A Finite Dominating Set Algorithm for a Dynamic Location Problem in the Plane *

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Abstract

A single facility problem in the plane is considered, where an optimal location has to be identified for each of finitely many time-steps with respect to time-dependent weights and demand points. It is shown that the median objective can be reduced to a special case of the static multifacility median problem such that results from the latter can be used to tackle the dynamic location problem. When using block norms as distance measure between facilities, a Finite Dominating Set (FDS) is derived. For the special case with only two time-steps, the resulting algorithm is analyzed with respect to its worst-case complexity. Due to the relation between dynamic location problems for T time periods and T-facility problems, this algorithm can also be applied to the static 2-facility location problem.

Keywords: optimal dynamic locations, optimal trajectory, block norm, public event

1 Introduction

The location problem presented in this paper is motivated by commercial and safety issues in the organization of public events. It is part of a larger project in which various conflicting objectives (safety, commercial success, visitor-friendliness, etc.) will be simultaneously optimized in multi-objective models. We consider, in particular, the presence of sales- and security personnel which we assume to be dependent on data changing over time.

As a first step in tackling this complex issue, we consider in this paper, single-commodity, single-facility dynamic location problems. In contrast to the existing literature, weights and demand points are time-dependent. Moreover, there will be penalties dependent on the distance for moving staff around.

In Section 2 we will give a short review of planar location problems, focusing on the time-dependent case. In Section 3 we will introduce our problems and show in Section 4 that the median version of the problem can be interpreted as special case of the static multifacility location problem. Optimality conditions are stated and applied to identify a Finite Dominating Set in Section 5. The worst-case complexity of the resulting algorithm is analyzed and an Algorithm is stated for the case with only two time-steps. In the last Section 6, possible improvements and future research are discussed.

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2 Literature Summary

Facility location is a major research topic in optimization and can be categorized in many subtopics. For a general overview of location problems see for example Love et al. [LMW88], Drezner [Dre95a] or Drezner and Hamacher [DH04].
In planar location theory, the distance is usually measured by norms, such as l_p-norms or block norms. In this paper we will focus on the latter. Block norms were introduced by Ward and Wendel in 1980 [WW80]. They showed that location problems with block norms can be formulated as linear programs [WW85] and are, therefore, solvable in polynomial time. As generalization of block norms, polyhedral gauges - in which the symmetry requirement of block norms are relaxed - have been considered for example in [ML87], [Dur90], [CCMMP97], [Nic98] and [Fli98]. The standard assumption in these models is that the data is constant over time.

In this paper planar location problems with time-dependent data are presented. To the best of our knowledge, Ballou [Bal68] was the first to consider such a dynamic location model. In his model he relocates a single warehouse to maximize the profit. He takes transportation and relocation costs into account and develops an approach based on dynamic programming. A discussion in the Journal of Marketing Research arose when Ballou stated that his approach could be easily applied to other location models. Lodish [Lod70] claimed that it is not possible to transfer the approach that easily, but Ballou [Bal70] defended himself in the same issue. However, Ballou's approach was later shown to be suboptimal by Sweeney and Tatham [ST76] who were also able to improve Ballou's solution approach. In 1973 Wesolowsky [Wes73] extended the single facility location model to one where the facility location can change within a given time interval. In this model, locations are predefined. Location problems are not only considered in optimization, but also in computational geometry where Atallah [Ata83] introduces various “location” problems with moving geometric objects.

Mediggo [Med86] works with moving demand points, however, not with changing the demand. He tries to find a single location in his two models which he calls the “global optimization” and the “steady state” problem. Moreover, the demand points and the location are coordinates in $\mathbb{R}^3$. Two years later Chand [Cha88] provided decision and forecast results for a dynamic relocation problem with a single facility. Campbell [Cam90] developed a myopic approach for a continuous location model of a freight carrier with increasing demand, terminal, transportation and relocation costs.

Drezner and Wesolowsky [DW91] introduced a multifacility model in which the facilities can relocate at so called “break points”. Their plan is to find an optimal location/ relocation plan for the corresponding break points. They investigate the minisum Weber and minimax location problem and propose algorithms. Later Drezner [Dre95b] considered the dynamic p-median problem where demand can change over time. For the Euclidean norm he gives a heuristic.

Planar dynamic location was rediscovered in 2008 by Farahani et al [FDA09]. In their article they consider weights given by a function of time. In their model the facilities can only be relocated at given break points. The latest work was done by Farahani et al. [FSG14] where they picked up the model of the previous paper. The location can only change once on a given time interval and their aim is to find the optimal relocation time.

3 Problem Formulations

The real-world problems sketched in Section 1 will subsequently be modeled by various versions of the *dynamic single facility problem in the plane*. The location of the sales-, security person, etc. at time $t \in \mathcal{T} := \{1, \ldots, T\}$ is a point $x_t \in \mathbb{R}^2$ in the plane. Here $\mathcal{T}$ is a given set of equidistant, discrete time steps with *time horizon* T. Since the set $X = (x_t)_{t\in\mathcal{T}} \in \mathbb{R}^{2T}$ of dynamic locations

can be interpreted as a *trajectory* in the plane, the dynamic single facility problem could also be coined *optimal trajectory problem.*
Each change of the location from time t to $t+1$ (for $t = 1, \ldots, T-1$) induces some cost $v_t \in \mathbb{R}_{>0}$ per unit distance between x_t and x_{t+1}. Moreover, for each $t \in \mathcal{T}$ there are M demand points $\mathcal{D}_t := \{d_{tm} \in \mathbb{R}^2, m \in \mathcal{M}\}$, each associated with weights $w_{tm} \in \mathbb{R}_{\geq 0}$, where $\mathcal{M} := \{1, \ldots, M\}$. The set of all demand points over time is denoted by $\mathcal{D} := \cup_{t\in\mathcal{T}} \mathcal{D}_t$. For the unweighted version all weights are equal to 1.
Depending on the real-world problem different objective functions may be used.

The *median objective*

$$\min_{X\in\mathbb{R}^{2T}} \sum_{t=1}^{T-1} v_t \|x_{t+1} - x_t\| + \sum_{t=1}^{T} \sum_{m=1}^{M} w_{tm} \|d_{tm} - x_t\|, \tag{1}$$

where $\|\cdot\|$ denotes a norm, is for instance useful in order to minimize the overall cost of a salesperson which is assumed to be proportional to the overall distance covered by moving along a trajectory and serving the customers.
For a safety patrol, objective (1) is in general not suitable, since it is more important to show presence during the public event rather than saving cost. In order to deter potential trouble-makers it is advisable that the patrol is able to see all visitors at some time during the periods $t \in \mathcal{T}$. This can be modeled by the objective function

$$\min_{X\in\mathbb{R}^{2T}} \max_{m\in\mathcal{M}} \min_{t\in\mathcal{T}} w_{tm} \|d_{tm} - x_t\|. \tag{2}$$

For police or medical services it is important to get as fast as possible to the emergency location over the whole time horizon. This is achieved by minimizing the maximum distance to any demand point for any time step, which gives us the classical *Center* objective:

$$\min_{X\in\mathbb{R}^{2T}} \max_{m\in\mathcal{M}} \max_{t\in\mathcal{T}} w_{tm} \|d_{tm} - x_t\|. \tag{3}$$

However, if at one time step the weight of a demand point is very big, or the demand points themselves are much farther away than at other time steps, then this time step might dominate all the others and it might not be important where the location at other time steps is. Therefore, instead of considering the maximum over all time steps, it might be better to find a path which minimizes the maximum distance over all demand points considering the whole time-horizon.

$$\min_{X\in\mathbb{R}^{2T}} \max_{m\in\mathcal{M}} \left\{ \sum_{t=1}^{T} w_{tm} \|d_{tm} - x_t\| \right\} + \sum_{t=1}^{T-1} v_t \|x_{t+1} - x_t\|. \tag{4}$$

There are several reasonable constraints on the choice of locations $X = (x_t)_{t\in\mathcal{T}} \in \mathbb{R}^{2T}$ which may be considered in the dynamic location model.

