of paying attention to the role of intermediate inputs used in production. They either suppose that law of one price (LOOP)[63] holds and that there is complete exchange rate pass through or that there are deviations from LOOP and exchange rate pass through is incomplete [64] . One example of these

[58] For other channels through which globalization may affect inflation dynamics see for example [119], [118] and [137].

[59] See [13].

[60] Leith and Malley (2003) (see [97]), Gali and Monacelli (2005) (see [53]), Balakrishnan and Lopez-salido (2002) (see [13]), Monacelli (2003) (see [109]), Holmberg (2006) (see [65]), Banerjee and Batini (2004) (see [15]) and Rumler (2005) (see [132]) are some examples of them.

[61] Examples of these models is the model developed by Balakrishnan and Lopez-Salido (2002) and Holmberg (2006) see [13] and [65] respectively.

[62] Example of this model is Rumler's (2005) model (see [132]).

[63] *According to the LOOP, in an efficient market identical commodities tend to have the same price heedless of where they are traded. If goods and services obey the LOOP then the exchange rate should be equivalent to the prices of traded goods and services sold in two or more countries when measured in the same currency. For instance, if the domestic price of good x is denoted by $p(x)$ and the foreign currency price of the same good is denoted by $p^*(x)$, then according to to the LOOP, $p(x) = nP^*(x)$ where n is the nominal exchange rate between the two countries measured as the domestic currency price of foreign exchange.*

[64] *Exchange rate pass through is defined as the percentage change in local currency import prices due to a one percent change in the exchange rate between the importing and exporting countries . We say complete pass through if local currency import prices change one to one with the exchange. If it happens that the local currency import prices are totally not sensitive to the exchange rate*

models is that which have been developed by Monacelli in (2003) (See [109]).

3.3.1 Derivation of IS-Curve and New Keynesian Phillips Curve (NKPC)

In the following we give a detailed (mathematical) derivation of the IS-Curve and the New Keynesian Phillips Curve[65]. For this, we first state some assumptions.

Assumptions

• Economy consists of a large number of identical consumer-producer households indexed by $j \in [0,1]$ (where Households supply labor, they purchase goods for consumptions and they hold money and bonds) and firms which produce and sell several differentiated goods (indexed along the unit interval) in monopolicitically competitive markets,
• Firms set their prices in a staggered way of Calvo (1983),
• Labor is supposed to be the only factor of production and firm's production function is assumed to follow the Cobb-Douglas production function:

$$C_{jt} = A_t N_{jt} \quad with \quad E(A_t) = 1, \tag{3.20}$$

where A_t is a random variable which denotes the economy-wide technology level (Total Factor Productivity [66]) (For simplicity we assume that the returns to scale technology is constant).
• Households seek to minimize their costs of buying the composite consumption good c_t and to maximize their expected utility,
• The objective of firms is to minimize their costs of production and to maximize the profits,
•The utility function of households is supposed to follow a constant relative risk aversion utility function[67] of the form:

$$u\left(C_t, N_t, \frac{M_t}{P_t}\right) = \frac{C_t^{1-\sigma}}{1-\sigma} + \frac{\bar{m}}{1-b}(\frac{M_t}{P_t})^{1-b} - \bar{n}\frac{N_t^{1+\eta}}{1+\eta}, \tag{3.21}$$

changes then we say that there is zero pass through. Something in between these two cases is called incomplete exchange rate pass through. (These definitions are referred to [1]).

[65] References are for example [115] and [85].

[66] *Total Factor Productivity is a variable which accounts for effects in total output not caused by traditionally measured inputs. If all inputs are accounted for, then the total factor productivity can be considered as a measure of an economy's long-run technological change or technological dynamism.*

[67] *The constant relative risk aversion utility function (CRRA-UF) is defined as* $u(C) = \begin{cases} \frac{1}{1-\gamma}c^{1-\gamma}, & if \;\; \gamma > 0, \gamma \neq 1 \\ \ln c, & if \;\; \gamma = 1 \end{cases}$, *where the elasticity of substitution between consumption at any two points in time is constant and equal to* $\frac{1}{\gamma}$*.* see [22] p.44.

where C_t, N_t and $\frac{M_t}{P_t}$ are consumption level, labor and real money holdings respectively,
$\bar{m}$ and $\bar{n}$ are positive real numbers,
$\sigma(\neq 1) > 0$ is the degree of relative risk aversion and $\frac{1}{\sigma}$ the elasticity of intertemporal substitution,
η denotes the inverse of the elasticity of labor supply regarding output
and $b \neq 1$ is the elasticity of money demand.
• The composite consumption good C_t (the sum of consumption of all goods j) is defined by the constant elasticity of substitution (CES) of Dixit and Stiglitz[68]:

$$C_t = \left[\int_0^1 C_{jt}^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}}, \tag{3.22}$$

where C_{jt} denotes differentiated goods produced by firms j,
$\theta \neq 1$ gives price elasticity of demand (which measures the responsiveness of demand after a change in price) for individual goods.

Household's Optimization Problems:

a) A cost minimization problem can be written mathematically as:

$min_{C_{jt}} \int_0^1 P_{jt} C_{jt} dj$

s.t $\left[\int_0^1 C_{jt}^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \geq C_t,$

where P_{jt} denotes prices of the individual goods.

b) Expected utility maximization problem defined as:

$\max_{C_t, M_t, N_t, B_t} E_t \sum_{l=0}^{\infty} \beta^l \left[\frac{C_{t+l}^{1-\sigma}}{1-\sigma} + \frac{\bar{m}}{1-b} (\frac{M_{t+l}}{P_{t+l}})^{1-b} - \bar{n} \frac{N_{t+l}^{1+\eta}}{1+\eta} \right],$

subject to the following household's period-by-period budget constraint:

$C_t + \frac{M_t}{P_t} + \frac{B_t}{P_t} = (\frac{W_t}{P_t}) N_t + \frac{M_{t-1}}{P_t} + (1 + i_{t-1}) \left(\frac{B_{t-1}}{P_t} \right) + T_t,$
where B_t, and T_t are one period bonds and real profits paid by firms respectively,
$\frac{W_t}{P_t}$ denotes real labor income,
$\frac{B_t}{P_t}$ is the real financial investments (bond purchases), β^l represents a discount factor, and $(1 + i_{t-1}) \frac{B_{t-1}}{P_t}$ represents the nominal interest gained from bond holdings from

[68] See [138] .

the previous period.

After formulating the optimization problems, we are now going to solve them using the Lagrangian method. Solving for Lagrange multiplier, we get the aggregate consumption price index.
Lagrangian function for the problem **a** reads:

$$\mathcal{L} := \int_0^1 P_{jt}C_{jt}dj - \Psi_t \left(\left[\int_0^1 C_{jt}^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} - C_t \right), \tag{3.23}$$

where Ψ_t is the Lagrange multiplier.
Taking the first order condition (FCO) for C_{jt} we get:

$$\frac{\partial \mathcal{L}}{\partial C_{jt}} \equiv P_{jt} - \Psi_t \underbrace{\frac{\theta}{\theta-1} \left[\int_0^1 C_{jt}^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}-1}}_{Outer\ derivative} \underbrace{\left[\frac{\theta-1}{\theta} C_{jt}^{\frac{\theta-1}{\theta}-1} \right]}_{Inner\ derivative} = 0. \tag{3.24}$$

This implies that:

$$P_{jt} = \Psi_t C_{jt}^{-\frac{1}{\theta}} \underbrace{\left[\int_0^1 C_{jt}^{\frac{\theta-1}{\theta}} dj \right]^{\frac{1}{\theta-1}}}_{C_t^{\frac{1}{\theta}}} \Leftrightarrow P_{jt} = \Psi_t C_{jt}^{-\frac{1}{\theta}} C_t^{\frac{1}{\theta}} \Leftrightarrow C_{jt} = C_t \left(\frac{P_{jt}}{\Psi_t} \right)^{-\theta}. \tag{3.25}$$

Substituting equation(3.25) in equation (3.22) we get:

$$C_t = \left\{ \int_0^1 \left[C_t \left(\frac{P_{jt}}{\Psi_t} \right)^{-\theta} \right]^{\frac{\theta-1}{\theta}} dj \right\}^{\frac{\theta}{\theta-1}} = C_t \left(\frac{1}{\Psi_t} \right)^{-\theta} \left(\int_0^1 P_{jt}^{1-\theta} dj \right)^{\frac{\theta}{\theta-1}}. \tag{3.26}$$

This implies that:

$$\left(\frac{1}{\Psi_t} \right)^{-\theta} \left(\int_0^1 P_{jt}^{1-\theta} dj \right)^{\frac{\theta}{\theta-1}} = 1 \Leftrightarrow \Psi_t = \left(\int_0^1 P_{jt}^{1-\theta} dj \right)^{\frac{1}{1-\theta}}. \tag{3.27}$$

The Lagrangian multiplier Ψ_t gives aggregate consumption price index P_t which is described by the following equation:

$$P_t := \left(\int_0^1 P_{jt}^{1-\theta} dj \right)^{\frac{1}{1-\theta}}. \tag{3.28}$$

Substituting Ψ_t [in equation(3.25)] by P_t we get the demand function for good j in the following form:

$$C_{jt} = C_t \left(\frac{P_{jt}}{P_t} \right)^{-\theta}. \tag{3.29}$$

From the equation (3.29), it is clear that when household knows prices and has made a decision on C_t, it also knows the quantity of good j to consume. Furthermore as $\theta \longrightarrow \infty$ the individual goods become closer substitutes and consequently the market power of individual firms decreases.
By solving the Problem **b** (also by means of Lagrangian method) one gets the relation (Euler equation) which determine an optimal decision on C_t.
The Lagrangian function for **b** is:

$$\mathcal{L} := E_t \sum_{l=0}^{\infty} \beta^l \Bigg\{ \left[\frac{C_{t+l}^{1-\sigma}}{1-\sigma} + \frac{\bar{m}}{1-b} \left(\frac{M_{t+l}}{P_{t+l}} \right)^{1-b} - \bar{n} \frac{N_{t+l}^{1+\eta}}{1+\eta} \right] -$$

$$\Lambda_{t+l} \left(C_{t+l} + \frac{M_{t+l}}{P_{t+l}} + \frac{B_{t+l}}{P_{t+l}} - \frac{W_{t+l}}{P_{t+l}} N_{t+l} - \frac{M_{t-1+l}}{P_{t+l}} - (1+i_{t-1+l}) \frac{B_{t-1+l}}{P_{t+l}} - T_{t+l} \right) \Bigg\}.$$

First order conditions give:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \frac{(1-\sigma) C_t^{-\sigma}}{(1-\sigma)} - \Lambda_t = 0 \Longleftrightarrow \Lambda_t = \frac{1}{C_t^{\sigma}}, \tag{3.30}$$

$$\frac{\partial \mathcal{L}}{\partial C_{t+1}} = \frac{(1-\sigma)\beta E_t \left(C_{t+1}^{-\sigma} \right)}{(1-\sigma)} - \beta \Lambda_{t+1} = 0 \Longleftrightarrow \Lambda_{t+1} = E_t \left(C_{t+1}^{-\sigma} \right), \tag{3.31}$$

$$\frac{\partial \mathcal{L}}{\partial M_t} = \frac{(1-b)\bar{m}}{(1-b)} \frac{1}{P_t} \left(\frac{M_t}{P_t} \right)^{-b} - \Lambda_t \frac{1}{P_t} - \beta E_t \left(-\Lambda_{t+1} \frac{1}{P_{t+1}} \right) = 0, \tag{3.32}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = \frac{(1+\eta)\bar{n} N_t^{\eta}}{(1+\eta)} - \Lambda_t \frac{W_t}{P_t} = 0, \tag{3.33}$$

and

$$\frac{\partial \mathcal{L}}{\partial B_t} = -\Lambda_t \frac{1}{P_t} + \beta E_t \left(\Lambda_{t+1} (1+i_t) \frac{1}{P_{t+1}} \right) = 0. \tag{3.34}$$

Using (3.30) and (3.31) in (3.34) gives:

$$-\frac{1}{C_t^{\sigma}} \frac{1}{P_t} + \beta E_t \left(\frac{1}{E_t C_{t+1}^{\sigma}} (1+i_t) \frac{1}{P_{t+1}} \right) = 0 \Leftrightarrow \frac{C_t^{-\sigma}}{P_t} \frac{1}{1+i_t} = \beta E_t \left(\frac{C_{t+1}^{-\sigma}}{P_{t+1}} \right). \tag{3.35}$$

Equation (3.35) is the Euler equation for the intertemporal allocation of consumption which will be log-linearized in order to determine C_t.
Combining equation(3.35) with (3.30) and (3.32), it follows that:

$$\bar{m} \frac{1}{P_t} \left(\frac{M_t}{P_t} \right)^{-b} - C_t^{-\sigma} \frac{1}{P_t} + \frac{C_t^{-\sigma}}{P_t} \frac{1}{1+i_t} = 0. \tag{3.36}$$

Rearranging this yields:

$$\bar{m} \left(\frac{M_t}{P_t} \right)^{-b} = \frac{i_t}{1+i_t}. \tag{3.37}$$

This is an equation of an optimal money holdings [69].
Substituting (3.30) in (3.33) gives

$$\bar{n}N_t^{\eta} - C_t^{-\sigma}\frac{W_t}{P_t} = 0 \Leftrightarrow \frac{\bar{n}N_t^{\eta}}{C_t^{-\sigma}} = \frac{W_t}{P_t}. \tag{3.38}$$

The relationship which is expressed by the equation (3.38) describes the optimal labor supply [70]

Log-linearization of Euler Equation

Log-linearization is an approximation technique used to convert a non-linear equation into an equation which is linear in terms of the log-deviations of the associated variables from their steady state values. The log-linearization of non-linear functions is commonly done by applying the first order Taylor approximation. According to Taylor's formula
(see for example [84] p.16), the first order Taylor expansion of a function of n real variables $f(x_1, x_2, \dots, x_n)$, which is differentiable at point $X = (x_1, x_2, \dots, x_n)$, around the point $X_0 = (x_{0,1}, x_{0,2}, \dots, x_{0,n})$ is given by:

$f(X) = f(X_0) + \sum_{i=1}^{n} \frac{\partial f(X_0)}{\partial x_i}(x_i - x_{0,i}) + o(R)$, where o denotes the Landau symbol, and $R = \sqrt{\sum_{i=1}^{n}(x_i - x_{0,i})^2}$ is the approximation error term for which $\lim_{R\to 0}\frac{o(R)}{R} = 0$ (i.e. $o(R)$ approaches 0 faster than R.). By omitting the error term, $f(X_0) + \sum_{i=1}^{n}\frac{\partial f(X_0)}{\partial x_i}(x_i - x_{0,i})$ will be the linear approximation of the function $f(x)$.

Now, let us first define π_{t+1} and $\bar{\rho}$ as follows :
$\pi_{t+1} := \frac{P_{t+1}-P_t}{P_t}$ and
$\bar{\rho} := \ln\beta$.
Noting that for small i_t and $\frac{P_{t+1}-P_t}{P_t}$, $\ln(1+i_t)$ can be linearly approximated by i_t, the linear approximation of $\ln\left(1 + \frac{P_{t+1}-P_t}{P_t}\right) = \ln\left(\frac{P_{t+1}}{P_t}\right) = \ln P_{t+1} - \ln P_t =: p_{t+1} - p_t$ is $\frac{P_{t+1}-P_t}{P_t} =: \pi_{t+1}$, and considering all mentioned above, the equation (3.35) can be written as:

$$1 \overset{x=e^{\ln x}}{=} E_t\left[e^{\bar{\rho}+i_t-\sigma(c_{t+1}-c_t)-\pi_{t+1}}\right]. \tag{3.39}$$

From equation(3.39) we can determine the value of $\bar{\rho}$ in the steady state [71] as follows:

$$0 = \bar{\rho} + i - \pi \Leftrightarrow \bar{\rho} = \pi - i. \tag{3.40}$$

[69] *This means that the intratemporal optimality condition setting the marginal rate of substitution between money and consumption is equal to the opportunity cost of holding money.*
[70] *Equation (3.38) shows that the intratemporal optimality condition setting the marginal rate of substitution between leisure and consumption is equal to the real wage.*
[71] *A steady state is a situation in which endogenous variables do not change anymore.*

Applying the first order Taylor approximation on the right side of the equation (3.39) we have:

$$1 = E_t \left[\underbrace{e^{\overbrace{(\pi - i)}^{\bar{\rho}} + i - \pi}}_{1} + 1.(i_t - i) - 1.\sigma(c_{t+1} - c) + 1.\sigma(c_t - c) - 1(\pi_{t+1} - \pi) \right] \tag{3.41}$$

$$\Updownarrow$$

$$0 = \underbrace{-i + \pi}_{\bar{\rho}} + i_t + \sigma c_t - \sigma E_t c_{t+1} - E_t \pi_{t+1} \tag{3.42}$$

$$\Updownarrow$$

$$c_t = E_t c_{t+1} - \frac{1}{\sigma}\left(i_t - E_t \pi_{t+1} - \bar{\rho}\right). \tag{3.43}$$

Assuming that $c_t = y_t$[72] it is possible to write:

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}\left(i_t - E_t \pi_{t+1} - \bar{\rho}\right). \tag{3.44}$$

The actual output can be different from the natural output due to the assumption of the nominal rigidities [Calvo's (1983)] and the monopolistic competition in this model. Hence, the output gap $\tilde{y}_t$ has the following form:
$\tilde{y}_t = y_t - y_t^n$.
Considering the Fisher's equation which states that real interest rate r_t is equal to the nominal interest rate i_t minus expected inflation rate $E_t \pi_{t+1}$ we have:

$$\tilde{y}_t = E_t \tilde{y}_{t+1} - \frac{1}{\sigma}\left(i_t - E_t \pi_{t+1} - r_t^n\right). \tag{3.45}$$

where r_t^n denotes the real natural interest rate.

