Alternative Formulations for the Ordered Weighted Averaging Objective*

André Chassein and Marc Goerigk†

Fachbereich Mathematik, Technische Universität Kaiserslautern, Germany

Abstract

The ordered weighted averaging objective (OWA) is an aggregate function over multiple optimization criteria which received increasing attention by the research community over the last decade. Different to the ordered weighted sum, weights are attached to ordered objective functions (i.e., a weight for the largest value, a weight for the second-largest value and so on). As this contains max-min or worst-case optimization as a special case, OWA can also be considered as an alternative approach to robust optimization.

For linear programs with OWA objective, compact reformulations exist, which result in extended linear programs. We present new such reformulation models with reduced size. A computational comparison indicates that these formulations improve solution times.

Keywords: multi-criteria optimization; ordered weighted averaging; linear programming

1 Introduction

We consider multi-criteria optimization problems of the form

$$\text{vec-max} \left\{ Cx \mid x \in \mathcal{X} \right\} \tag{1}$$

where $C \in \mathbb{R}^{k \times n}$ is a matrix of linear objective functions and $\mathcal{X} \subseteq \mathbb{R}^n$ denotes some set of feasible solutions; e.g., for linear programs, we have

$$\mathcal{X} = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$$

for a coefficient matrix $A \in \mathbb{R}^{m \times n}$ and a right-hand size $b \in \mathbb{R}^m$.

To formulate the ordered weighted averaging (OWA) aggregate function as introduced by Yager [Yag88], we further consider the ordering map $\Theta : \mathbb{R}^k \to \mathbb{R}^k$

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†Corresponding author. Email: goerigk@mathematik.uni-kl.de

with $\Theta(y) = (\theta_1(y), \theta_2(y), \ldots, \theta_k(y))$ that permutes the vector components of y such that $\theta_i(y) \leq \theta_{i+1}(y)$. Given a weight vector $w \in \mathbb{R}^k$, the OWA problem is then defined as

$$\max \left\{ \sum_{i\in[k]} w_i\theta_i(Cx) \mid x \in \mathcal{X} \right\} \tag{2}$$

where we use the notation $[k] := \{1, \ldots, k\}$. As frequently done, we make the assumptions of *equitability*, which is given if $w_1 > w_2 > \ldots > w_k > 0$ (see also [KOW04]). In [OŚ03], this case is considered for linear programs, and reformulations of (2) are presented. They show that the problem is equivalent to

$$\begin{aligned}
&\max\ z && && (3)\\
&\text{s.t. } Cx = y && && (4)\\
&\quad z \leq \sum_{i\in[k]} w_{\tau(i)} y_i && \forall \tau \in \Pi && (5)\\
&\quad x \in \mathcal{X} && && (6)
\end{aligned}$$

where Π denotes all permutations of $[k]$. The variables y are redundant, but often improve solver performance. Dualizing this problem, one can use column generation over the dual variables associated with the permutations. Furthermore, they present the following compact model:

$$\begin{aligned}
&\max\ \sum_{j\in[k]} j w'_j r_j - \sum_{i\in[k]} \sum_{j\in[k]} w'_j d_{ij} && && (7)\\
&\text{s.t. } Cx = y && && (8)\\
&\quad d_{ij} \geq r_j - y_i && \forall i, j \in [k] && (9)\\
&\quad d_{ij} \geq 0 && \forall i, j \in [k] && (10)\\
&\quad x \in \mathcal{X} && && (11)
\end{aligned}$$

where $w'_j = w_j - w_{j+1}$ for all $j = 1, \ldots, k-1$ and $w'_k = w_k$. A model of this type has been applied, e.g., to the multi-objective spanning tree problem in [GS12], and to facility location problems in [KNPRC10]. In the following we present a new model that can be useful to any of these application.

