Vector bundles on elliptic curves and factors of automorphy

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Preface

The idea of a vector bundle is one of the key notions of contemporary mathematics. It appears in algebraic geometry, algebraic topology, differential geometry, in the theory of partial differential equations.

The theory of vector bundles and the mathematical formalism developed over the years for the study of vector-bundle related concepts leads to the clarification or solution of many mathematical problems. The language of vector bundles is extremely fruitful. The notion of a vector bundle is very natural in physics. For example it appears in field theory: various field theories are described in terms of connections on principal vector bundles. Transformation groups of vector bundles arise in many physical problems, in asymptotic methods of partial differential equations with small parameter, in elliptic operator theory, in mathematical methods of classical mechanics and in mathematical methods in economics.

Some of vector-bundle related concepts are generalizations of well-known classical notions. For instance the notion of a section of a vector bundle over a space X is a generalization of vector valued functions on X.

One of the important problems is the problem of classification of bundles. The problem of classification is of great importance for mathematics and for applications.

The problem of classification of vector bundles over an elliptic curve has been started and completely solved by Atiyah in [At]. Atiyah showed that for a fixed indecomposable vector bundle E of rank r and degree d every other indecomposable vector bundle of the same rank and degree is of the form $E \otimes L$, where $L \in \mathrm{Pic}^0(X)$.

The aim of this paper is to interpret some results obtained by Atiyah in [At] in the language of factors of automorphy. Namely, this papers aims to prove Proposition 1 in [Pol], which states that all vector bundles on an elliptic curve over $\mathbb{C}$ are obtained as push forwards of vector bundles of certain type with respect to finite coverings.

Every algebraic subvariety of an algebraic variety over $\mathbb{C}$ defines an associated analytic subvariety. In the case of a projective algebraic variety the converse is also true. It is known that the classification of holomorphic vector bundles on a projective variety over $\mathbb{C}$ is equivalent to the classification of algebraic vector bundles.[1] Therefore, we can use the results obtained by Atiyah(in the algebraic situation) in the analytic case. So we do not make any difference between algebraic and analytic cases. The considerations in this paper are mainly analytic.

The paper is organized as follows. In order to give the necessary background we present in Section 1 some necessary statements about vector bundles and complex

[1] For more detailed information see [Serre] and [Gr].

tori. In Section 2 the results of Atiyah are considered. The proofs of main theorems are also presented.

Section 3 is devoted to the classification of vector bundles on a complex torus in terms of factors of automorphy. We develop the language of factors of automorphy, prove the statement of Polishchuk and Zaslow in [Pol] mentioned above, and give a description of vector bundles of fixed rank and degree in terms of factors of automorphy:

An r-dimensional factor of automorphy on a complex manifold Y is a holomorphic function $f : \Gamma \times Y \to \mathrm{GL}_r(\mathbb{C})$ satisfying $f(\lambda\mu, y) = f(\lambda, \mu y) f(\mu, y)$. Here Γ is a group acting on Y. We introduce an equivalence relation on the set of r-dimensional factors of automorphy on Y. We consider coverings $Y \to X$. For a normal covering $p : Y \to X$ and for Γ equal to the group of deck transformations $\mathrm{Deck}(Y/X)$ there is a one-to-one correspondence between equivalence classes of r-dimensional factors of automorphy on Y and isomorphism classes of vector bundles on X with trivial pull back with respect to p. If all vector bundles on Y are trivial, which is for example the case for non-compact Riemann surfaces, we obtain a one-to-one correspondence between isomorphism classes of vector bundles of rank r on X and equivalence classes of r-dimensional factors of automorphy on Y. This gives a description of vector bundles on X in terms of factors of automorphy. The main result is stated in Theorem 3.37, where we give a classification of vector bundles of fixed rank and degree in terms of factors of automorphy on $\mathbb{C}^*$ ($\mathbb{C}$). It gives also a normal form for factors of automorphy.

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Contents

1 Basic notions and facts

1.1 Complex manifolds and sheaves

We assume the notions of sheaf and complex manifold to be known. Cohomology theory of sheaves is assumed to be known also. Here we only recall two important results.

Definition 1.1. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on a ringed space X is called locally free of rank n if it is locally isomorphic to the sheaf $\mathcal{O}_X^n := \bigoplus_1^n \mathcal{O}_X$, i. e., if there exists an open covering $\{U_i\}_{i\in\mathcal{I}}$ of X such that $\mathcal{F}|_{U_i} \simeq \mathcal{O}_X^n|_{U_i}$.

Theorem 1.1. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_Y$-modules. Then there exists a natural isomorphism (projection formula):*

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \simeq f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}.$$

Theorem 1.2 (Serre's duality)**.** *Let X be an n-dimensional complex manifold. Let $\mathcal{F}$ be a locally free sheaf on X. Let K_X be a canonical sheaf on X. Then there are canonical isomorphisms*

$$H^i(X, \mathcal{F}) \simeq H^{n-i}(X, K_X \otimes \mathcal{F}^*)^*.$$

In particular it follows that $H^i(X, \mathcal{F})$ and $H^{n-i}(X, K_X \otimes \mathcal{F}^*)$ have the same dimension.

1.2 Vector bundles

All the spaces here are complex manifolds.

Definition 1.2. A holomorphic map $p : E \to X$ of complex manifolds is called a complex vector bundle of rank n if it satisfies the following conditions.

1) For any point $x \in X$ the preimage $E_x := p^{-1}(x)$ has a structure of n-dimensional $\mathbb{C}$-vector space.

2) p is locally trivial, i. e., for any point $x \in X$ there exist an open neighbourhood U containing x and a biholomorphic map $\varphi_U : p^{-1}(U) \to U \times \mathbb{C}^n$ such that the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi_U"] \arrow[dr, "p"'] & & U \times \mathbb{C}^n \arrow[dl, "\mathrm{pr}_U"] \\ & U & \end{tikzcd}$$

commutes. Moreover, $\varphi_U|_{\{y\}} : E_y \to \{y\} \times \mathbb{C}^n$ is an isomorphism of vector spaces for any point $y \in U$.

Notation. Following Atiyah's paper [At] we denote by $\mathcal{E}(r, d) = \mathcal{E}_X(r, d)$ the set of isomorphism classes of indecomposable vector bundles over X of rank r and degree d.

Definition 1.3. Let U be an open set in X. A holomorphic map $s : U \to E$ is called a holomorphic section of E over U if $ps = \mathrm{id}_U$. The set of all holomorphic sections of E over U is denoted by $\Gamma(U, E)$. Sections over X are called global sections of E. The set of global holomorphic sections of E is also denoted by $\Gamma(E)$.

Definition 1.4. Let $p : E \to X$ and $p' : E' \to X$ be two complex vector bundles on X. A holomorphic map $f : E \to E'$ is called a morphism of vector bundles if the diagram

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

commutes and for each point $x \in X$ the map $f|_{E_x} : E_x \to E'_x$ is a homomorphism of vector spaces.

Consider a vector E bundle of rank n. Let $\{U_i\}$ be an open covering such that E is trivial over each U_i. This means that there exist biholomorphic maps

$$\varphi : p^{-1}(U_i) \to U \times \mathbb{C}^n.$$

For each pair (i, j) consider the map

$$\varphi_i \varphi_j^{-1} : U_i \cap U_j \times \mathbb{C}^n \to U_i \cap U_j \times \mathbb{C}^n, \quad (x, v) \mapsto (x, g_{ij}(x)v).$$

We get a family of holomorphic maps $g_{ij} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{C})$ satisfying the cocycle conditions:

i) $g_{ii}(x) = \mathrm{id}_{\mathbb{C}^n}, \quad x \in U_i$;

ii) $g_{ij}g_{jk}(x) = g_{ik}(x), \quad x \in U_i \cap U_j \cap U_k$.

Vice versa, any cocycle defines a vector bundle. Suppose we have a cocycle $\{g_{ij}\}$. Define $E = \bigsqcup (U_i \times \mathbb{C}^n)/_\sim$, where $(x, v) \sim (x, g_{ij}(v))$. Since the cocycle conditions guarantee that E is a Hausdorff space, one concludes that E is a complex manifold. It is easy to see that E is trivial over U_i. Therefore, E is a complex vector bundle.

Theorem 1.3. *Let E and E' be vector bundles over X. Let $\{U_i\}$ be an open covering such that E and E' are trivial over U_i for each i. Then there is a one-to-one correspondence between morphisms $f : E \to E'$ and sets of holomorphic functions $\{f_i : U_i \to \mathrm{Mat}_{r' \times r}(\mathbb{C})\}$ satisfying*

$$g'_{ij} f_j|_{U_i \cap U_j} = f_i g_{ij}, \tag{1}$$

where g_{ij} and g'_{ij} are cocycles defining E and E' respectively.

Proof. Let $f : E \to E'$ be a map of vector bundles over X. Let $\{U_i\}$ be an open covering such that E and E' are trivial over U_i for each i. Then for each i there is a map

$$U_i \times \mathbb{C}^r \to U_i \times \mathbb{C}^{r'}, \quad (x, v) \mapsto (x, f_i(x)v).$$

It is easy to see that holomorphic maps $f_i : U_i \to \mathrm{Mat}_{r' \times r}(\mathbb{C})$ satisfy (1).

Vice versa, suppose a set of holomorphic functions $\{f_i : U_i \to \mathrm{Mat}_{r' \times r}(\mathbb{C})\}$ satisfies (1). Then these functions define a map between the vector bundles defined by g_{ij} and g'_{ij} respectively. □

Remark. Note that a set of functions $\{f_i\}$ defines an isomorphism of vector bundles if and only if $f_i(x)$ are invertible matrices for all i and x.

Thus we obtain the following:

Theorem 1.4. *Two n-dimensional cocycles $\{g_{ij}\}$ and $\{g'_{ij}\}$ corresponding to the open covering $\{U_i\}_{i\in\mathcal{I}}$ of X define isomorphic vector bundles if and only if there exist holomorphic functions $f_i : U_i \to \mathrm{GL}_n(\mathbb{C})$ such that for each pair $(i,j) \in \mathcal{I}\times\mathcal{I}$*

$$g_{ij}(x)f_j(x) = f_i(x)g'_{ij}(x), \quad x \in U_i \cap U_j. \tag{2}$$

Proof. If $\{g_{ij}\}$ and $\{g'_{ij}\}$ define isomorphic vector bundles, then by previous theorem and remark there exist holomorphic functions $f_i : U_i \to \mathrm{GL}_n(\mathbb{C})$ satisfying (2).

Vice versa, let two cocycles $\{g_{ij}\}$ and $\{g'_{ij}\}$ corresponding to an open covering $\{U_i\}$ satisfy (2). Then the functions f_i define an isomorphism between vector bundles defined by $\{g_{ij}\}$ and $\{g'_{ij}\}$ respectively.

□

Corollary 1.5. *A cocycle $\{g_{ij} : U_i \cap U_j \to \mathrm{GL}_r(\mathbb{C})\}$ defines a trivial vector bundle if and only if there exist holomorphic functions $f_i : U_i \to \mathrm{GL}_n(\mathbb{C})$ such that for each pair $(i,j) \in \mathcal{I}\times\mathcal{I}$*

$$g_{ij}(x) = f_i(x)f_j(x)^{-1}, \quad x \in U_i \cap U_j.$$

Theorem 1.6. *Let E be a vector bundle of rank n over X, let $\{U_i\}$ be an open covering of X such that E is trivial over U_i, and let g_{ij} be a cocycle defining E. Then any global holomorphic section s of E defines a set of holomorphic functions $s_i : U_i \to \mathbb{C}^n$ that satisfy*

$$s_i(x) = g_{ij}(x)s_j(x), \quad x \in U_i \cap U_j. \tag{3}$$

Vice versa, any such a set of functions defines a global holomorphic section of E.

Proof. Let $s : X \to E$ be a holomorphic section of E. Let $\varphi_i : E|_{U_i} \to U_i \times \mathbb{C}^n$ be a trivialization corresponding to the cocycle g_{ij}. Then $\varphi_i s(x) = (x, s_i(x))$, where s_i is a holomorphic function. Consider $x \in U_i \cap U_j$. Then one obtains

$$\varphi_i^{-1}(x, s_i(x)) = \varphi_i^{-1}\varphi_i s(x) = \varphi_j^{-1}\varphi_j s(x) = \varphi_j^{-1}(x, s_j(x)).$$

Applying φ_i, we obtain $(x, s_i(x)) = \varphi_i\varphi_j^{-1}(x, s_j(x)) = (x, g_{ij}(x)s_j(x))$ or in other words $s_i(x) = g_{ij}(x)s_j(x)$.

Vice versa, let $s_i : U_i \to \mathbb{C}^n$ be holomorphic functions satisfying (3). We define $s : X \to E$ by $s(x) := \varphi_i^{-1}(x, s_i(x))$ for $x \in U_i$. From $(x, s_i(x)) = (x, g_{ij}(x)s_j(x))$ applying φ_i^{-1} we obtain $\varphi_i^{-1}(x, s_i(x)) = \varphi_j^{-1}(x, s_j(x))$, which means that s is well-defined. It is clear that s is a holomorphic global section of E. □

Theorem 1.7. *There is a one-to-one correspondence between isomorphism classes of vector bundles of rank n and isomorphism classes of locally free sheaves of rank n.*

Proof. Consider a vector bundle $p : E \to X$ of rank n. For any open set $U \subset X$ define $\mathcal{E}(U)$ to be a set of holomorphic sections of E over U. $\mathcal{E}$ is clearly a sheaf of $\mathcal{O}_X$-modules.

By the definition of a vector bundle, there exists an open covering $\{U_i\}_{i\in\mathcal{I}}$ of X such that E is trivial over U_i for each $i \in \mathcal{I}$. Therefore, $\Gamma(U_i, E) \simeq \mathcal{O}_X^n|_{U_i}$ for each $i \in \mathcal{I}$. This proves that $\mathcal{E}$ is a locally free sheaf of rank n. It is clear that two isomorphic vector bundles have isomorphic sheaves of holomorphic sections.

Vice versa, let $\mathcal{E}$ be a locally free sheaf of rank n. This means that there exist an open covering $\{U_i\}_{i\in\mathcal{I}}$ of X and isomorphisms of sheaves

$$\varphi_i : \mathcal{E}|_{U_i} \to \mathcal{O}_X^n|_{U_i}.$$

For each pair i, j we define

$$g_{ij}(x) = (\varphi_i\varphi_j^{-1})(x) : \mathbb{C}^n \to \mathbb{C}^n.$$

Clearly $g_{ij} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{C})$ defines a cocycle, i. e., $g_{ii}(x) = \mathrm{id}_{\mathbb{C}^n}$ for $x \in U_i$, $g_{ij}g_{jk}(x) = g_{ik}(x)$ for $x \in U_i \cap U_j \cap U_k$. This defines a complex vector bundle of rank n.