The most obvious one is the restriction to the area in which the public event takes place and the space which is available inside this area. If $F \subset \mathbb{R}^2$ is the feasible set from which the locations x_t can be chosen, then the dynamic location problem is restricted (see, e.g., [HN95] for the static case).

Often, the start point and/or end point of the path is predefined, i.e. for two points in the plane $p_1, p_T \in \mathbb{R}^2$:

$$x_1 = p_1 \text{ and/or } \quad x_T = p_T, \tag{5}$$

or the trajectory of the staff is repeating after a time interval, hence

$$x_1 = x_T \, (= p_1) \, . \tag{6}$$

Further interesting constraints are limits on the distance of the staff by considering lower and upper bounds L and U, respectively (which may also be time-dependent) for the distance covered between two consecutive time steps.

$$L \leq \|x_{t+1} - x_t\| \leq U, \qquad t = 1, \dots, T-1, \tag{7}$$

or during the complete time horizon

$$L \leq \sum_{t=1}^{T-1} \|x_{t+1} - x_t\| \leq U. \tag{8}$$

4 Dynamic Single Facility Location Problem with Median Objective

Interrelation Between Dynamic and Static Location Problems

The *unconstrained single-facility dynamic location problem in the plane with median objective* is to find $X = (x_t)_{t\in\mathcal{T}} \in \mathbb{R}^{2T}$ minimizing

$$F(X) := \sum_{t=1}^{T-1} v_t \|x_{t+1} - x_t\| + \sum_{t=1}^{T} \sum_{m=1}^{M} w_{tm} \|d_{tm} - x_t\| \tag{1FDynLoc}$$

Using the classification scheme of Hamacher et al. [HNS96], we denote the problem also as $1/P/dyn/\bullet/\Sigma$. Its set of optimal solutions is denoted by $\mathcal{X}^*$, or more extensively by $\mathcal{X}^*(1/P/dyn/\bullet/\Sigma)$. The possible optimal values for one time-step t is defined by $\mathcal{X}^*_t := \{x \in \mathbb{R}^2 : \exists X^* \in \mathcal{X}^*, x = x^*_t\}$.
We will first show that this problem can be interpreted as special case of the well-known multi-facility location problem introduced next (see Love et al. [LMW88]).

Given a set of N facilities with $\mathcal{N} := \{1, \dots, N\}$. Between each pair of facilities, there are weights $\tilde{v}_{kl} \geq 0$ for all $k, l \in \mathcal{N} := \{1, \dots, N\}$. Let $\widetilde{\mathcal{D}} := \{\tilde{d}_m \in \mathbb{R}^2 : m \in \widetilde{\mathcal{M}} := \{1, \dots, \tilde{M}\}\}$ be the set of demand points with corresponding weights between the facilities and the demand points $\tilde{w}_{lm} \geq 0$ for all $l \in \mathcal{N}, m \in \widetilde{M}$.
The *(static) multifacility Weber problem in the plane*, denoted $N/P/static/\bullet/\Sigma$, is to find $X = (x_n)_{n\in\mathcal{N}} \in \mathbb{R}^{2N}$ minimizing

$$\widetilde{F}(X) = \sum_{k=1}^{N} \sum_{l=1}^{N} \tilde{v}_{kl} \|x_l - x_k\| + \sum_{m=1}^{\tilde{M}} \sum_{l=1}^{N} \tilde{w}_{lm} \|\tilde{d}_m - x_l\|. \tag{9}$$

Theorem 4.1. *The single-facility dynamic location problem with objective function* (1FDynLoc) *and time horizon T is a special case of the static multi-facility location problem.*

Proof. Any given instance of $1/P/dyn/\bullet/\Sigma$ can be reformulated as an equivalent multi-facility static Weber problem with

- $\mathcal{N} = \mathcal{T}$,
- $\widetilde{\mathcal{M}} = \{(t, m) : t \in \mathcal{T}, m \in \mathcal{M}\}$,
- $\widetilde{\mathcal{D}} = \cup_{t\in\mathcal{T}} \mathcal{D}_t$,

and weights

$$\tilde{w}_{t(\hat{t},m)} := \begin{cases} w_{tm}, & \text{if } t = \hat{t}, \\ 0, & \text{else} \end{cases}$$

$$\tilde{v}_{t\hat{t}} = \begin{cases} v_t, & \text{if } \hat{t} = t+1, \\ 0, & \text{else.} \end{cases}$$

□

Optimality Conditions

In general (1FDynLoc) is not differentiable whenever $x_t = x_{t+1}$ for some $t \in \mathcal{T}$. Plastria [Pla92] derived optimality conditions using the subdifferential of the objective function. Lefebvre et al. [LMP90] gave a geometric interpretation of the optimality conditions. Before stating these conditions one has to be familiar with the concept of dual norms and normal cones.
The *dual* or also called *polar norm* of $\|\cdot\|$ with unit ball B is given by

$$\|p\|^\circ := \max\{\langle p, x\rangle : \|x\| \leq 1\},$$

with $\langle \cdot, \cdot \rangle$ denoting the dot product. Its associated unit ball is denoted by B°.
The *normal cone* $K(p)$ to B° at $p \in B^\circ$ is defined by

$$K(p) = \{x \in \mathbb{R}^n : \langle x, q - p\rangle \leq 0 \ \forall q \in B^\circ\}.$$

The following Theorem is stated for $1/P/dyn/\bullet/\Sigma$, however, was first developed by Lefebvre et al. [LMP90] for $N/P/static/\bullet/\Sigma$

Theorem 4.2 (Lefebvre et al. [LMP90]). *(i) If X is an optimal solution to $1/P/dyn/\bullet/\Sigma$, then there exists vectors $p_{tm} \in \mathbb{R}^2$ and $\tilde{p}_t$ for $t \in \mathcal{T}$, $m \in \mathcal{M}$, whereas $\tilde{p}_0 = \tilde{p}_T = 0$, satisfying the conservative constraints:*

$$\sum_{m \in \mathcal{M}} w_{tm} p_{tm} + v_t \hat{p}_t - v_{t-1} \hat{p}_{t-1} = 0, \qquad \forall t \in \mathcal{T} \tag{10}$$

and the cone and ball conditions:

$$\begin{aligned} & x_t \in d_{tm} + K(p_{tm}), && t \in \mathcal{T}, m \in \mathcal{M}, && (11)\\ & x_t \in x_{t+1} + K(\tilde{p}_t), && t \in \mathcal{T} \setminus T, && (12)\\ & p_{tm} \in B^\circ, && t \in \mathcal{T}, m \in \mathcal{M}, && (13)\\ & \tilde{p}_t \in B^\circ, && t \in \mathcal{T} \setminus T. && (14) \end{aligned}$$

(ii) Let $p = (p_{tm})_{m \in \mathcal{M}, t \in \mathcal{T}}$, $\tilde{p} = (\tilde{p}_t)_{t \in \mathcal{T}}$ and $P = (p, \tilde{p})$ be a vector satisfying the conservation constraints (10) and the ball conditions (13) and (14) on p and $\tilde{p}$. If there exists a $\tilde{X}$ such that the pair $(\tilde{X}, P)$ also satisfies the cone conditions, then $\tilde{X}$ is optimal. Moreover, X' is also an optimal solution if and only if (X', P) also satisfies these cone conditions.