2 Alternative Models

We present different approaches to reformulate problem (2). Starting from the observation that

$$\sum_{i\in[k]} w_i\theta_i(Cx) = \min_{\tau\in\Pi} \sum_{i\in[k]} w_{\tau(i)}\theta_i(Cx)$$

as presented in [OŚ03], we reconsider the model

$$\begin{aligned}
&\max\ \min_{\tau\in\Pi} \sum_{i\in[k]} w_{\tau(i)} y_i && (12)\\
&\text{s.t. } Cx = y && (13)\\
&\quad x \in \mathcal{X} && (14)
\end{aligned}$$

Note that instead of considering all permutations in Π, one can also relax the objective function (12) and use the convex hull of permutations $conv(\Pi)$ instead, i.e., the permutahedron P^{Π}. This does not change the objective value of the inner minimum, as there is always a corner of P^{Π} at which the optimum is attained. Therefore, we can rewrite the inner optimization problem

$$\min_{\tau \in \Pi} \sum_{i \in [k]} w_{\tau(i)} y_i$$

for fixed values of y as

$$\begin{aligned}
\min \ & \sum_{i \in [k]} \sum_{j \in [k]} w_j y_i p_{ij} && && (15)\\
\text{s.t.} \ & \sum_{i \in [k]} p_{ij} = 1 && \forall j \in [k] && (16)\\
& \sum_{j \in [k]} p_{ij} = 1 && \forall i \in [k] && (17)\\
& p_{ij} \geq 0 && \forall i, j \in [k] && (18)
\end{aligned}$$

the dual of which is

$$\begin{aligned}
\max \ & \sum_{i \in [k]} (\alpha_i + \beta_i) && && (19)\\
\text{s.t.} \ & \alpha_i + \beta_j \leq w_j y_i && \forall i, j \in [k] && (20)
\end{aligned}$$

Using this reformulation of the inner problem, we get the following new compact formulation for problem (2):

$$\begin{aligned}
\max \ & \sum_{i \in [k]} (\alpha_i + \beta_i) && && (21)\\
\text{s.t.} \ & Cx = y && && (22)\\
& \alpha_i + \beta_j \leq w_i y_j && \forall i, j \in [k] && (23)\\
& x \in \mathcal{X} && && (24)
\end{aligned}$$

This formulation needs $3k$ additional variables compared to the original problem (1), and $k^2 + k$ new constraints. In comparison, model (7–11) requires $k^2 + 2k$ additional variables and $k^2 + k$ new constraints. Thus, the formulation we propose needs an order of k variables less. Furthermore, it is numerically better tractable.

Alternatively, for linear programming problems with $\mathcal{X} = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$ one might also use an extended description of P^{Π} to do column generation in the primal problem (note that in [OŚ03], column generation needed the dual). To this end, we write for the inner optimization problem as

$$\begin{aligned}
\min \ & \sum_{i \in [k]} y_i p_i && && (25)\\
\text{s.t.} \ & \sum_{i \in S} p_i \geq \varphi_{|S|} && \forall \emptyset \subsetneq S \subsetneq [k] && (26)
\end{aligned}$$

$$\sum_{i \in [k]} p_i = \sum_{i \in [k]} w_i \tag{27}$$

where $\varphi_\ell = \sum_{j=k+1-\ell}^{k} w_j$, i.e., φ_ℓ is equal to the sum of the ℓ smallest weights w (see, e.g., [Pos09] for this description of the permutahedron based on Rado's inequality [Rad52]).

Via dualization, this description gives the following problem formulation:

$$\max \sum_{S \in P([k])} \varphi_{|S|} \alpha_S \tag{28}$$

$$\text{s.t. } Ax = b \tag{29}$$

$$Cx = y \tag{30}$$

$$\sum_{\substack{S \in P([k]) \\ i \in S}} \alpha_S = y_i \qquad \forall i \in [k] \tag{31}$$

$$x_j \geq 0 \qquad \forall j \in [n] \tag{32}$$

$$\alpha_S \geq 0 \qquad \forall S \in P([k]) \setminus \{[k]\} \tag{33}$$

where the exponentially many variables α_S can be generated as required using column generation. The pricing problem can be easily solved by sorting the dual variables of Constraint (31). Then for $\ell = 1, \ldots, k$ one needs to check if the sum of the ℓ smallest dual variables is less or equal to φ_ℓ. If that is the case, the variable corresponding this subset is added to the master problem.