If $\varphi_i' : \mathcal{E}|_{U_i} \to \mathcal{O}_X^n|_{U_i}$ is another trivialization of $\mathcal{E}$, then there exist invertible $n \times n$ matrices of holomorphic functions $f_i(x)$ such that $\varphi_i' = f_i\varphi_i$. Therefore, the cocycle corresponding to φ_i' is

$$g_{ij}'(x) = (\varphi_i'\varphi_j'^{-1})(x) = (f_i\varphi_i\varphi_j^{-1}f_j^{-1})(x) = (f_ig_{ij}f_j^{-1})(x).$$

The last means that the cocycles g_{ij} and g_{ij}' define the same up to isomorphism vector bundle. We obtained a well-defined map from the set of isomorphism classes of locally free sheaves of rank n to the set of isomorphism classes of vector bundles of rank n.

It is easy to see that the described correspondences between isomorphism classes of vector bundles and isomorphism classes of locally free sheaves are inverse to each other. $\square$

Notation. Usually we denote by $\mathcal{E}$ the sheaf corresponding to a vector bundle E.

Let V and W be finite dimensional vector spaces. Let $n = \dim W$ and $m = \dim V$. Then the set $\mathrm{Hom}(V, W) = \mathrm{Hom}_{\mathbb{C}}(V, W)$ has a structure of a complex manifold since it can be identified with the set of all $n \times m$ matrices.

By Vect we denote the category of finite dimensional vector spaces. The category of vector bundles of finite rank over X is denoted by Vectb_X.

Definition 1.5. Let $\mathcal{F} : \mathsf{Vect}^n \to \mathsf{Vect}$ be a functor. $\mathcal{F}$ is called a holomorphic functor if the map

$$\mathrm{Hom}(V_1, W_1) \times \cdots \times \mathrm{Hom}(V_1, W_1) \to \mathrm{Hom}(\mathcal{F}(V_1, \ldots, V_n), \mathcal{F}(W_1, \ldots, W_n))$$

is a holomorphic map for any $V_i, W_j \in \mathrm{Ob}\,\mathsf{Vect}$.

Note that there is an obvious embedding of categories

$$\mathsf{Vect} \to \mathsf{Vectb}_X, \quad V \mapsto (X \times V \xrightarrow{\mathrm{pr}} X).$$

Theorem 1.8. *Let $\mathcal{F} : \mathsf{Vect}^n \to \mathsf{Vect}$ be a holomorphic functor. Then the functor $\mathcal{F}$ can be canonically extended to the category Vectb_X. In other words there exists a functor $\mathcal{F}_X : \mathsf{Vectb}_X^n \to \mathsf{Vectb}_X$ such that the restriction of $\mathcal{F}_X$ to Vect coincides with $\mathcal{F}$ and $\mathcal{F}_X(E_1, \ldots, E_n)_x = \mathcal{F}(E_{1x}, \ldots, E_{nx})$. By abuse of notation $\mathcal{F}_X$ is also denoted by $\mathcal{F}$.*

Proof. We give the construction of $\mathcal{F}_X$. Let $p_\alpha : E_\alpha \to X$, $\alpha = \overline{1,n}$ be vector bundles with rank $E_\alpha = r_\alpha$. One can choose an open covering $\{U_i\}$ of X such that each E_α is trivial over U_i for all i. Let $\{g_{ij}^\alpha : U_i \cap U_j \to \mathrm{GL}_{r_\alpha}(\mathbb{C})\}$ be a set of transition functions defining E_α. Suppose that the functor $\mathcal{F}$ is covariant in the first p variables and contravariant in the last $n-p$ variables. One defines

$$g_{ij}(x) := \mathcal{F}(g_{ij}^1, \ldots, g_{ij}^p, g_{ji}^{p+1}, \ldots, g_{ji}^n)(x).$$

Clearly, for any vector spaces $V_1, \ldots, V_n$ the dimension of $\mathcal{F}(V_1, \ldots, V_n)$ is a function only of dimensions of $V_1, \ldots, V_n$. Since $g_{ij}^\alpha(x)$ are isomorphisms, one concludes that $g_{ij}(x)$ is also an isomorphism. Combining these two observations and using that $\mathcal{F}$ is a holomorphic functor, one obtains that all the g_{ij} are holomorphic maps $g_{ij} : U_i \cap U_j \to \mathrm{GL}_r(\mathbb{C})$ for some r. Moreover, g_{ij} is also a cocycle since

$$\begin{aligned} g_{ij}(x)g_{jk}(x) =& F(g_{ij}^1, \ldots, g_{ij}^p, g_{ji}^{p+1}, \ldots, g_{ji}^n)(x) F(g_{jk}^1, \ldots, g_{jk}^p, g_{kj}^{p+1}, \ldots, g_{kj}^n)(x) = \\ & F(g_{ij}^1 g_{jk}^1, \ldots, g_{ij}^p g_{jk}^p, g_{kj}^{p+1} g_{ji}^{p+1}, \ldots, g_{kj}^n g_{ji}^n)(x) = \\ & F(g_{ik}^1, \ldots, g_{ik}^p, g_{ki}^{p+1}, \ldots, g_{ki}^n)(x) = g_{ik}(x). \end{aligned}$$

Then $\mathcal{F}_X(E_1, \ldots, E_n)$ is the vector bundle defined by g_{ij}.

Let $f = \{f_i^\alpha : U_i \to \mathrm{Mat}_{r'_\alpha \times r_\alpha}(\mathbb{C})\}_{\alpha=1}^n$ be a morphism in Vectb_X^n. Then one defines

$$\mathcal{F}_X(f) := \{\mathcal{F}(f_i^1, \ldots, f_i^n) : U_i \to \mathrm{Hom}(\mathcal{F}(\mathbb{C}^{r_1}, \ldots, \mathbb{C}^{r_n}), \mathcal{F}(\mathbb{C}^{r'_1}, \ldots, \mathbb{C}^{r'_n}))\}.$$

It remains to show that this definition of $\mathcal{F}_X$ is independent of the choice of covering $\{U_i\}$ and that $\mathcal{F}_X$ is a functor. □

Definition 1.6. Let $f : X \to Y$ be a morphism of complex manifolds. Let $p : E \to Y$ be a vector bundle over Y. Then

$$f^*E = X \times_Y E = \{(x, e) \in X \times E |\ f(x) = p(e)\}$$

is a vector bundle over X. Moreover: if $\{U_i\}$ is an open covering of Y such that E is trivial over U_i, then f^*E is trivial over $f^{-1}(U_i)$. If $g_{ij}^Y : U_i \cap U_j \to \mathrm{GL}_r(\mathbb{C})$ is a cocycle defining E, then

$$g_{ij}^X : f^{-1}(U_i) \cap f^{-1}(U_j) \to GL_r(\mathbb{C}), \quad g_{ij}^X(x) = g_{ij}^Y f(x)$$

is a cocycle defining f^*E.

Theorem 1.9. *Let $\mathcal{E}$ be a locally free sheaf on Y, let $f : X \to Y$ be a morphism of complex manifolds. Then $f^*\mathcal{E}$ is a locally free sheaf.*

Proof. Let U be an open set in Y such that $\mathcal{E}|_U \simeq \mathcal{O}_Y^r|_U$. Let $\varphi : \mathcal{E}|_U \to \mathcal{O}_Y^r$ be an isomorphism. Then $f^*\varphi : f^*\mathcal{E}|_{f^{-1}(U)} \to f^*\mathcal{O}_Y^r|_{f^{-1}(U)}$ is also an isomorphism of sheaves. But $f^*\mathcal{O}_Y^r$ is isomorphic to $\mathcal{O}_X^r$ via

$$f^*\mathcal{O}_Y^r = f^{-1}\mathcal{O}_Y^r \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \to \mathcal{O}_X^r, \quad s \otimes h \mapsto h \cdot (s \circ f).$$

Therefore, we obtain an isomorphism

$$\Phi : f^*\mathcal{E}|_{f^{-1}(U)} \to \mathcal{O}_X^r|_{f^{-1}(U)}, \quad \sigma \otimes h \mapsto h \cdot (\varphi(\sigma) \circ f).$$

□

Remark. Note that if $\mathcal{E}$ is the sheaf corresponding to a vector bundle E, then f^*E is exactly the vector bundle corresponding to $f^*\mathcal{E}$.

Proof. We prove that the morphism of sheaves

$$\eta : f^*\mathcal{E} \to \mathcal{G}, \quad \sigma \otimes h \mapsto h \cdot (\sigma \circ f)$$

is an isomorphism.

Let U be an open set such that there is an isomorphism $\varphi : \mathcal{E}|_U \simeq \mathcal{O}_Y^r|_U$. Then there is an isomorphism

$$\Phi : f^*\mathcal{E}|_{f^{-1}(U)} \to \mathcal{O}_X^r|_{f^{-1}(U)}, \quad \sigma \otimes h \mapsto h \cdot (\varphi(\sigma) \circ f).$$

On the level of vector bundles the map corresponding to φ is

$$\phi : E \to U \times \mathbb{C}^r, \quad e_y \mapsto (y, \varphi(y)e_y),$$

where $e_y \in E_y$.

Let $\mathcal{G}$ be the sheaf of sections of the vector bundle f^*E. Then there is an isomorphism of sheaves

$$\Psi : \mathcal{G}|_{f^{-1}(U)} \to \mathcal{O}_X^r|_{f^{-1}(U)}, \quad (x \mapsto s(x)) \mapsto (x \mapsto \varphi(f(x))s(x)),$$

where we identify sections of f^*E over $f^{-1}(U)$ with maps $s : f^{-1}(U) \to E$ such that $s(x) \in E_{f(x)}$. The inverse Ψ^{-1} is

$$\Psi^{-1} : \mathcal{O}_X^r|_{f^{-1}(U)} \to \mathcal{G}|_{f^{-1}(U)}, \quad g \mapsto (\ x \mapsto \varphi^{-1}(f(x))(g(x))\ ).$$

The composition $\Psi^{-1}\Phi : f^*\mathcal{E}|_{f^{-1}(U)} \to \mathcal{G}|_{f^{-1}(U)}$ is

$$\sigma \otimes h \mapsto h(\varphi(\sigma) \circ f) \mapsto (\ x \mapsto \varphi^{-1}(f(x))(\ h(x)\varphi(\sigma)(f(x))\ ) = h(x)\sigma(f(x))\ ),$$

therefore, $\Psi^{-1}\Phi(\sigma \otimes h) = h \cdot (\sigma \circ f) = \eta(\sigma \otimes h)$. We conclude that η is locally an isomorphism. Therefore, the morphism

$$\eta : f^*\mathcal{E} \to \mathcal{G}, \quad \sigma \otimes h \mapsto h \cdot (\sigma \circ f).$$

is an isomorphism of sheaves. □

Remark. Note that for a locally free sheaf $\mathcal{E}$ on Y and for a morphism $f : Y \to X$ the push forward $f_*\mathcal{E}$ is not a locally free sheaf in general. This is true only by some additional assumptions on f.

In particular the following theorem holds true.

Theorem 1.10. *Let $Y \to X$ be a finite covering, i. e., a morphism of complex manifolds such that for any point $x \in X$ there exists an open neigbourhoud U of x with the property that $f^{-1}(U) = \bigsqcup_{i=1}^{n} V_i$ and $f|_{V_i} : V_i \to U$ is a biholomorphism for all V_i. Let $\mathcal{E}$ be a locally free sheaf of rank r on Y. Then $f_*\mathcal{E}$ is a locally free sheaf of rank rn on X.*

Proof. Consider an open set $U \subset X$ such that $f^{-1}(U) = \bigsqcup_{i=1}^{n} V_i$, f restricted to each V_i is a biholomorphism, and $\mathcal{E}|_{V_i}$ is isomorphic to $\mathcal{O}_Y^r|_{V_i}$ for each V_i.

Since V_i is biholomorphic to U, we get an isomorphism of sheaves

$$f|_{V_i}^{\#} : \mathcal{O}_X|_U \to \mathcal{O}_Y|_{V_i}, \quad s \mapsto (sf)|_{V_i}.$$

Therefore,

$$f_*\mathcal{E}(U) = \mathcal{E}(f^{-1}(U)) = \mathcal{E}(\bigsqcup V_i) = \oplus\mathcal{E}(V_i) \simeq \oplus\mathcal{O}_Y^r|_{V_i} \simeq \oplus\mathcal{O}_X^r|_U \simeq \mathcal{O}_X^{rn}.$$

□

Definition 1.7. A sequence of morphisms of vector bundles over a space X

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$$

is an exact sequence of vector bundles if

$$0 \to E'_x \xrightarrow{f_x} E_x \xrightarrow{g_x} E''_x \to 0$$

is an exact sequence of vector spaces for all $x \in X$. E' is called a sub-bundle of E, and E'' is called a factor bundle of E.

We say also that an exact sequence

$$0 \to E' \to E \to E'' \to 0$$

is an extension of E'' by E'. In this case E is also called an extension of E'' by E'.

Remark. Note that $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles if and only if $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is an exact sequence of locally free sheaves, where $\mathcal{E}'$, $\mathcal{E}$, $\mathcal{E}''$ denote the sheaves of holomorphic sections of E', E, and E'' respectively.

Let $0 \to \mathcal{E}' \xrightarrow{f} \mathcal{E}$ be an injective morphism of locally free sheaves. Let E' and E be the corresponding to $\mathcal{E}'$ and $\mathcal{E}$ vector bundles. Then f induces an injective morphism of vector bundles $0 \to E' \to E$ (or equivalently E' is a subbundle of E) if and only if the cokernel of f is also a locally free sheaf. In other words if there exists an exact sequence

$$0 \to \mathcal{E}' \xrightarrow{f} \mathcal{E} \to \mathcal{E}'' \to 0,$$

where $\mathcal{E}''$ is a locally free sheaf.

Equivalently, $0 \to \mathcal{E}' \xrightarrow{f} \mathcal{E}$ defines a subbundle E' in E if and only if the corresponding morphism of dual sheaves $\mathcal{E}^* \xrightarrow{f^*} \mathcal{E}'$ is surjective.

Definition 1.8. Two extensions E_1 and E_2 of E'' by E' are called equivalent if there is an isomorphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & E' \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d] & E'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E' \arrow[r] & E_2 \arrow[r] & E'' \arrow[r] & 0.
\end{tikzcd}$$

If the exact sequence

$$0 \to E' \to E \to E'' \to 0$$

splits we say that the extension is trivial. Clearly trivial extensions are exactly the extensions equivalent to a direct sum

$$0 \to E' \to E' \oplus E'' \to E'' \to 0.$$

The following theorem gives a parametrization of the equivalence classes of extensions of E'' by E'.

Theorem 1.11. *The equivalence classes of extensions of E'' by E' are in a one-to-one correspondence with the elements of $H^1(X, \mathcal{H}om(\mathcal{E}'', \mathcal{E}'))$, where $\mathcal{E}'$, $\mathcal{E}''$ are the sheaves corresponding to the vector bundles E' and E'' respectively. The trivial extension corresponds to the zero element.*[2]

Definition 1.9. A vector bundle of rank 1 is called a line bundle.

