Geometric Ergodicity of Binary Autoregressive Models with Exogenous Variables

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Abstract

In this paper we introduce a binary autoregressive model. In contrast to the typical autoregression framework, we allow the conditional distribution of the observed process to depend on past values of the time series and some exogenous variables. Such processes have potential applications in econometrics, medicine and environmental sciences. In this paper, we establish stationarity and geometric ergodicity of these processes under suitable conditions on the parameters of the model. Such properties are important for understanding the stability properties of the model as well as for deriving the asymptotic behavior of the parameter estimators.

1 Introduction and Model Definition

Binary time series are important in many areas of applications, e.g., econometrics, medical sciences and meteorology. They typically occur if one is observing whether a certain event has or has not occurred within a given time frame. Wilks and Wilby [6] for example observe, whether it has been raining on a specific day, Kauppi and Saikkonen [3] and Startz [4] observe whether the US economy were in the recession in a given month.

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Let us consider the Binary autoregressive -BAR- process with exogenous variables defined via its conditional distribution given the past realizations of the observed process as well as those of the exogenous variables,

$$Y_t \mid Y_{t-1}, Y_{t-2}, \ldots, U_{t-1}, U_{t-2}, \ldots, \sim \ Bern(\pi_t(\beta)) \tag{1.1}$$

with $Z_t = (Y_t, \ldots, Y_{t-p+1}, U_t, \ldots, U_{t-q+1})'$. It then follows $\beta \in \mathbb{R}^{p+q}$. Additionally,

$$\pi_t(\beta) : \mathbb{R} \longrightarrow [0, 1],$$

for example,

$$\pi_t(\beta) = g^{-1}(\beta' Z_{t-1})$$

with the canonical link function

$$g(x) = \log\left(\frac{x}{1-x}\right)$$

as popular choice.

The model defined here can be purely autoregressive or exogenous based or a mixture of both. The purely autoregressive case $p \geq 1, q = 0$, is analyzed by Wang and Li [5]. For sake of simplicity and for illustrating the challenges we have to deal with, while dealing with the more general model, we focus on the case $p = q = 1$ here. However, the results presented can be extended to the higher order processes, i.e., $p, q \geq 1$.

Let us postulate a standard first order autoregressive model on the exogenous component,

$$U_t = \alpha U_{t-1} + \varepsilon_t \tag{1.2}$$

and introduce the following assumptions.

C. 1. (Model Assumptions)

1. $Y_t \mid \varepsilon_t, Y_{t-1}, U_{t-1} \sim Y_t \mid Y_{t-1}, U_{t-1}$ and ε_t i.i.d., $(0, \sigma_\varepsilon^2)$ random variables, with continuous probability density function f_ε which is positive everywhere on the real line.
2. g^{-1} is a continuous function, satisfying for all $x \in \mathbb{R}$,

$$g^{-1}(x) > 0$$

The assumption on g^{-1} is satisfied, for example, by the canonical link function, which is the function considered in the BAR model without exogenous variable investigated in Wang and Li [5].

The aim of this paper is to establish stationarity and geometric ergodicity of these processes under suitable conditions on the parameters of the model. In particular, proving that Z_t is geometric ergodic, implies Z_t is β-mixing with exponential rate (see e.g, Davydov [1]). Such properties are important for understanding the stability properties of the model as well as for deriving the asymptotic behavior of various statistics and model parameters. Furthermore, the theory we develop here is primarily motivated by, for example, deriving the NULL asymptotic of various statistics in the change point set up for BAR models.

2 Stability of the Model

Although it is obvious to see that $\{Y_t\}$ alone is not a homogenous Markov, our main result still relies on the stability theory of Markov chains. Therefore, we first design the Markov chain from which we will derive the desired properties

Lemma 2.1. *Under the model assumptions C.1, $\{Z_t = (Y_t, U_t)\}$ is a homogenous first order Markov chain with the Feller property.*

Proof. For proving that the extended process Z_t is a Markov chain, we need, for example, to compute the conditional distribution of Z_t given Z_{t-1}. Indeed,

$$\begin{aligned}
&P(Y_t = s, U_t \in du \mid Y_{t-1} = y_{t-1}, U_{t-1} = u_{t-1}) \\
&= P(Y_t = s \mid U_t \in du, y_{t-1}, u_{t-1}) P(U_t \in du \mid y_{t-1}, u_{t-1}) \\
&= P(Y_t = s \mid \varepsilon_t, y_{t-1}, u_{t-1}) P(U_t \in du \mid y_{t-1}, u_{t-1})) \\
&= P(Y_t = s \mid y_{t-1}, u_{t-1}) P(U_t \in du \mid y_{t-1}, u_{t-1})) \\
&= g^{-1}(\beta' z_{t-1}) f_\varepsilon (u - \alpha u_{t-1})
\end{aligned}$$

The first equality is derived by applying twice the definition of the conditional probability, for the third one, we use the conditional independence of Y_t and ε_t and for the last equality we use the model definition in (1.1) and (1.2) .

Remark 2.1. From the proof above, it obvious to see that the one step transition kernel $g^{-1}(\beta' z_{t-1}) f_\varepsilon (u - \alpha u_{t-1})$ is positive every where on the real line, using the model assumptions C.1.