Firm's Optimization Problems

1) Cost (of productions) minimization problem described by the following equation:

$$min_{N_t} \left(\frac{W_t}{P_t}\right) N_t \quad s.t \quad C_{jt} = A_t N_t. \tag{3.46}$$

This can be solved by means of Lagrangian method as it has been done before.
Lagrangian function $\mathcal{L}$ is given by:

$$\mathcal{L} := \left(\frac{W_t}{P_t} N_t\right) + \Psi_t \left(C_{jt} - A_t N_{jt}\right). \tag{3.47}$$

[72] *In case $c_t \neq y_t$ one can add the other aggregate demand components a an additional shock, called demand shock.*

First order condition yields:

$$\frac{W_t}{P_t} - \Psi_t A_t = 0 \Leftrightarrow \Psi_t = \frac{W_t/P_t}{A_t}, \tag{3.48}$$

where Ψ_t in this case represents firm's real marginal costs [73].

2) Firm' profit maximization problem expressed by the following representation:

$$max_{P_{jt}} E_t \left\{ \sum_{l=0}^{\infty} \alpha^l \overbrace{\Delta_{l,t+l}}^{discountfactor} \left[\overbrace{\frac{P_{jt}}{P_{t+l}} C_{jt+l}}^{\text{real revenues (assuming that no price change happened)}} - \overbrace{\Psi_{t+l} C_{jt+l}}^{\text{real production cost}} \right] \right\} \tag{3.49}$$

$$s.t \;\; C_{jt} = \left(\frac{P_{jt}}{P_t} \right)^{-\theta} C_t, \tag{3.50}$$

and the assumption of Calvo' s pricing [74] where $\Delta_{l,t+l} = \beta^l \left(\frac{C_{t+l}}{C_t} \right)^{-\sigma}$ (because firms must consider the furure demand when they set prices) and α denotes a measure of rigidity.
Substituting equation (3.50) in Equation (3.49) we come to the following expression:

$$max_{P_{jt}} E_t \left\{ \sum_{l=0}^{\infty} \alpha^l \Delta_{l,t+l} \left[\frac{P_{jt}}{P_{t+l}} \left(\frac{P_{jt}}{P_{t+l}} \right)^{-\theta} C_{t+l} - \Psi_{t+l} \left(\frac{P_{jt}}{P_{t+l}} \right)^{-\theta} C_{t+l} \right] \right\} \tag{3.51}$$

$$\Updownarrow$$

$$max_{P_{jt}} E_t \left\{ \sum_{l=0}^{\infty} \alpha^l \Delta_{l,t+l} \left[\left(\frac{P_{jt}}{P_{t+l}} \right)^{1-\theta} - \Psi_{t+l} \left(\frac{P_{jt}}{P_{t+l}} \right)^{-\theta} \right] C_{t+l} \right\}. \tag{3.52}$$

Here the price P_{jt} is assumed to be not changed to P_{jt+l} because firms choose their price in the period t expecting that they are not allowed to change this price in future periods.
For $P_{jt} := P_t^*$ the first order condition gives:

$$E_t \left\{ \sum_{l=0}^{\infty} \alpha^l \Delta_{l,t+l} \left[(1-\theta) \frac{1}{P_{t+l}} \left(\frac{P_t^*}{P_{t+l}} \right)^{-\theta} + \theta \Psi_{t+l} \frac{1}{P_{t+l}} \left(\frac{P_t^*}{P_{t+l}} \right)^{-\theta-1} \right] C_{t+l} \right\} = 0 \tag{3.53}$$

$$\Updownarrow$$

[73] *Equation (3.48) means that the firm's marginal costs in a flexible price equilibrium is equal to the ratio of real wage and marginal product of labor A_t.*
[74] See the page 33 of this work.

$$E_t \left\{ \sum_{l=0}^{\infty} \alpha^l \Delta_{l,t+l} \left[(1-\theta)\frac{1}{P_{t+l}} + \theta\Psi_{t+l}\frac{1}{P_{t+l}} \left(\frac{P_t^*}{P_{t+l}}\right)^{-1} \right] \left(\frac{P_t^*}{P_{t+l}}\right)^{-\theta} C_{t+l} \right\} = 0 \quad (3.54)$$

$$\Updownarrow$$

$$E_t \left\{ \sum_{l=0}^{\infty} \alpha^l \Delta_{l,t+l} \left[(1-\theta)\frac{P_t^*}{P_{t+l}} + \theta\Psi_{t+l} \overbrace{\frac{1}{P_{t+l}} \left(\frac{1}{P_{t+l}}\right)^{-1}}^{1} \right] \left(\frac{1}{P_t^*}\right) \left(\frac{P_t^*}{P_{t+l}}\right)^{-\theta} C_{t+l} \right\} = 0 \quad (3.55)$$

$$\Updownarrow$$

$$E_t \left\{ \sum_{l=0}^{\infty} \alpha^l \Delta_{l,t+l} \left[(1-\theta)\frac{P_t^*}{P_{t+l}} + \theta\Psi_{t+l} \right] \left(\frac{1}{P_t^*}\right) \left(\frac{P_t^*}{P_{t+l}}\right)^{-\theta} C_{t+l} \right\} = 0. \quad (3.56)$$

Substituting the value of $\Delta_{l,t+l}$ in equation (3.56) yields:

$$E_t \left\{ \sum_{l=0}^{\infty} \alpha^l \beta^l \left(\frac{C_{t+l}}{C_t}\right)^{-\sigma} C_{t+l} \left[(1-\theta)\frac{P_t^*}{P_{t+l}} + \theta\Psi_{t+l} \right] \left(\frac{1}{P_t^*}\right) \left(\frac{P_t^*}{P_{t+l}}\right)^{-\theta} \right\} = 0 \quad (3.57)$$

$$\Updownarrow$$

$$\begin{aligned} &E_t \sum_{l=0}^{\infty} \alpha^l \beta^l \left(\frac{C_{t+l}}{C_t}\right)^{-\sigma} C_{t+l}(\theta-1)\frac{P_t^*}{P_{t+l}} \left(\frac{1}{P_t^*}\right) \left(\frac{P_t^*}{P_{t+l}}\right)^{-\theta} \\ &= \theta E_t \sum_{l=0}^{\infty} \alpha^l \beta^l \left(\frac{C_{t+l}}{C_t}\right)^{-\sigma} C_{t+l}\Psi_{t+l} \left(\frac{1}{P_t^*}\right) \left(\frac{P_t^*}{P_{t+l}}\right)^{-\theta} \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} &(\theta-1)\frac{1}{C_t^{-\sigma}} \left(P_t^*\right)^{-\theta} E_t \sum_{l=0}^{\infty} \alpha^l \beta^l C_{t+l}^{1-\sigma} \frac{1}{P_{t+l}} \left(\frac{1}{P_{t+l}}\right)^{-\theta} \\ &= \theta \frac{1}{C_t^{-\sigma}} \left(P_t^*\right)^{-1-\theta} E_t \sum_{l=0}^{\infty} \alpha^l \beta^l C_{t+l}^{1-\sigma} \Psi_{t+l} \left(\frac{1}{P_{t+l}}\right)^{-\theta} \end{aligned} \quad (3.58)$$

$$\Updownarrow$$

$$\frac{1}{P_t}\frac{\left(P_t^*\right)^{-\theta}}{\left(P_t^*\right)^{-1-\theta}} = \frac{\theta}{\theta-1} \frac{E_t \sum_{l=0}^{\infty} \alpha^l \beta^l C_{t+l}^{1-\sigma} \Psi_{t+l} P_{t+l}^{\theta}}{E_t \sum_{l=0}^{\infty} \alpha^l \beta^l C_{t+l}^{1-\sigma} P_{t+l}^{\theta-1}} \frac{1}{P_t} \quad (3.59)$$

$$\Updownarrow$$

$$\frac{P_t^*}{P_t} = \frac{\theta}{\theta-1}\frac{E_t\sum_{l=0}^{\infty}\alpha^l\beta^l C_{t+l}^{1-\sigma}\Psi_{t+l}P_{t+l}^{\theta}}{E_t\sum_{l=0}^{\infty}\alpha^l\beta^l C_{t+l}^{1-\sigma}P_{t+l}^{\theta-1}P_t\frac{P_t^{\theta-1}}{P_t^{\theta-1}}}. \tag{3.60}$$

This implies that:

$$\frac{P_t^*}{P_t} = \frac{\theta}{\theta-1}\frac{E_t\sum_{l=0}^{\infty}\alpha^l\beta^l C_{t+l}^{1-\sigma}\Psi_{t+l}\left(\frac{P_{t+l}}{P_t}\right)^{\theta}}{E_t\sum_{l=0}^{\infty}\alpha^l\beta^l C_{t+l}^{1-\sigma}\left(\frac{P_{t+l}}{P_t}\right)^{\theta-1}}. \tag{3.61}$$

Equation (3.61) represents the price setting rule for firms facing sticky prices.

Considering the case where the measure of regidity α is equal to zero, we can derive the relation which describes the natural output (also called flexible price equilibrium output). In this case the problem **2** collapses to a one period problem and the equation (3.61) becomes:

$$\frac{P_t^*}{P_t} = \frac{\theta}{\theta-1}\frac{\beta^0 C_t^{1-\sigma}\Psi_t\left(\frac{P_t}{P_t}\right)^{\theta}}{\beta^0 C_t^{1-\sigma}\left(\frac{P_t}{P_t}\right)^{\theta-1}} = \frac{\theta}{\theta-1}\Psi_t = \mu\Psi_t, \tag{3.62}$$

where μ denotes mark-up.
Knowing that under flexible prices all firms set the same price (i.e $P_t^* = P_t$ and thus $\Psi_t = \frac{1}{\mu}$) and combining this with the equation (3.48) we get:

$$\frac{W_t/P_t}{A_t} = \frac{1}{\mu} \Leftrightarrow \frac{W_t}{P_t} = \frac{A_t}{\mu}. \tag{3.63}$$

Substituting the equation (3.38) in equation (3.63) we come to the following relation:

$$\frac{\bar{n}N_t^{\eta}}{C_t^{-\sigma}} = \frac{A_t}{\mu} \Leftrightarrow 1 = \frac{\mu\bar{n}N_t^{\eta}}{A_t C_t^{-\sigma}}. \tag{3.64}$$

This equation shows that in a flexible price equilibrium a marginal rate of substitution between leisure and consumption is equal to the ratio of marginal product and mark-up.

Log-linearization of the Equation (3.64)

Following the same procedure as before, we have:

$$1 = \frac{\mu\bar{n}N_t^{\eta}}{A_t C_t^{-\sigma}} \Leftrightarrow 1 = \underbrace{e^{\ln\mu+\ln\bar{n}+\eta\ln N_t-\ln A_t+\sigma\ln C_t}}_{V}. \tag{3.65}$$

From equation (3.65), in the steady state $\ln \bar{n} = -\ln \mu - \eta \ln N + \ln A - \sigma \ln C$, and then the right hand side of equation (3.65) is equal to one in the steady state.

As $\frac{\partial V}{\partial N_t} = \eta \frac{1}{N_t} V$, $\frac{\partial V}{\partial A_t} = -\frac{1}{A_t} V$ and $\frac{\partial V}{\partial C_t} = \sigma \frac{1}{C_t} V$, the application of the first order Taylor approximation around the steady state yields:

$$1 = e^0 + \eta \frac{1}{N} e^0 (N_t - N) - \frac{1}{A} e^0 (A_t - A) + \sigma \frac{1}{C} e^0 (C_t - C). \tag{3.66}$$

This implies that:

$$\underbrace{\frac{A_t - A}{A}}_{:=a_t} = \eta \underbrace{\frac{N_t - N}{N}}_{:=n_t} + \sigma \underbrace{\frac{C_t - C}{C}}_{:=c_t} \Leftrightarrow a_t = \eta n_t + \sigma c_t. \tag{3.67}$$

The production function ($C_{jt} = A_t N_{jt}$) can also be approximated in the same way and we get the following relation in flexible price equilibrium:

$$c_t^n = n_t^n + a_t^n \Leftrightarrow n_t^n = c_t^n - a_t^n. \tag{3.68}$$

Noting that the natural output y_t^n is equal to the consumption c_t^n and considering the relations expressed by the equations (3.67) and (3.68), the following relation holds:

$$\eta(y_t^n - a_t^n) + \sigma y_t^n = a_t^n \Leftrightarrow (\eta + \sigma) y_t^n = (1 + \eta) a_t^n. \tag{3.69}$$

Consequently, the output y_t^n is described by:

$$y_t^n = \frac{1 + \eta}{\eta + \sigma} a_t^n. \tag{3.70}$$

Consideration of sticky price (i.e., $\alpha > 0$)

From the equation (3.28) we can define the price index in period t as follows [74]:

$$P_t = \left[\int_0^{1-\alpha} \left(P_{jt}^*\right)^{1-\theta} dj + \int_{1-\alpha}^{1} \underbrace{\left(P_{jt-1}\right)^{1-\theta}}_{\text{Price of non-adjusting firms in t}} dj \right]^{\frac{1}{1-\theta}}. \tag{3.71}$$

Considering that non-adjusters were randomly selected (and hence $P_{jt-1} = P_{t-1}$ on average) and calculating this integral we get the average price for all firms in t:

$$P_t^{1-\theta} = (1 - \alpha) \left(P_{jt}^*\right)^{1-\theta} + \left[\left(P_{t-1}\right)^{1-\theta} (1) - \left(P_{t-1}\right)^{1-\theta} (1 - \alpha) \right] \tag{3.72}$$

[74]Note that in equation (3.71) prices of adjusting firms in period t are all identical due to the assumption of identical producers.

$$\Updownarrow$$

$$P_t^{1-\theta} = (1-\alpha)\left(P_{jt}^*\right)^{1-\theta} + \alpha\left(P_{t-1}\right)^{1-\theta} \tag{3.73}$$

$$\Updownarrow$$

$$1 = \left[(1-\alpha)\left(\frac{P_t^*}{P_t}\right)^{1-\theta} + \alpha\left(\frac{P_{t-1}}{P_t}\right)^{1-\theta}\right] = e^{\ln\left((1-\alpha)\left(\frac{P_t^*}{P_t}\right)^{1-\theta} + \alpha\left(\frac{P_{t-1}}{P_t}\right)^{1-\theta}\right)}. \tag{3.74}$$

Log-linearization of the Equation (3.74)

Due to the fact that θ is not equal to one, as it has already been assumed, it is clear from the equation (3.74) that the ratio of P_t^* and P_t is equal to one in the steady state.
Applying first order Taylor approximation around the steady state on the equation

$$1 = e^{\ln\left((1-\alpha)\left(\frac{P_t^*}{P_t}\right)^{1-\theta} + \alpha\left(\frac{P_{t-1}}{P_t}\right)^{1-\theta}\right)}, \tag{3.75}$$

we get:

$$(1-\theta)(1-\alpha)\frac{Q_t - Q}{Q} = (1-\theta)\alpha\left(\frac{P_t}{P_{t-1}} - 1\right) \Rightarrow q_t = \frac{\alpha}{1-\alpha}\pi_t, \tag{3.76}$$

where $\frac{P_t^*}{P_t} := Q_t$.