Proof. [LMP90] □

An example of the optimality conditions will be given later using a block norm which will be defined in the next section.

5 1FDynLoc with Block Norms: Finite Dominating Set Result

In the following we will consider (1FDynLoc) under block norms. Let $B \subset \mathbb{R}^2$ be a bounded, convex, symmetric polytope with 0 as center. Then a *block norm* is defined by $\gamma(v) := \inf\{\mu > 0 : v \in \mu B\}$ and the objective is denoted by:

$$F_\gamma(X) = \sum_{t=1}^{T-1} v_t \gamma(x_{t+1} - x_t) + \sum_{t=1}^{T} \sum_{m=1}^{M} w_{tm} \gamma(d_{tm} - x_t), \tag{F_γ}$$

Considering the cone conditions (11), then the sets

$$\bigcap_{m \in \mathcal{M}} d_{tm} + K(p_{tm})$$

define geometric objects, more precisely polyhedral cells, edges and extreme points of those cells. The extreme points of those cells will be denoted by $\mathcal{I}(\mathcal{D}_t)$, respectively $\mathcal{I}(S)$ for $S \subseteq \mathcal{D}$ if we consider the intersection points of a specific subset of $\mathcal{D}$.
The extreme points of B, denoted by $\mathsf{Ext}(B) = \{b_{\pm 1}, \dots, b_{\pm R}\}$ with $b_r = -b_{-r}$ as B was assumed to be symmetric, define fundamental directions $f_r = \{\lambda b_r : \lambda \geq 0\}$ for $r \in \mathcal{R} := \{\pm 1, \dots, \pm R\}$. The construction line defined by f_r at point d is given by:

$$L^r(d) := \{x \in \mathbb{R}^2 : x = d + f_r\}.$$

For a finite set $\mathcal{D} \subset \mathbb{R}^2$ define the set of construction lines of $\mathcal{D}$ by

$$\mathcal{CL}(\mathcal{D}) := \bigcup_{d \in \mathcal{D}} \bigcup_{r \in \mathcal{R}} L^r(d).$$

The following example will illustrate the optimal conditions of Theorem 4.2 using a block norm.

Example 5.1
Given the unit ball with extreme points $(0, {}^2\!/_3\sqrt{2})$, $({}^1\!/_3, {}^1\!/_3)$, $({}^2\!/_3\sqrt{2}, 0)$, $({}^1\!/_3, -{}^1\!/_3)$, $(0, -{}^2\!/_3\sqrt{2})$, $(-{}^1\!/_3, -{}^1\!/_3)$, $(-{}^2\!/_3\sqrt{2}, 0)$, $(-{}^1\!/_3, {}^1\!/_3)$ and its dual ball with extreme points $(1,2)$, $(2,1)$, $(2,-1)$, $(1,-2)$, $(-1,-2)$, $(-2,-1)$, $(-2,1)$, $(-1,2)$. Consider the following set of demand points with according weights:

$\mathcal{T} \setminus \mathcal{M}$	1	2	3	4
1	$(1,15)$	$(3,13)$	$(1,13)$	$(3,15)$
2	$(11,15)$	$(10,18)$	$(6,8)$	$(9,17)$
3	$(7,5)$	$(9,3)$	$(7,3)$	$(9,5)$

demand points: d_{tm}

$\mathcal{T} \setminus \mathcal{M}$	1	2	3	4
1	6	6	8	4
2	4	4	3	3
3	6	6	8	4

weights: w_{tm}

and weights for moving the location per unit distance $v_1 = 6$, $v_2 = 6$.
By simple calculation, one can see that the following flow P is satisfies the conservation constraints 10:

$\mathcal{T} \setminus \mathcal{M}$	1	2	3	4
1	$\begin{pmatrix} 1.5 \\ -1.5 \end{pmatrix}$	$\begin{pmatrix} -1.5 \\ 1.5 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -1 \\ -2 \end{pmatrix}$
2	$\begin{pmatrix} -2 \\ -1 \end{pmatrix}$	$\begin{pmatrix} -1 \\ -2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ -2 \end{pmatrix}$
3	$\begin{pmatrix} 1.5 \\ -1.5 \end{pmatrix}$	$\begin{pmatrix} -1.5 \\ 1.5 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	$\begin{pmatrix} -2 \\ -1 \end{pmatrix}$

flow: $p_{tm} \in B^0$,

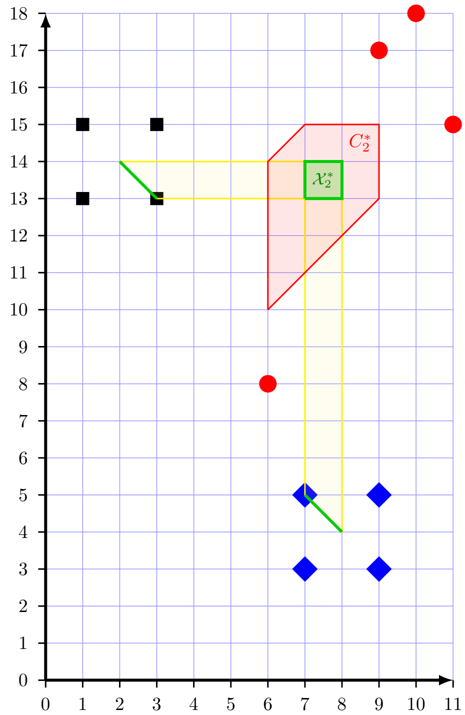
$\tilde{p}_1$	$\begin{pmatrix} -2 \\ 0 \end{pmatrix}$
$\tilde{p}_2$	$\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$\tilde{p}_t \in B^0$.

Clearly, the ball conditions are satisfied, hence, if we find a solution X such that the pair (X, P) satisfies the cone conditions, then we know by Theorem 4.2 that X is optimal. In addition, by the second part of the Theorem, each optimal solution must satisfy the cone condition considering the flow P.
One can see in the Figure that the set of optimal solution is given for $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$:

$$x_1 = (2 + \alpha, 14 - \alpha),$$
$$x_2 = (7 + \beta, 14 - \alpha),$$
$$x_3 = (7 + \beta, 5 - \beta).$$

Note that $C_2^* = \bigcap_{m \in \mathcal{M}} d_{2m} + K(p_{2m})$ is the cell defined by the cone conditions (11) of the demand points.



5.1 1FDynLoc with minimal dimension

To derive general geometrical properties, we first have to consider a special case of 1FDynLoc: The dynamic location problem (1FDynLoc) has *minimal dimension* if there exists no optimal solution $X \in \mathcal{X}^*$, such that $x_t = x_{t+1}$ for all $t \in \mathcal{T} \setminus T$. Throughout this section, we will assume that 1FDynLoc has minimal dimension. The interior of a set $S \subset \mathbb{R}^2$ will be denoted by $int(S)$ and the boundary by $bd(S)$.