Remark. The set $\mathrm{Pic}(X)$ of isomorphism classes of line bundles over a complex manifold X is a group with respect to the operation of tensoring. The trivial line bundle is the neutral element. For any line bundle L its dual L^* is the inverse, i. e., $L \otimes L^* \simeq 1$. We write $L^{-1} := L^*$.

Note also that for an open covering $\{U_i\}$, the set $Z^1(\{U_i\}, \mathcal{O}_X^*)$ of 1-cocycles g_{ij} corresponding to this covering is an abelian group. The map mapping a cocycle g_{ij} to the vector bundle defined by g_{ij} is therefore a homomorphism of groups. By Corollary 1.5 the kernel of this homomorphism is the set $B^1(\{U_i\}, \mathcal{O}_X^*)$ of cocycles of the type f_i/f_j, where $f_i, f_j \in \mathcal{O}_X^*$. Therefore, $\mathrm{Pic}(X)$ is isomorphic to $Z^1(\{U_i\}, \mathcal{O}_X^*)/B^1(\{U_i\}, \mathcal{O}_X^*)$. But the last is by definition $H^1(X, \mathcal{O}_X^*)$. We obtain the following:

Theorem 1.12. *There is an isomorphism*

$$H^1(X, \mathcal{O}_X^*) \simeq \mathrm{Pic}(X), \quad [g_{ij}] \mapsto \textit{vector bundle defined by } g_{ij}.$$

1.3 Vector bundles over Riemann surfaces

The following result will be important in the sequel.

Theorem 1.13. *Any holomorphic vector bundle on a non-compact Riemann surface is trivial.*[3]

Definition 1.10. Let X be a Riemann surface. A divisor D on X is an element of the free abelian group $\mathrm{Div}(X)$ generated by all the points of X. In other words, D is a finite linear combination $\sum a_i x_i$, where $a_i \in \mathbb{Z}$, $x_i \in X$.

Divisors can also be thought as functions with finite support from X to $\mathbb{Z}$.

A divisor $D = \sum a_i x_i$ is called effective if $a_i \geqslant 0$ for all i. We write $D \geqslant 0$ for D effective.

Let f be a meromorphic function on X having finitely many poles and zeros on X. We define $\mathrm{div}(f)(x) := \mathrm{ord}_x(f)$. This divisor is a divisor associated to the meromorphic function f. Divisors of the type $\mathrm{div}(f)$ are called principal divisors.

[2]see [At2], Proposition 2.

[3]see [Forst], page 204, Satz **30.4**

Remark. Note that a holomorphic function always has finitely many poles and zeros if X is a compact Riemann surface.

Theorem 1.14. *Let $\mathcal{M}_X^*$ denote the multiplicative sheaf of meromorphic functions on X, which are not identically zero. Let $\mathcal{O}_X^*$ be the subsheaf of holomorphic functions that are nowhere zero. Then* $\mathrm{Div}(X) \simeq \Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$*, i. e., divisors are global sections the sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$.*

Proof. Consider a global section of $\mathcal{M}_X^*/\mathcal{O}_X^*$ given by an open covering $\{U_i\}$ and meromorphic functions $f_i \neq 0$ in U_i with $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$. Then for any point $x \in U_i \cap U_j$ we have $\mathrm{ord}_x(f_i) = \mathrm{ord}_x(f_j)$. We can define the divisor

$$D = \sum_{x \in X} \mathrm{ord}_x(f_{i(x)})x,$$

where for each $x \in X$ we choose $i(x)$ such that $x \in U_{i(x)}$.

Vice versa, let $D = \sum a_i x_i$ be a divisor on X. One can find an open covering $\{U_j\}$ of X such that in each U_j for every x_i there exists a local defining function $g_{ij} \in \mathcal{O}_X(U_j)$, i. e., $x_i \cap U_j$ is the set of zeros of g_{ij}. (Note that locally any point x has a defining function $z - x$, where z is a local coordinate at x.)

Define $f_j := \prod_i g_{ij}^{a_i}$. Since $(f_j/f_k)|_{U_j \cap U_k} \in \mathcal{O}_X^*(U_j \cap U_k)$, the set of functions $\{f_j\}$ defines a section of $\mathcal{M}_X^*/\mathcal{O}_X^*$.

Clearly the described correspondences are inverse to each other and define an isomorphism of abelian groups $\mathrm{Div}(X)$ and $\Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$ □

Definition 1.11. Let D be a divisor on X and let $\{f_i\}$ be functions defining the corresponding to D section of $\mathcal{M}_X^*/\mathcal{O}_X^*$. Then f_i's are called local defining functions of D.

Let D be a divisor on X with local defining functions $\{f_i\}$ over some open covering $\{U_i\}$. Then $g_{ij} = f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$. Moreover $g_{ij}g_{jk} = \frac{f_i}{f_j}\frac{f_j}{f_k} = \frac{f_i}{f_k} = g_{ik}$, i. e., g_{ij} is a cocycle.

If $\{f_i'\}$ are other defining functions of D, then clearly $h_i = f_i/f_i' \in \mathcal{O}_X^*(U_i)$ and $g_{ij} = \frac{f_i}{f_j} = \frac{h_i f_i'}{h_j f_j'} = \frac{h_i}{h_j} g_{ij}'$, which means that g_{ij} and g_{ij}' define isomorphic line bundles.

The line bundle given by g_{ij} is called the associated line bundle of D. Its isomorphism class is denoted by $[D]$. By the above this definition is well-defined. The sheaf corresponding to $[D]$ is denoted by $\mathcal{O}_X(D)$.

If D and D' are two divisors given by local defining functions $\{f_i\}$ and $\{f_i'\}$ respectively, then clearly $D + D'$ is defined by $f_i f_i'$.

On the other hand, $[D] \otimes [D']$ is defined by the cocycle $g_{ij}g_{ij}' = \frac{f_i}{f_j}\frac{f_i'}{f_j'}$, which implies that $[D] \otimes [D']$ is defined by $D + D'$. Therefore, $[D + D'] = [D] \otimes [D']$ and the map

$$[\ ] : \mathrm{Div}(X) \to \mathrm{Pic}(X)$$

is a homomorphism of abelian groups.

Theorem 1.15. *The line bundle associated to a divisor* $[D]$ *is trivial if and only if* D *is a principal divisor, i. e., there exists a meromorphic function* f *on* X *such that* $D = \mathrm{div}(f)$.

Proof. If $D = \mathrm{div}(f)$, take as local defining functions for D for any covering $\{U_i\}$ $f_i = f|_{U_i}$. Then $f_i/f_j = 1$ and $[D]$ is trivial.

Vice versa, if $[D]$ is trivial and D is given by local defining functions $\{f_i\}$ for an open covering $\{U_i\}$, then there exist functions $h_i \in \mathcal{O}_X^*(U_i)$ such that $g_{ij} = f_i/f_j = h_i/h_j$. But this means $f_i/h_i = f_j/h_j$. Therefore, $f(x) = f_i(x)/h_i(x)$ for $x \in U_i$ is a well-defined global meromorphic function on X with divisor D. □

Remark. Let D be a divisor. One can show that $\mathcal{O}_X(D)$ is isomorphic to the sheaf

$$\mathcal{F}(U) = \{f | f \text{ is a meromorphic function on } U \text{ with } \mathrm{div}(f) + D \geqslant 0\}.$$

Definition 1.12. Let L be a line bundle given by a cocycle g_{ij} over an open covering $\{U_i\}$. A meromorphic section of L over an open set U is a section of the sheaf $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$ over U.

As in Theorem 1.6 we obtain that global meromorphic sections of L are given by meromorphic functions $s_i \in \mathcal{M}_X(U_i)$ such that $s_i(x) = g_{ij}(x)s_j(x)$ for $x \in U_i \cap U_j$. This can be also taken as a definition of a holomorphic section of L.

Definition 1.13. Let L be a line bundle given by a cocycle g_{ij} over $\{U_i\}$. Let s be a global meromorphic not identically zero (non-trivial) section of L given by meromorphic functions $s_i \in \mathcal{M}_X(U_i)$. Then $\frac{s_i}{s_j}|_{U_i \cap U_j} = g_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$. Therefore, $\mathrm{ord}_x(s_i) = \mathrm{ord}_x(s_j)$ for all $x \in U_i \cap U_j$, and the divisor

$$\mathrm{div}(s)(x) := \mathrm{ord}_x(s_i), \quad x \in U_i$$

is well-defined. It is called a divisor associated to the meromorphic section $s = \{s_i\}$.

Theorem 1.16. *For a divisor* D *there exists a non-trivial meromorphic section* s *of* $[D]$ *with* $\mathrm{div}(s) = D$.

For any non-trivial meromorphic section s *of* L, $L \simeq [\mathrm{div}(s)]$.

Proof. Let D be a divisor given by local defining functions $f_i \in \mathcal{M}_X^*(U_i)$. Then $[D]$ is defined by the cocycle $g_{ij} = \frac{f_i}{f_j}$. Since $f_i = g_{ij} f_j$, we obtain that the functions f_i define a meromorphic section of $[D]$.

Conversely, let L be a line bundle defined by a cocycle g_{ij}. Let $s = \{s_i\}$ be a global meromorphic section of L. Then $\frac{s_i}{s_j}|_{U_i \cap U_j} = g_{ij}$, which means that $\mathrm{div}(s)$ defines L, i. e., $L \simeq [\mathrm{div}(s)]$. □

Corollary 1.17. *Let L be a line bundle over X. Then L is of the form $[D]$ for some divisor $D \in \mathrm{Div}(X)$ if and only if L has a non-trivial meromorphic section.*

Theorem 1.18. *Any vector bundle over a compact Riemann surface has a non-trivial meromorphic section.*[4]

From this theorem using Corollary 1.17 we obtain the following corollary.

Theorem 1.19. *On a compact Riemann surface any line bundle is of the form $[D]$ for some divisor $D \in \mathrm{Div}(X)$.*

It is known that on a compact Riemann surface X any meromorphic function has the same number of zeros and poles counted with multiplicity.[5] Therefore, $\deg D = 0$ for any principal divisor D on X. Since by Theorem 1.15 a line bundle $[D]$ is trivial if and only if D is a principal divisor, we conclude that $[D] = [D']$ implies $\deg D = \deg D'$. Therefore, the following definition makes sense.

Definition 1.14. Let L be a line bundle on a compact Riemann surface. We define the degree of L by $\deg L := \deg D$, where D is a divisor on X such that $L = [D]$.

For a vector bundle E of rank $r > 1$ one defines the degree of E by $\deg E := \deg(\Lambda^r E)$.

Definition 1.15. Let X be a compact Riemann surface. Then $H^1(X, \mathcal{O}_X)$ is finite dimensional. The number $g := \dim H^1(X, \mathcal{O}_X)$ is called the genus of X.

Theorem 1.20 (Riemann-Roch)**.** *Let E be a vector bundle of rank r over a compact Riemann surface X of genus g. Denote $h^i(X, \mathcal{E}) := \dim H^i(X, \mathcal{E})$. Then*

$$h^0(X, \mathcal{E}) - h^1(X, \mathcal{E}) = r(1 - g) + \deg E,$$

where $\mathcal{E}$ denotes the sheaf of holomorphic sections of E.

1.4 Some facts about tori

Definition 1.16. Consider two linear independent over $\mathbb{R}$ complex numbers ω_1 and ω_2. Consider the additive subgroup Γ in $\mathbb{C}$ generated by ω_1 and ω_2, i. e., $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Consider the quotient group $\mathbb{C}/\Gamma$ and the canonical projection $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$. We equip $\mathbb{C}/\Gamma$ with the quotient topology, i. e., a set $U \subset \mathbb{C}/\Gamma$ is open if and only if its preimage $\pi^{-1}(U)$ under π is an open set in $\mathbb{C}$.

For each $a \in \mathbb{C}$ consider the set $V_a = \{a + \alpha\omega_1 + \beta\omega_2 |\ \alpha, \beta \in \mathbb{R},\ 0 < \alpha, \beta < 1\}$. V_a is called the standard parallelogram constructed at point a. Clearly,

$$\pi|_{V_a} : V_a \to \pi(V_a) =: U_a$$

[4] see [Forst], Corollar **29.17**.
[5] see [Forst], Corollar **4.25**.

is a bijection. It is moreover a homeomorphism. Define the maps

$$\varphi_a := (\pi|_{V_a})^{-1} : U_a \to V_a \subset \mathbb{C}.$$

We claim that $\{\varphi_a : U_a \to V_a\}_{a\in\mathbb{C}}$ is a complex atlas of $\mathbb{C}/\Gamma$. Clearly, for each $a, b \in \mathbb{C}$ we have $U_a \cap U_b \neq \emptyset$. For the map

$$\phi_{ab} := \varphi_a \varphi_b^{-1}|_{\varphi_b(U_a\cap U_b)} : \varphi_b(U_a \cap U_b) \to \varphi_a(U_a \cap U_b)$$

we have $\pi\phi_{ab}(x) = \pi\varphi_a\varphi_b^{-1}(x) = \varphi_b^{-1}(x) = \pi(x)$. Therefore, $\phi_{ab}(x) = x + \gamma(x)$, where $\gamma(x) \in \Gamma$. Since ϕ_{ab} is a continuous map, we conclude that $\gamma(x)$ should be constant on each connected component of $U_a \cap U_b$. This means that ϕ_{ab} is a locally constant map. Therefore, it is holomorphic. This proves that $\{\varphi_a : U_a \to V_a\}_{a\in\mathbb{C}}$ is a complex atlas. Thus $\mathbb{C}/\Gamma$ is a Riemann surface (1-dimensional complex manifold). Riemann surfaces $\mathbb{C}/\Gamma$ are called complex tori.

Remark. Note that for any $u \in \mathbb{C}^*$ the lattices $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $u\Gamma = \mathbb{Z}u\omega_1 + \mathbb{Z}u\omega_2$ define isomorphic in complex sense tori. Therefore, after rescaling it is always possible to choose $\omega_1 = 1$. Since $\mathbb{Z}\omega_2 = \mathbb{Z}(-\omega_2)$, we can choose moreover $\operatorname{Im}\omega_2 > 0$.

Remark. It is known that complex tori correspond to elliptic curves, i. e., to non-singular projective curves of genus one.

Theorem 1.21 (Abel)**.** *Let X be a complex torus, let $D = \sum a_i x_i$ be a divisor. Then a meromorphic function f on X such that $\operatorname{div}(f) = D$ exists if and only if $\deg D = 0$ and $\sum a_i x_i = 0$ as an element of X.*

Theorem 1.22. *Let L be a line bundle of degree $d \neq 0$ on a torus X. Then there exists a unique point $p \in X$ such that $L = [dp]$.*

Proof. By Theorem 1.19 there exists a divisor $D = \sum a_i x_i$ such that $L = [D]$. Since X is a divisible group, there exist a unique solution p of the equation

$$D = \sum a_i x_i = dp$$

in X. The divisor $D' := D - dp$ has degree zero and $\sum a_i x_i - dp = 0$ in X. Therefore, by Abel's theorem there exists a meromorphic function f with $\operatorname{div}(f) = D'$. By Theorem 1.15 $[D']$ is a trivial bundle. Therefore, $L = [D] = [D' + dp] = [D'] \otimes [dp] = [dp]$.