Log-linearization of the Equation (3.61)

$$\frac{P_t^*}{P_t} = \frac{\theta}{\theta-1}\frac{E_t \sum_{l=0}^{\infty} \alpha^l \beta^l C_{t+l}^{1-\sigma} \Psi_{t+l}\left(\frac{P_{t+l}}{P_t}\right)^{\theta}}{E_t \sum_{l=0}^{\infty} \alpha^l \beta^l C_{t+l}^{1-\sigma}\left(\frac{P_{t+l}}{P_t}\right)^{\theta-1}} \tag{3.77}$$

$$\Updownarrow$$

$$0 = \underbrace{Q_t\left[E_t \sum_{l=0}^{\infty} \alpha^l \beta^l C_{t+l}^{1-\sigma}\left(\frac{P_{t+l}}{P_t}\right)^{\theta-1}\right]}_{I} - \mu\underbrace{\left[E_t \sum_{l=0}^{\infty} \alpha^l \beta^l C_{t+l}^{1-\sigma}\Psi_{t+l}\left(\frac{P_{t+l}}{P_t}\right)^{\theta}\right]}_{II}. \tag{3.78}$$

Using the first order Taylor approximation as before and considering that $Q_t = Q = 1$ in the steady state, the log-linearization of the equation (3.78) can be done as follows [75]:

$$I = E_t \sum_{l=0}^{\infty} \alpha^l \beta^l C^{1-\sigma} + E_t \sum_{l=0}^{\infty} \alpha^l \beta^l C^{1-\sigma} q_t + E_t \sum_{l=0}^{\infty} \alpha^l \beta^l (1-\sigma) C^{-\sigma} C c_{t+l}$$

$$+ E_t \sum_{l=0}^{\infty} \alpha^l \beta^l C^{1-\sigma} (\theta - 1) p_{t+l} - E_t \sum_{l=0}^{\infty} \alpha^l \beta^l C^{1-\sigma} (\theta - 1) p_t$$

$$= \sum_{l=0}^{\infty} \alpha^l \beta^l C^{1-\sigma} + E_t \sum_{l=0}^{\infty} \alpha^l \beta^l C^{1-\sigma} q_t - E_t \sum_{l=0}^{\infty} \alpha^l \beta^l C^{1-\sigma} (\theta - 1) p_t \tag{3.79}$$

$$+ E_t \sum_{l=0}^{\infty} \alpha^l \beta^l C^{1-\sigma} (\theta - 1) p_{t+l} + E_t \sum_{l=0}^{\infty} \alpha^l \beta^l (1-\sigma) C^{1-\sigma} c_{t+l}$$

$$= C^{1-\sigma} E_t \sum_{l=0}^{\infty} \alpha^l \beta^l \left[1 + q_t - (\theta - 1) p_t + (\theta - 1) p_{t+l} + (1-\sigma) c_{t+l}\right].$$

$$II = E_t \sum_{l=0}^{\infty} \alpha^l \beta^l C^{1-\sigma} \Psi + E_t \sum_{l=0}^{\infty} \alpha^l \beta^l (1-\sigma) \Psi C^{-\sigma} C c_{t+l} + E_t \sum_{l=0}^{\infty} \alpha^l \beta^l C^{1-\sigma} \Psi \psi_{t+l}$$

$$+ E_t \sum_{l=0}^{\infty} \alpha^l \beta^l C^{1-\sigma} \Psi \theta p_{t+l} - E_t \sum_{l=0}^{\infty} \alpha^l \beta^l C^{1-\sigma} \Psi \theta p_t$$

$$= C^{1-\sigma} \Psi E_t \sum_{l=0}^{\infty} \alpha^l \beta^l \left[1 - \theta p_t + \theta p_{t+l} + \psi_{t+l} + (1-\sigma) c_{t+l}\right]. \tag{3.80}$$

Substituting I and II in the equation (3.78) , we have:

$$\begin{aligned} 0 = E_t \sum_{l=0}^{\infty} \alpha^l \beta^l \left[1 + q_t - (\theta - 1) p_t + (\theta - 1) p_{t+l} + (1-\sigma) c_{t+l}\right] \\ -\Psi E_t \sum_{l=0}^{\infty} \alpha^l \beta^l \left[1 - \theta p_t + \theta p_{t+l} + \psi_{t+l} + (1-\sigma) c_{t+l}\right]. \end{aligned} \tag{3.81}$$

[75]Note that there will not be change in our result if the constant μ is ignored by doing approximation.

Noting that $|\alpha\beta| < 1$ (and hence $\sum_{l=0}^{\infty} \alpha^l \beta^l$ is a geometric series which converges to the value $\frac{1}{1-\alpha\beta}$) the equation (3.81) becomes:

$$\begin{aligned} 0 = \frac{1}{1-\alpha\beta} + \frac{1}{1-\alpha\beta} q_t + \sum_{l=0}^{\infty} \alpha^l \beta^l \left[(\theta - 1) \left(E_t p_{t+l} - p_t \right) + (1-\sigma) E_t c_{t+l} \right] \\ -\Psi \left\{ \frac{1}{1-\alpha\beta} + \sum_{l=0}^{\infty} \alpha^l \beta^l \left[\theta \left(E_t p_{t+l} - p_t \right) + E_t \psi_{t+l} + (1-\sigma) E_t c_{t+l} \right] \right\}. \end{aligned} \tag{3.82}$$

In approximation the constant Ψ and can be suppressed and we have:

$$\begin{aligned} 0 = \frac{1}{1-\alpha\beta} q_t + \sum_{l=0}^{\infty} \alpha^l \beta^l \left[(\theta - 1) \left(E_t p_{t+l} - p_t \right) + (1-\sigma) E_t c_{t+l} \right] \\ - \sum_{l=0}^{\infty} \alpha^l \beta^l \left[\theta \left(E_t p_{t+l} - p_t \right) + E_t \psi_{t+l} + (1-\sigma) E_t c_{t+l} \right] \end{aligned} \tag{3.83}$$

$$\Updownarrow$$

$$\begin{aligned} \frac{1}{1-\alpha\beta} q_t + \sum_{l=0}^{\infty} \alpha^l \beta^l \left[\theta \left(E_t p_{t+l} - p_t \right) - \left(E_t p_{t+l} - p_t \right) + (1-\sigma) E_t c_{t+l} \right] \\ = \sum_{l=0}^{\infty} \alpha^l \beta^l \left[\theta \left(E_t p_{t+l} - p_t \right) + E_t \psi_{t+l} + (1-\sigma) E_t c_{t+l} \right] \end{aligned} \tag{3.84}$$

$$\Updownarrow$$

$$\frac{1}{1-\alpha\beta} q_t + \sum_{l=0}^{\infty} \alpha^l \beta^l (-1) \left(E_t p_{t+l} - p_t \right) = \sum_{l=0}^{\infty} \alpha^l \beta^l E_t \psi_{t+l} \tag{3.85}$$

$$\Updownarrow$$

$$\frac{1}{1-\alpha\beta} \left(q_t + p_t \right) = \sum_{l=0}^{\infty} \alpha^l \beta^l E_t p_{t+l} + \sum_{l=0}^{\infty} \alpha^l \beta^l E_t \psi_{t+l} \tag{3.86}$$

$$\Updownarrow$$

$$\left(q_t + p_t \right) = (1-\alpha\beta) \left(\sum_{l=0}^{\infty} \alpha^l \beta^l E_t p_{t+l} + \sum_{l=0}^{\infty} \alpha^l \beta^l E_t \psi_{t+l} \right). \tag{3.87}$$

Note that since $Q_t := \frac{P_t^*}{P_t}$, $P_t^* = Q_t P_t$ holds and consequently we have:

$$p_t^* = q_t + p_t. \tag{3.88}$$

In addition, the equation (3.87) can be written in a two period framework, where quadratic terms are ignored. This yields:

$$p_t + q_t = (1-\alpha\beta)(\psi_t + p_t) + \alpha\beta(E_t q_{t+1} + E_t p_{t+1}) \tag{3.89}$$

$$\Updownarrow$$

$$\begin{aligned} q_t &= (1-\alpha\beta)\psi_t + \alpha\beta(E_t q_{t+1} + E_t p_{t+1} - p_t) \\ &= (1-\alpha\beta)\psi_t + \alpha\beta(E_t q_{t+1} + E_t \pi_{t+1}). \end{aligned} \tag{3.90}$$

Substituting equation (3.76) in equation (3.90) we get:

$$\begin{aligned} \frac{\alpha}{1-\alpha}\pi_t &= (1-\alpha\beta)\psi_t + \alpha\beta\left(\frac{\alpha}{1-\alpha}E_t\pi_{t+1} + E_t\pi_{t+1}\right) \\ &= (1-\alpha\beta)\psi_t + \frac{\alpha^2\beta}{1-\alpha}E_t\pi_{t+1} + \alpha\beta\frac{1-\alpha}{1-\alpha}E_t\pi_{t+1} \\ &= (1-\alpha\beta)\psi_t + \frac{\alpha\beta}{1-\alpha}E_t\pi_{t+1} \end{aligned} \tag{3.91}$$

$$\Updownarrow$$

$$\pi_t = \beta E_t\pi_{t+1} + \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}\psi_t. \tag{3.92}$$

This equation is the standard New Keynesian Phillips Curve in closed economy framework. Note that, according to the theoretical derivation of the NKPC proposed by Rotemberg (1982) and Calvo (1983) , no error term tacked onto the NKPC equation. However, in much of literature, researchers add a stochastic error term to the equation of NKPC. They argue that the error term may capture for example: approximation errors which can result from linearization of the theoretical model, measurement errors or shocks to desired markups (see [1] and [19]).

New Keynesian Phillips Curve in terms of output gap

First note that, under the assumption of a flexible price, the following equations hold : [76]
$\frac{P_t^*}{P_t} = \mu\Psi_t$, $\frac{W_t}{P_t} = \frac{A_t}{\mu} = \frac{\bar{n}N_t^\eta}{C_t^{-\sigma}}$ and $\frac{W_t/P_t}{A_t} = \Psi_t$.
Considering their log-linearization, we have:
$w_t - p_t = \eta n_t \sigma y_t$,

[76] See equations 3.62, 3.63, 3.38 and 3.48.

$\pi_t = (w_t - p_t) - a_t$, and from the production function
$y_t = n_t + a_t$.
Combining these we get:

$$\pi_t = \eta n_t + \sigma y_t - (y_t - n_t) \tag{3.93}$$

$$\Updownarrow$$

$$\begin{aligned} \psi_t &= [\eta(y_t - a_t) + \sigma y_t] - (y_t - y_t + a_t) \\ &= \eta y_t - \eta a_t + \sigma y_t - a_t \\ &= (\eta + \sigma) y_t - (1 + \eta) a_t \\ &= (\eta + \sigma) \left[y_t - \frac{1+\eta}{\eta+\sigma} a_t \right]. \end{aligned} \tag{3.94}$$

Substituting equation (3.70) in equation (3.94) yields:

$$\psi_t = (\eta + \sigma)\left(y_t - y_t^n\right). \tag{3.95}$$

Substituting equation (3.95) in equation (3.92) we get the New Keynesian Phillips Curve in terms of output gap described by the following equation:

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(1-\alpha)(1-\alpha\beta)(\eta+\sigma)}{\alpha} \tilde{y}_t, \tag{3.96}$$

where $\tilde{y}_t := y_t - y_t^n$ is the output gap.

3.3.2 Continuous time analogue of the New Keynesian Phillips Curve (NKPC)

New Keynesian Phillips curve models are developed in many literatures in discrete time. In this framework it is mostly assumed that their underlying period length is a quarter. Moreover one supposes that agents make decisions discontinuously and all transactions of certain class take place in the same synchronized rhythm. But in this section we will derive the continuous time version of the NKPC in order to make the model technology accessible for optimal control methods [77]. To obtain a continuous time form of the NKPC we first transform it into its high-frequency equivalent by referring to the works done by Sacht (see [135]), and Franke and Sacht

[77] In a discrete time framework inflation rates are observed at fixed intervals of time (t) and they do not change between these observation points whereas in continuous time inflation can be measured at any time t and they can take different values at any time.

(see [50]). By calculating the limit of this high-frequency version of the NKPC as the length of period (defined as $0 < h = \frac{1}{f} < 1$, where f represents the frequency of decision making) shrinks to zero, we will find that it converges to a well-defined continuous time formulation of the New Keynesian Phillips curve model.

High-Frequency Version of the New Keynesian Phillips Curve (NKPC)

The high-frequency economy version of NKPC (denoted as h-economy NKPC) is defined as a model version with a shorter period length than that in its original formulation. In other words, an h-economy NKPC is constituted by a period of length $0 < h = \frac{1}{f} < 1$ [78], if the period of the benchmark model is considered as the time unit which is fixed. A h-economy NKPC can be achieved by adapting the frequency-dependent parameters and variables of the NKPC to the period length h. Impact of this transformation on dynamic properties of the NKPC in quarterly magnitudes has been investigated by Franke and Sacht (2010) (See [50]). By comparing the Impulse-Response Functions (IRFs) based on different values of h they found that there are in general qualitative and quantitative dissimilarities between those IRFs. But by checking the limiting behavior of h-economy NKPC as the frequency of decisions tends to infinity (i.e $h \longrightarrow 0$) they showed that in limit as h shrinks to zero, one must arrive at a well defined continuous time formulation of NKPC. This result emphasizes the conventional procedure to convert a discrete time model to its continuous time version which is applied since the works done for example by Foley (1975) and May (1970) on the relation between continuous time and discrete time models (See [14] p.2). According to this procedure, one resizes all the key equations describing the economy along any sub-interval $[t, t+h]$ and lets the period length h shrinks to zero in case the limit of this equations exists.
In the following we will use the transition rules[79] given by Sacht (2014) in order to transform our model into a h-economy model and then we determine the corresponding model in continuous time by computing the limit of its h-economy version as mentioned before.
The model which is going to be transformed is:

$$\pi_t = p_t - p_{t-1} = \beta E_t(\pi_{t+1}) + \tilde{\lambda}(y_t - y_t^n), \tag{3.97}$$

where

$$\tilde{\lambda} = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}(\eta + \sigma), \tag{3.98}$$

$0 < \alpha < 1$ is the Calvo's fraction of the firms which do not adjust their prices,
$0 < \beta < 1$ denotes the discount factor, as it has been presented in previous sections of this chapter, η and σ are positive constant parameters,
and $p_t := logP_t$ with P_t denotes the price index.

[78] Frequency f expresses the number of transactions made by agents over the length of period.
[79] These rules will be summarized in appendix A. More details can be found in [135] P.19-20.

The evolution of the output gap $(y_t - y_t^n)$ will be assumed to be determined by an Investiment-Savings (IS) curve of the following form (see derivation of IS curve in subsection 3.2.1):

$$y_t - y_t^n = E_t(y_{t+1} - y_{t+1}^n) - \frac{1}{\sigma}(i_t - E_t\pi_{t+1} - r_t^n), \tag{3.99}$$

where the monetary policy is supposed to operate according to the interest rate given by the following equation:

$$i_t = r_t^n + \pi^T + a\tilde{y}_t + b(\pi_t - \pi^T), \tag{3.100}$$

where $a = \frac{\phi_{\tilde{y}}}{\sigma\phi_i}$, $b = \frac{\tilde{\lambda}\phi_\pi}{\sigma\phi_i}$,
$\phi_i > 0$ reflects the central bank's aversion to deviations of i_t from its target $i_t^n = r_t^n + \pi^T$[80],
r_t^n denotes the natural (or equilibrium) real interest rate,
π^T symbolizes inflation target for the central bank,
ϕ_π and $\phi_{\tilde{y}}$ are positive coefficients given by the central bank, that express the strength of the interest rate response to the fluctuations of inflation and the output gap from their target levels respectively,
and π_t, $\tilde{y}_t := y_t - y_t^n$ are inflation rate and output gap respectively.
Interest rate rule given by the equation (3.100) will be derived in appendix A of this work by solving an optimal control problem of a flexible inflation-targeting[81] central bank where costs related to changes in interest rate are assumed to be equal to zero. In this case the control (interest rate represented by the equation 3.100) may be applied every time instant and it would lead to the optimal trajectory over time of output gap and inflation variables. However, in real world, the rule mentioned above tends to give inefficient stabilization of output and inflation for the following reasons:

[80] See for example [5].

[81] According to the European Central Bank (see the the work of the European Central Bank with the title " the monetary policy of the ECB" 2004), inflation-targeting is a central bank's strategy aimed at maintaining price stability by focusing on deviations in published inflation forecasts from an announced inflation target. This strategy can be divided into two types, namely strict and flexible inflation-targeting, depending on the objectives of the central banks. The difference is that a flexible inflation-targeting central bank is concerned not only by the stability of inflation around the inflation targeting but also by the stability of the real economic variables whereas in strict inflation- targeting strategy, it is only concerned about the stability of inflation rate. Following macro-econometric literature, see for example [96],[18],[155] and the work of Svensson entitled "Optimal Inflation Targeting: Further Developments of Inflation Targeting", an inflation-targeting framework would employ a general loss function L_t expressed by :

$$L_t = E_t \sum_{\tau=0}^{\infty} \frac{1}{2}\delta^\tau \left[\phi_\pi(\pi_{t+\tau} - \pi^T)^2 + \phi_{\tilde{y}}(y_{t+\tau} - y_{t+\tau}^n)^2 + \phi_i(i_{t+\tau} - i_{t+\tau}^n)^2\right], \tag{3.101}$$

where E_t is the expectation conditional on information available in period t and δ denotes the discount factor.