Theorem 5.2. *Given $1/P/dyn/\gamma/\Sigma$ with minimal dimension. Assume we have a pair (X, P) satisfying the optimality conditions in Theorem 4.2, then there exists a solution $X^* \in \mathcal{X}^*$ satisfying the following aspects:*

(i) if there exist a $t' \in \mathcal{T}$ with $x^*_{t'} \in int(C_{t'})$ for a cell $C_{t'} := \bigcap_{m \in \mathcal{M}} d_{t'm} + K(p_{t'm})$, then there exists t_1, t_2 with $t_1 < t' < t_2$ such that $x^*_{t_1} \in \mathcal{I}(\mathcal{D}_{t_1})$, $x^*_{t_2} \in \mathcal{I}(\mathcal{D}_{t_2})$ and moreover, for all time-steps t between t_1 and t' as well as between t' and t_2 holds: $x^*_t \in \mathcal{CL}(\mathcal{D}_t)$,

(ii) there exists at least one $t \in \mathcal{T}$ with $x^*_t \in \mathcal{I}(\mathcal{D}_t)$,

(iii) let $\bar{t}_1$ and $\bar{t}_2$ be the smallest t_1, resp. the biggest t_2 in property (i) or $\bar{t}_1 = \bar{t}_2 = t$ in property (ii) if (i) does not apply, then $x^*_t \in \mathcal{CL}(\mathcal{D}_t)$ for $t = 1, \ldots, t_1, t_2 \ldots T$

Proof. The proof is constructive and returns a solution $X^* \in \mathcal{X}^*$ fulfilling the desired property:
Since x_1 must be an optimal solution to the static single facility problem with fixed $x_2, \ldots, x_t$, we can assume that $x_1 \in \mathcal{I}(\mathcal{D}_1 \cup x_2)$. The same holds for x_T. Hence, without loss of generality x_1 and x_T lie on the construction lines $\mathcal{CL}(\mathcal{D}_1)$, resp. $\mathcal{CL}(\mathcal{D}_T)$.
Recall that $G_t := \bigcap_{m \in \mathcal{M}} d_{tm} + K(p_{tm})$ are geometric objects formed by the cone conditions. Define r_t as the ray with initial point x_t and direction given by x_t and x_{t+1} and r_0 as the line lying on a boundary of G_1 going through x_1 and r_T respectively with G_T and x_T.

In the following, only the initial points of the rays will change, the direction will stay the same. Hence, by moving x_t along r_{t-1} in any direction for which G_t is bounded and adjusting x_{t+1} accordingly on r_t and r_{t+1} (see Figure 1) until x_t or x_{t+1} reaches its object boundary $bd(G_t)$, resp. $bd(G_{t+1})$ (see Figure 2), yields another optimal solution as the optimality conditions are still fulfilled.

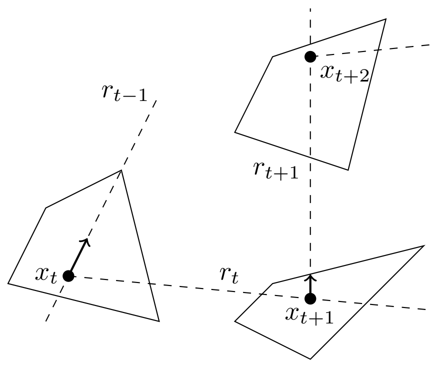


Figure 1: Modifying original optimal solution

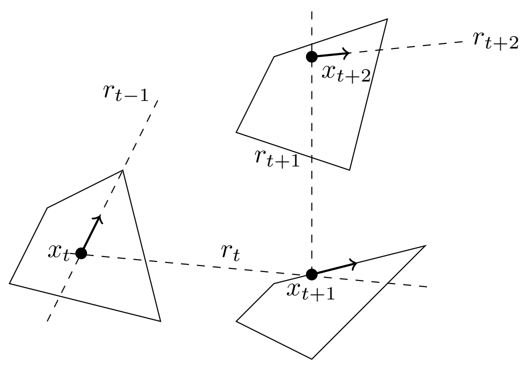


Figure 2: Another optimal solution

Now assume x_{t+1} reaches its boundary and, hence, moving x_t further and x_{t+1} along r_{t+1} violates the cone conditions (11). The only possibility is to move x_{t+1} along its cell boundary of G_t, like indicated in Figure 2 and adjusting x_{t+2} along r_{t+2} until it reaches its boundary (see Figure 3) or one of the previous time-steps reaches an extreme point. Note, that since no direction is modified, the cone conditions are still satisfied and hence, this can be iteratively repeated until moving x_t any further results in violating the optimality conditions. Since all cells are bounded in at least one direction, this will definitely happen.

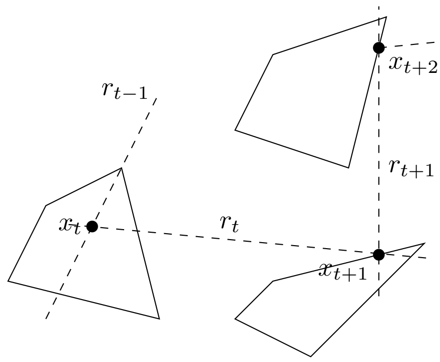


Figure 3: Moving the optimal solution on the boundaries

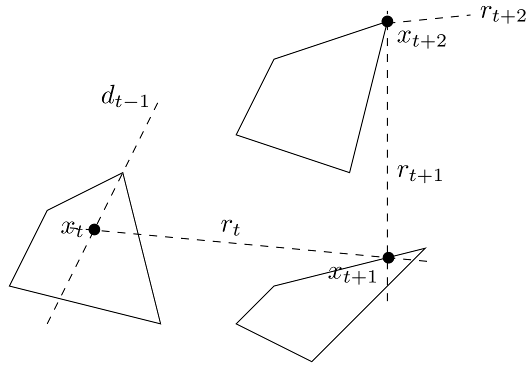
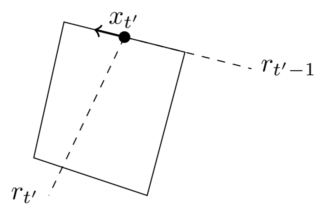


Figure 4: x_{t+2} reached an extreme point of its cell

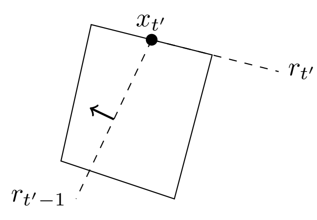
There are three cases to distinguish:

Case 1: Point x_t cannot be moved anymore, since it itself or one of the succeeding points reached an extreme point (like shown in Figure 4).

Case 2: There is a t' for which the boundary of the cell $G_{t'}$ lies on $r_{t'-1}$. In this case, we cannot keep moving x_t in the desired direction and no intersection point has been found yet. Instead, we choose a direction for $x_{t'}$ and move it along $r_{t'-1}$ and continue the process.



Case 3: There is a t' for which the boundary of the cell $G_{t'}$ lies on $r_{t'}$ or $t' = T$. We can move $x_{t'}$ along the boundary without adjusting $x_{t'+1}$. Hence, by continuing moving the solution, we must reach Case 1.



Since we only moved x_t along r_{t-1} all the optimality conditions for the previous time-steps are still satisfied.