3 Computational Experiments

In this section we compare the compact model of [OŚ03] with our model. Note that the purpose here is not to find the fastest solution algorithm: To this end, other approaches (such as column generation or constraint generation) should be used with careful consideration of the application at hand. Rather, our intention it to show the advantage of using $O(k)$ additional variables instead of $O(k^2)$.

We use Cplex v.12.6. All experiments were conducted on a computer with a 16-core Intel Xeon E5-2670 processor, running at 2.60 GHz with 20MB cache, and Ubuntu 12.04. Processes were pinned to one core.

For benchmark instances, we consider the same simplified portfolio optimization problems as [OŚ03], that is,

$$\max \left\{ Cx : \sum_{i \in [n]} x_i = 1, x \geq 0 \right\}$$

Problem parameters are generated in the same way. Objective values c_{ij} are generated by choosing a random value r_j uniformly distributed in $[0.05, 0.15]$ for every column $j \in [n]$. Then, for every row we choose $c_{ij} \in [-0.75 r_j, r_j]$ uniformly distributed. Weights w_j for the OWA objective are generated so that $w_k = 1$, and the difference between two subsequent values is uniformly distributed in $[1, 2]$, except for 5 values on average, which have a difference in $[1, k/3]$. Note that weights generated this way fulfill the equitability criterion.

		# items (n)									
		Old model					New model				
		20	40	60	80	100	20	40	60	80	100
# objectives (k)	40	0.09	0.12	0.15	0.16	0.17	0.06	0.07	0.08	0.08	0.09
	50	0.23	0.32	0.39	0.44	0.44	0.15	0.17	0.17	0.18	0.19
	60	0.46	0.65	0.77	0.86	0.89	0.28	0.32	0.35	0.36	0.37
	70	0.84	1.13	1.40	1.51	1.68	0.48	0.53	0.58	0.61	0.63
	80	2.36	3.15	3.53	3.65	4.04	0.35	0.42	0.46	0.50	0.54
	90	4.30	5.36	6.24	6.96	7.21	0.51	0.62	0.70	0.74	0.79
	100	6.19	8.04	9.51	10.03	10.55	0.73	0.88	0.99	1.06	1.14

Table 1: Primal simplex. Average computation times in seconds.

		# items (n)									
		Old model					New model				
		20	40	60	80	100	20	40	60	80	100
# objectives (k)	40	0.07	0.09	0.10	0.10	0.11	0.03	0.03	0.04	0.04	0.05
	50	0.16	0.20	0.22	0.24	0.25	0.06	0.06	0.07	0.08	0.09
	60	0.31	0.39	0.45	0.48	0.51	0.11	0.11	0.14	0.16	0.17
	70	0.56	0.71	0.81	0.89	0.92	0.18	0.20	0.23	0.26	0.27
	80	1.03	1.22	1.33	1.51	1.68	0.08	0.16	0.26	0.31	0.41
	90	1.66	1.96	2.21	2.60	2.88	0.11	0.20	0.33	0.43	0.50
	100	1.44	1.70	2.96	3.20	3.20	0.13	0.23	0.40	0.56	0.73

Table 2: Dual simplex. Average computation times in seconds.

We compare formulation (7–11) ("old model") with formulation (21–24) ("new model") using the primal simplex method in Table 1 and the dual simplex method in Table 2, averaging over 50 instances of each size.

As can be seen, the new formulation can be solved up to nearly 10 times faster for the primal simplex, and up to around 5 times faster using the dual simplex method. Furthermore, the new model scales better with k than the old model.

4 Conclusion

We proposed a new and simple model to solve multi-criteria problems which use the ordered weighted averaging aggregation. While our compact model reduces the required number of variables by a factor k compared to a formulation currently in use, an extended formulation allows for column generation in the primal problem instead of the dual problem as before. Furthermore, experimental experience suggests that our formulation is considerably easier to solve.

Further research should include if similar progress can be made on problems with the even harder weighted ordered weighted averaging (WOWA) criterion, see [OŚ09].

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