Input variables that occur in this rule are very difficult to measure in practice. As shown by Billi (2011) (see [20] and [21]) and Orphanides (1998) (see [116]), flawed estimates of those variables (i.e., Output gap and natural real interest rate) will lead the central bank to set inappropriate policy rates which can cause unnecessary fluctuations in output gap and inflation. Moreover, as illustrated by Sami et al. 2013 (see[66]) and King et al.2005 (see [82]), this type of rules exacerbate output gap and inflation fluctuations mainly when the central bank alters interest rate in a discontinuous way.
In order to determine a more realistic interest rate, we will consider interest rate adjustment costs and we assume that fluctuations in output gap and inflation are represented by the Gaussian white noises. This brings us to the chapter 4 where we will use stochastic control techniques to control the continuous time version of equation 3.97 and 3.99 which will be formulated in the following.
In model considered above, the frequency-dependent parameters which have to be adjusted by $0 < h < 1$ are α, β and hence $\tilde{\lambda}$.
As we want to consider the limiting case ($h \longrightarrow 0$), these parameters can be transformed as follow:
In h-economy, the time preference discount rate δ becomes $h\delta$ because the household is less discounting future changes in the utility over interval of time $0 < h < 1$. Therefore the discount factor $\beta(h) = \frac{1}{1+h\delta}$.
$\alpha(h) = 1 - h(1-\alpha)$ since in period of length $0 < h < 1$ the probability for resetting the price of the firms will be $h(1-\alpha)$.
By substituting $\beta(h)$ and $\alpha(h)$ in equation (3.98) we get:

$$\begin{aligned}
\tilde{\lambda}(h) &= \frac{\{1-[1-h(1-\alpha)]\}\left\{1-\frac{[1-h(1-\alpha)]}{1+h\delta}\right\}}{[1-h(1-\alpha)]}(\eta+\sigma) \\
&= \frac{h(1-\alpha)\,[1+h\delta-1+h(1-\alpha)]}{(1+h\delta)\,[1-h(1-\alpha)]}(\eta+\sigma) \\
&= \frac{h(1-\alpha)\,[h\delta+h(1-\alpha)]}{(1+h\delta)\,[1-h(1-\alpha)]}(\eta+\sigma) \\
&= \frac{h^2(1-\alpha)\,[\delta+1-\alpha]}{(1+h\delta)\,[1-h(1-\alpha)]}(\eta+\sigma).
\end{aligned} \tag{3.102}$$

π_t , i_t and r_t^n are the only variables which must be adjusted by h. In h-economy π_t will be $h\pi_t$ (i.e inflation rate is given by $\pi_t = \frac{p_t - p_{t-h}}{h}$), i_t will change to hi_t and r_t^n will fall to hr_t^n.
Output gap $\tilde{y}_t$ which is measured by $log\left(\frac{GDP_a}{GDP_p}\right)$ (where GDP_a is actual output and GDP_p symbolizes potential output) is obviously the log of ratio of two flow

magnitudes. Consequently it will not be adjusted by h since it is a contemporaneous adjustment rate.
When we substitute $\beta(h)$ and $\tilde{\lambda}(h)$ in equation 3.97 we get a h-economy NKPC which is expressed by the following equation:

$$\pi_t^h = h\pi_t = p_t - p_{t-h} = \frac{1}{1+h\delta}E_t(p_{t+h} - p_t) + \frac{h^2(1-\alpha)\left[\delta + 1 - \alpha\right]}{(1+h\delta)\left[1 - h(1-\alpha)\right]}(\eta+\sigma)\tilde{y}_t \tag{3.103}$$

$$\Updownarrow$$

$$\begin{aligned}\pi_t &= \frac{1}{1+h\delta}E_t\left(\frac{p_{t+h} - p_t}{h}\right) + \frac{h^2(1-\alpha)\left[\delta + 1 - \alpha\right]}{h(1+h\delta)\left[1 - h(1-\alpha)\right]}(\eta+\sigma)\tilde{y}_t \\ &= \frac{1}{1+h\delta}E_t(\pi_{t+h}) + \frac{h(1-\alpha)\left[\delta + 1 - \alpha\right]}{(1+h\delta)\left[1 - h(1-\alpha)\right]}(\eta+\sigma)\tilde{y}_t.\end{aligned} \tag{3.104}$$

By definition (in the context of rational expectation),

$$E_t\pi_{t+h} = \pi_{t+h} + \epsilon_{t+h}, \tag{3.105}$$

where $E_t\epsilon_{t+h} = 0$ and ϵ_{t+h} is the error term of rational expectation.
In this work we will consider the error term ϵ_{t+h} as a zero mean Gaussian white noise with a finite and constant variance σ_π^2 which is customary modeled by increments of the Brownian motions on infinitesimal intervals (see for example [38] and [141]). Therefore the term ϵ_{t+h} can be represented by the following equation:

$$\epsilon_{t+h} = \sigma_\pi\left(B_{t+h} - B_t\right), \ \textit{for a very small } h \ (i.e., h \longrightarrow 0^+), \tag{3.106}$$

where B_t is a standard Brownian motion.
By substituting equation 3.105 and 3.106 in 3.104 we obtain:

$$\pi_t = \frac{1}{1+h\delta}\pi_{t+h} + \frac{\sigma_\pi}{1+h\delta}\left(B_{t+h} - B_t\right) + \frac{h(1-\alpha)\left[\delta + 1 - \alpha\right]}{(1+h\delta)\left[1 - h(1-\alpha)\right]}(\eta+\sigma)\tilde{y}_t. \tag{3.107}$$

The equation 3.107 can be rearranged algebraically into the following:

$$\pi_t(1 - \frac{1}{1+h\delta}) = \frac{1}{1+h\delta}\left(\pi_{t+h} - \pi_t\right) + \frac{\sigma_\pi}{1+h\delta}\left(B_{t+h} - B_t\right) + \frac{h(1-\alpha)\left[\delta + 1 - \alpha\right]}{(1+h\delta)\left[1 - h(1-\alpha)\right]}(\eta+\sigma)\tilde{y}_t \tag{3.108}$$

$$\Updownarrow$$

$$\frac{h\delta\pi_t}{1+h\delta} = \frac{1}{1+h\delta}\left(\pi_{t+h} - \pi_t\right) + \frac{\sigma_\pi}{1+h\delta}\left(B_{t+h} - B_t\right) + \frac{h(1-\alpha)\left[\delta + 1 - \alpha\right]}{(1+h\delta)\left[1 - h(1-\alpha)\right]}(\eta+\sigma)\tilde{y}_t \tag{3.109}$$

$$\Updownarrow$$

$$h\delta\pi_t = (\pi_{t+h} - \pi_t) + \sigma_\pi (B_{t+h} - B_t) + h\frac{(1-\alpha)[\delta+1-\alpha]}{[1-h(1-\alpha)]}(\eta+\sigma)\tilde{y}_t \tag{3.110}$$

$$\Updownarrow$$

$$(\pi_{t+h} - \pi_t) = h\left\{\delta\pi_t - \frac{(1-\alpha)[\delta+1-\alpha]}{[1-h(1-\alpha)]}(\eta+\sigma)\tilde{y}_t\right\} - \sigma_\pi (B_{t+h} - B_t). \tag{3.111}$$

In h-economy equation (3.99) becomes

$$\tilde{y}_t = E_t\tilde{y}_{t+h} - \frac{1}{\sigma}(hi_t - hE_t\pi_{t+h} - hr_t^n). \tag{3.112}$$

And equation (3.100) will be

$$hi_t = hr_t^n + h\pi^T + hb(\pi_t - \pi^T) + ha\tilde{y}_t. \tag{3.113}$$

Combining equation (3.112) with equation (3.113) yields

$$\tilde{y}_t = E_t\tilde{y}_{t+h} - \frac{h}{\sigma}\left(\pi^T + a(\tilde{y}_t) + b(\pi_t - \pi^T) - E_t\pi_{t+h}\right). \tag{3.114}$$

Knowing that $\pi_t - E_t\pi_{t+h} = h\frac{(1-\alpha)(\delta+1-\alpha)(\eta+\sigma)}{[1-h(1-\alpha)]}\tilde{y}_t - h\delta\pi_t$ from the equation (3.104) and substituting this value in equation (3.114) we have

$$\tilde{y}_t = E_t\tilde{y}_{t+h} - \frac{h}{\sigma}\left\{(b-1)\pi_t + (1-b)\pi^T + a\tilde{y}_t + \frac{h(1-\alpha)(\delta+1-\alpha)(\eta+\sigma)}{[1-h(1-\alpha)]}\tilde{y}_t - h\delta\pi_t\right\} \tag{3.115}$$

$$\Updownarrow$$

$$\tilde{y}_t = E_t\tilde{y}_{t+h} - h\left\{\frac{(b-1)}{\sigma}\pi_t + \frac{(1-b)}{\sigma}\pi^T + \frac{a}{\sigma}\tilde{y}_t + \frac{h(1-\alpha)(\delta+1-\alpha)(\eta+\sigma)}{\sigma[1-h(1-\alpha)]}\tilde{y}_t - \frac{h}{\sigma}\delta\pi_t\right\}. \tag{3.116}$$

Assuming rational expectations as before and substituting equation (3.106) for $\tilde{y}$ in equation (3.116), it results in the following stochastic difference equation

$$\tilde{y}_{t+h} - \tilde{y}_t = h\left\{\frac{(b-1)}{\sigma}\pi_t + \frac{(1-b)}{\sigma}\pi^T + \frac{a}{\sigma}\tilde{y}_t + \frac{h(1-\alpha)(\delta+1-\alpha)(\eta+\sigma)}{\sigma[1-h(1-\alpha)]}\tilde{y}_t - \frac{h}{\sigma}\delta\pi_t\right\} - \sigma_{\tilde{y}}(\bar{B}_{t+h} - \bar{B}_t). \tag{3.117}$$

In the field of applied mathematics, the continuous-time interpretation of stochastic difference equations (3.111) and (3.117) for $h \longrightarrow 0$ are the following stochastic differential equations[82] :

$$d\pi(t) = (\delta\pi(t) - (1-\alpha)(\delta + 1 - \alpha)(\eta + \sigma)\tilde{y}(t))\, dt - \sigma_\pi dB(t) \tag{3.118}$$

and

$$d\tilde{y}(t) = \left(\frac{(1-b)}{\sigma}\pi^T + \frac{(b-1)}{\sigma}\pi(t) + \frac{a}{\sigma}\tilde{y}(t)\right) dt - \sigma_{\tilde{y}} d\bar{B}(t), \tag{3.119}$$

where $b = \frac{\phi_\pi \tilde{\lambda}}{\sigma\phi_i}$, $a = \frac{\phi_{\tilde{y}}}{\sigma\phi_i}$, $\tilde{\lambda} = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}(\eta + \sigma)$, $B(t)$ and $\bar{B}(t)$ are standard correlated Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$ with correlation coeffiecient $\hat{\rho} \in (-1, 1)$,
$\sigma_{\tilde{y}}$ and σ_π denote standard deviations of output gap and inflation respectively.

For simplicity, we transform the correlated Brownian motions $B(t)$ and $\bar{B}(t)$ as follows [83]:

$$B(t) = B^\pi(t) \tag{3.120}$$

and

$$\bar{B}(t) = \hat{\rho}B^\pi(t) + \sqrt{1-\hat{\rho}^2}B^{\tilde{y}}(t), \tag{3.121}$$

where $B^\pi(t)$ and $B^{\tilde{y}}(t)$ are two independent standard Brownian motions which are also defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$.
Substituting equation (3.120) and (3.121) in equation (3.118) and (3.119) respectively we get the following system of equations:

$$dY(t) = (q + AY(t))\, dt + Cd\hat{B}(t), \tag{3.122}$$

where $dY(t) = \begin{pmatrix} d\pi(t) \\ d\tilde{y}(t) \end{pmatrix}$, $A = \begin{pmatrix} \delta & -(1-\alpha)(\delta+1-\alpha)(\eta+\sigma) \\ \frac{(b-1)}{\sigma} & \frac{a}{\sigma} \end{pmatrix}$, $q = \begin{pmatrix} 0 \\ \frac{(1-b)}{\sigma}\pi^T \end{pmatrix}$,

$Y(t) = \begin{pmatrix} \pi(t) \\ \tilde{y}(t) \end{pmatrix}$, $C = \begin{pmatrix} -\sigma_\pi & 0 \\ -\sigma_{\tilde{y}}\hat{\rho} & -\sigma_{\tilde{y}}\sqrt{1-\hat{\rho}^2} \end{pmatrix}$ and $d\hat{B}(t) = \begin{pmatrix} dB^\pi(t) \\ dB^{\tilde{y}}(t) \end{pmatrix}$

[82] For discussions on these convergences see for example [126] and its references, [122], [8] and [29].
[83] For the proof see [150].

Proposition 3.2.2.1: *Assume that the matrix A has 2 linear independent eigenvectors P_1 and P_2 which correspond to two distinct real eigenvalues $\hat{\lambda}_1$ and $\hat{\lambda}_2$ respectively. Then the equation(3.122) has an explicit analytic and unique solution which is expressed by the following equation:*

$$Y(t) = e^{tA}\left(Y(0) + \int_0^t e^{-sA} q ds + \int_0^t e^{-sA} C d\hat{B}(s)\right) \tag{3.123}$$

$$\Updownarrow$$

$$Y(t) = e^{tA}Y(0) + (e^{tA} - I)A^{-1}q + e^{tA}\int_0^t e^{-sA} C d\hat{B}(s), \tag{3.124}$$

where $e^{tA} = P \begin{pmatrix} e^{t\hat{\lambda}_1} & 0 \\ 0 & e^{t\hat{\lambda}_2} \end{pmatrix} P^{-1}$ *(P is a matrix of 2 column vectors P_1 and P_2) is solution of the homogeneous part of equation(3.122).*

Proof: Define $X(t) = Y(0) + \int_0^t e^{-sA} q ds + \int_0^t e^{-sA} C d\hat{B}(t)$,

the following stochastic differential equation holds:

$dX(t) = e^{-tA}\left(qdt + Cd\hat{B}(t)\right)$.

Applying Ito's formula for $Y(t) = u\left(t, X(t)\right) = e^{tA}X(t)$,

$$\begin{aligned}
dY(t) &= \left[u_t\left(t, X(t)\right) + u_X\left(t, X(t)\right)e^{-tA}q + \frac{1}{2}u_{XX}\left(t, X(t)\right)e^{-tA}Ce^{-tA}C\right]dt + \\
&\quad u_X\left(t, X(t)\right)e^{-tA}Cd\hat{B}(t) \\
&= \left[Ae^{tA}X(t) + e^{tA}e^{-tA}q + 0\right]dt + e^{tA}e^{-tA}Cd\hat{B}(t) \\
&= \left[AY(t) + q\right]dt + Cd\hat{B}(t).
\end{aligned}$$

The uniqueness of this solution follows from the theorem 2.1.1 and the fact that the following verifications indicate that the conditions a)-c) are fulfilled.

a)

$$\begin{aligned}\|b(t,Y)\| + \|\sigma(t,Y)\| &= \|q + AY\| + \|C\| \\ &\leq \|q\| + \|C\| + \|A\|\|Y\| = F_1 + F_2\|Y\| \\ &\leq max\,(F_1, F_2)\,(1 + \|Y\|) = F(1 + \|Y\|),\end{aligned}$$

where $F_1 = \|q\| + \|C\|$ and $F_2 = \|A\|$ are constants.

b)

$$\begin{aligned}\|b(t,Y) - b(t,X)\| + \|\sigma(t,Y) - \sigma(t,X)\| &= \|q + AY - q - AX\| + \|C - C\| \\ &= \|AY - AX\| = \|A(Y - X)\| \\ &\leq \|A\|\|Y - X\| = D\|Y - X\|,\end{aligned}$$

Where D is a constant.

c) $\xi = \begin{pmatrix} \pi(0) \\ \tilde{y}(0) \end{pmatrix}$ is independent of $\hat{B}(t)$ and it is most reasonable to assume that $E\left[\|\xi\|^2\right] < \infty$. $\square$

Explicit expression for the exponential of the matrix (tA)

Assumption for the matrix A ensures that the matrix A is diagonalizable. From the definition of e^A and the fact that A is diagonalizable, we have:

$$e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!}(tA)^k = \sum_{k=0}^{\infty} \frac{1}{k!}(PtDP^{-1})^k = P\left(\sum_{k=0}^{\infty} \frac{1}{k!}(tD)^k\right)P^{-1} = Pe^{tD}P^{-1}, \tag{3.125}$$

where $e^{tD} = tI + tD + \frac{1}{2!}(tD)^2 + \frac{1}{3!}(tD)^3 \ldots = diag(e^{t\hat{\lambda}_1}, e^{t\hat{\lambda}_2})$ and $P = (P_1 \vdots P_2)$ is a matrix of 2 column eigenvectors P_1 and P_2 of the matrix A.