When iteratively applying this procedure to all time-steps from 1 to T, we get the desired result for all parts (i),(ii) and (iii). When there is a $\mathcal{X}_t^* \subset int(G_t)$, then this procedure has already found a $t_1 < t$ in the previous iterations and in the subsequent, it must find t_2, which proves part (i). Part (ii) and (iii) immediately follow as one of the three cases have to occur. $\square$

Theorem 5.3. *Given $1/P/dyn/\bullet/\Sigma$ with minimal dimension and a pair (X, P) satisfying the optimality conditions in Theorem 4.2, then there exists an optimal solution $X^* \in \mathcal{X}^*$ such that:*

(i) For all $x_{t'}^* \in \mathcal{CL}(\mathcal{D}_{t'}) \setminus \mathcal{I}(\mathcal{D}_{t'})$ there exists a $\bar{t} \in \mathcal{T} \setminus \{t'\}$ with $x_{\bar{t}}^* \in \mathcal{I}(\mathcal{D}_{\bar{t}})$ and

$$x_t^* \in \begin{cases} \mathcal{I}(\mathcal{D}_t \cup x_{t+1}) \ \forall t = t', \dots, \bar{t}-1, & \text{if } t' < \bar{t}, \\ \mathcal{I}(\mathcal{D}_t \cup x_{t-1}) \ \forall t = \bar{t}+1, \dots, t', & \text{if } t' > \bar{t}. \end{cases}$$

(ii) If $x_{t'}^* \in int(C_{t'})$ with a cell $C_{t'} := \bigcap_{m \in \mathcal{M}} d_{t'm} + K(p_{t'm})$ for a $t' \in \mathcal{T}$, then there exists t_1, t_2 with $t_1 < t < t_2$, such that $x_{t_1} \in \mathcal{I}(\mathcal{D}_{t_1})$, $x_{t_2} \in \mathcal{I}(\mathcal{D}_{t_2})$ and

$$\begin{aligned} & x_t \in \mathcal{I}(\mathcal{D}_t \cup x_{t-1}) && \text{for } t = t_1+1, \dots, t'-1, \\ & x_t \in \mathcal{I}(\mathcal{D}_t \cup x_{t+1}) && \text{for } t = t'+1, \dots, t_2-1, \\ & x_{t'} \in \mathcal{I}(x_{t'-1} \cup x_{t'+1}). \end{aligned}$$

Proof. The proof is based on the same argumentation as in the previous proof. Assume we have an optimal solution $X \in \mathcal{X}^*$ satisfying all conditions in Theorem 5.2. Note, that there is at least one location $x_t \in \mathcal{I}(\mathcal{D}_t)$ and if there is a $x_t \in int(C_t)$ then it is surrounded by only locations on the construction lines and two locations on the intersection points.

(i) By symmetry we start at $x_1 \in \mathcal{X}_1^*$ and by Theorem 5.2 assume that there is a $x_{t'} \in \mathcal{I}(\mathcal{D}_{t'})$ s.t. $x_t \in \mathcal{CL}(\mathcal{D}_t)$ for $t = 1, \dots, t'$. Define the rays r_t as above. If the direction of r_1 is not the same as one of the fundamental directions defined by the extreme points $\mathsf{Ext}(B)$, then the optimality conditions (12) must yield a cone and not a ray. Hence, we can move x_1 along the boundary of its geometric object $G_1 := \bigcap_{m \in \mathcal{M}} d_{1m} + K(p_{1m})$ until it either reaches an extreme point, or the boundary of the cone $K(\tilde{p}_1)$ (see Figure 5). If it reaches the boundary of $K(\tilde{p}_1)$, then $x_1 \in x_2 + f_r$ for one of the fundamental directions f_r. If x_1 reaches an intersection point, then we can go to x_2 and do the same thing as with x_1; in the case that it reaches the boundary of the cone, we set the direction of r_1 to f_r and continue moving x_1 (see Figure 6). Like done in the proof of Theorem 5.2 we now have to adjust x_2 accordingly to x_1, but instead of adjusting x_3 right away, we will only adjust it when x_2 reaches the boundary of $K(\tilde{p}_2)$. Then we iteratively continue this process, until one of the points reaches an extreme point of its cell or all r_t are extreme points of the block norm. The case, where there is a x_t in the interior of its cell is covered in (ii). Note: When one the locations reaches an extreme point and we continue the procedure with the next location, we might have to adjust the previous locations accordingly.

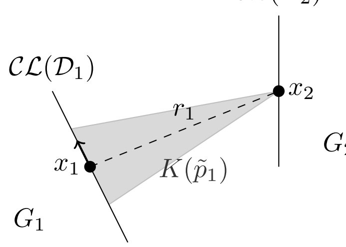


Figure 5: Moving x_1

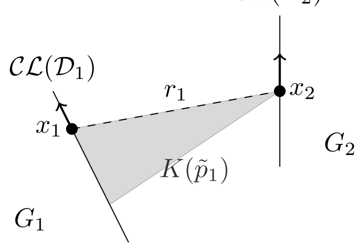


Figure 6: Moving x_2

(ii) Assume that we have a solution with property (i) and a point $x_{t'} \in int(C_{t'})$. Note that x_t has to be located in the cell $C := C_{t'} \cap (x_{t'+1} + K(\tilde{p}_{t'+1})) \cap (x_{t'-1} - K(\tilde{p}_{t'}))$ to fulfill the optimality conditions. This cell must contain an intersection point of $\mathcal{I}(\mathcal{D}_{t'} \cup x_{t'-1} \cup x_{t'+1})$. Hence, even if $\mathcal{X}^*_{t'} \subset int(C_{t'})$, there must be a point such that $x_{t'}$ fulfills the desired property, like shown in Figure 7.

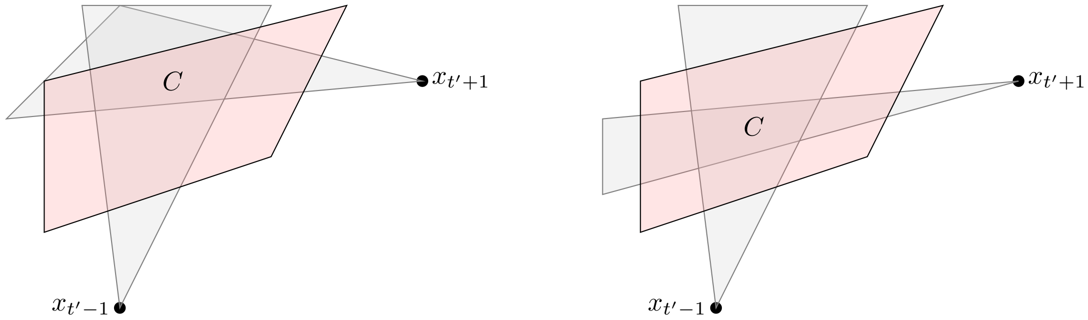


Figure 7: Different illustrations of (ii)

□

5.2 1FDynLoc General Result

So far we have considered 1FDynLoc with minimal dimension. However, the argumentation in the proofs won't work out if two locations for consecutive time-steps coincide. In this case, if $x_{t'} = x_{t'+1}$, then both can lie on $\mathcal{I}(\mathcal{D}_{t'} \cup \mathcal{D}_{t'+1})$. However, knowing that two time-steps $x_{t'} = x_{t'+1}$ coincide, we can reduce the size of the problem the following way:
Consider the parts in the objective function F, where $x_{t'}, x_{t'+1}$ appears:

1. The part measuring the distances between the new locations:

$$\begin{aligned} \dots + v_{t'-1}\|x_{t'} - x_{t'-1}\| + v_{t'}\|x_{t'+1} - x_{t'}\| + v_{t'+1}\|x_{t'+2} - x_{t'+1}\| + \dots \\ = \dots + v_{t'-1}\|x_{t'} - x_{t'-1}\| + v_{t'+1}\|x_{t'+2} - x_{t'}\| + \dots \end{aligned}$$

2. the part measuring the distances between the new location and the facility:

$$\begin{aligned} &\dots + \sum_{m=1}^{M} w_{t'm}\|x_{t'} - d_{t'm}\| + w_{(t'+1)m}\|x_{t'+1} - d_{(t'+1)m}\| + \dots \\ &= \dots + \underbrace{\sum_{m=1}^{M} w_{t'm}\|x_{t'} - d_{t'm}\| + w_{(t'+1)m}\|x_{t'} - d_{(t'+1)m}\|}_{\text{2M facilities at time step } t'} + \dots \end{aligned}$$

Hence, after rearranging and defining the weights and demands accordingly, one can reduce the dimension of the dynamic single facility location problem with time horizon T to one with time horizon $T-1$. However, the demand points at time step $t'+1$ join the demand points at time t' and we have $2M$ demand points.
By iteratively applying the problem reduction we get the following Theorem:

Theorem 5.4. *There exists an optimal solution (X, P) that satisfies the optimality conditions in Theorem 4.2 and that has $k \leq T$ non-coincident consecutive facilities (i.e., for a sequence of time-steps $1 = \tau_1 < \tau_2 < \dots < \tau_k < \tau_{k+1} := T+1$ holds $x_{\tau_i} \neq x_{\tau_{i+1}}$ for all $i = 1, \dots, k-1$ and $x_{\tau_i} = \dots = x_{(\tau_{i+1}-1)}$ for all $i = 1, \dots, k$) such that further:*

(i) There exists at least one τ_i with:

$$x_{\tau_i} \in \mathcal{I}(\bigcup_{t\in\Pi_i} \mathcal{D}_t),$$

with $\Pi_i := \{\tau_i, \ldots, \tau_{i+1} - 1\}$.

(ii) for all $x^*_{\tau_i} \in \mathcal{CL}(\bigcup_{t\in\Pi_i} \mathcal{D}_t) \setminus \mathcal{I}(\bigcup_{t\in\Pi_i} \mathcal{D}_t)$ there exists an $m \in \{1, \ldots, k\} \setminus \{i\}$ with $x^*_{\tau_m} \in \mathcal{I}(\bigcup_{t\in\Pi_m} \mathcal{D}_t)$ and

$$x^*_{\tau_l} \in \begin{cases} \mathcal{I}(\bigcup_{t\in\Pi_l} \mathcal{D}_t \cup x_{\tau_{l+1}})\ \forall l = i, \ldots, m-1, & \textit{if } i < m, \\ \mathcal{I}(\bigcup_{t\in\Pi_l} \mathcal{D}_t \cup x_{\tau_{l-1}})\ \forall l = m+1, \ldots, i, & \textit{if } i > m. \end{cases}$$

(iii) Moreover, if $x^*_{\tau_i} \in int(C)$ for a cell

$$C := \bigcap_{t\in\Pi_i} \bigcap_{m\in\mathcal{M}} d_{tm} + K(p_{tm}),$$

then there exists $m, n \in \{1, \ldots, k\}$ with $m < i < n$, such that $x_{\tau_m} \in \mathcal{I}(\bigcup_{t\in\Pi_m} \mathcal{D}_t)$, $x_{\tau_n} \in \mathcal{I}(\bigcup_{t\in\Pi_n} \mathcal{D}_t)$ and

$$\begin{aligned} x_{\tau_l} &\in \mathcal{I}(\bigcup_{t\in\Pi_l} \mathcal{D}_t \cup x_{\tau_{l-1}}) && \textit{for } l = m+1, \ldots, i-1, \\ x_{\tau_l} &\in \mathcal{I}(\bigcup_{t\in\Pi_l} \mathcal{D}_t \cup x_{\tau_{l+1}}) && \textit{for } l = i+1, \ldots, n-1, \\ x_{\tau_i} &\in \mathcal{I}(x_{\tau_{i-1}} \cup x_{\tau_{i+1}}). \end{aligned}$$

Proof. The proof immediately follows from the Theorems 5.2 and 5.3 and the dimension reduction introduced in this Section. □

The previous result gives us an FDS result for $1/P/dyn/\bullet/\Sigma$. However, although the structure of 1FDynLoc is much simpler than $N/P/\bullet/\gamma/\Sigma$ calculating the FDS is exponential in T.

5.3 Construction Line Algorithm

Algorithm for $\mathbf{1/P/dyn, T = 2/\gamma/\Sigma}$

To understand the general case, first have a look when there are only two time-steps. By the previous result, one has to distinguish the following cases:

- $x_1 \in \mathcal{I}(\mathcal{D}_1)$ and $x_2 \in \mathcal{I}(\mathcal{D}_2)$:
  In this case, we only have to check
  $$(x_1^*, x_2^*) = \arg \min_{(x_1,x_2)\in\mathbb{R}^4} \{F(x_1, x_2) : x_1 \in \mathcal{I}(\mathcal{D}_1), x_2 \in \mathcal{I}(\mathcal{D}_2)\}.$$
- $x_1 = x_2 \in \mathcal{I}(\mathcal{D})$:
  When $x_1 = x_2$, then it is possible that the optimal solution lies of the construction lines of $\mathcal{I}(\mathcal{D}_1 \cup \mathcal{D}_2)$, therefore we have to check
  $$(x_1^*, x_2^*) = \arg \min_{(x,x)\in\mathbb{R}^4} \{F(x, x) : x \in \mathcal{I}(\mathcal{D})\}.$$
- x_1, x_2 lie on the construction lines of each other:
  Excluding the first case, either $x_1 \in \mathcal{I}(\mathcal{D}_1)$ and $x_2 \in \mathcal{I}(\mathcal{D}_2 \cup x_1) \setminus \mathcal{I}(\mathcal{D}_2)$ or $x_2 \in \mathcal{I}(\mathcal{D}_2)$ and $x_1 \in \mathcal{I}(\mathcal{D}_1 \cup x_2) \setminus \mathcal{I}(\mathcal{D}_1)$:
  $$\begin{aligned} S_1 &:= \{(x_1, x_2) \in \mathbb{R}^4 : x_1 \in \mathcal{I}(\mathcal{D}_1), x_2 \in \mathcal{I}(\mathcal{D}_2 \cup x_1) \setminus \mathcal{I}(\mathcal{D}_2)\}, \\ S_2 &:= \{(x_1, x_2) \in \mathbb{R}^4 : x_2 \in \mathcal{I}(\mathcal{D}_2), x_1 \in \mathcal{I}(\mathcal{D}_1 \cup x_2) \setminus \mathcal{I}(\mathcal{D}_1)\} \end{aligned}$$

and, therefore,

$$(x_1^*, x_2^*) = \arg \min_{(x_1,x_2)\in\mathbb{R}^4} \{F(x_1, x_2) : (x_1, x_2) \in S_1 \cup S_2\}.$$

The complete procedure is shown in Algorithm 1.