To show that Z_t is a Feller chain, we need to consider a bounded continuous function

$$h_{bc} : \{0,1\} \times \mathbb{R} \longrightarrow \mathbb{R}$$

and prove that

$$\mathbb{E}\left(h_{bc}(Z_t) \mid Z_{t-1} = (v,w)'\right)$$

is bounded and continuous. In fact, using the definition of the conditional expectation,

$$\begin{aligned}
&\mathbb{E}\left(h_{bc}(Z_t) \mid Z_{t-1} = (v,w)'\right) \\
&= \int h_{bc}(s,u) P(Y_t = s, U_t \in du \mid Y_{t-1} = v, U_{t-1} = w) duds \\
&= \int h_{bc}(s,u) g^{-1}(\beta' Z_{t-1}) f_\varepsilon (u - \alpha w) \, duds \\
&= g^{-1}(\beta' Z_{t-1}) \int h_{bc}(s,u) f_\varepsilon (u - \alpha w) \, duds
\end{aligned}$$

which is obviously bounded continuous. Indeed, g^{-1} is bounded by definition and continuous by assumption and, by mean of the dominated convergence theorem, $\int h_{bc}(s,u) f_\varepsilon (u - \alpha w) \, duds$ is bounded and continuous in w, therefore, in (v,w). ■

The next step toward our main result is to prove that the designed Markov chain is irreducible.

Lemma 2.2. *Under the model assumptions C.1 and for a suitable measure λ, Z_t is λ-irreducible.*

Proof. Let us consider $\lambda = \lambda_1 \otimes \lambda_2$ with λ_1 any probability measure on $\{0,1\}$ and λ_2 the Lebesgue measure on $\mathbb{R}$. We further consider $A \in \mathcal{P} \otimes \mathcal{B}$ (for simplicity, $\mathcal{P}$ is the partition of $\{0,1\}$ and $\mathcal{B}$ the Borel σ-algebra on $\mathbb{R}$) with $\lambda(A) > 0$. It is enough to prove, for example,

$$P^2(Z, A) = P(Z_2 \in A \mid Z_0 = z_0) > 0.$$

Indeed,

$$\begin{aligned}
&P(Z_2 \in A \mid Z_0 = z_0) \\
&= \int_A P(Z_2 \mid Z_1) P(Z_1 \mid Z_0 = z_0) dZ_1 dZ_2 \\
&= \int_A g^{-1}(\beta' Z_1) g^{-1}(\beta' z_0) f_\epsilon \left(u_2 - \alpha u_1\right) f_\epsilon \left(u_1 - \alpha u_0\right) du_1 du_2 dy_1 \\
&> 0,
\end{aligned}$$

due the fact, see Remark 2.1, that the one step transition kernel is strictly positive over the real line. ∎

Theorem 2.1. *Under the model assumptions C.1 and if additionally $|\alpha| < 1$, then, Z_t is geometrically ergodic.*

Remark 2.2. In the purely exogenous based model in (1.1), Theorem 2.1 confirms the intuition that we only need the exogenous variable to be geometric ergodic in order to achieve the geometric ergodicty property of the binary process. In fact, $|\alpha|$ is the standard parameter constraint for the geometric ergodicity of first order AR. Furthermore, depending on the choice of the V function (see Theorem 1 of Feigin and Tweedie [2] below), we will automatically derive the existence of moments of certain order for the exogenous variables.

To conclude with the stability property of our process, we need to make use of Theorem 1 of Feigin and Tweedie [2], that we rephrase here for sake of completeness.

Theorem 2.2. *(Feigin and Tweedie, 1985, Theorem 1)*
Suppose Z_t is a Feller Chain, that there exist a measure λ and a compact set A with $\lambda(A) > 0$ such that

1. *Z_t is λ-irreducible*
2. *there exists a non-negative continuous function $V : \{0,1\} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfying*

$$V(x) \geq 1, \ \forall\, x \in A \tag{2.1}$$

and for some $0 < \beta < 1$

$$\mathbb{E}[V(Z_t) \mid Z_{t-1} = z] \leq (1-\beta) V(z) \text{ for } z \in A^c. \tag{2.2}$$

Then, Z_t is geometrically ergodic.

Proof. For the proof of Theorem 2.1, let us first note that Lemma 2.1 and Lemma 2.2 already prove Z_t is an irreducible Feller chain. Now, to use Theorem 2.2, we define,

$$V(Z_t) = 1 + Y_t^2 + U_t^2.$$

It is obvious that V satisfies (2.1) in Theorem 2.2. Additionally, let us compute

$$\begin{aligned} \mathbb{E}\left(V(Z_t) \mid Z_{t-1} = (y,u)'\right) &= 1 + g^{-1}(\beta'(y,u)') + \mathbb{E}\left(U_t^2 \mid Z_{t-1} = (y,u)'\right) \\ &= 1 + g^{-1}(\beta' z) + \alpha^2 u^2 + \sigma_\varepsilon^2 \\ &\leq 1 + \alpha^2 u^2 + (1 + \sigma_\varepsilon^2) \end{aligned}$$

since $0 < g^{-1}(\beta' z) \leq 1$. Finally, for $0 < \alpha^2 < \delta < 1$ we have

$$\mathbb{E}\left(V(Z_t) \mid Z_{t-1} = (y,u)'\right) \leq 2 + \alpha^2 u^2 + \sigma_\varepsilon^2 < \delta(1 + u^2) \leq \delta V(y,u)$$

for all $u \in A_2^c$, with

$$A_2 = \left\{ u \in \mathbb{R} : |u|^2 \leq \frac{2 - \delta + \sigma_\varepsilon^2}{\delta - \alpha^2} \right\}$$

Consequently, (2.2) follows for $\beta = 1 - \delta$ and

$$A = \{0,1\} \times A_2.$$

■

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