Eigenvalues of matrix A

The characteristic equation of A is:

$$(\delta - \hat{\lambda})(\tfrac{\phi_{\tilde{y}}}{\sigma^2\phi_i} - \hat{\lambda}) + \tfrac{(b-1)}{\sigma}\underbrace{(1-\alpha)(\delta+1-\alpha)(\eta+\sigma)}_{:=\eta_1} = 0$$

$$\Updownarrow$$

$$\hat{\lambda}^2 - \hat{\lambda}\underbrace{\left(\delta + \frac{\phi_{\tilde{y}}}{\sigma^2\phi_i}\right)}_{:=\Gamma_1} + \underbrace{\frac{(b-1)}{\sigma}\eta_1 + \delta\frac{\phi_{\tilde{y}}}{\sigma^2\phi_i}}_{\Gamma_2} = 0.$$

This implies that :

Eingenvalue $\hat{\lambda}_1 = \frac{\Gamma_1 + \sqrt{\Gamma_1^2 - 4\Gamma_2}}{2}$ and

Eingenvalue $\hat{\lambda}_2 = \frac{\Gamma_1 - \sqrt{\Gamma_1^2 - 4\Gamma_2}}{2}$ where $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are two distinct real eigenvalues of A from the assumption for the matrix A (i.e. $\Gamma_1^2 - 4\Gamma_2 > 0$).

Eigenvectors of matrix A

Let $P_1 = \begin{pmatrix} P_{11} \\ P_{21} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ be 2 vectors corresponding to $\hat{\lambda}_1$ and $\hat{\lambda}_2$ respectively. Then , according to the definition of eigenvector,

$\left(A - \hat{\lambda}_1 I\right) P_1 = 0$ and $\left(A - \hat{\lambda}_2 I\right) P_2 = 0$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. These mean that:

$$\left[\begin{pmatrix} \delta & -\eta_1 \\ \frac{(b-1)}{\sigma} & \frac{\phi_{\tilde{y}}}{\sigma^2\phi_i} \end{pmatrix} - \begin{pmatrix} \hat{\lambda}_1 & 0 \\ 0 & \hat{\lambda}_1 \end{pmatrix}\right]\begin{pmatrix} P_{11} \\ P_{21} \end{pmatrix} = 0 \Longrightarrow \begin{cases} (\delta - \hat{\lambda}_1)P_{11} - \eta_1 P_{21} = 0 \\ \frac{(b-1)}{\sigma}P_{11} + \left(\frac{\phi_{\tilde{y}}}{\sigma^2\phi_i} - \hat{\lambda}_1\right) P_{21} = 0 \end{cases},$$

and

$$\left[\begin{pmatrix} \delta & -\eta_1 \\ \frac{(b-1)}{\sigma} & \frac{\phi_{\tilde{y}}}{\sigma^2\phi_i} \end{pmatrix} - \begin{pmatrix} \hat{\lambda}_2 & 0 \\ 0 & \hat{\lambda}_2 \end{pmatrix}\right]\begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix} = 0 \Longrightarrow \begin{cases} (\delta - \hat{\lambda}_2)P_{12} - \eta_1 P_{22} = 0 \\ \frac{(b-1)}{\sigma}P_{12} + \left(\frac{\phi_{\tilde{y}}}{\sigma^2\phi_i} - \hat{\lambda}_2\right) P_{22} = 0 \end{cases}.$$

Choosing $P_{21} = 1$ and $P_{22} = 1$ we get two eigenvectors:

$$P_1 = \begin{pmatrix} \frac{\eta_1}{\left(\delta-\hat{\lambda}_1\right)} \\ \\ 1 \end{pmatrix}$$

and

$$P_2 = \begin{pmatrix} \frac{\eta_1}{\left(\delta-\hat{\lambda}_2\right)} \\ \\ 1 \end{pmatrix}$$ of matrix A, which are linear independent from the assumption for the matrix A.

Using the formula for the inverse of 2×2 matrix we have that,

for $P = \begin{pmatrix} \frac{\eta_1}{\left(\delta-\hat{\lambda}_1\right)} & \frac{\eta_1}{\left(\delta-\hat{\lambda}_2\right)} \\ \\ 1 & 1 \end{pmatrix}$, inverse of matrix P is $P^{-1} = \begin{pmatrix} \frac{(\delta-\hat{\lambda}_1)(\delta-\hat{\lambda}_2)}{\eta_1(\hat{\lambda}_1-\hat{\lambda}_2)} & -\frac{(\delta-\hat{\lambda}_1)}{(\hat{\lambda}_1-\hat{\lambda}_2)} \\ \\ -\frac{(\delta-\hat{\lambda}_1)(\delta-\hat{\lambda}_2)}{\eta_1(\hat{\lambda}_1-\hat{\lambda}_2)} & \frac{(\delta-\hat{\lambda}_2)}{(\hat{\lambda}_1-\hat{\lambda}_2)} \end{pmatrix}$.

From the equation (3.125) we have:

$$\begin{aligned} e^{tA} &= \begin{pmatrix} \frac{\eta_1}{\left(\delta-\hat{\lambda}_1\right)} & \frac{\eta_1}{\left(\delta-\hat{\lambda}_2\right)} \\ \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{t\hat{\lambda}_1} & 0 \\ \\ 0 & e^{t\hat{\lambda}_2} \end{pmatrix} \begin{pmatrix} \frac{(\delta-\hat{\lambda}_1)(\delta-\hat{\lambda}_2)}{\eta_1(\hat{\lambda}_1-\hat{\lambda}_2)} & -\frac{(\delta-\hat{\lambda}_1)}{(\hat{\lambda}_1-\hat{\lambda}_2)} \\ \\ -\frac{(\delta-\hat{\lambda}_1)(\delta-\hat{\lambda}_2)}{\eta_1(\hat{\lambda}_1-\hat{\lambda}_2)} & \frac{(\delta-\hat{\lambda}_2)}{(\hat{\lambda}_1-\hat{\lambda}_2)} \end{pmatrix} \\ &= \begin{pmatrix} \frac{(\delta-\hat{\lambda}_2)e^{t\hat{\lambda}_1}-(\delta-\hat{\lambda}_1)e^{t\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2} & \frac{-\eta_1 e^{t\hat{\lambda}_1}+\eta_1 e^{t\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2} \\ \\ \frac{(\delta-\hat{\lambda}_1)(\delta-\hat{\lambda}_2)}{\eta_1(\hat{\lambda}_1-\hat{\lambda}_2)}(e^{t\hat{\lambda}_1}-e^{t\hat{\lambda}_2}) & \frac{-(\delta-\hat{\lambda}_1)e^{t\hat{\lambda}_1}+(\delta-\hat{\lambda}_2)e^{t\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2} \end{pmatrix}. \end{aligned} \tag{3.126}$$

Substituting equation (3.126) into equation (3.124) we have:

$$\begin{pmatrix} \pi(t) \\ \tilde{y}(t) \end{pmatrix} = \underbrace{e^{tA}Y(0)}_{:=i} + \underbrace{\left(e^{tA}-I\right)A^{-1}q}_{:=ii} + \underbrace{\int_0^t e^{(t-s)A}Cd\hat{B}(s)}_{:=iii}, \tag{3.127}$$

where

$$i = \begin{pmatrix} \left(\frac{(\delta-\hat{\lambda}_2)e^{t\hat{\lambda}_1}-(\delta-\hat{\lambda}_1)e^{t\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2}\right)\pi(0) + \left(\frac{-\eta_1 e^{t\hat{\lambda}_1}+\eta_1 e^{t\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2}\right)\tilde{y}(0) \\ \\ \left[\frac{(\delta-\hat{\lambda}_1)(\delta-\hat{\lambda}_2)}{\eta_1(\hat{\lambda}_1-\hat{\lambda}_2)}(e^{t\hat{\lambda}_1}-e^{t\hat{\lambda}_2})\right]\pi(0) + \left[\frac{-(\delta-\hat{\lambda}_1)e^{t\hat{\lambda}_1}+(\delta-\hat{\lambda}_2)e^{t\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2}\right]\tilde{y}(0) \end{pmatrix},$$

$$ii = \begin{pmatrix} \frac{(\delta-\hat{\lambda}_2)e^{t\hat{\lambda}_1}-(\delta-\hat{\lambda}_1)e^{t\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2} - 1 & \frac{-\eta_1 e^{t\hat{\lambda}_1}+\eta_1 e^{t\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2} \\ \frac{(\delta-\hat{\lambda}_1)(\delta-\hat{\lambda}_2)}{\eta_1(\hat{\lambda}_1-\hat{\lambda}_2)}(e^{t\hat{\lambda}_1} - e^{t\hat{\lambda}_2}) & \frac{-(\delta-\hat{\lambda}_1)e^{t\hat{\lambda}_1}+(\delta-\hat{\lambda}_2)e^{t\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2} - 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\phi_{\tilde{y}}}{\left[(\phi_{\tilde{y}}\delta+\eta_1\sigma\phi_i)(b-1)\right]} & \frac{\eta_1\sigma^2\phi_i}{\left[(\phi_{\tilde{y}}\delta+\eta_1\sigma\phi_i)(b-1)\right]} \\ -\frac{\sigma\phi_i(b-1)}{\left[(\phi_{\tilde{y}}\delta+\eta_1\sigma\phi_i)(b-1)\right]} & \frac{\sigma^2\phi_i\delta}{\left[(\phi_{\tilde{y}}\delta+\eta_1\sigma\phi_i)(b-1)\right]} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{(1-b)}{\sigma}\pi^T \end{pmatrix}$$

$$= \begin{pmatrix} \left[\left(\frac{(\delta-\hat{\lambda}_2)e^{t\hat{\lambda}_1}-(\delta-\hat{\lambda}_1)e^{t\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2} - 1\right)\left(\frac{\eta_1\sigma^2\phi_i}{\left[(\phi_{\tilde{y}}\delta+\eta_1\sigma\phi_i)(b-1)\right]}\right) + \right. \\ \left. \left(\frac{-\eta_1 e^{t\hat{\lambda}_1}+\eta_1 e^{t\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2}\right)\left(\frac{\sigma^2\phi_i\delta}{\left[(\phi_{\tilde{y}}\delta+\eta_1\sigma\phi_i)(b-1)\right]}\right)\right]\frac{(1-b)}{\sigma}\pi^T \\ \\ \left[\frac{(\delta-\hat{\lambda}_1)(\delta-\hat{\lambda}_2)}{\eta_1(\hat{\lambda}_1-\hat{\lambda}_2)}(e^{t\hat{\lambda}_1} - e^{t\hat{\lambda}_2})\frac{\eta_1\sigma^2\phi_i}{\left[(\phi_{\tilde{y}}\delta+\eta_1\sigma\phi_i)(b-1)\right]} + \right. \\ \left. \left(\frac{-(\delta-\hat{\lambda}_1)e^{t\hat{\lambda}_1}+(\delta-\hat{\lambda}_2)e^{t\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2} - 1\right)\frac{\sigma^2\phi_i\delta}{\left[(\phi_{\tilde{y}}\delta+\eta_1\sigma\phi_i)(b-1)\right]}\right]\frac{(1-b)}{\sigma}\pi^T \end{pmatrix}$$

and

$$iii = \int_0^t \begin{pmatrix} \frac{(\delta-\hat{\lambda}_2)e^{(t-s)\hat{\lambda}_1}-(\delta-\hat{\lambda}_1)e^{(t-s)\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2} & \frac{-\eta_1 e^{(t-s)\hat{\lambda}_1}+\eta_1 e^{(t-s)\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2} \\ \frac{(\delta-\hat{\lambda}_1)(\delta-\hat{\lambda}_2)}{\eta_1(\hat{\lambda}_1-\hat{\lambda}_2)}(e^{(t-s)\hat{\lambda}_1} - e^{(t-s)\hat{\lambda}_2}) & \frac{-(\delta-\hat{\lambda}_1)e^{(t-s)\hat{\lambda}_1}+(\delta-\hat{\lambda}_2)e^{(t-s)\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2} \end{pmatrix}$$

$$\begin{pmatrix} -\sigma_\pi & 0 \\ -\sigma_{\tilde{y}}\hat{\rho} & -\sigma_{\tilde{y}}\sqrt{1-\hat{\rho}^2} \end{pmatrix} \begin{pmatrix} dB^\pi(s) \\ dB^{\tilde{y}}(s) \end{pmatrix}$$

$$
= \int_0^t \left(\begin{array}{c}
\left(-\sigma_\pi \frac{(\delta-\hat{\lambda}_2)e^{(t-s)\hat{\lambda}_1}-(\delta-\hat{\lambda}_1)e^{(t-s)\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2} - \sigma_{\tilde{y}}\hat{\rho}\frac{-\eta_1 e^{(t-s)\hat{\lambda}_1}+\eta_1 e^{(t-s)\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2}\right) dB^{\pi}(s) \\
+\left(-\sigma_{\tilde{y}}\sqrt{1-\hat{\rho}^2}\left(\frac{-\eta_1 e^{(t-s)\hat{\lambda}_1}+\eta_1 e^{(t-s)\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2}\right)\right) dB^{\tilde{y}}(s) \\
\\
\left(-\sigma_\pi \frac{(\delta-\hat{\lambda}_1)(\delta-\hat{\lambda}_2)}{\eta_1(\hat{\lambda}_1-\hat{\lambda}_2)}(e^{(t-s)\hat{\lambda}_1}-e^{(t-s)\hat{\lambda}_2}) - \sigma_{\tilde{y}}\hat{\rho}\frac{-(\delta-\hat{\lambda}_1)e^{(t-s)\hat{\lambda}_1}+(\delta-\hat{\lambda}_2)e^{(t-s)\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2}\right) dB^{\pi}(s) \\
+\left(-\sigma_{\tilde{y}}\sqrt{1-\hat{\rho}^2}\left(\frac{-(\delta-\hat{\lambda}_1)e^{(t-s)\hat{\lambda}_1}+(\delta-\hat{\lambda}_2)e^{(t-s)\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2}\right)\right) dB^{\tilde{y}}(s)
\end{array} \right)
$$

Then i + ii + iii give:

$$
\begin{aligned}
\pi(t) = &\left(\frac{(\delta-\hat{\lambda}_2)e^{t\hat{\lambda}_1}-(\delta-\hat{\lambda}_1)e^{t\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2}\right)\pi(0) + \left(\frac{-\eta_1 e^{t\hat{\lambda}_1}+\eta_1 e^{t\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2}\right)\tilde{y}(0) \\
&+\left[\left(\frac{(\delta-\hat{\lambda}_2)e^{t\hat{\lambda}_1}-(\delta-\hat{\lambda}_1)e^{t\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2}-1\right)\left(\frac{\eta_1\sigma^2\phi_i}{[(\phi_{\tilde{y}}\delta+\eta_1\sigma\phi_i)(b-1)]}\right)\right]\frac{(1-b)}{\sigma}\pi^T \\
&+\left[\left(\frac{-\eta_1 e^{t\hat{\lambda}_1}+\eta_1 e^{t\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2}\right)\left(\frac{\sigma^2\phi_i\delta}{[(\phi_{\tilde{y}}\delta+\eta_1\sigma\phi_i)(b-1)]}\right)\right]\frac{(1-b)}{\sigma}\pi^T \\
&+\int_0^t\left(-\sigma_\pi\frac{(\delta-\hat{\lambda}_2)e^{(t-s)\hat{\lambda}_1}-(\delta-\hat{\lambda}_1)e^{(t-s)\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2}-\sigma_{\tilde{y}}\hat{\rho}\frac{-\eta_1 e^{(t-s)\hat{\lambda}_1}+\eta_1 e^{(t-s)\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2}\right)dB^{\pi}(s) \\
&+\left(-\sigma_{\tilde{y}}\sqrt{1-\hat{\rho}^2}\left(\frac{-\eta_1 e^{(t-s)\hat{\lambda}_1}+\eta_1 e^{(t-s)\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2}\right)\right)dB^{\tilde{y}}(s).
\end{aligned}
\tag{3.128}
$$

$$\tilde{y}(t) = \left[\frac{(\delta-\hat{\lambda}_1)(\delta-\hat{\lambda}_2)}{\eta_1(\hat{\lambda}_1-\hat{\lambda}_2)}(e^{t\hat{\lambda}_1}-e^{t\hat{\lambda}_2})\right]\pi(0) + \left[\frac{-(\delta-\hat{\lambda}_1)e^{t\hat{\lambda}_1}+(\delta-\hat{\lambda}_2)e^{t\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2}\right]\tilde{y}(0)$$

$$+\left[\frac{(\delta-\hat{\lambda}_1)(\delta-\hat{\lambda}_2)}{\eta_1(\hat{\lambda}_1-\hat{\lambda}_2)}(e^{t\hat{\lambda}_1}-e^{t\hat{\lambda}_2})\frac{\eta_1\sigma^2\phi_i}{[(\phi_{\tilde{y}}\delta+\eta_1\sigma\phi_i)(b-1)]}\right]\frac{(1-b)}{\sigma}\pi^T$$

$$+\left[\left(\frac{-(\delta-\hat{\lambda}_1)e^{t\hat{\lambda}_1}+(\delta-\hat{\lambda}_2)e^{t\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2}-1\right)\frac{\sigma^2\phi_i\delta}{[(\phi_{\tilde{y}}\delta+\eta_1\sigma\phi_i)(b-1)]}\right]\frac{(1-b)}{\sigma}\pi^T$$

$$+\int_0^t\Bigg(-\sigma_\pi\frac{(\delta-\hat{\lambda}_1)(\delta-\hat{\lambda}_2)}{\eta_1(\hat{\lambda}_1-\hat{\lambda}_2)}(e^{(t-s)\hat{\lambda}_1}-e^{(t-s)\hat{\lambda}_2})-$$

$$\sigma_{\tilde{y}}\hat{\rho}\frac{-(\delta-\hat{\lambda}_1)e^{(t-s)\hat{\lambda}_1}+(\delta-\hat{\lambda}_2)e^{(t-s)\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2}\Bigg)dB^\pi(s)$$

$$+\left(-\sigma_{\tilde{y}}\sqrt{1-\hat{\rho}^2}\left(\frac{-(\delta-\hat{\lambda}_1)e^{(t-s)\hat{\lambda}_1}+(\delta-\hat{\lambda}_2)e^{(t-s)\hat{\lambda}_2}}{\hat{\lambda}_1-\hat{\lambda}_2}\right)\right)dB^{\tilde{y}}(s). \tag{3.129}$$