Algorithm 1: Solving $1/P/dyn, T = 2/\gamma/\Sigma$ ($\mathcal{O}(M^5R^4 \log R)$)

---

input : $1/P/dyn, T = 2/\gamma/\Sigma$
output: Optimal Solution $X \in \mathbb{R}^{2T}$

```
// Calculate FDS:
```
$FDS_1 := \{(x_1, x_2) : x_1 \in \mathcal{I}(\mathcal{D}_1), x_2 \in \mathcal{I}(\mathcal{D}_2)\}$;
$FDS_2 := \{(x, x) : x \in \mathcal{I}(\mathcal{D})\}$;
$FDS_3 := \{(x_1, x_2) : x_1 \in \mathcal{I}(\mathcal{D}_1), x_2 \in \mathcal{I}(\mathcal{D}_2 \cup x_1) \setminus \mathcal{I}(\mathcal{D}_2)\}$;
$FDS_4 := \{(x_1, x_2) : x_2 \in \mathcal{I}(\mathcal{D}_2), x_1 \in \mathcal{I}(\mathcal{D}_1 \cup x_2) \setminus \mathcal{I}(\mathcal{D}_1)\}$;
$FDS := \bigcup_{i=1}^4 FDS_i$;
```
// Check for minimum objective function value:
```
$(x_1^*, x_2^*) := \arg\min_{(x_1,x_2)\in\mathbb{R}^4}\{F(x_1, x_2) : x_1, x_2 \in FDS\}$

Theorem 5.5. *Algorithm 1 runs in $\mathcal{O}(M^5R^4 \log R)$ using complete enumeration, given $\mathcal{O}(MR)$ space and $\mathcal{O}(R \log R)$ preprocessing, using a suitable implementation.*

Proof. For the preprocessing we sort the extreme points of B in clockwise order, which can be done in $\mathcal{O}(R \log R)$. This realizes the possibility to calculate the norm of a vector in $\mathcal{O}(\log R)$ using suitable search structures like binary search. For the actual algorithm we distinguish the following two cases:

Case 1 ($\mathbf{x_1^* = x_2^*}$): When $x := x_1 = x_2$ we know the problem reduces to the static single facility location problem $1/P/static/\gamma/\Sigma$ and hence the optimal solution is one of the intersection points of $\mathcal{I}(\mathcal{D})$. The number of intersection points is bounded by $\frac{2MR(2MR-1)}{2}$.

Case 2 ($\mathbf{x_1^* \neq x_2^*}$): In case of $x_t \in \mathcal{I}(\mathcal{D}_t)$ for both $t = 1, 2$, the number of possible solutions is bounded by $\left(\frac{MR(MR-1)}{2}\right)^2$, however, as one of the solutions can be on the construction line made by the other time-step, we also have to calculate the intersection points in S_1 and S_2. As there are at most $\frac{MR(MR-1)}{2}$ intersection points for one time-step, each having R lines going through it possibly intersecting each of the MR construction lines from the other time-step, we have an upper bound of $2\frac{MR(MR-1)}{2} \cdot R \cdot MR + \left(\frac{MR(MR-1)}{2}\right)^2 \in \mathcal{O}(M^4R^4)$.

Both cases together we have $\mathcal{O}(M^4R^4)$ intersection points to check for optimality.
However, for calculating $F(x_1, x_2)$ we have to calculate the distance of $2M + 1$ vectors. Using prepossessing, the distance can be calculated in $\mathcal{O}(\log R)$.
As the FDS can be calculated on the spot while checking for the best objective value gives the space complexity and finishes the proof. □

General Case: 1/P/dyn/γ/Σ

As one can see, constructing the Finite Dominating Set for $1/P/dyn/\gamma/\Sigma$ is exponential in T as there are already $\mathcal{O}((MR)^{2T})$ possible solutions lying on the intersection points $\mathcal{I}(\mathcal{D}_t)$ for $t \in \mathcal{T}$. Jumping back to a problem with minimal dimension, there are three possibilities for an optimal location at a time-step t:

1. $x_t \in \mathcal{I}(\mathcal{D}_t)$,
2. $x_t \in \mathcal{I}(\mathcal{D}_t \cup x_{t-1}) \setminus \mathcal{I}(\mathcal{D}_t)$ or $x_t \in \mathcal{I}(\mathcal{D}_t \cup x_{t+1}) \setminus \mathcal{I}(\mathcal{D}_t)$,
3. $x_t \in \mathcal{I}(x_{t-1} \cup x_{t+1})$.

By the same argumentation as in the case for $T = 2$, we know that the complexity of the first case dominates the second one, and by similar argumentation it also dominates the third case.
To get a valid upper bound for the general case, assume $t + 1$ facilities coincide ($t = 0, \dots, T - 1$). Overall, there are $\binom{T-1}{t}$ possibilities for these facilities to coincide. More precisely, these facilities may not necessarily coincide consecutively.
Assume for a fixed t we have K separate components of coincident facilities, each component contains $k_i \in \mathbb{N}_{\geq 2}$ coincident facilities ($i \leq K$). Therefore, we must have $\sum_{i=1}^{K} k_i = t + 1$.
Overall, the number of optimal points for $t + 1$ coincident locations is bounded by

$$\begin{aligned}\mathcal{O}&\left(\prod_{i=1}^{K} \frac{k_i MR(k_i MR - 1)}{2} \cdot \left(\frac{MR(MR-1)}{2}\right)^{T-(t+1)}\right)\\ &= \mathcal{O}\left(\left(\prod_{i=1}^{K} k_i\right)^2 \cdot \left(\frac{M^2R^2}{2}\right)^{T-(t+1)+K}\right),\end{aligned}$$

where K can vary.
Now, we want to find an upper bound for the first part of the product. Therefore, consider

$$\begin{aligned}\max_{K\in\mathbb{N}} \quad & \prod_{i=1}^{K} k_i\\ \text{s.t.} \quad & \sum_{i=1}^{K} k_i = t + 1\\ & k_i \in \mathbb{N}_{\geq 2}.\end{aligned}$$

In the following, we will show that this is bounded by $4 \cdot 3^{\lfloor t+1/3 \rfloor}$:
Assume there is a $k_i > 4$ which is supposed to be optimal. Then, by introducing an additional variable $k_{K+1} := k_i - 2$ and setting $k_i = 2$ we still have a feasible solution. However, $k_i \cdot k_{K+1} = 2k_{K+1} = 2k_i - 4 > k_i$ as $k_i > 4$. Therefore, no k_i can be bigger than 4. Moreover, as $2+2 = 2\cdot 2 = 4$ there exists an optimal solution of the form $2^a 3^b$. More precisely, as $2+2+2 = 3+3$ and $2^3 < 3^2$, a must be less than 3, i.e. $a \leq 2$, which gives us an upper bound of $2^2 3^b$. By the constraints, we also know that $3b \leq t + 1$, otherwise the IP would be infeasible. Therefore $b \leq \lfloor (t+1)/3 \rfloor$, which finishes the claim.
Applying the binomial Theorem, we have

$$\begin{aligned}\mathcal{O}&\left(\left[\sum_{t=0}^{T-1} \binom{T-1}{t} 9^{t/3}\right] \cdot (MR)^{2T}\right)\\ &= \mathcal{O}\left(\left(1 + \sqrt[3]{9}\right)^T (MR)^{2T}\right),\end{aligned}$$

including the possibilities when the facilities lie on the construction lines of each other. Calculating the objective value takes additional $\mathcal{O}(M \log R)$, which gives us a total approximated upper bound of

$$\mathcal{O}\left(3.08^T M^{2T+1} R^{2T} \log R\right)$$

which might become unusable in praxis for very big T if we use complete enumeration. Note that one might be able to get a better bound by considering the $(1/2)^{T-(t+1)+K}$, which we have thrown away as $T - (t + 1) + K \geq 0$.