Suppose that there exists another point p' such that $L = [dp] = [dp']$. Then $1 = [dp] \otimes [-dp'] = [dp - dp']$. By Theorem 1.15 there is a meromorphic function h with $\operatorname{div}(h) = dp - dp'$. By Abel's theorem we conclude $dp = dp'$ in X, hence $p = p'$. □

Recall that for a complex manifold X we denote by $\mathrm{Pic}^0(X) = \mathcal{E}(1,0)$ the set of isomorphism classes of line bundles over X of degree zero. This set is a subgroup in the group $\mathrm{Pic}(X)$ of isomorphism classes of line bundles over X.

For any point $o \in X$ we can "put" the zero of X in o. Namely we define the group operation by $x + y := x + y - o$. The neutral element is then o, the inverse of x is $-x$. When we say that we fix a point o we usually consider X with this changed group structure.

Theorem 1.23. *For any fixed point $o \in X$ we define a map from X to $\mathrm{Pic}^0(X)$*

$$X \to \mathrm{Pic}^0(X), \quad x \mapsto [x - o].$$

It is an isomorphism of groups (with respect to the group structure with o as the neutral element).

Proof. Clearly this map is a homomorphism. Suppose now that $[p - o] = 1$. Then by Abel's theorem $p = o$. This proves the injectivity.

Let L be a line bundle of degree zero. Then $L \otimes [o]$ is a line bundle of degree 1. By Theorem 1.22 there exists a unique point $x \in X$ such that $L \otimes [o] = [x]$. Therefore, $L = [x - o]$. This proves the surjectivity. $\square$

This allows us to introduce a structure of a complex manifold on $\mathrm{Pic}^0(X)$.

Since X is a divisible group it follows that $\mathrm{Pic}^0(X)$ is also a divisible group. The last means that for any $L' \in \mathcal{E}(1,0)$ and for any $n \in \mathbb{N}$ there exists an $L \in \mathcal{E}(1,0)$ such that $L^n \simeq L'$.

For $x \in X$ there is the map of translation by x:

$$t_x : X \to X, \quad y \mapsto x + y.$$

Theorem 1.24. *Let $A \in \mathcal{E}(1,1)$. Then for any $L' \in \mathcal{E}(1,0)$ there exists a unique $x \in X$ such that $L' \simeq t_x^*(A) \otimes A^{-1}$.*

Proof. Any $A \in \mathcal{E}(1,1)$ is of the form $A \simeq \mathcal{O}_X(o)$ for a unique $o \in X$. We know that $L' \simeq \mathcal{O}_X([y - o])$ for a unique point $y \in X$. Take $x = o - y \in X$. Since $t_x y = o$ we obtain $t_x^*(\mathcal{O}_X(o)) \simeq \mathcal{O}_X(y)$. This implies

$$L' \simeq \mathcal{O}_X([y - o]) \simeq \mathcal{O}_X(y) \otimes \mathcal{O}_X(o)^{-1} \simeq t_x^*(\mathcal{O}_X(o)) \otimes A^{-1} \simeq t_x^*(A) \otimes A^{-1}.$$

$\square$

Theorem 1.25. *Let $A \in \mathcal{E}(1,1)$ and let $L' \in \mathcal{E}(1,d)$. Then there exists a unique $L \in \mathcal{E}(1,0)$ such that $L' \simeq A^d \otimes L$.*

Proof. Define $L = L' \otimes A^{-d}$. L has degree zero. $\square$

2 Results of Atiyah

In this section we present results of Atiyah. Here X is an elliptic curve over an algebraically closed field of characteristic zero. Recall that by $\mathcal{E}(r,d) = \mathcal{E}_X(r,d)$ we denote the set of isomorphism classes of indecomposable vector bundles over X of rank r and degree d.

2.1 Necessary definitions and lemmas

Definition 2.1. Let E be a vector bundle on X. One says that E has sufficient sections if the canonical homomorphism $\Gamma(E) \to E_x$ is an epimorphism for all $x \in X$.

Consider an embedding $X \to \mathbb{P}^2$ of X in $\mathbb{P}^2$. Recall that the sheaf $\mathcal{O}_{\mathbb{P}^2}(1)$ is defined by the cocycle $g_{ij} = z_j/z_i$ corresponding to the open covering $\{U_0, U_1, U_2\}$ of $\mathbb{P}^2$, where

$$U_i = \{(z_0 : z_1 : z_2) \in \mathbb{P}^2 | z_i \neq 0\}, \quad i = 0, 1, 2.$$

Let H be a line bundle corresponding to $\mathcal{H} = \mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}^2}(1)|_X$. For a vector bundle E on X we denote $E(n) = E \otimes H^n$, where $H^n := \bigotimes\limits_1^n H$. Atiyah uses the following:

Theorem A. *For sufficiently large n(depending on E) the vector bundle $E(n)$ has sufficient sections.*[6]

Moreover, if E has sufficient sections, then E/E' also has sufficient sections for all subbundles $E' \subset E$.

Definition 2.2. Let E be a vector bundle of rank r. Suppose there exists a series of subbundles of E:

$$0 = E_0 \subset E_1 \subset E_2 \subset \ldots \subset E_r = E,$$

where $L_i = E_i/E_{i-1}$ is a line bundle. Such a series is called a splitting of E. We write $E = (L_1, L_2, \ldots, L_r)$.

The results obtained by Atiyah in [At] are based on the existence of a splitting for a vector bundle E over X.

Lemma 2.1. *Let E be a vector bundle over X, then there exists a splitting of E.*

[6]see [Serre].

Proof. By Theorem A $E(n)$ has sufficient sections for sufficiently large n. If $E(n) = (L_1, \ldots, L_r)$, then $E = (L_1(-n), \ldots, L_r(-n))$. Therefore, it is enough to construct a splitting in case when E has sufficient sections.

Assume that E has sufficient sections. Consider a non-zero section $\phi \in \Gamma(E)$. It defines the morphism of sheaves

$$\phi : \mathcal{O}_X \to \mathcal{E}, \quad 1 \mapsto \phi,$$

which induces the morphism of dual sheaves

$$\phi^* : \mathcal{E}^* \to \mathcal{O}_X^*, \quad (s : \mathcal{E} \to \mathcal{O}_X) \mapsto s\phi.$$

The cokernel of ϕ^* is $\mathcal{O}_{Z(\phi)}$, where $Z(\phi)$ denotes the set of zeros of ϕ counted with multiplicities. We get an exact sequence

$$\mathcal{E}^* \xrightarrow{\phi^*} \mathcal{O}_X^* \xrightarrow{\eta} \mathcal{O}_{Z(\phi)} \to 0.$$

The kernel of η (the ideal sheaf of $\mathcal{O}_{Z(\phi)}$) is isomorphic to $\mathcal{O}_X(-\operatorname{div}\phi) \simeq \mathcal{O}_X(\operatorname{div}\phi)^*$. Therefore, the image of ϕ^* is $\mathcal{O}_X(\operatorname{div}\phi)^*$. We get an exact sequence

$$\mathcal{E}^* \to \mathcal{O}_X(\operatorname{div}\phi)^* \to 0,$$

which means that the sheaf $\mathcal{O}_X(\operatorname{div}\phi)$ defines a subbundle of E. We denote this subbundle by $[\phi]$.

Since E has sufficient sections, one gets that $E' = E/[\phi]$ also has sufficient sections. Proceeding by induction we obtain a series of subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_r = E,$$

where $L_i = E_i/E_{i-1}$ is a line bundle ($L_1 = [\phi]$). $\square$

We denote $\deg\phi := \deg[\phi] = \deg(\operatorname{div}\phi)$. Clearly $\phi(X) \subset [\phi]$ and so ϕ can be treated as a section of $[\phi]$.

For a fixed splitting $0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$ of E there exists i such that $[\phi] \subset E_i$ and $[\phi] \not\subset E_{i-1}$. Therefore, we obtain a non-zero morphism $[\phi] \to L_i = E_i/E_{i-1}$ and thus by Riemann-Roch theorem $\deg[\phi] \leqslant \deg L_i$. We get that the integers $\deg\phi$ are bounded above by $\sup\limits_i(\deg L_i)$.

Definition 2.3. If $0 \neq \phi \in \Gamma(E)$ has the maximal degree we call ϕ a maximal section and $[\phi]$ a maximal line bundle of E.

Definition 2.4. A splitting $(L_1, L_2, \ldots, L_r)$ is a maximal splitting if
(i) L_1 is a maximal line bundle of E,
(ii) $(L_2, \ldots, L_r)$ is a maximal splitting of E/L_1.

Lemma 2.2. *If a vector bundle E over X has sufficient sections, then there exists a maximal splitting of E.*

Proof. The proof is the same as the proof of Lemma 2.1. One should only take maximal sections on each step. □

Lemma 2.3. *Let E an indecomposable vector bundle over X, and let $0 \to E_1 \to E \to E_2 \to 0$ be an exact sequence. Then $\Gamma \operatorname{Hom}(E_1, E_2) \neq 0$.*

Proof. By Theorem 1.11 the classes of extensions of E_2 by E_1 are in one-to-one correspondence with elements of $H^1(X, \mathcal{H}om(\mathcal{E}_2, \mathcal{E}_1))$. By Serre duality this is dual to the vector space $\Gamma \operatorname{Hom}(E_1, E_2 \otimes K)$, where K is the canonical line bundle on X. But $K = 1$, since X is an elliptic curve. As E is indecomposable, we conclude that $\Gamma \operatorname{Hom}(E_1, E_2) \neq 0$. □

Recall that there is a relation $\geqslant$ on the set of vector bundles: $E_1 \geqslant E_2$ if there exists a non-trivial morphism $E_2 \to E_1$. Note that for line bundles $L_1 \geqslant L_2$ by Riemann-Roch theorem implies $\deg L_1 \geqslant \deg L_2$.

Lemma 2.4. *Let E be an indecomposable vector bundle on X with $\Gamma(E) \neq 0$. Then E has a maximal splitting $(L_1, \ldots, L_r)$ with $L_i \geqslant L_1 \geqslant 1$.*

Proof. Since $\Gamma(E) \neq 0$, one concludes that E has a maximal line bundle $L_1 \geqslant 1$. We proceed now inductively. Suppose there exists a sequence

$$0 = E_0 \subset E_1 \subset \cdots \subset E_i,$$

where $L_j = E_j/E_{j-1}$ is a maximal line bundle of E/E_{j-1} for $j = 1, \ldots, i$, and such that all the L_j's satisfy the requirements of the lemma. Denote $E_i' = E/E_i$, then by Lemma 2.3 we get a non-zero element $f \in \operatorname{Hom}(E_i, E_i')$. There exists an integer j, $1 \leqslant j \leqslant i$ such that $f(E_{j-1}) = 0$ and $f(E_j) \neq 0$. We obtain a non-zero morphism $\bar{f} : L_j \to E_i'$. By inductive hypothesis there exists a non-zero section ϕ_j of L_j. Thus $\bar{f}\phi_j$ is a non-zero section of E_i' and $\operatorname{div} \bar{f}\phi_j \geqslant \operatorname{div} \phi_i$, which implies $[\bar{f}\phi] \geqslant L_j$.

If $\bar{f}\phi$ is a maximal section we take $L_{j+1} = [\bar{f}\phi]$. If not let L_{j+1} be a maximal line bundle of E_i'. Then $\deg L_{j+1} > \deg[\bar{f}\phi]$ and thus $\deg([\bar{f}\phi]^* \otimes L_{j+1}) > 0$. Since X is an elliptic curve, by Riemann-Roch theorem we get $\dim \Gamma([\bar{f}\phi]^* \otimes L_{j+1}) > 0$. This implies $L_{i+1} \geqslant [\bar{f}\phi]$. Using $[\bar{f}\phi] \geqslant L_j$ and inductive hypothesis, we obtain $L_{i+1} \geqslant L_j \geqslant L_1$. So we have a maximal line bundle L_{i+1} in E_i'. Take $E_{i+1} \subset E$ such that $E_{i+1}/E_i = L_{i+1}$. This establishes the induction and proves the lemma. □

Lemma 2.5. *Let $E = (L_1, \ldots, L_r)$, then*

$$\dim \Gamma(E) \leqslant \sum_{i=1}^{r} \dim \Gamma(L_i).$$

Proof. We have a series of subbundles $0 = E_0 \subset E_1 \subset E_2 \subset \ldots \subset E_r = E$ such that $L_i \simeq E_i/E_{i-1}$. We proceed by induction.

From the exact sequence

$$0 \to L_1 \to E_2 \to L_2 \to 0$$

we obtain the exact sequence

$$0 \to \Gamma(L_1) \to \Gamma(E_2) \to \Gamma(L_2).$$

Therefore, $\dim \Gamma(E_2) \leqslant \dim \Gamma(L_1) + \dim \Gamma(L_2)$.

Suppose now that we proved that $\dim \Gamma(E_j) \leqslant \sum\limits_{i=1}^{j} \dim \Gamma(L_i)$. From the exact sequence

$$0 \to E_j \to E_{j+1} \to L_{j+1} \to 0$$

as above we obtain the exact sequence

$$0 \to \Gamma(E_j) \to \Gamma(E_{j+1}) \to \Gamma(L_{j+1})$$

and $\dim \Gamma(E_{j+1}) \leqslant \dim \Gamma(E_j) + \dim \Gamma(L_{j+1})$. Using inductive hypothesis one gets

$$\dim \Gamma(E_{j+1}) \leqslant \dim \Gamma(E_j) + \dim \Gamma(L_{j+1}) \leqslant \sum_{i=1}^{j+1} \dim \Gamma(L_i),$$

which proves the statement of the lemma. □

Lemma 2.6. *Let $E \in \mathcal{E}(r,d)$, $0 \leqslant d < r$, and let $s = \dim \Gamma(E) > 0$. Then E has a trivial subbundle I_s.*

Proof. By Lemma 2.4 E has a maximal splitting $(L_1, \ldots, L_r)$ with $L_i \geqslant L_1 \geqslant 1$. If $\deg L_1 > 0$, then

$$d = \deg E = \sum_1^r \deg L_i \geqslant r \deg L_1 \geqslant r,$$

which is a contradiction. So $\deg L_1 = 0$, and since $L_1 \geqslant 1$, we get $L_1 = 1$. Since L_1 is a maximal line bundle we obtain $\deg \phi = 0$ for all $\phi \in \Gamma(E)$. But $\operatorname{div}(\phi) \geqslant 0$ for all $x \in X$. Therefore, $\operatorname{div} \phi = 0$ for all $\phi \in \Gamma(E)$. Hence $\Gamma(E) \to E_x$ is a monomorphism for all x and $\Gamma(E)$ generates a trivial subbundle I_s of E. □

Lemma 2.7. *Let $E \in \mathcal{E}(r,r)$. Then E has a splitting $(L, L, \ldots, L)$ where $\deg L = 1$.*

Proof. By Riemann-Roch theorem

$$s = \dim \Gamma(E) = \dim H^1(X, \mathcal{E}) + \deg E = \dim H^1(X, \mathcal{E}) + r \geqslant r.$$

By Lemma 2.4 there exists a maximal splitting $E = (L_1, \ldots, L_r)$ with $L_i \geqslant L_1 \geqslant 1$. If $\deg L_1 = 0$ as in Lemma 2.6 we obtain that E has a trivial subbundle I_s. Since $s \geqslant r$, this implies $E = I_s$, a contradiction, since E is indecomposable. So $\deg L_1 \geqslant 1$. But from

$$r = \deg E = \sum_1^r \deg L_i \geqslant r \deg L_1 \geqslant r,$$

one gets $\deg L_i = \deg L_1 = 1$ for all i. From $L_i \geqslant L_1$ we get $L_i = L_1$. This proves the statement of the lemma. □

The following two key lemmas are used in the sequel to establish a one-to-one correspondence between $\mathcal{E}(r, d)$ and $\mathcal{E}(r - d, d)$.