Chapter 4

Inflation Controlling

Nowadays many central banks have adopted inflation-targeting strategy to control inflation rate.
In order to achieve their target, inflation-targeting central banks can typically change domestic nominal interest rates or manipulate money supply through insurance or withdrawal of domestic government securities. In doing so, central banks can face some costs arising from these changes, on the one hand, and the losses stemming from deviations of inflation from their target on the other hand. In practice, costs which arise from changes in interest rate stimulate incentive of central banks of not responding immediately to current small deviations from the target. Even in the case where the current loss due to output gap and inflation gap exceeds these costs, they find it not optimal to react rapidly hoping that inflation gap and output gap will revert back towards the target due to other future macroeconomic shocks.
This induces a relatively large range of inaction around the official inflation target and output target for which it holds that interest rate will be maintained constant as long as the actual rate of inflation and output gap are inside of this band. In this case the central bank's problem boils down to an optimal control problem where one has to determine threshold levels for the inflation gap and output gap which will trigger an interest rate manipulation and an optimal strategy (namely find an optimal intervention time and an optimal additional interest rate $\Delta\zeta$) to push back output gap and inflation processes into the inaction region when they hit the borders.
Impulse control method is useful to optimal control problems of this type by assuming that the change in interest rate $\Delta\zeta$ would lead to a direct change in output gap $\Delta\tilde{y} = \gamma_{\tilde{y}}\Delta\zeta$ and in inflation rate $\Delta\pi = \gamma_\pi\Delta\zeta$ for $\gamma_{\tilde{y}}, \gamma_\pi \in \mathbb{R}$.
In the following we will formulate this central bank's problem and we will apply the QVI-approach to solve the special case of this problem where the central bank has to choose an optimal control to minimize the output gaps (i.e., the case in which inflation is considered to be on target) and the choice of parameters $\phi_{\tilde{y}}$ and ϕ_i respect

the equation $\frac{\phi_{\tilde{y}}}{\phi_i} = \delta\sigma^2$.

4.1 Problem Formulation

We assume that the dynamics of inflation rate and output gap processes, in the absence of control, are given by unique strong solutions π_t and $\tilde{y}_t$ of equation 3.122 which are driven by standard Brownian motions B_t^{π} and $B_t^{\tilde{y}}$ defined on a filtered probability space $((\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P}))$.
We consider an inflation-targeting central bank with an interest rate as one monetary tool available and which is allowed to choose intervention times τ_i where it can shift directly the output gap and inflation processes to other values by adjusting interest rate (i.e., change in interest rate $\Delta\zeta$ reads to an instantaneous change in output gap $\Delta\tilde{y} = \gamma_{\tilde{y}}\Delta\zeta$ and in inflation rate $\Delta\pi = \gamma_{\pi}\Delta\zeta$ as it has been assumed above). Additionally we suppose that the cost of raising interest rate and the costs of lowering it are equal.
In the same way as presented in chapter 2, the infinite-horizon optimal control problem consists of minimizing the expected discounted controlling cost and running costs [84] over the set of admissible impulse control strategies $\mathcal{A}$ and it is described by the following equations:

$$\min_{\{(\tau_i,\Delta\zeta_i), i\in\mathbb{N}\}\in\mathcal{A}} E^S_{\pi(0),\tilde{y}(0)} \Bigg\{ \int_0^{\infty} \frac{1}{2} e^{-\delta t} \left[\phi_{\pi} \left(\pi(t) - \pi^T\right)^2 + \phi_{\tilde{y}} \left(\tilde{y}(t)\right)^2 \right] dt + \\ \sum_{i=1}^{\infty} e^{-\delta\tau_i} \left(K + k|\Delta\zeta_i|\right) 1_{\{\tau_i<\infty\}} \Bigg\} \tag{4.1}$$

Subject to:

$$d\pi(t) = \left[\delta\pi(t^-) - (1-\alpha)(1-\alpha+\delta)(\eta+\sigma)\tilde{y}(t^-)\right] dt - \sigma_{\pi} dB^{\pi}(t) - \gamma_{\pi} \sum_{i=1}^{\infty} \Delta\zeta_i 1_{\{\tau_i=t\}} \tag{4.2}$$

and

[84]The running cost considered here is the one period loss function given by the equation 3.101, where inflation-targeting central bank is only concerned about the deviations of inflation rate and output gap from their target.

$$d\tilde{y}(t) = \left[\frac{1-b}{\sigma}\left(\pi^T - \pi(t^-)\right) + \frac{\phi_{\tilde{y}}}{\sigma^2\phi_i}\tilde{y}(t^-)\right]dt - \sigma_{\tilde{y}}\underbrace{\left(\hat{\rho}dB^{\pi}(t) + \sqrt{1-\hat{\rho}^2}dB^{\tilde{y}}(t)\right)}_{d\bar{B}(t)} -$$

$$\gamma_{\tilde{y}}\sum_{i=1}^{\infty}\Delta\zeta_i 1_{\{\tau_i = t\}}, \tag{4.3}$$

where $\Delta\zeta_i = (\gamma_\pi\Delta\zeta_i, \gamma_{\tilde{y}}\Delta\zeta_i)$ denotes the two-dimenional vector of the desired amounts to push the processes back to the continuation region,
$E^S_{\pi(0),\tilde{y}(0)}(.)$ is the expectation when processes $\pi(t)$ and $\tilde{y}(t)$ start with initial value $\pi(0)$ and $\tilde{y}(0)$ respectively, and the strategy S is selected by the controller,
the constant $K \in (0,\infty)$ denotes the fix cost per intervention,
$k \in (0,\infty)$ represents the proportional cost per intervention,
δ is a constant discount rate and hence $e^{-\delta t}$ denotes the discount factor.

4.2 Application of QVI-approach: Optimal Control of the Output Gap Dynamics.

Now we have to determine the optimal impulse control that optimally minimizes the output gaps by solving the following equation:

$$V(\tilde{y}) := \min_{\{(\tau_i,\Delta\tilde{y}_i), i\in\mathbb{N}\}\in\mathcal{A}} E^S_{\tilde{y}(0)}\left\{\frac{1}{2}\int_0^\infty e^{-\delta t}\phi_{\tilde{y}}\left(\tilde{y}(t)\right)^2 dt + \sum_{i=1}^{\infty} e^{-\delta\tau_i}\left(K + k|\Delta\tilde{y}_i|\right)1_{\{\tau_i<\infty\}}\right\} \tag{4.4}$$

Subject to:

$d\tilde{y}(t) = \frac{\phi_{\tilde{y}}}{\sigma^2\phi_i}\tilde{y}(t^-)dt - \sigma_{\tilde{y}}d\bar{B}(t) - \sum_{i=1}^{\infty}\Delta\tilde{y}_i 1_{\{\tau_i=t\}}$.

The following is the strategy that we will use to find an optimal solution:
we propose an optimal impulse control band S characterized by four parameters $-\infty < a < \alpha \leq \beta < b < \infty$ as illustrated in figure 2.2.1.
Next, we use it to construct a solution V^* of the QVI for which S is the QVI-control.
As we know from the chapter 2 that a candidate for value function must meet the continuous and smooth pasting conditions, we will assume that these requirements hold for our solution V^* and we will use them to calculate the unknown parameters (a, α, β and b) and specify our solution V^*.
Finally, we will verify that the hypothesis of the verification theorem are fulfilled for

V^* and its corresponding QVI-control is admissible. If all of these assumptions are satisfied, then we apply the theorem 2.2.1 to conclude that our control strategy S is optimal and the solution V^* obtained is the value function for our problem.

Now, the definition 2.2.4 and remark 2.2.1 indicate that
• for every $\tilde{y}$ in intervention region we have

$$\begin{aligned} V^*(\tilde{y}) &= V^*[\tilde{y} - (\tilde{y} - \alpha)] + K + k|\tilde{y} - \alpha| \\ &= V^*(\alpha) + K + k(\alpha - \tilde{y}) \quad \forall \tilde{y} \in (-\infty, a], \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} V^*(\tilde{y}) &= V^*[\tilde{y} - (\tilde{y} - \beta)] + K + k|\tilde{y} - \beta| \\ &= V^*(\beta) + K + k(\tilde{y} - \beta) \quad \forall \tilde{y} \in [b, \infty). \end{aligned} \tag{4.6}$$

Equation 4.5 - 4.6 together with the continuity assumption for V^* imply that:

$$V^*(a) = V^*(\alpha) + K + k(\alpha - a), \tag{4.7}$$

and

$$V^*(b) = V^*(\beta) + K + k(b - \beta). \tag{4.8}$$

Furthermore, the consideration of equation 4.5 -4.6 and smooth pasting assumption for V^* in $\{a, b\}$ (i.e. the continuity condition for the first derivative of V^* in $\{a, b\}$) yields that:

$$V^{*'}(a) = -k \quad and \quad V^{*'}(b) = k, \tag{4.9}$$

and lastly from the optimality assumption for our impulse control band and knowing from definition 2.2.3 and (2.10) that $\Delta\tilde{y} = b - \beta$ is the value for which the infimum holds for $V^*(\tilde{y} - \Delta\tilde{y}) + K + k|\Delta\tilde{y}|$, the necessary conditions for optimality of actions in $\{a, b\}$ requires that:

$$At\ b: \quad 0 = \frac{\partial}{\partial \Delta}\left(V^*[b + \Delta] + K - k\Delta\right)|_{\Delta = \beta - b} \tag{4.10}$$

$$\Updownarrow$$

$$V^{*'}(\beta) = k. \tag{4.11}$$

The same procedure yields that:

$$At\ a: \quad V^{*'}(\alpha) = -k. \tag{4.12}$$

- for every $\tilde{y} \in (a, b)$,

$$
\begin{aligned}
0 = \mathcal{L}V^*(\tilde{y}) + f(\tilde{y}) &= \frac{1}{2}\left(-\sigma_{\tilde{y}}\right)^2 V^{*''}(\tilde{y}) + \frac{\phi_{\tilde{y}}}{\sigma^2\phi_i}\tilde{y}V^{*'}(\tilde{y}) - \delta V^*(\tilde{y}) + \frac{1}{2}\phi_{\tilde{y}}\tilde{y}^2 \\
&= V^{*''}(\tilde{y}) + \frac{2\phi_{\tilde{y}}}{\sigma_{\tilde{y}}^2\sigma^2\phi_i}\tilde{y}V^{*'}(\tilde{y}) - \frac{2\delta}{\sigma_{\tilde{y}}^2}V^*(\tilde{y}) + \frac{\phi_{\tilde{y}}}{\sigma_{\tilde{y}}^2}\tilde{y}^2
\end{aligned}
\tag{4.13}
$$

$$\Updownarrow$$

$$
V^{*''}(\tilde{y}) + \underbrace{\frac{2\phi_{\tilde{y}}}{\sigma_{\tilde{y}}^2\sigma^2\phi_i}}_{=C_1}\tilde{y}V^{*'}(\tilde{y}) - \underbrace{\frac{2\delta}{\sigma_{\tilde{y}}^2}}_{=C_2}V^*(\tilde{y}) = -\underbrace{\frac{\phi_{\tilde{y}}}{\sigma_{\tilde{y}}^2}}_{=C_3}\tilde{y}^2. \tag{4.14}
$$

Since the choice of parameters $\phi_{\tilde{y}}$ and ϕ_i must satisfy the equation $\frac{\phi_{\tilde{y}}}{\phi_i} = \delta\sigma^2$ (i.e., $C_2 = C_1$) as previously assumed, we deal with the inhomogeneous differential equation

$$
V^{*''}(\tilde{y}) + C_1\tilde{y}V^{*'}(\tilde{y}) - C_1V^*(\tilde{y}) = -C_3\tilde{y}^2. \tag{4.15}
$$

This equation can be solved as follows:

Particular solution $V_p^*(\tilde{y})$ of equation 4.15.

As we have seen in chapter 2, the particular solution of this equation has the form $V_p^*(\tilde{y}) = c + b\tilde{y} + d\tilde{y}^2$. Then $V_p^{*'}(\tilde{y}) = b + 2d\tilde{y}$ and $V_p^{*''}(\tilde{y}) = 2d$.
Substituting these in equation 4.15 we get:
$0 = 2d + C_1\tilde{y}(b + 2d\tilde{y}) - C_1(c + b\tilde{y} + d\tilde{y}^2) + C_3\tilde{y}^2 = 2d - C_1c + \tilde{y}(C_1b - C_1b) + \tilde{y}^2(2C_1d - C_1d + C_3)$.
This implies that:
b can be any number of $\mathbb{R}$(here we choose $b = 0$), $c = -\frac{2C_3}{C_1^2}$ and $d = -\frac{C_3}{C_1}$.
Then the particular solution is given by:

$$
V_p^*(\tilde{y}) = -\frac{2C_3}{C_1^2} - \frac{C_3}{C_1}\tilde{y}^2. \tag{4.16}
$$

Homogeneous solution $V_h^*(\tilde{y})$ of equation 4.15.

$V_{h1}^*(\tilde{y}) = \tilde{y}$.

From the remark 2.1.1 we have:

$$\begin{aligned} V_{h2}^{*}(\tilde{y}) &= V_{h1}^{*}(\tilde{y})\int^{\tilde{y}}\left[\frac{1}{(V_{h1}(t))^{2}}e^{-\frac{C_1}{2}t^{2}}\right]dt \\ &= V_{h1}^{*}(\tilde{y})\int^{\tilde{y}}\frac{\exp\left(-\int^{t}C_1 u du\right)}{\left(V_{h1}^{*}(\tilde{y})\right)^{2}}dt \\ &= \tilde{y}\int\left[\frac{1}{\tilde{y}^{2}}e^{-\frac{C_1}{2}\tilde{y}^{2}}\right]d\tilde{y}. \end{aligned} \tag{4.17}$$

Using integration by part method which says $\int UV'dx = UV - \int U'Vdx$, and by assuming that $V' = \frac{1}{\tilde{y}^2}$ and $U = e^{-\frac{C_1}{2}\tilde{y}^2}$ we have :

$$\begin{aligned} V_{h2}^{*}(\tilde{y}) = \tilde{y}\int\left[\frac{1}{\tilde{y}^{2}}e^{-\frac{C_1}{2}\tilde{y}^{2}}\right]d\tilde{y} &= -\frac{\tilde{y}}{\tilde{y}}e^{-\frac{C_1}{2}\tilde{y}^{2}} - \tilde{y}C_1\int e^{-\frac{C_1}{2}\tilde{y}^{2}}d\tilde{y} \\ &= -e^{-\frac{C_1}{2}\tilde{y}^{2}} - \tilde{y}C_1\int e^{-\frac{C_1}{2}\tilde{y}^{2}}d\tilde{y}, \end{aligned} \tag{4.18}$$

and hence the homogeneous solution $V_h^*(\tilde{y})$ is

$$V_h^{*}(\tilde{y}) = A\tilde{y} + B\left\{-e^{-\frac{C_1}{2}\tilde{y}^{2}} - \tilde{y}C_1\int e^{-\frac{C_1}{2}\tilde{y}^{2}}d\tilde{y}\right\}. \tag{4.19}$$

From the proposition 2.1.2, the general solution $V_g^*(\tilde{y})$ of the equation 4.15 has the form:

$$\begin{aligned} V_g^{*}(\tilde{y}) &= A\tilde{y} + B\left\{-e^{-\frac{C_1}{2}\tilde{y}^{2}} - \tilde{y}C_1\int e^{-\frac{C_1}{2}\tilde{y}^{2}}d\tilde{y}\right\} - \frac{2C_3}{C_1^{2}} - \frac{C_3}{C_1}\tilde{y}^{2} \\ &= A\tilde{y} - Be^{-\frac{C_1}{2}\tilde{y}^{2}} - BC_1\tilde{y}\left[\frac{\sqrt{\pi}}{2\sqrt{\frac{C_1}{2}}}\operatorname{erf}\left(\tilde{y}\sqrt{\frac{C_1}{2}}\right)\right] - \frac{2C_3}{C_1^{2}} - \frac{C_3}{C_1}\tilde{y}^{2}. \end{aligned} \tag{4.20}$$