6 Conclusion and Future Research

In a first step to model security guards, first-aider and other staff at public events we have introduced different objectives for single facility, single objective location problems with time depending data. We have shown for the dynamic median version that the problem is a special case of the static multifacility median problem. The main result is the derivation of a finite dominating set which yields an efficient solution algorithm whenever the number of time periods is assumed to be fixed. The same idea can be used to deal with the more general case of static, planar location problems for T new facilities.

Various ideas for improvement and extensions are possible (see [Mai16]):
The running time of the algorithm can be improved by adapting coincidence results (see [Pla92], [LMP90] or [FMP98]) of multifacility location problems. Using the result of Durier and Michelot [DM85] that the optimal solution lies in the metric hull of the demand points, the size of the FDS can be reduced. Instead of enumerating all candidates in the FDS a suitable "search direction" can be used to speed up the algorithm. The block norms considered in this paper can be extended to polyhedral gauges, since the symmetry property has not been used in any of the proofs (see, for instance, Lefebvre et al. [LMP90] and Plastria [Pla92]).
For the dynamic multifacility case, it will be more suitable to consider a location-allocation model within each time-step. This is justified by our real-world problem in public events, since it is, for instance, sufficient to have a single patrol around a potential danger site.
Another research topic is to consider the objective functions under constraints, for example in public events security guards often walks in elliptic curves. Therefore, we might have constraints like

$$x_t = (r_1 \sin(t), r_2 \cos(t)),$$

for the major radius $r_1 \in \mathbb{R}_{>0}$ and the minor radius $r_2 \in \mathbb{R}_{>0}$. If we have restricted areas, like a lake or anything else, it is not possible to apply current known algorithms for the multifacility location problem with restricted areas, as the security guard possibly might "jump" around the lake, which does not yield a meaningful trajectory in our case. Therefore, dynamic location problems with barriers, in which trespassing of forbidden regions is part of the model, is an interesting modeling extension.

References

[Ata83] M. J. Atallah. Dynamic computational geometry. *Foundations of Computer Science, 24th Annual Symposium*, pages 92–99, 1983.

[Bal68] Ronald H. Ballou. Dynamic warehouse location analysis. *Journal of Marketing Research (JMR)*, 5:271–276, 1968.

[Bal70] Ronald H. Ballou. Computational limitations of dynamic programming for warehouse location: A comment. *Journal of Marketing Research (JMR)*, 7:263–264, 1970.

[Cam90] James F. Campbell. Locating transportation terminals to serve an expanding demand. *Transportation Research Part B: Methodological*, 24(3):173 – 192, 1990.

[CCMMP97] E. Carrizosa, E. Conde, M. Munoz-Márquez, and J. Puerto. Simpson points in planar problems with locational con- straints. the polyhedral-gauge case. *Mathematics of Operations Research*, 22(2):291–300, 1997.

[Cha88] S. Chand. Decision/forecast horizon results for a single facility dynamic location/relocation problem. *Operations Research Letters*, 7:247–251, 1988.

[DH04] Z. Drezner and H.W. Hamacher. *Facility Location: Applications and Theory.* Springer series in operations research. Springer, 2004.

[DM85] Roland Durier and Christian Michelot. Geometrical properties of the fermat-weber problem. *European Journal of Operational Research*, 20(3):332–343, 1985.

[Dre95a] Z. Drezner. *Facility Location: A Survey of Applications and Methods*. Springer Series in Operations Research and Financial Engineering. Springer New York, 1995.

[Dre95b] Zvi Drezner. Dynamic facility location: the progressive p-median problem. *Location Science*, 3(1):1–7, 1995.

[Dur90] R. Durier. On pareto optima, the fermat-weber problem, and polyhedral gauges. *Mathematical Programming*, 47(1):65–79, 1990.

[DW91] Z. Drezner and G. O. Wesolowsky. Facility location when demand is time dependent. *Naval Research Logistics*, 38:763–777, 1991.

[FDA09] Reza Zanjirani Farahani, Zvi Drezner, and Nasrin Asgari. Single facility location and relocation problem with time dependent weights and discrete planning horizon. *Annals of Operations Research*, 167(1):353–368, 2009.

[Fli98] J. Fliege. Solving convex location problems with gauges in polynomial time. Technical report, Fakultät für Mathematik, TU Dortmund, 1998. Ergebnisberichte des Instituts für Angewandte Mathematik, Nummer 158.

[FMP98] J. Fliege, Michelot, and Plastria. A polynomial time coincidence detector for multifacility location problems. Technical report, Fakultät für Mathematik, TU Dortmund, 1998. Ergebnisberichte des Instituts für Angewandte Mathematik, Nummer 152.

[FSG14] Reza Zanjirani Farahani, W.Y. Szeto, and Saeed Ghadimi. The single facility location problem with time-dependent weights and relocation cost over a continuous time horizon. *Journal of the Operational Research Society*, 66:265–277, 2014.

[HN95] H. W. Hamacher and S. Nickel. Restricted planar location problems and applications. *Naval Research Logistics (NRL)*, 42(6):967–992, 1995.

[HNS96] Horst W. Hamacher, Stefan Nickel, and Anja Schneider. Classification of location problems. Technical report, Location Science, 1996.

[LMP90] O. Lefebvre, C. Michelot, and F. Plastria. Geometric interpretation of the optimality conditions in multifacility location and applications. *Journal of Optimization Theory and Applications*, 65(1):85–101, 1990.

[LMW88] R.F. Love, J.J.G. Morris, and G.O. Wesolowsky. *Facilities Location: Models & Methods*. Publications in Operations Research. North-Holland, 1988.

[Lod70] Leonard M. Lodish. Computational limitations of dynamic programming for warehouse location. *Journal of Marketing Research (JMR)*, 7:263–264, 1970.

[Mai16] A. Maier. *Location Problems with Time-Dependent Data (Working title)*. PhD Thesis, in progress, University of Kaiserslautern. 2016.

[Med86] N. Mediggo. Dynamic facility location. *Annals of Operation Research*, 6:313–319, 1986.

[ML87] C. Michelot and O. Lefebvre. A primal-dual algorithm for the fermat-weber problem involving mixed gauges. *Mathematical Programming*, 39(3):319–335, 1987.

[Nic98] S. Nickel. Restricted center problems under polyhedral gauges. *European Journal of Operational Research*, 104(2):343–357, 1998.

[Pla92] Frank Plastria. When facilities coincide: Exact optimality conditions in multifacility location. *Journal of Mathematical Analysis and Applications*, 169(2):476 – 498, 1992.

[ST76] Dennis J. Sweeney and Ronad L. Tatham. An improved long-run model for multiple warehouse location. *Management Science*, 22(7):748–758, 1976.

[Wes73] G. O. Wesolowsky. Dynamic facility location. *Management Science*, 19:1241–1248, 1973.

[WW80] J. E. Ward and R. E. Wendell. A new norm for measuring distance which yields linear location problems. *Operations Research*, 28:836–844, 1980.

[WW85] J. E. Ward and R. E. Wendell. Using block norms for location modeling. *Operations Research*, 33(5):1074–1090, 1985.