Lemma 2.8. *Let $E \in \mathcal{E}(r, d)$, $d \geqslant 0$. Then*
(i) $s = \dim \Gamma(E) = \begin{cases} d & \textit{if } d > 0 \\ 0 \textit{ or } 1 & \textit{if } d = 0; \end{cases}$
(ii) if $d < r$, E contains a trivial sub-bundle I_s and $E' = E/I_s$ is indecomposable; moreover $\dim \Gamma(E') = s$.

Lemma 2.9. *Let $E' \in \mathcal{E}(r', d)$, $d \geqslant 0$, and if $d = 0$ we suppose $\Gamma(E') \neq 0$. Then there exists a vector bundle $E \in \mathcal{E}(r, d)$, unique up to isomorphism, given by an extension $0 \to I_s \to E \to E' \to 0$, where $r = r' + s$ and* $s = \begin{cases} d & \textit{if } d > 0 \\ 1 & \textit{if } d = 0. \end{cases}$

2.2 Main results

Theorem 2.10. *(i) There exists a vector bundle $F_r \in \mathcal{E}(r, 0)$, unique up to isomorphism, with $\Gamma(F_r) \neq 0$. There exists an exact sequence*

$$0 \to 1 \to F_r \to F_{r-1} \to 0,$$

which uniquely determines F_r.
(ii) Let $E \in \mathcal{E}(r, 0)$, then there exists a unique up to isomorphism line bundle L such that $E \simeq L \otimes F_r$. Moreover $\det E \simeq L^r$.

Proof. (i) Denote by $\bar{\mathcal{E}}(r, 0)$ the set of isomorphism classes of vector bundles $E \in \mathcal{E}(r, 0)$ with $\Gamma(E) \neq 0$. We proceed by induction on r. For $r = 1$ the set $\bar{\mathcal{E}}(r, 0)$ contains only the trivial bundle, because every non-trivial section has no zeros.

Suppose we proved the statement for all $r' < r$. By Lemma 2.9 there is a unique up to isomorphism vector bundle F_r given by the extension

$$0 \to 1 \to F_r \to F_{r-1} \to 0.$$

Therefore, the set $\bar{\mathcal{E}}(r,0)$ is not empty. It remains to prove that there is no other bundle in $\bar{\mathcal{E}}(r,0)$. But by Lemma 2.8 for any $E \in \bar{\mathcal{E}}(r,0)$ there exists an exact sequence

$$0 \to 1 \to E \to E' \to 0,$$

where $E' \in \bar{\mathcal{E}}(r-1,0)$. By inductive hypothesis one obtains $E \simeq F_{r-1}$ and thus by Lemma 2.9 $E \simeq F_r$.

(ii) Let $E \in \mathcal{E}(r,0)$. Then $E \otimes A \in \mathcal{E}(r,r)$. By Lemma 2.7

$$E \otimes A = (L_1, \ldots, L_1)$$

for some line bundle $L_1 \in \mathcal{E}(1,1)$. Therefore,

$$E \otimes A \otimes L_1^* = (1, \ldots, 1),$$

and hence $E \otimes A \otimes L_1^*$ has sections. We obtain $E \otimes A \otimes L_1^* \in \bar{\mathcal{E}}(r,0)$ and by (i) $E \otimes A \otimes L_1^* \simeq F_r$. Therefore, $E \simeq F_r \otimes L_1 \otimes A^* = F_r \otimes L$, where $L = L_1 \otimes A^*$. Since F_r is an extension of F_{r-1} by the trivial line bundle, one gets $\det F_r \simeq \det F_{r-1}$ and by induction $\det F_r \simeq 1$. We obtain

$$\det E = \det(F_r \otimes L) = \det(F_r) \otimes L^r \simeq L^r.$$

Suppose $E \simeq F_r \otimes L \simeq F_r \otimes L'$. We shall show $L \simeq L'$. Since $F_r \otimes L \simeq F_r \otimes L'$ is equivalent to $F_r \simeq F_r \otimes L' \otimes L^*$ it is enough to prove that $F_r \otimes L \simeq F_r$ implies $L \simeq 1$.

As $F_r = (1, \ldots, 1)$, one obtains $F_r \otimes L = (L, \ldots, L)$. Clearly $\deg L = 0$. If L is not a trivial bundle, then $\Gamma(L) = 0$. Using Lemma 2.5, one gets

$$\dim \Gamma(F_r) = \dim \Gamma(F_r \otimes L) \leqslant r \dim \Gamma(L) = 0.$$

This is a contradiction, since $\Gamma(F_r) \neq 0$. Therefore, L is the trivial line bundle. □

Theorem 2.11. *Let A be a line bundle of degree 1 on X. Then A determines a bijective map $\alpha_{r,d} : \mathcal{E}(h,0) \to \mathcal{E}(r,d)$, where $h = (r,d)$. Moreover, $\det \alpha_{r,d}(E) \simeq \det E \otimes A^d$ and $\alpha_{r,d}$ is uniquely determined by the following properties:*
(i) $\alpha_{r,0} = 1$,
(ii) $\alpha_{r,d+r}(E) \simeq \alpha_{r,d}(E) \otimes A$,
(iii) if $0 < d < r$, *there exists an exact sequence* $0 \to I_d \to \alpha_{r,d}(E) \to \alpha_{r-d,d}(E) \to 0$.

Proof. Note, that A determines a bijection

$$\mathcal{E}(r,d) \to \mathcal{E}(r,d+r), \quad E \mapsto E \otimes A.$$

Therefore, it is sufficient to consider $0 \leqslant d < r$. Clearly $h = r$ if $d = 0$. In this case we put $\alpha_{r,d} = \alpha_{r,0} = 1$. If $d > 0$, then by Lemma 2.8 for each $E \in \mathcal{E}(r,d)$ there exists an exact sequence

$$0 \to I_d \to E \to E' \to 0,$$

where $E' \in \mathcal{E}(r-d,d)$.

Conversely let $E' \in \mathcal{E}(r-d,d)$. By Lemma 2.9 there is a unique up to isomorphism vector bundle $E \in \mathcal{E}(r,d)$ which is given by the exact sequence

$$0 \to I_d \to E \to E' \to 0.$$

This gives a bijection between $\mathcal{E}(r,d)$ and $\mathcal{E}(r-d,d)$. Using this together with (ii) one obtains a bijection $\alpha_{r,d} : \mathcal{E}(h,0) \to \mathcal{E}(r,d)$. The procedure given here corresponds to the Euclidean algorithm for determining the greatest common divisor of r and d.

If $d = qr + d'$, $0 \leqslant d' < d$, then repeating $|q|$ times (ii) we obtain a one-to-one correspondence

$$\mathcal{E}(r,d) \leftrightarrow \mathcal{E}(r,d-r) \leftrightarrow \ldots \leftrightarrow \mathcal{E}(r,d-qr) = \mathcal{E}(r,d').$$

If $r = qd + r'$, $0 \leqslant r' < r$, then repeating $|q|$ times (iii) we get a one-to-one correspondence

$$\mathcal{E}(r,d) \leftrightarrow \mathcal{E}(r-d,d) \leftrightarrow \ldots \leftrightarrow \mathcal{E}(r-qd,d) = \mathcal{E}(r',d).$$

Proceeding according to the divisions with rest from Euclidean algorithm, we obtain the required bijection $\alpha_{r,d} : \mathcal{E}(h,0) \to \mathcal{E}(r,d)$. □

Following Atiyah we put $E_A(r,d) := \alpha_{r,d}(F_h)$.

Theorem 2.12. *Let X be an elliptic curve, let A correspond to a fixed point on X, i. e., $A \in \mathcal{E}(1,1)$. We may regard X as an abelian variety with A as the zero element. Each $\mathcal{E}(r,d)$ can be identified with X in such a way that the map*

$$\det : \mathcal{E}(r,d) \to \mathcal{E}(1,d)$$

corresponds to the map

$$H : X \to X, \quad x \mapsto hx,$$

where $h = (r,d)$.

Proof. By Theorem 2.11 we have a bijection $\alpha_{r,d} : \mathcal{E}(h,0) \to \mathcal{E}(r,d)$. By Theorem 2.10 (ii) there is a bijection

$$\beta : \mathcal{E}(h,0) \to \mathcal{E}(1,0), \quad \mathcal{F}_h \otimes L \mapsto L.$$

But $\mathcal{E}(1,0)$ can be identified with X itself by the map

$$X \to \mathcal{E}(1,0), \quad x \mapsto [x] \otimes A^{-1}.$$

The map H under this identification corresponds to the map

$$H' : \mathcal{E}(1,0) \to \mathcal{E}(1,0), \quad L \mapsto L^h$$

Combining β and $\alpha_{r,d}$ we obtain a bijection between $\mathcal{E}(r,d)$ and X. Now the statement of the theorem is clear from the commutative diagram

$$\begin{tikzcd}
\mathcal{E}(r,d) \arrow[d, "\det"'] & \mathcal{E}(h,0) \arrow[l, "\alpha_{r,d}"'] \arrow[r, "\beta"] \arrow[d, "\det"'] & \mathcal{E}(1,0)\, , \arrow[d, "H'"'] \\
\mathcal{E}(1,d) & \mathcal{E}(1,0) \arrow[l, "\alpha_{1,d}"'] \arrow[r, equal] & \mathcal{E}(1,0)
\end{tikzcd}
\qquad
\begin{tikzcd}
\alpha_{r,d}(F_h \otimes L) \arrow[d, mapsto, "\det"'] & F_h \otimes L \arrow[l, mapsto, "\alpha_{r,d}"'] \arrow[r, mapsto, "\beta"] \arrow[d, mapsto, "\det"'] & L \arrow[d, mapsto, "H'"'] \\
L^h \otimes A^d & L^h \arrow[l, mapsto, "\alpha_{1,d}"'] \arrow[r, equal] & L^h.
\end{tikzcd}$$

□

Corollary 2.13. *Let $h = (r,d) = 1$. Then if $E \in \mathcal{E}(r,d)$*
(i) The map $\det : \mathcal{E}(r,d) \to \mathcal{E}(1,d)$ is a bijection,
(ii) $E \simeq E_A(r,d) \otimes L$ for some $L \in \mathcal{E}(1,0)$,
(iii) $E_A(r,d) \otimes L \simeq E_A(r,d)$ if and only if $L^r \simeq 1$,
(iv) $E_A(r,d)^* \simeq E_A(r,-d)$.

Proof. (i) is a consequence of Theorem 2.12, because the map H in this case is the identity.

(ii) Since $\det(E_A(r,d) \otimes L) \simeq A^d \otimes L^r$ and since there exists an L such that $\det E \simeq L^r \otimes A^d$, we conclude $\det E \simeq \det(E_A(r,d) \otimes L)$ and by (i) $E \simeq E_A(r,d) \otimes L$.

(iii) $E_A(r,d) \otimes L \simeq E_A(r,d)$ if and only if $\det(E_A(r,d) \otimes L) \simeq \det(E_A(r,d))$. But the last is the same as $A^d \otimes L^r \simeq A^d$ or as $L^r \simeq 1$.

(iv) is clear, since

$$\det(E_A(r,d)^*) \simeq (\det E_A(r,d))^* \simeq (A^d)^* \simeq A^{-d} \simeq \det E_A(r,-d).$$

□

We need two important lemmas.

Lemma 2.14. *Let $E \in \mathcal{E}(r,d)$, where $(r,d) = 1$. Then $\operatorname{End} E \simeq \bigoplus_{i=1}^{r^2} L_i$, where the L_i are all the line bundles of order dividing r, i. e., $L^r \simeq 1$ and $L_i \not\simeq L_j$ for $i \neq j$.*

Lemma 2.15. *Let* $(r,d)=1$*, then* $E_A(r,d)\otimes F_h$ *is indecomposable.*

Theorem 2.16. *Let* $(r,d)=1$*, then* $E_A(r,d)\otimes F_h \simeq E_A(rh,dh)$*.*

Proof. We prove the theorem by double induction on r and h. More precisely we assume the theorem true (for $h, r \geqslant 2$)
(i) for $h-1$ and all r,
(ii) for h and all $s \leqslant r-1$.

First we observe that if $h=1$, then $F_h = 1$, and so the lemma is true for all r. Also if $r=1$, $E_A(r,d)=A^d$, $E_A(rh,dh)=A^d\otimes F_h$ by definition. This starts the induction.