And

$$V_g^{*\prime}(\tilde{y}) = A - B\left\{-C_1\tilde{y}e^{-\frac{C_1}{2}\tilde{y}^{2}} + C_1\left[\int e^{-\frac{C_1}{2}\tilde{y}^{2}}d\tilde{y} + \tilde{y}\frac{d}{d\tilde{y}}\left(\int e^{-\frac{C_1}{2}\tilde{y}^{2}}d\tilde{y}\right)\right]\right\} - \frac{2C_3}{C_1}\tilde{y}. \tag{4.21}$$

Applying the second fundamental theorem of calculus , we have

$$\begin{aligned} V_g^{*'}(\tilde{y}) &= A - B\left\{-C_1\tilde{y}e^{-\frac{C_1}{2}\tilde{y}^2} + C_1\left[\int e^{-\frac{C_1}{2}\tilde{y}^2}d\tilde{y} + \tilde{y}e^{-\frac{C_1}{2}\tilde{y}^2}\right]\right\} - \frac{2C_3}{C_1}\tilde{y} \\ &= A - BC_1\left[\frac{\sqrt{\pi}}{2\sqrt{\frac{C_1}{2}}}\operatorname{erf}\left(\tilde{y}\sqrt{\frac{C_1}{2}}\right)\right] - \frac{2C_3}{C_1}\tilde{y}. \end{aligned} \tag{4.22}$$

Finally our function $V^*(\tilde{y})$ for all $\tilde{y}$ becomes

$$V^*(\tilde{y}) = \begin{cases} V^*(\alpha) + K + k(\alpha - \tilde{y}) & \text{for } \tilde{y} \leq a, \\ A\tilde{y} - Be^{-\frac{C_1}{2}\tilde{y}^2} - BC_1\tilde{y}\left[\frac{\sqrt{\pi}}{2\sqrt{\frac{C_1}{2}}}\operatorname{erf}\left(\tilde{y}\sqrt{\frac{C_1}{2}}\right)\right] - \frac{2C_3}{C_1^2} - \frac{C_3}{C_1}\tilde{y}^2 & \text{for } a < \tilde{y} < b, \\ V^*(\beta) + K + k(\tilde{y} - \beta) & \text{for } \tilde{y} \geq b. \end{cases} \tag{4.23}$$

And

The parameters A,B,a, α, β and b such that $-\infty < a < \alpha \leq \beta < b < \infty$ can be determined by solving the system of equations (4.7), (4.8), (4.9), (4.11) and (4.12) which imply that

$$\begin{aligned} &Aa - Be^{-\frac{C_1}{2}a^2} - BC_1a\left[\frac{\sqrt{\pi}}{2\sqrt{\frac{C_1}{2}}}\operatorname{erf}\left(a\sqrt{\frac{C_1}{2}}\right)\right] - \frac{C_3}{C_1}a^2 = \\ &A\alpha - Be^{-\frac{C_1}{2}\alpha^2} - BC_1\alpha\left[\frac{\sqrt{\pi}}{2\sqrt{\frac{C_1}{2}}}\operatorname{erf}\left(\alpha\sqrt{\frac{C_1}{2}}\right)\right] - \frac{C_3}{C_1}\alpha^2 + K + k(\alpha - a), \end{aligned} \tag{4.24}$$

$$\begin{aligned} &Ab - Be^{-\frac{C_1}{2}b^2} - BC_1b\left[\frac{\sqrt{\pi}}{2\sqrt{\frac{C_1}{2}}}\operatorname{erf}\left(b\sqrt{\frac{C_1}{2}}\right)\right] - \frac{C_3}{C_1}b^2 = \\ &A\beta - Be^{-\frac{C_1}{2}\beta^2} - BC_1\beta\left[\frac{\sqrt{\pi}}{2\sqrt{\frac{C_1}{2}}}\operatorname{erf}\left(\beta\sqrt{\frac{C_1}{2}}\right)\right] - \frac{C_3}{C_1}\beta^2 + K + k(b - \beta), \end{aligned} \tag{4.25}$$

$$A - BC_1\left[\frac{\sqrt{\pi}}{2\sqrt{\frac{C_1}{2}}}\operatorname{erf}\left(a\sqrt{\frac{C_1}{2}}\right)\right] - \frac{2C_3}{C_1}a = -k, \tag{4.26}$$

$$A - BC_1 \left[\frac{\sqrt{\pi}}{2\sqrt{\frac{C_1}{2}}} \operatorname{erf}\left(\alpha \sqrt{\frac{C_1}{2}} \right) \right] - \frac{2C_3}{C_1}\alpha = -k, \tag{4.27}$$

$$A - BC_1 \left[\frac{\sqrt{\pi}}{2\sqrt{\frac{C_1}{2}}} \operatorname{erf}\left(\beta \sqrt{\frac{C_1}{2}} \right) \right] - \frac{2C_3}{C_1}\beta = k, \tag{4.28}$$

and

$$A - BC_1 \left[\frac{\sqrt{\pi}}{2\sqrt{\frac{C_1}{2}}} \operatorname{erf}\left(b \sqrt{\frac{C_1}{2}} \right) \right] - \frac{2C_3}{C_1}b = k. \tag{4.29}$$

The following proposition verfies that $V^*(\tilde{y})$ is equal to $V(\tilde{y})$ presented in equation (4.4) and that our proposed impulse control strategy S is optimal.

Proposition 4.1.1. *Let A,B,a, α, β and b, with $-\infty < a < \alpha \leq \beta < b < \infty$ be a solution of the system of equations (4.24) - (4.29). Consider the function V^* expressed by the equation (4.23).*
If for all $\tilde{y} \in [b, \infty)$,

$$-\frac{1}{2}BC_1 e^{-\frac{C_1}{2}\beta^2} + \frac{\sigma_{\tilde{y}}^2 C_3}{C_1} + 2\delta\beta k + \frac{1}{2}\phi_{\tilde{y}}\beta^2 - \delta K + \frac{1}{2}\phi_{\tilde{y}}\tilde{y}^2 > 0, \tag{4.30}$$

and for all $\tilde{y} \in (-\infty, a]$

$$-\frac{1}{2}BC_1 e^{-\frac{C_1}{2}\alpha^2} + \frac{\sigma_{\tilde{y}}^2 C_3}{C_1} - 2\delta\alpha k + \frac{1}{2}\phi_{\tilde{y}}\alpha^2 - \delta K + \frac{1}{2}\phi_{\tilde{y}}\tilde{y}^2 > 0, \tag{4.31}$$

then the function V^* coincides with the value function defined by the equation (4.4), and the QVI-control associated with V^* is optimal.

Proof. We have to check that the QVI-control corresponding to V^* is admissible, to verify that the growth conditions (2.19) and (2.20) are satisfied and to show that the function V^* satisfies the QVI.
From the definition of the proposed impulse control (see figure 2.2.1), we observe that

$$\lim_{n\to\infty} \tau_n = \infty \quad \textit{almost surely.} \tag{4.32}$$

From the assumption that the drift and the volatility of our process satisfy the global Lipschitz conditions, the discount factor ensures that $\int_0^\infty e^{-\delta t} \underbrace{\frac{1}{2}\phi_{\tilde{y}} \left(\tilde{y}(t)\right)^2}_{f(\tilde{y})} dt < \infty$ and thus

$$E_x^S\left[\int_0^\infty e^{-\delta t}\underbrace{\frac{1}{2}\phi_{\tilde{y}}\left(\tilde{y}(t)\right)^2 dt}_{f(\tilde{y})}\right] < \infty. \tag{4.33}$$

Furthermore, since $e^{-\delta t}\tilde{y}(t)$ is bounded, we can apply the theorem 2.1.2 (DCT) on $\lim_{T\to\infty} E_x^S\left[e^{-\delta T}\tilde{y}(T)\right]$ and we get

$$\lim_{T\to\infty} E_x^S\left[e^{-\delta T}\tilde{y}(T)\right] = E_x^S\left[\lim_{T\to\infty} e^{-\delta T}\tilde{y}(T)\right] = 0. \tag{4.34}$$

Thus, from the definition 2.2.1, equations (4.32), (4.33) and (4.34) imply that the QVI-control associated with V^* is admissible.

In addition, it is clear that the function V^* defined by the equation (4.23) is twice continuous differentiable in (a, b), once continuously differentiable and linear in $(-\infty, a] \cup [b, \infty)$. Moreover, $V^{*'}$ is indeed continuous in $[a, b]$ and is constant in $(-\infty, a] \cup [b, \infty)$. From the extrem values theorem [85], V^* and $V^{*'}$ are bounded on the interval $[a, b]$. So the equation (4.32), the linearity of V^* in $(-\infty, a] \cup [b, \infty)$ and its boundedness imply that

$$\lim_{T\to\infty} E_x^S\left[e^{-\delta T} V^*\left(\tilde{y}(T)\right)\right] = 0. \tag{4.35}$$

and

from the fact that σ is a finite constant together with the equation (4.31) and $V^{*'}$ is bounded we have

$$E_x^S\left[\int_0^\infty \left(e^{-\delta t}\sigma V^{*'}\tilde{y}(t)\right)^2 dt\right] < \infty. \tag{4.36}$$

As V^* satisfies the growth conditions (2.19) and (2.20), and the QVI-control corresponding to V^* is admissible, it remains to show that V^* obeys the following conditions for all $\tilde{y} \in \mathbb{R}$

$$\mathcal{L}V^*(\tilde{y}) + \frac{1}{2}\phi_{\tilde{y}}\tilde{y}^2 \geq 0, \tag{4.37}$$

$$V^*(\tilde{y}) \leq MV^*(\tilde{y}), \tag{4.38}$$

and

[85]The Extrem values theorem states that:
If a function f is continuous on the closed interval $[a, b]$, then it is bounded on $[a, b]$ (i.e. there exists points c and d in $[a, b]$ such that $f(c) \leq f(x) \leq f(d)$ $\forall x \in [a, b]$). (Proof of this theorem can be found in [26]P.151).

$$(\mathcal{L}V^*(\tilde{y}) + \frac{1}{2}\phi_{\tilde{y}}\tilde{y}^2)\,(V^*(\tilde{y}) - MV^*(\tilde{y})) = 0. \tag{4.39}$$

First inequality: By construction of V^*,

$$\mathcal{L}V^*(\tilde{y}) + \frac{1}{2}\phi_{\tilde{y}}\tilde{y}^2 = 0$$

for all $\tilde{y} \in (a, b)$. So we have to show that

$$\mathcal{L}V^*(\tilde{y}) + \frac{1}{2}\phi_{\tilde{y}}\tilde{y}^2 := \frac{1}{2}\sigma_{\tilde{y}}^2 V^{*''}(\tilde{y}) + \delta\tilde{y}V^{*'}(\tilde{y}) - \delta V^*(\tilde{y}) + \frac{1}{2}\phi_{\tilde{y}}\tilde{y}^2 > 0$$

for all $\tilde{y} \in (-\infty, a] \cup [b, \infty)$.
Considering $b \leq \tilde{y} < \infty$, we have

$$\begin{aligned}
\mathcal{L}V^*(\tilde{y}) &\overset{(4.6)\text{ and }(4.11)}{=} \delta\tilde{y}k - \delta\left[V^*(\beta) + K + k(\tilde{y} - \beta)\right] \\
&= \delta\tilde{y}k - \delta V^*(\beta) - \delta K - \delta k(\tilde{y} - \beta) \\
&\overset{(4.13)\text{ and }(4.11)}{=} \delta\tilde{y}k + \frac{1}{2}\sigma_{\tilde{y}}^2 V^{*''}(\beta) + \delta\beta k + \frac{1}{2}\phi_{\tilde{y}}\beta^2 - \delta K - \delta k(\tilde{y} - \beta) \\
&= \frac{1}{2}\sigma_{\tilde{y}}^2 V^{*''}(\beta) + \delta\beta k + \frac{1}{2}\phi_{\tilde{y}}\beta^2 - \delta K + \delta k\beta \\
&= -\frac{1}{2}BC_1 e^{-\frac{C_1}{2}\beta^2} + \frac{\sigma_{\tilde{y}}^2 C_3}{C_1} + 2\delta\beta k + \frac{1}{2}\phi_{\tilde{y}}\beta^2 - \delta K.
\end{aligned} \tag{4.40}$$

From the equation (4.30) we have $\mathcal{L}V^*(\tilde{y}) + \frac{1}{2}\phi_{\tilde{y}}\tilde{y}^2 > 0$.

For $\tilde{y} \in (-\infty, a]$, we have

$$\begin{aligned}
\mathcal{L}V^*(\tilde{y}) &\overset{(4.5)\text{ and }(4.12)}{=} -\delta\tilde{y}k - \delta\left[V^*(\alpha) + K + k(\alpha - \tilde{y})\right] \\
&= -\delta\tilde{y}k - \delta V^*(\alpha) - \delta K - \delta k(\alpha - \tilde{y}) \\
&\overset{(4.13)\text{ and }(4.12)}{=} -\delta\tilde{y}k + \frac{1}{2}\sigma_{\tilde{y}}^2 V^{*''}(\alpha) - \delta\alpha k + \frac{1}{2}\phi_{\tilde{y}}\alpha^2 - \delta K - \delta k(\alpha - \tilde{y}) \\
&= \frac{1}{2}\sigma_{\tilde{y}}^2 V^{*''}(\alpha) - \delta\alpha k + \frac{1}{2}\phi_{\tilde{y}}\alpha^2 - \delta K - \delta k\alpha \\
&= -\frac{1}{2}BC_1 e^{-\frac{C_1}{2}\alpha^2} + \frac{\sigma_{\tilde{y}}^2 C_3}{C_1} - 2\delta\alpha k + \frac{1}{2}\phi_{\tilde{y}}\alpha^2 - \delta K.
\end{aligned} \tag{4.41}$$

From the equation (4.31) we have $\mathcal{L}V^*(\tilde{y}) + \frac{1}{2}\phi_{\tilde{y}}\tilde{y}^2 > 0$.

Second inequality: By construction $V^*(\tilde{y}) = MV^*(\tilde{y})$ for all $\tilde{y} \in (-\infty, a] \cup [b, \infty)$. Because $-k \leq V^{*'}(\tilde{y}) \leq k$ for all $\tilde{y} \in [\alpha, \beta]$, it is not optimal to intervene (i.e., the process will not move but incurs the fixed costs K which means that $MV^*(\tilde{y}) = V^*(\tilde{y}) + K$) in this interval. This implies that $MV^*(\tilde{y}) - V^*(\tilde{y}) = K > 0$.
For $\tilde{y} \in (\beta, b]$, the argument of the operator M is $\tilde{y} - \beta$. So the construction of β and b implies that

$$V^*(\tilde{y}) \leq V^*(\beta) + K + k|\tilde{y} - \beta| = MV^*(\tilde{y}).$$

For $\tilde{y} \in [a, \alpha)$ we also have

$$V^*(\tilde{y}) \leq V^*(\alpha) + K + k|\tilde{y} - \alpha| = MV^*(\tilde{y})$$

because the argment of the operator M is $\tilde{y} - \alpha$ in this interval.
The equation (4.39) holds as a result of the first and the second inequalities.
Applying the verification theorem (theorem 2.2.1), we can conclude that V^* is the value function of the problem (4.4) and the corresponding QVI-control is optimal.$\square$

4.2.1 Numerical example

In this part, we provide the numerical solution to the system of equations $4.24 - 4.29$ using the Matlab nonlinear system solver "fsolve". In addition, we study the effect of the parameters K, k, $\sigma_{\tilde{y}}$ and $\phi_{\tilde{y}}$ on the optimal two band policy. These sensitivity analysis will be tested as follows.
We choose a parameter from other parameters, which are considered to be fixed, of the system and we observe how the optimal band policy changes when the value of the chosen parameter decreases or increases. The test results are summarized in the table $4.1 - 4.4$ below.

From the results presented in the tables 4.1, 4.2 and 4.4, we observe that the optimal band width $(b - a)$ increases as one of the parameters K, k and $\sigma_{\tilde{y}}$ increases. In this case, the central bank waits longer to intervene.

In contrast, as it has been illustrated in the table 4.3, the optimal band width $(b - a)$ reaches its maximum as the value of the parameter $\phi_{\tilde{y}}$ decreases.