Now we have the exact sequence

$$0 \to 1 \to F_h \to F_{h-1} \to 0. \tag{4}$$

(4) $\otimes E_A(r,d)$ gives the exact sequence

$$0 \to E_A(r,d) \to E_A(r,d)\otimes F_h \to E_A(r,d)\otimes F_{h-1} \to 0. \tag{5}$$

Since $E_A(r,d)\otimes A \simeq E_A(r,r+d)$ it suffices to consider the range $0<d<r$. Then also $0<dh<rh$ and $0<d(h-1)<r(h-1)$. For brevity we write (5) as

$$0 \to E_1 \to E_2 \to E_3 \to 0,$$

and we put $d_i = \deg E_i$, $r_i = \dim E_i$. Then $0<d_i<r_i$ and by Lemma 2.15 $E_i \in \mathcal{E}(r_i,d_i)$. Hence, by Lemma 2.8 $\dim\Gamma(E_i)=d_i$ and $\Gamma(E_i)$ generates a trivial subbundle I_{d_i} of E_i; moreover, if $E_i' = E_i/I_{d_i}$ then $E_i' \in \mathcal{E}(r_i-d_i,d_i)$, and $E_i \simeq E_A(r_i,d_i)$ if and only if $E_i' \simeq E_A'(r_i-d_i,d_i)$(this follows from Lemma 2.9 and from the construction of the map $\alpha_{r,d}$ in Theorem 2.11). Thus we have an exact sequence diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I_{d_1} \arrow[r] \arrow[d] & I_{d_2} \arrow[r] \arrow[d] & I_{d_3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & E_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E_1' \arrow[r] \arrow[d] & E_2' \arrow[r] \arrow[d] & E_3' \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Applying inductive hypothesis (i) in the last column of the diagram, we find $E_3 \simeq E_A(r_3, d_3)$, hence $E_3' \simeq E_A(r_3 - d_3, d_3)$ and so again by (i)

$$E_3' \simeq E_A(r-d,d) \otimes F_{h-1}.$$

We have also $E_1' \simeq E_A(r-d,d)$, hence

$$\dim \Gamma \operatorname{Hom}(E_1', E_3') = \dim \Gamma(\operatorname{End} E_A(r-d,d) \otimes F_{h-1}) = 1$$

by Lemma 2.14, Theorem 2.10, and Lemma 2.8. In fact, by Lemma 2.14 we have that $\operatorname{End} E_A(r-d,d)$ is a direct sum of line bundles L_i. Since $L_i^r \simeq 1$, one concludes that all the L_i have degree zero. Therefore, $\operatorname{End} E_A(r-d,d) \otimes F_{h-1}$ is the direct sum of vector bundles $F_{h-1} \otimes L_i$ of degree zero, only one of which (namely the tensor product of F_{h-1} with the trivial line bundle) by Theorem 2.10 has sections. By Lemma 2.8, dimension of $\Gamma(F_{h-1})$ is 1.

Now, by Theorem 1.11, the extension classes of E_3' by E_1' correspond to the elements of $H^1(X, \mathcal{H}om(\mathcal{E}_3', \mathcal{E}_1'))$, and the extensions corresponding to a, λa, where $a \in H^1(X, \mathcal{H}om(\mathcal{E}_3', \mathcal{E}_1'))$ and λ is a non-zero constant, define isomorphic vector bundles. In the present case this vector space is of dimension one, because by Serre duality $H^1(X, \mathcal{H}om(\mathcal{E}_3', \mathcal{E}_1')) \simeq \Gamma \operatorname{Hom}(E_1', E_3')$. So any two non-trivial extensions of E_3' by E_1' define isomorphic vector bundles. Now the bottom row of the diagram is one extension and $(4) \otimes E_A(r-d,d)$ is another. Moreover, both are non-trivial extensions since E_2' and $E_A(r-d,d) \otimes F_h$ are indecomposable (the latter by Lemma 2.15 or by (ii)). Hence $E_2' \simeq E_A(r-d,d) \otimes F_h \simeq E_A(h(r-d), hd)$ by inductive hypothesis (ii). Thus $E_2 \simeq E_A(rh, dh)$, and the induction is established. □

Lemma 2.17. *Let $(r,d) = 1$, $0 < d < r$ and let L be a line bundle of degree zero. Then we have an exact sequence*

$$0 \to I_{dh} \to E_A(rh, dh) \otimes L \to E_A(rh - dh, dh) \otimes L' \to 0,$$

where L' is any line bundle such that $L'^{(r-d)} \simeq L^r$.

Theorem 2.18. *Let $E \in \mathcal{E}(r,d)$. Then there exists a line bundle L such that $E \simeq E_A(r,d) \otimes L$.*

Proof. We proceed by induction on r. One can assume $0 < d < r$, since for $d = 0$ we already have Theorem 2.10. By Lemma 2.8 there exists an exact sequence

$$0 \to I_d \to E \to E' \to 0.$$

By inductive hypothesis $E' \simeq E_A(r-d,d) \otimes L'$ for some L'. Let L be any line bundle such that $L^{rh} \simeq L'^{(r-d)/h}$, where $h = (r,d)$. Then by Lemma 2.17 we have the exact sequence

$$0 \to I_d \to E_A(r,d) \otimes L \to E' \to 0.$$

By Lemma 2.9 $E \simeq E_A(r,d) \otimes L$. □

In particular this theorem means that for any two indecomposable vector bundles E_1 and E_2 of rank r and degree d there exists a line bundle L of degree zero such that $E_1 = E_2 \otimes L$.

3 Language of factors of automorphy

3.1 Basic correspondence between vector bundles and factors of automorphy

Let X be a complex manifold and let $p : Y \to X$ be a covering of X. Let $\Gamma \subset \mathrm{Deck}(Y/X)$ be a subgroup in the group of deck transformations $\mathrm{Deck}(Y/X)$ such that for any two points y_1 and y_2 with $p(y_1) = p(y_2)$ there exists an element $\gamma \in \Gamma$ such that $\gamma(y_1) = y_2$. In other words, Γ acts transitively in each fiber. We call this property (**T**) .

Remark. Note that for any two points y_1 and y_2 there can be only one $\gamma \in \mathrm{Deck}(Y/X)$ with $\gamma(y_1) = y_2$ (see [Forst], Satz **4.8**). Therefore, $\Gamma = \mathrm{Deck}(Y/X)$ and the property (**T**) simply means that $p : Y \to X$ is a normal (Galois) covering.

We have an action of Γ on Y:

$$\Gamma \times Y \to Y, \quad y \mapsto \gamma(y) =: \gamma y.$$

Definition 3.1. A holomorphic function $f : \Gamma \times Y \to \mathrm{GL}_r(\mathbb{C})$, $r \in \mathbb{N}$ is called an r-dimensional factor of automorphy if it satisfies the relation

$$f(\lambda\mu, y) = f(\lambda, \mu y) f(\mu, y).$$

Denote by $Z^1(\Gamma, r)$ the set of all r-dimensional factors of automorphy.

We introduce the relation $\sim$ on $Z^1(\Gamma, r)$. We say that f is equivalent to f' if there exists a holomorphic function $h : Y \to \mathrm{GL}_r(\mathbb{C})$ such that

$$h(\lambda y) f(\lambda, y) = f'(\lambda, y) h(y).$$

We write in this case $f \sim f'$.

Claim. *The relation $\sim$ is an equivalence relation on $Z^1(\Gamma, r)$.*

Proof. f is clearly equivalent to itself because $h(\lambda y) f(\lambda, y) = f(\lambda, y) h(y)$ for the constant map $h(y) = I_r$. Therefore, reflexivity holds true.

Consider $f \sim f'$, i. e., $h(\lambda y) f(\lambda, y) = f'(\lambda, y) h(y)$ for some h. Define $h' = h^{-1}$, then

$$h'(\lambda y) f'(\lambda, y) = f(\lambda, y) h'(y),$$

which means $f' \sim f$.

Let $f \sim f'$ with $h(\lambda y) f(\lambda, y) = f'(\lambda, y) h(y)$ and let $f' \sim f''$ with $h'(\lambda y) f'(\lambda, y) = f''(\lambda, y) h'(y)$. Multiplying the first equality by $h'(\lambda, y)$ from the left one obtains

$$h'(\lambda, y) h(\lambda y) f(\lambda, y) = h'(\lambda, y) f'(\lambda, y) h(y) = f''(\lambda, y) h'(y) h(y),$$

which means that f is equivalent to f'' and proves that $\sim$ is symmetric. $\square$

We denote the set of equivalence classes of $Z^1(\Gamma, r)$ with respect to $\sim$ by $H^1(\Gamma, r)$.

Consider $f \in Z^1(\Gamma, r)$ and a trivial vector bundle $Y \times \mathbb{C}^r \to Y$. Define a holomorphic action of Γ on $Y \times \mathbb{C}^r$:

$$\begin{aligned} \Gamma \times Y \times \mathbb{C}^r &\to Y \times \mathbb{C}^r \\ (\lambda, y, v) &\mapsto (\lambda y, f(\lambda, y)v) =: \lambda(y, v). \end{aligned}$$

Denote $E(f) = Y \times \mathbb{C}^r/\Gamma$ and note that for two equivalent points $(y, v) \sim_\Gamma (y', v')$ with respect to the action of Γ on $Y \times \mathbb{C}^r$ it follows that $p(y) = p(y')$. In fact, $(y, v) \sim_\Gamma (y', v')$ implies in particular that $y = \gamma y'$ for some $\gamma \in \Gamma$ and by the definition of deck transformations $p(y) = p(\gamma y') = p(y')$. Hence the projection $Y \times \mathbb{C}^r \to Y$ induces the map

$$\begin{aligned} \pi : E(f) &\to X \\ [y, v] &\mapsto p(y). \end{aligned}$$

We equip $E(f)$ with the quotient topology.

Theorem 3.1. *$E(f)$ inherits a complex structure from $Y \times \mathbb{C}^r$ and the map $\pi : E(f) \to X$ is a holomorphic vector bundle on X.*

Proof. First we prove that π is a topological vector bundle. Clearly π is a continuous map. Consider the commutative diagram

$$\begin{tikzcd} Y \times \mathbb{C}^r \arrow[r] \arrow[d] & E(f) \arrow[d, "\pi"] \\ Y \arrow[r, "p"] & X. \end{tikzcd}$$

Let x be a point of X. Since p is a covering, one can choose an open neighbourhood U of x such that its preimage is a disjoint union of open sets biholomorphic to U, i. e., $p^{-1}(U) = \bigsqcup_{i \in \mathcal{I}} V_i$, $p_i := p|_{V_i} : V_i \to U$ is a biholomorphism for each $i \in \mathcal{I}$. For each pair $(i, j) \in \mathcal{I} \times \mathcal{I}$ there exists a unique $\lambda_{ij} \in \Gamma$ such that $\lambda_{ij} p_j^{-1}(x) = p_i^{-1}(x)$ for all $x \in U$. This follows from the property $(\mathbf{T})$.

We have $\pi^{-1}(U) = ((\bigsqcup_{i \in \mathcal{I}} V_i) \times \mathbb{C}^r)/\Gamma$.

Choose some $i_U \in \mathcal{I}$. Consider the holomorphic map

$$\varphi'_U : (\bigsqcup_{i \in \mathcal{I}} V_i) \times \mathbb{C}^r \to U \times \mathbb{C}^r, \quad (y_i, v) \mapsto (p(y_i), f(\lambda_{i_U i}, y_i)v), \; y_i \in V_i.$$

Suppose that $(y_i, v') \sim_\Gamma (y_j, v)$. This means

$$(y_i, v') = \lambda_{ij}(y_j, v) = (\lambda_{ij} y_j, f(\lambda_{ij}, y_j)v).$$

Therefore,

$$\begin{aligned}\varphi'_U(y_i, v') =&(p(y_i), f(\lambda_{i_U i}, y_i)v') = (p(\lambda_{ij}y_j), f(\lambda_{i_U i}, \lambda_{ij}y_j)f(\lambda_{ij}, y_j)v) = \\ &(p(y_j), f(\lambda_{i_U j}, y_j)v) = \varphi'_U(y_j, v).\end{aligned}$$

Thus φ'_U factorizes through $((\bigsqcup_{i\in\mathcal{I}} V_i) \times \mathbb{C}^r)/\Gamma$, i. e., the map

$$\varphi_U : ((\bigsqcup_{i\in\mathcal{I}} V_i) \times \mathbb{C}^r)/\Gamma \to U \times \mathbb{C}^r, \quad [(y_i, v)] \mapsto (p(y_i), f(\lambda_{i_U i}, y_i)v),\ y_i \in V_i$$

is well-defined and continuous. We claim that φ_U is bijective.

Suppose $\varphi_U([(y_i, v')]) = \varphi_U([(y_j, v)])$, where $y_i \in V_i$, $y_j \in V_j$. By definition this is equivalent to $(p(y_i), f(\lambda_{i_U i}, y_i)v') = (p(y_j), f(\lambda_{i_U j}, y_j)v)$, which means $y_i = \lambda_{ij}y_j$ and

$$\begin{aligned}f(\lambda_{i_U i}, \lambda_{ij}y_j)v' =&f(\lambda_{i_U i}, y_i)v' = f(\lambda_{i_U j}, y_j)v = \\ &f(\lambda_{i_U i}\lambda_{ij}, y_j)v = f(\lambda_{i_U i}, \lambda_{ij}y_j)f(\lambda_{ij}, y_j)v.\end{aligned}$$

We conclude $v' = f(\lambda_{ij}, y_j)v$ and $[(y_i, v')] = [(y_j, v)]$, which means injectivity of φ_U.

At the same time for each element $(y, v) \in U \times \mathbb{C}^r$ one has

$$\begin{aligned}&\varphi_U([(p_i^{-1}(y), f(\lambda_{i_U i}, p_i^{-1}(y))^{-1}v)]) = \\ &\quad (pp_i^{-1}(y), f(\lambda_{i_U i}, p_i^{-1}(y))f(\lambda_{i_U i}, p_i^{-1}(y))^{-1}v) = (y, v),\end{aligned}$$

i. e., φ_U is surjective and we obtain that φ_U is a bijective map.

This means, that φ_U is a trivialization for U and that $\pi : E(f) \to X$ is a (continuous) vector bundle. If U and V are two neighbourhoods of X defined as above for which $E(f)|_U$, $E(f)|_V$ are trivial, then the corresponding transition function is

$$\begin{aligned}&\varphi_U\varphi_V^{-1} : (U \cap V) \times \mathbb{C}^r \to (U \cap V) \times \mathbb{C}^r \\ &(x, v) \mapsto (x, g_{UV}(x)v),\end{aligned}$$

where $g_{UV} : U \cap V \to \mathrm{GL}_r(\mathbb{C})$ is a cocycle defining $E(f)$. But from the construction of φ_U it follows that

$$g_{UV}(x) = f(\lambda_{i_U i_V}, p_{i_V}^{-1}(x)).$$

Therefore, g_{UV} is a holomorphic map, hence $\varphi_U\varphi_V^{-1}$ is also a holomorphic map. Thus the maps φ_U give $E(f)$ a complex structure. Since π is locally a projection, one sees that π is a holomorphic map.

□

Remark. Note that $p^*E(f)$ is isomorphic to $Y \times \mathbb{C}^r$. An isomorphism can be given by the map

$$\begin{aligned} p^*E(f) &\to Y \times \mathbb{C}^r \\ (y, [\tilde{y}, v]) &\mapsto (y, f(\lambda, \tilde{y})v),\ \lambda\tilde{y} = y. \end{aligned}$$

Now we have the map from $Z^1(\Gamma, r)$ to the set $K_r = \{[E] | p^*(E) \simeq Y \times \mathbb{C}^r\}$ of isomorphism classes of vector bundles of rank r over X with trivial pull back with respect to p.

$$\phi' : Z^1(\Gamma, r) \to K_r; \quad f \mapsto [E(f)].$$

Theorem 3.2. *Let K_r denote the set of isomorphism classes of vector bundles of rank r on X with trivial pull back with respect to p. Then the map*

$$H^1(\Gamma, r) \to K_r, \quad [f] \mapsto [E(f)]$$

is a bijection.

Proof. [7] Consider the map $\phi' : Z^1(\Gamma, r) \to K_r$ and let f and f' be two equivalent r-dimensional factors of automorphy. It means that there exists a holomorphic function $h : Y \to \mathrm{GL}_r(\mathbb{C})$ such that

$$f'(\lambda, y) = h(\lambda y) f(\lambda, y) h(y)^{-1}.$$

Therefore, for two neighbourhoods U, V constructed as above we have the following relation for cocycles corresponding to f and f'.

$$\begin{aligned} g'_{UV}(x) =& f'(\lambda_{UV}, p_{i_V}^{-1}(x)) = h(\lambda_{UV} p_{i_V}^{-1}(x)) f(\lambda_{UV}, p_{i_V}^{-1}(x)) h(p_{i_V}^{-1}(x))^{-1} = \\ & h(p_{i_U}^{-1}(x)) g_{UV}(x) h(p_{i_V}^{-1}(x))^{-1} = h_U(x) g_{UV}(x) h_V(x)^{-1}, \end{aligned}$$

where $\lambda_{UV} = \lambda_{i_U i_V}$, $h_U(x) = h(p_{i_U}^{-1}(x))$ and $h_V(x) = h(p_{i_V}^{-1}(x))$. We obtained

$$g'_{UV} = h_U g_{UV} h_U^{-1},$$

which is exactly the condition for two cocycles to define isomorphic vector bundles. Therefore, $E(f) \simeq E(f')$ and it means that ϕ' factorizes through $H^1(\Gamma, r)$, i. e., the map

$$\phi : H^1(\Gamma, r) \to K_r; \quad [f] \mapsto [E(f)]$$

is well-defined.

[7] This proof generalizes the proof from [Lange] given only for line bundles.

It remains to construct the inverse map. Suppose $E \in K_r$, in other words $p^*(E)$ is the trivial bundle of rank r over Y. Let $\alpha : p^*E \to Y \times \mathbb{C}^r$ be a trivialization. The action of Γ on Y induces a holomorphic action of Γ on p^*E :

$$\lambda(y, e) := (\lambda y, e) \text{ for } (y, e) \in p^*E = Y \times_X E.$$

Via α we get for every $\lambda \in \Gamma$ an automorphism ψ_λ of the trivial bundle $Y \times \mathbb{C}^r$. Clearly ψ_λ should be of the form

$$\psi_\lambda(y, v) = (\lambda y, f(\lambda, y)v),$$

where $f : \Gamma \times Y \to \mathrm{GL}_r(\mathbb{C})$ is a holomorphic map. The equation for the action $\psi_{\lambda\mu} = \psi_\lambda \psi_\mu$ implies that f should be an r-dimensional factor of automorphy.

Suppose α' is an another trivialization of p^*E. Then there exists a holomorphic map $h : Y \to \mathrm{GL}_r(\mathbb{C})$ such that $\alpha'\alpha^{-1}(y, v) = (y, h(y)v)$. Let f' be a factor of automorphy corresponding to α'. From

$$\begin{aligned}(\lambda y, f'(\lambda, y)v) =&\psi'_\lambda(y, v) = \alpha'\lambda\alpha'^{-1}(y, v) = \alpha'\alpha^{-1}\alpha\lambda\alpha^{-1}\alpha\alpha'^{-1}(y, v) = \\ &\alpha'\alpha^{-1}\psi_\lambda(\alpha'\alpha^{-1})^{-1}(y, v) = \alpha'\alpha^{-1}\psi_\lambda(y, h(y)^{-1}v) = \\ &\alpha'\alpha^{-1}(\lambda y, f(\lambda, y)h(y)^{-1}v) = (\lambda y, h(\lambda y)f(\lambda, y)h(y)^{-1}),\end{aligned}$$

we obtain $f'(\lambda, y) = h(\lambda y)f(\lambda, y)h(y)^{-1}$. The last means that $[f] = [f']$, in other words,

the class of a factor of automorphy in $H^1(\Gamma, r)$ does not depend on the trivialization and we get a map $K_r \to H^1(\Gamma, r)$. This map is the inverse of ϕ. □

Let X be a connected complex manifold, let $p : \tilde{X} \to X$ be a universal covering of X, and let $\Gamma = \mathrm{Deck}(Y/X)$. Since universal coverings are normal coverings, Γ satisfies the property (**T**) .[8] Moreover, Γ is isomorphic to the fundamental group $\pi_1(X)$ of X.[9] An isomorphism is given as follows.

Fix $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$ with $p(\tilde{x}_0) = x_0$. We define a map

$$\Phi : \mathrm{Deck}(\tilde{X}/X) \to \pi_1(X, x_0)$$

as follows. Let $\sigma \in \mathrm{Deck}(\tilde{X}/X)$ and $v : [0; 1] \to \tilde{X}$ be a curve with $v(0) = \tilde{x}_0$ and $v(1) = \sigma(\tilde{x}_0)$. Then a curve

$$pv : [0 : 1] \to X, \quad t \mapsto pv(t)$$

is such that $pv(0) = pv(1) = x_0$. Define $\Phi(\sigma) := [pv]$, where $[pv]$ denotes a homotopy class of pv. The map Φ is well defined and is an isomorphism of groups.

[8] see [Forst], Satz **5.6**.

[9] see [Forst], Satz **5.6**.

So we can identify Γ with $\pi_1(X)$. Therefore, we have an action of $\pi_1(X)$ on $\tilde{X}$ by deck transformations.

Consider an element $[w] \in \pi_1(X, x_0)$ represented by a path $w : [0;1] \to X$. We denote $\sigma = \Phi^{-1}([w])$. Consider any $\tilde{x}_0 \in X$ such that $p(\tilde{x}_0) = w(0) = w(1)$, then the path w can be uniquely lifted to the path

$$v : [0;1] \to \tilde{X}$$

with $v(0) = \tilde{x}_0$ (see [Forst], Satz **4.14**). Denote $\tilde{x}_1 = v(1)$. Then σ is a unique element in $\mathrm{Deck}(\tilde{X}/X)$ such that $\sigma(\tilde{x}_0) = \tilde{x}_1$. This gives a description of the action of $\pi_1(X, x_0)$ on $\tilde{X}$.

Now we have a corollary to Theorem 3.2.

Corollary 3.3. *Let X be a connected complex manifold, let $p : \tilde{X} \to X$ be the universal covering, let Γ be the fundamental group of X naturally acting on $\tilde{X}$ by deck transformations. As above, $H^1(\Gamma, r)$ denotes the set of equivalence classes of r-dimensional factors of automorphy*

$$\Gamma \times \tilde{X} \to \mathrm{GL}_r(\mathbb{C}).$$

Then there is a bijection

$$H^1(\Gamma, r) \to K_r, \quad [f] \mapsto E(f),$$

where K_r denotes the set of isomorphism classes of vector bundles of rank r on X with trivial pull back with respect to p.

3.2 Properties of factors of automorphy

Definition 3.2. Let $f : \Gamma \times Y \to \mathrm{GL}_r(\mathbb{C})$ be an r-dimensional factor of automorphy. A holomorphic function $s : Y \to \mathbb{C}^r$ is called an f-theta function if it satisfies

$$s(\gamma y) = f(\gamma, y)s(y) \text{ for all } \gamma \in \Gamma,\ y \in Y.$$

Theorem 3.4. *Let $f : \Gamma \times Y \to \mathrm{GL}_r(\mathbb{C})$ be an r-dimensional factor of automorphy. Then there is a one-to-one correspondence between sections of $E(f)$ and f-theta functions.*

Proof. Let $\{V_i\}_{i\in\mathcal{I}}$ be a covering of Y such that p restricted to V_i is a homeomorphism. Denote $\varphi_i := (p|_{V_i})^{-1}$, $U_i := p(V_i)$. Then $\{U_i\}$ is a covering of X such that $E(f)$ is trivial over each U_i.

Consider a section of $E(f)$ given by functions $s_i : U_i \to \mathbb{C}^r$ satisfying

$$s_i(x) = g_{ij}(x)s_j(x) \text{ for } x \in U_i \cap U_j,$$

where

$$g_{ij}(x) = f(\lambda_{U_iU_j}, \varphi_j(x)), \quad x \in U_i \cap U_j$$

is a cocycle defining $E(f)$ (see the proof of Theorem 3.2). Define $s : Y \to \mathbb{C}^r$ by $s(\varphi_i(x)) := s_i(x)$. To prove that this is well-defined we need to show that $s_i(x) = s_j(x)$ when $\varphi_i(x) = \varphi_j(x)$. But since $\varphi_i(x) = \varphi_j(x)$ we obtain $\lambda_{U_iU_j} = 1$. Therefore,

$$s_i(x) = g_{ij}(x)s_i(x) = f(\lambda_{U_iU_j}, \varphi_j(x))s_j(x) = f(1, \varphi_j(x))s_j(x) = s_j(x).$$

For any $\gamma \in \Gamma$ for any point $y \in Y$ take $i, j \in \mathcal{I}$ and $x \in X$ such that $y = \varphi_j(x)$ and $\gamma y = \gamma\varphi_j(x) = \varphi_i(x)$. Thus $\gamma = \lambda_{U_iU_j}$ and one obtains

$$\begin{aligned} s(\gamma y) =& s(\varphi_i(x)) = s_i(x) = g_{ij}(x)s_j(x) = \\ & f(\lambda_{U_iU_j}, \varphi_j(x))s_j(x) = f(\gamma, y)s(\varphi_j(x)) = f(\gamma, z)s(y). \end{aligned}$$

In other words, s is an f-theta function.

Vice versa, let $s : Y \to \mathbb{C}^r$ be an f-theta function. We define $s_i : U_i \to \mathbb{C}^r$ by $s_i(x) := s(\varphi_i(x))$. Then for a point $x \in U_i \cap U_j$ we have

$$\begin{aligned} s_i(x) =& s(\varphi_i(x)) = s(\lambda_{U_iU_j}\varphi_j(x)) = \\ & f(\lambda_{U_iU_j}, \varphi_j(x))s(\varphi_j(x)) = g_{ij}(x)s_j(x), \end{aligned}$$

which means that the functions s_i define a section of $E(f)$. The described correspondences are clearly inverse to each other. □

The following statement will be useful in the sequel.

Theorem 3.5. *Let* $f(\lambda, y) = \begin{pmatrix} f'(\lambda,y) & \tilde{f}(\lambda,y) \\ 0 & f''(\lambda,y) \end{pmatrix}$ *be an* $r'+r''$*-dimensional factor of automorphy, where* $f'(\lambda,y) \in \mathrm{GL}_{r'}(\mathbb{C})$*,* $f''(\lambda,y) \in \mathrm{GL}_{r''}(\mathbb{C})$*. Then*

(a)$f' : \Gamma \times Y \to \mathrm{GL}_{r'}(\mathbb{C})$ *and* $f'' : \Gamma \times Y \to \mathrm{GL}_{r''}(\mathbb{C})$ *are* r' *and* r''*-dimensional factors of automorphy respectively;*

(b) there is an extension of vector bundles

$$0 \longrightarrow E(f') \xrightarrow{\ i\ } E(f) \xrightarrow{\ \pi\ } E(f'') \longrightarrow 0\,.$$

Proof. The statement of (a) follows from straightforward verification. To prove (b) we define maps i and π as follows.

$$i : E(f') \to E(f), \quad [y,v] \mapsto [y, \begin{pmatrix} v \\ 0 \end{pmatrix}], \quad v \in \mathbb{C}^{r'}, \quad \begin{pmatrix} v \\ 0 \end{pmatrix} \in \mathbb{C}^{r'+r''}$$

$$\pi : E(f) \to E(f''), \quad [y, \begin{pmatrix} v \\ w \end{pmatrix}] \to [y,w], \quad v \in \mathbb{C}^{r'}, \quad w \in \mathbb{C}^{r''}$$

Since $[\lambda y, f'(\lambda,y)v]$ is mapped via i to $[\lambda y, \begin{pmatrix} f'(\lambda,y)v \\ 0 \end{pmatrix}] = [\lambda y, f(\lambda,y)\begin{pmatrix} v \\ 0 \end{pmatrix}]$, one concludes that i is well-defined. Analogously, since $[\lambda y, f''(\lambda,y)w] = [y,w]$ one sees that π is well-defined. Using the charts from the proof of (3.1) one easily sees that the defined maps are holomorphic.

Notice that i and π respect fibers, i is injective and π is surjective in each fiber. This proves the statement. □

Now we recall one standard construction from linear algebra. Let A be an $m \times n$ matrix. It represents some morphism $\mathbb{C}^n \to \mathbb{C}^m$ for fixed standard bases in $\mathbb{C}^n$ and $\mathbb{C}^m$.

Let $\mathcal{F} : \mathsf{Vect}^p \to \mathsf{Vect}$ be a covariant functor. Let $A_1, \ldots, A_p$ be the matrices representing morphisms $\mathbb{C}_1^n \xrightarrow{f_1} \mathbb{C}_1^m, \ldots, \mathbb{C}_p^n \xrightarrow{f_p} \mathbb{C}_p^m$ in standard bases.

If for each object $\mathcal{F}(\mathbb{C}^m)$ we fix some basis, then the matrix corresponding to the morphism $\mathcal{F}(f_1, \ldots, f_p)$ is denoted by $\mathcal{F}(A_1, \ldots, A_p)$. Clearly it satisfies

$$\mathcal{F}(A_1B_1, \ldots, A_pB_p) = \mathcal{F}(A_1, \ldots, A_p)\mathcal{F}(B_1, \ldots, B_p).$$

In this way $A \otimes B$, $S^q(A)$, $\Lambda^q(A)$ can be defined. As $\mathcal{F}$ one considers

$$\begin{aligned} \_ \otimes \_ &: \mathsf{Vect}^2 \to \mathsf{Vect}, \\ S^n &: \mathsf{Vect} \to \mathsf{Vect}, \\ \Lambda &: \mathsf{Vect} \to \mathsf{Vect} \end{aligned}$$

respectively.

Theorem 3.6. *Let $\mathcal{F} : \mathsf{Vect}^n \to \mathsf{Vect}$ be a covariant holomorphic functor. Let $f_1, \dots, f_n$ be r_i-dimensional factors of automorphy. Then $f = \mathcal{F}(f_1, \dots, f_n)$ is a factor of automorphy defining $\mathcal{F}(E(f_1), \dots, E(f_n))$.*

Proof.

$$\begin{aligned}\mathcal{F}(f_1, \dots, f_n)(\lambda\mu, y) =& \mathcal{F}(f_1(\lambda\mu, y), \dots, f_n(\lambda\mu, y)) = \\ & \mathcal{F}(f_1(\lambda, \mu y) f_1(\mu, y), \dots, f_n(\lambda, \mu y) f_n(\mu, y)) = \\ & F(f_1(\lambda, \mu y), \dots, f_n(\lambda, \mu y)) F(f_1(\mu, y), \dots, f_n(\mu, y)) = \\ & \mathcal{F}(f_1, \dots, f_n)(\lambda, \mu y) \mathcal{F}(f_1, \dots, f_n)(\mu, y).\end{aligned}$$

Since $(f_1, \dots, f_n)$ represents an isomorphism in Vect^n, $\mathcal{F}(f_1, \dots, f_n)$ also represents an isomorphism $\mathbb{C}^r \to \mathbb{C}^r$ for some $r \in \mathbb{N}$. Therefore, f is an r-dimensional factor of automorphy.