Table 4.1: Effect of the parameter K for fixed values $\phi_{\tilde{y}} = 0.6$, $\sigma = 0.1$, $\phi_i = 0.1$, $\sigma_{\tilde{y}} = 0.05$ and $k = 20$.

case	K=5	K=0.1	K=0.15
a	-100.0045	-14.1474	-17.3257
α	-0.0041	-0.0049	-0.0048
β	0.0041	0.0049	0.0048
b	100.0045	14.1474	17.3257
A	-0.0000	-0.0000	0.0000
B	-0.0231	-0.0230	-0.0231
b - a	200.0090	28.2948	34.6514

Table 4.2: Effect of the parameter k for fixed values $\phi_{\tilde{y}} = 0.6$, $\sigma = 0.1$, $\phi_i = 0.1$, $\sigma_{\tilde{y}} = 0.05$ and $K = 5$.

case	k=20	k=0.20	k=0.40
a	-100.0045	-100.0022	-100.0025
α	-0.0041	-0.0014	-0.0018
β	0.0041	0.0014	0.0018
b	100.0045	100.0022	100.0025
A	-0.0000	0.0000	0.0000
B	-0.0231	-0.0003	-0.0006
b - a	200.0090	200.0044	200.0050

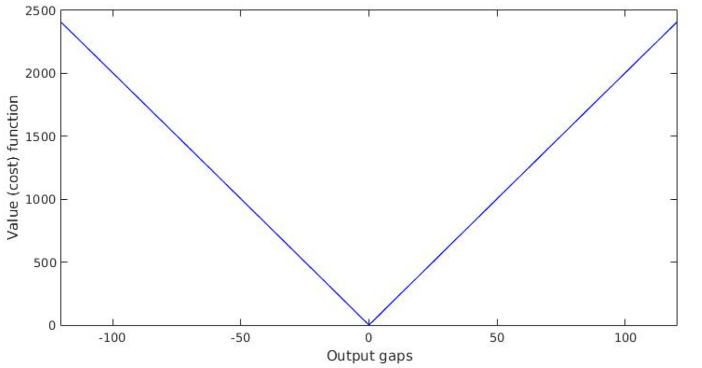
Table 4.3: Effect of the parameter $\phi_{\tilde{y}}$ for fixed values $k = 20$, $\sigma = 0.1$, $\phi_i = 0.1$, $\sigma_{\tilde{y}} = 0.05$ and $K = 5$.

case	$\phi_{\tilde{y}} = 0.6$	$\phi_{\tilde{y}} = 0.50$	$\phi_{\tilde{y}} = 0.80$
a	-100.0045	-100.0049	-100.0039
α	-0.0041	-0.0044	-0.0035
β	0.0041	0.0044	0.0035
b	100.0045	100.0049	100.0039
A	-0.0000	0.0000	0.0000
B	-0.0231	-0.0254	-0.0200
b - a	200.0090	200.0098	200.0078

Table 4.4: Effect of the parameter $\sigma_{\tilde{y}}$ for fixed values $k = 20$, $\sigma = 0.1$, $\phi_{\tilde{y}} = 0.6$, $\phi_i = 0.1$ and $K = 5$.

case	$\sigma_{\tilde{y}} = 0.05$	$\sigma_{\tilde{y}} = 0.1$	$\sigma_{\tilde{y}} = 0.2$
a	-100.0045	-100.0090	-100.0179
α	-0.0041	-0.0081	-0.0162
β	0.0041	0.0081	0.0162
b	100.0045	100.0090	100.0179
A	-0.0000	-0.0000	0.0000
B	-0.0231	-0.0463	-0.0926
b - a	200.0090	200.0180	200.0358

Figure 4.1: The Value function presented by the equation (4.23) for $k = 20$, $\sigma_{\tilde{y}} = 0.05$, $\phi_{\tilde{y}} = 0.6$, $\phi_i = 0.1$, $\sigma = 0.20$ and $K = 5$



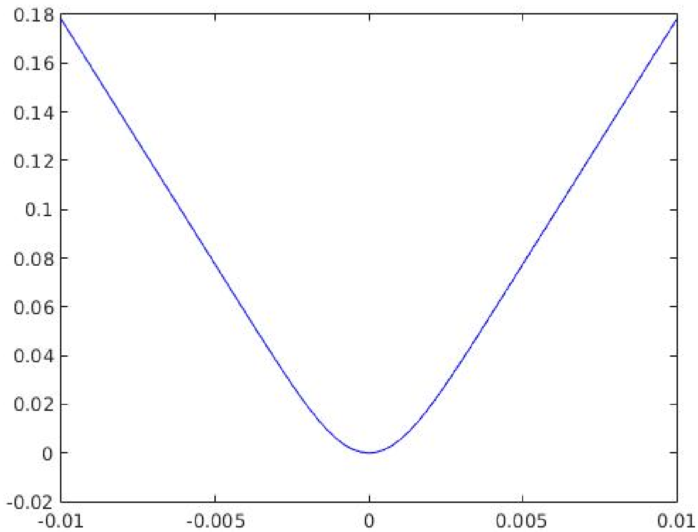


Figure 4.2: Zoom in toward the point $(0, V^*(0))$

Appendix A

Transition rules

Firstly, the baseline model with $h = 1$ must be formulated. As a main assumption, it is assumed that the functional form of the model will not change accross different frequences [86].
Secondly, the frequency-dependent components of the model have to be suitably transformed under the consideration of h. For instance, the discount rates must be adjusted, while fractions remain unchanged. With respect to the variables, growth rates exhibit a certain time dimension and hence they must be divided by h, while contemporaneous adjustment rates (e.g. output gap) have no time dimension.
Finally, the variables have to be normalized. Note that a variable without time-dimension such as the output gap have not to be normalized (or quarterized). Normalization refers to a specific aggregation technique applied on high-frequency variables relative to the benchmark period length. The skip sampling aggregation scheme based on the deviation of the corresponding variables by h is one among others, which exist in econometric literature, that will be used in this work.

A.1 Heuristic Derivation of Interest Rate given by the Equation 3.100: Lagrangian Method

The optimal short-term nominal interest rate i_t can be realized from the first order condition of the solution to the inflation-targeting central bank's problem of minimizing the loss function of the form:

$$L_t = E_t \sum_{\tau=0}^{\infty} \frac{1}{2}\delta^{\tau} \left[\phi_{\pi}(\pi_{t+\tau} - \pi^T)^2 + \phi_{\tilde{y}}(\tilde{y}_{t+\tau})^2 + \phi_i(i_{t+\tau} - i^n_{t+\tau})^2\right]$$

[86] It has been shown that the functional form of the NKPC will remain unchanged when the period length is reduced (see [145]).

subject to the constraints:
$\pi_t = \beta E_t \pi_{t+1} + \tilde{\lambda}\tilde{y}_t$,
$\tilde{y}_t = E_t \tilde{y}_{t+1} - \frac{1}{\sigma} i_t + \frac{1}{\sigma} E_t \pi_{t+1} + \frac{1}{\sigma} r_t^n$.

Its Lagrangian expression is:

$$\mathcal{L} = E_t \sum_{\tau=0}^{\infty} \frac{1}{2}\delta^\tau \left[\phi_\pi(\pi_{t+\tau} - \pi^T)^2 + \phi_{\tilde{y}}(\tilde{y}_{t+\tau})^2 + \phi_i(i_{t+\tau} - i_{t+\tau}^n)^2\right]$$

$$+ \lambda_1 \left(\beta E_t \pi_{t+1} + \tilde{\lambda}\tilde{y}_t - \pi_t\right) + \lambda_2 \left(E_t \tilde{y}_{t+1} - \frac{1}{\sigma} i_t + \frac{1}{\sigma} E_t \pi_{t+1} + \frac{1}{\sigma} r_t^n - \tilde{y}_t\right).$$

The first order conditions for $\tau = 0$ are:

$$\frac{\partial \mathcal{L}}{\partial \tilde{y}_t} = \phi_{\tilde{y}} \tilde{y}_t - \lambda_1 \tilde{\lambda} - \lambda_2 = 0, \tag{A.1}$$

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = \phi_\pi \left(\pi_t - \pi^T\right) - \lambda_1 = 0 \Longrightarrow \lambda_1 = \phi_\pi \left(\pi_t - \pi^T\right), \tag{A.2}$$

$$\frac{\partial \mathcal{L}}{\partial i_t} = \phi_i \left(i_t - i_t^n\right) - \frac{\lambda_2}{\sigma} = 0, \tag{A.3}$$

Substituting equation (A.2) into equation (A.1) we have:

$$\lambda_2 = \phi_{\tilde{y}} \tilde{y}_t + \tilde{\lambda}\phi_\pi \left(\pi_t - \pi^T\right). \tag{A.4}$$

Inserting the value of λ_2 in equation (A.3) we get:

$$\phi_i \left(i_t - i_t^n\right) - \tfrac{\phi_{\tilde{y}}}{\sigma} \tilde{y}_t - \tfrac{\phi_\pi \tilde{\lambda}}{\sigma} \left(\pi_t - \pi^T\right) = 0 \Longrightarrow i_t = i_t^n + \underbrace{\frac{\phi_{\tilde{y}}}{\sigma \phi_i}}_{:=a} \tilde{y}_t + \underbrace{\frac{\phi_\pi \tilde{\lambda}}{\sigma \phi_i}}_{:=b} \left(\pi_t - \pi^T\right)$$

$$\Longleftrightarrow i_t = r_t^n + \pi^T + a\tilde{y}_t + b\left(\pi_t - \pi^T\right).$$

A.2 Proof of the Verification Theorem

Let $S = \{(\tau_m, \Delta X_m)\}_{m \in \mathbb{N}}$ be an admissible impulse control strategy with $\tau_0 = 0$. Define $\theta_m = t \wedge \tau_m := \min t, \tau_m$ for every $t > 0$ and $m \in \mathbb{N}$. Then we have

$$\begin{aligned} e^{-\rho\theta_m} V^*\left(X(\theta_m)\right) - V^*(x) = &\sum_{i=1}^{m} \underbrace{\left(e^{-\rho\theta_i} V^*\left(X(\theta_i^-)\right) - e^{-\rho\theta_{i-1}} V^*\left(X(\theta_{i-1})\right)\right)}_{(*)} \\ &+ \sum_{i=1}^{m} 1_{\{\tau_i \le t\}} \underbrace{e^{-\rho\tau_i}\left(V^*\left(X(\theta_i)\right) - V^*\left(X(\theta_i^-)\right)\right)}_{**}. \end{aligned} \tag{A.5}$$

For the term (*) of the first summand in (A.5), the application of the Itô's formula presented in chapter 2 leads to the following equation:

$$(*) = \int_{\theta_{i-1}}^{\theta_i} e^{-\rho u} \underbrace{\left[-\rho V^*\left(X(u)\right) + b\left(X(u)\right) V_x^*\left(X(u)\right) + \frac{1}{2}\sigma\left(X(u)\right) V_{xx}^*\left(X(u)\right)\right]}_{=\mathcal{L}V^*(X(u))} du$$
$$+ \int_{\theta_{i-1}}^{\theta_i} e^{-\rho u} V_x^*\left(X(u)\right)\sigma\left(X(u)\right) dB(u). \tag{A.6}$$

According to inequality (2.15) we get:

$$e^{-\rho\theta_i} V^*\left(X(\theta_i^-)\right) - e^{-\rho\theta_{i-1}} V^*\left(X(\theta_{i-1})\right) \geq \int_{\theta_{i-1}}^{\theta_i} e^{-\rho u}\left(-f\left(X(u)\right)\right) du$$
$$+ \int_{\theta_{i-1}}^{\theta_i} e^{-\rho u} V_x^*\left(X(u)\right)\sigma\left(X(u)\right) dB(u). \tag{A.7}$$

For the term (**) of the second summand in (A.5), we have for $\{\tau_i \leq t\}$ [87]:

$$e^{-\rho\theta_i}\left(V^*\left(X(\theta_i)\right) - V^*\left(X(\theta_i^-)\right)\right) \geq -e^{-\rho\theta_i}\left(K + k|\Delta X_i|\right). \tag{A.8}$$

Substituting (A.8) and (A.7) into (A.5) we get:

$$\begin{aligned} V^*(x) - e^{-\rho\theta_m} V^*\left(X(\theta_m)\right) \leq & \sum_{i=1}^{m} \int_{\theta_{i-1}}^{\theta_i} e^{-\rho u}\left(f\left(X(u)\right)\right) du \\ & - \sum_{i=1}^{m} \int_{\theta_{i-1}}^{\theta_i} e^{-\rho u} V_x^*\left(X(u)\right)\sigma\left(X(u)\right) dB(u) \\ & + \sum_{i=1}^{m} 1_{\{\tau_i \leq t\}} e^{-\rho\tau_i}\left(K + k|\Delta X_i|\right). \end{aligned} \tag{A.9}$$

[87] Inequality presented in (A.8) holds beacause our $X(\theta_i) = X(\theta_i^-) - \Delta X_i$ according to the definition 2.2.1 (iV) and the definition 2.2.3 and (2.10) imply that

$$V^*\left(X(\theta_i^-)\right) \leq V^*\left[\underbrace{X(\theta_i^-) - \Delta X_i}_{:=X(\theta_i)}\right] + K + k|\Delta X_i|$$

which also implies that:
$V^*\left(X(\theta_i)\right) - V^*\left(X(\theta_i^-)\right) \geq V^*\left(X(\theta_i)\right) - V^*\left(X(\theta_i)\right) - K - k|\Delta X_i|$

$$\Updownarrow$$

$e^{-\rho\theta_i}\left[V^*\left(X(\theta_i)\right) - V^*\left(X(\theta_i^-)\right)\right] \geq -e^{-\rho\theta_i}\left(K + k|\Delta X_i|\right)$.

Application of the expectation operator E_x^S on the both sides of (A.9) yields

$$V^*(x) - E_x^S\left[e^{-\rho\theta_m}V^*\left(X(\theta_m)\right)\right] \leq E_x^S\left[\sum_{i=1}^{m}\int_{\theta_{i-1}}^{\theta_i} e^{-\rho u}\left(f\left(X(u)\right)\right)du\right]$$
$$- E_x^S\left[\sum_{i=1}^{m}\int_{\theta_{i-1}}^{\theta_i} e^{-\rho u}V_x^*\left(X(u)\right)\sigma\left(X(u)\right)dB(u)\right]$$
$$+ E_x^S\left[\sum_{i=1}^{m}1_{\{\tau_i\leq t\}}e^{-\rho\tau_i}\left(K+k|\Delta X_i|\right)\right]. \tag{A.10}$$

Taking limit as $m \longrightarrow \infty$, the left hand side of (A.10) becomes[88]

$$\lim_{m\to\infty}\left\{V^*(x) - E_x^S\left[e^{-\rho\theta_m}V^*\left(X(\theta_m)\right)\right]\right\} = V^*(x) - E_x^S\left[e^{-\rho t}V^*\left(X(t)\right)\right], \tag{A.11}$$

together with the condition (2.19) which implies that the expectation of the stochastic integral vanishes, the inequality expressed by (A.10) simplifies to

$$V^*(x) - E_x^S\left[e^{-\rho t}V^*\left(X(t)\right)\right] \leq E_x^S\left[\sum_{i=1}^{\infty}\int_{\theta_{i-1}}^{\theta_i} e^{-\rho u}\left(f\left(X(u)\right)\right)du\right]$$
$$+ E_x^S\left[\sum_{i=1}^{\infty}1_{\{\tau_i\leq t\}}e^{-\rho\tau_i}\left(K+k|\Delta X_i|\right)\right]. \tag{A.12}$$

Taking the limit as $t \longrightarrow \infty$ and considering the condition (2.20), the left hand side of (A.12) will be equal to $V^*(x)$. Together with the definition of θ_m and the theorem 2.1.2 we can write the right-hand side of (A.12) as

$$\lim_{t\to\infty}E_x^S\left[\sum_{i=1}^{\infty}\int_{\theta_{i-1}}^{\theta_i} e^{-\rho u}\left(f\left(X(u)\right)\right)du\right] + \lim_{t\to\infty}E_x^S\left[\sum_{i=1}^{\infty}1_{\{\tau_i\leq t\}}e^{-\rho\tau_i}\left(K+k|\Delta X_i|\right)\right]$$
$$= E_x^S\left[\int_0^{\infty} e^{-\rho u}\left(f\left(X(u)\right)\right)du + \sum_{i=1}^{\infty}1_{\{\tau_i<\infty\}}e^{-\rho\tau_i}\left(K+k|\Delta X_i|\right)\right].$$

And hence

$$V^*(x) \leq E_x^S\left[\int_0^{\infty} e^{-\rho u}\left(f\left(X(u)\right)\right)du + \sum_{i=1}^{\infty}1_{\{\tau_i<\infty\}}e^{-\rho\tau_i}\left(K+k|\Delta X_i|\right)\right]. \tag{A.13}$$

[88] This follows from the definition 2.2.1(V) which implies that $\tau_\infty = \infty$ and the definition of θ_m.

Therefore, for every $S = \{(\tau_m, \Delta X_m)\}_{m\in\mathbb{N}} \in Z$, where $Z(x)$ denotes a the set of admissible impulse control strategies for the starting state x,

$$V^*(x) \leq V(x). \tag{A.14}$$

If there exists an admissible QVI-control associated with V^* , then it is an optimal impulse control and then $V^*(x) = V(x)$ for every $x \in \mathbb{R}$.$\square$

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