Since $f = \mathcal{F}(f_1, \dots, f_n)$, the same holds for cocycles defining the corresponding vector bundles, i. e., $g_{U_1 U_2} = \mathcal{F}(g_{1 U_1 U_2}, \dots, g_{n U_1 U_2})$, where $g_{i U_1 U_2}$ is a cocycle defining $E(f_i)$. This is exactly the condition $E(f) = \mathcal{F}(E(f_1), \dots, E(f_n))$. □

For example for $\mathcal{F} = \_ \otimes \_ : \mathsf{Vect}^2 \to \mathsf{Vect}$ we get the following obvious corollary.

Corollary 3.7. *Let $f' : \Gamma \times Y \to \mathrm{GL}_{r'}(\mathbb{C})$ and $f'' : \Gamma \times Y \to \mathrm{GL}_{r''}(\mathbb{C})$ be two factors of automorphy. Then $f = f' \otimes f'' : \Gamma \times Y \to \mathrm{GL}_{r'r''}(\mathbb{C})$ is also a factor of automorphy. Moreover, $E(f) \simeq E(f') \otimes E(f'')$.*

It is not essential that the functor in Theorem 3.6 is covariant. The following theorem is a generalization of Theorem 3.6.

Theorem 3.8. *Let $\mathcal{F} : \mathsf{Vect}^n \to \mathsf{Vect}$ be a holomorphic functor. Let $\mathcal{F}$ be covariant in k first variables and contravariant in $n - k$ last variables. Let $f_1, \dots, f_n$ be r_i-dimensional factors of automorphy. Then $f = \mathcal{F}(f_1, \dots, f_k, f_{k+1}^{-1}, \dots, f_n^{-1})$ is a factor of automorphy defining $\mathcal{F}(E(f_1), \dots, E(f_n))$.*

Proof. The proof is analogous to the proof of Theorem 3.6. □

3.3 Vector bundles on complex tori

3.3.1 One dimensional complex tori

Let X be a complex torus, i. e., $X = \mathbb{C}/\Gamma$, $\Gamma = \mathbb{Z}\tau + \mathbb{Z}$, $\operatorname{Im}\tau > 0$. Then the universal covering is $\tilde{X} = \mathbb{C}$, namely

$$\mathrm{pr} : \mathbb{C} \to \mathbb{C}/\Gamma, \quad x \mapsto [x].$$

We have an action of Γ on $\mathbb{C}$:

$$\Gamma \times \mathbb{C} \to \mathbb{C}, \quad (\gamma, y) \mapsto \gamma + y.$$

Clearly Γ acts on $\mathbb{C}$ by deck transformations and satisfies the property $(\mathbf{T})$.

Since $\mathbb{C}$ is a non-compact Riemann surface, by Theorem 1.13, there are only trivial bundles on $\mathbb{C}$. Therefore, we have a one-to-one correspondence between classes of isomorphism of vector bundles of rank r on X and equivalence classes of factors of automorphy

$$f : \Gamma \times \mathbb{C} \to \mathrm{GL}_r(\mathbb{C}).$$

As usually, V_a denotes the standard parallelogram constructed at point a, U_a is the image of V_a under the projection, $\varphi_a : U_a \to V_a$ is the local inverse of the projection.

Remark. Let f be an r-dimensional factor of automorphy. Then

$$g_{ab}(x) = f(\varphi_a(x) - \varphi_b(x), \varphi_b(x))$$

is a cocycle defining $E(f)$. This follows from the construction of the cocycle in the proof of Theorem 3.2.

Example. There are factors of automorphy corresponding to classical theta functions. For any theta-characteristic $\xi = a\tau + b$, where $a, b \in \mathbb{R}$, there is a holomorphic function $\theta_\xi : \mathbb{C} \to \mathbb{C}$ defined by

$$\theta_\xi(z) = \sum_{n\in\mathbb{Z}} \exp(\pi i(n+a)^2\tau)\exp(2\pi i(n+a)(z+b)),$$

which satisfies

$$\theta_\xi(\gamma + z) = \exp(2\pi i a\gamma - \pi i p^2\tau - 2\pi i p(z+\xi))\theta_\xi(z) = e_\xi(\gamma, z)\theta_\xi(z),$$

where $\gamma = p\tau + q$ and $e_\xi(\gamma, z) = \exp(2\pi i a\gamma - \pi i p^2\tau - 2\pi i p(z+\xi))$. Since

$$e_\xi(\gamma_1 + \gamma_2, z) = e_\xi(\gamma_1, \gamma_2 + z)e_\xi(\gamma_2, z),$$

we conclude that $e_\xi(\gamma, z)$ is a factor of automorphy.

By Theorem 3.4 $\theta_\xi(z)$ defines a section of $E(e_\xi(\gamma, z))$.

More on classical theta functions see in [Mum].

Theorem 3.9. $\deg E(e_\xi) = 1$.

Proof. We know that sections of $E(e_\xi)$ correspond to e_ξ - theta functions. The classical e_ξ-theta function $\theta_\xi(z)$ defines a section s_ξ of $E(e_\xi)$. Since θ_ξ has only simple zeros and the set of zeros of $\theta_\xi(z)$ is $\frac{1}{2} + \frac{\tau}{2} + \xi + \Gamma$, we conclude that s_ξ has exactly one zero at point $p = [\frac{1}{2} + \frac{\tau}{2} + \xi] \in X$. By Theorem 1.16 we get $E(e_\xi) \simeq [p]$ and thus $\deg E(e_\xi) = 1$. □

Theorem 3.10. *Let ξ and η be two theta-characteristics. Then*

$$E(e_\xi) \simeq t^*_{[\eta-\xi]} E(e_\eta),$$

where $t_{[\eta-\xi]} : X \to X, \quad x \mapsto x + [\eta - \xi]$ is the translation by $[\eta - \xi]$.

Proof. As in the proof of Theorem 3.9 $E(e_\xi) \simeq [p]$ and $E(e_\eta) = [q]$ for $p = [\frac{1}{2} + \frac{\tau}{2} + \xi]$ and $q = [\frac{1}{2} + \frac{\tau}{2} + \eta]$. Since $t_{[\eta-\xi]} p = q$, we get

$$E(e_\xi) \simeq [p] \simeq t^*_{[\eta-\xi]}[q] \simeq t^*_{[\eta-\xi]} E(e_\eta).$$

□

Now we are going to investigate the extensions of the type

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{O}_X \to 0$$

or equivalently $0 \to X \times \mathbb{C} \to E \to X \times \mathbb{C} \to 0$. In this case the transition functions are given by matrices of the type

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

and E is isomorphic to $E(f)$ for some factor of automorphy f of the form $f(\lambda, \tilde{x}) = \begin{pmatrix} 1 & \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix}$. Note that the condition for f to be a factor of automorphy in this case is equivalent to the condition

$$\mu(\lambda + \lambda', \tilde{x}) = \mu(\lambda, \lambda' + \tilde{x}) + \mu(\lambda', \tilde{x}),$$

where we use the additive notation for the group operation since Γ is commutative.

Theorem 3.11. *f defines trivial bundle if and only if $\mu(\lambda, \tilde{x}) = \xi(\lambda\tilde{x}) - \xi(\tilde{x})$ for some holomorphic function $\xi : \mathbb{C} \to \mathbb{C}$.*

Proof. We know that E is trivial if and only if $h(\lambda\tilde{x}) = f(\lambda,\tilde{x})h(\tilde{x})$ for some holomorphic function $h : \tilde{X} \to \mathrm{GL}_2(\mathbb{C})$. Let $h = \begin{pmatrix} a(\tilde{x}) & b(\tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix}$, then the last condition is

$$\begin{pmatrix} a(\lambda\tilde{x}) & b(\lambda\tilde{x}) \\ c(\lambda\tilde{x}) & d(\lambda\tilde{x}) \end{pmatrix} = \begin{pmatrix} 1 & \mu(\lambda,\tilde{x}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a(\tilde{x}) & b(\tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix} = \begin{pmatrix} a(\tilde{x}) + c(\tilde{x})\mu(\lambda,\tilde{x}) & b(\tilde{x}) + d(\tilde{x})\mu(\lambda,\tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix}.$$

In particular it means $c(\lambda\tilde{x}) = c(\tilde{x})$ and $d(\lambda\tilde{x}) = d(\tilde{x})$, i. e., c and d are doubly periodic functions on $\tilde{X} = \mathbb{C}$, so they should be constant, i. e., $c(\lambda,\tilde{x}) = c \in \mathbb{C}$, $d(\lambda,\tilde{x}) = d \in \mathbb{C}$.

Now we have

$$\begin{aligned} a(\tilde{x}) + c\mu(\lambda,\tilde{x}) &= a(\lambda\tilde{x}) \\ b(\tilde{x}) + d\mu(\lambda,\tilde{x}) &= b(\lambda\tilde{x}) \end{aligned}$$

which implies

$$\begin{aligned} c\mu(\lambda,\tilde{x}) &= a(\lambda\tilde{x}) - a(\tilde{x}) \\ d\mu(\lambda,\tilde{x}) &= b(\lambda\tilde{x}) - b(\tilde{x}). \end{aligned}$$

Since $\det h(\tilde{x}) \neq 0$ for all $\tilde{x} \in \tilde{X} = \mathbb{C}$ one of the numbers c and d is not equal to zero. Therefore, one concludes that $\mu(\lambda,\tilde{x}) = \xi(\lambda\tilde{x}) - \xi(\tilde{x})$ for some holomorphic function $\xi : \tilde{X} = \mathbb{C} \to \mathbb{C}$.

Now suppose $\mu(\lambda,\tilde{x}) = \xi(\lambda\tilde{x}) - \xi(\tilde{x})$ for some holomorphic function $\xi : \mathbb{C} \to \mathbb{C}$. Clearly for $h(\tilde{x}) = \begin{pmatrix} 1 & \xi(\tilde{x}) \\ 0 & 1 \end{pmatrix}$ one has that $\det h(\tilde{x}) = 1 \neq 0$ and

$$f(\lambda,\tilde{x})h(\tilde{x}) = \begin{pmatrix} 1 & \mu(\lambda,\tilde{x}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \xi(\tilde{x}) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \xi(\tilde{x}) + \mu(\lambda,\tilde{x}) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \xi(\lambda\tilde{x}) \\ 0 & 1 \end{pmatrix} = h(\lambda\tilde{x}).$$

We have shown, that f defines the trivial bundle. This proves the statement of the theorem. □

Theorem 3.12. *Two factors of automorphy* $f(\lambda,\tilde{x}) = \begin{pmatrix} 1 & \mu(\lambda,\tilde{x}) \\ 0 & 1 \end{pmatrix}$ *and* $f'(\lambda,\tilde{x}) = \begin{pmatrix} 1 & \nu(\lambda,\tilde{x}) \\ 0 & 1 \end{pmatrix}$ *defining non-trivial bundles are equivalent if and only if*

$$\mu(\lambda,\tilde{x}) - k\nu(\lambda,\tilde{x}) = \xi(\lambda\tilde{x}) - \xi(\tilde{x}), \quad k \in \mathbb{C}, \quad k \neq 0$$

for some holomorphic function $\xi : \mathbb{C} = \tilde{X} \to \mathbb{C}$.

Proof. Suppose the factors of automorphy $f(\lambda, \tilde{x}) = \begin{pmatrix} 1 & \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix}$ and $f'(\lambda, \tilde{x}) = \begin{pmatrix} 1 & \nu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix}$ are equivalent. This means that $f(\lambda, \tilde{x})h(\tilde{x}) = h(\lambda\tilde{x})f(\lambda, \tilde{x})$ for some holomorphic function $h : \mathbb{C} = \tilde{X} \to \mathrm{GL}_2(\mathbb{C})$. Let

$$h(\tilde{x}) = \begin{pmatrix} a(\tilde{x}) & b(\tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix}.$$

The condition for equivalence of f and f' can be rewritten in the following way:

$$\begin{pmatrix} 1 & \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a(\tilde{x}) & b(\tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix} = \begin{pmatrix} a(\lambda\tilde{x}) & b(\lambda\tilde{x}) \\ c(\lambda\tilde{x}) & d(\lambda\tilde{x}) \end{pmatrix} \begin{pmatrix} 1 & \nu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} a(\tilde{x}) + c(\tilde{x})\mu(\lambda, \tilde{x}) & b(\tilde{x}) + d(\tilde{x})\mu(\lambda, \tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix} = \begin{pmatrix} a(\lambda\tilde{x}) & a(\lambda\tilde{x})\nu(\lambda, \tilde{x}) + b(\lambda\tilde{x}) \\ c(\lambda\tilde{x}) & c(\lambda\tilde{x})\nu(\lambda, \tilde{x}) + d(\lambda\tilde{x}) \end{pmatrix}.$$

This leads to the system of equations

$$\begin{cases} a(\tilde{x}) + c(\tilde{x})\mu(\lambda, \tilde{x}) = a(\lambda\tilde{x}) \\ b(\tilde{x}) + d(\tilde{x})\mu(\lambda, \tilde{x}) = a(\lambda\tilde{x})\nu(\lambda, \tilde{x}) + b(\lambda\tilde{x}) \\ c(\tilde{x}) = c(\lambda\tilde{x}) \\ d(\tilde{x}) = c(\lambda\tilde{x})\nu(\lambda, \tilde{x}) + d(\lambda\tilde{x}). \end{cases}$$

The third equation means that c is a double periodic function. Therefore, c should be a constant function.

If $c \neq 0$ from the first and the last equations using Theorem 3.11 one concludes that f and f' define the trivial bundle.

In the case $c = 0$ one has

$$\begin{cases} a(\tilde{x}) = a(\lambda\tilde{x}) \\ b(\tilde{x}) + d(\tilde{x})\mu(\lambda, \tilde{x}) = a(\lambda\tilde{x})\nu(\lambda, \tilde{x}) + b(\lambda\tilde{x}) \\ d(\tilde{x}) = d(\lambda\tilde{x}), \end{cases}$$

i. e., as above, a and d are constant and both not equal to zero since $\det(h) \neq 0$. Finally one concludes that

$$d\mu(\lambda, \tilde{x}) - a\nu(\lambda, \tilde{x}) = b(\lambda\tilde{x}) - b(\tilde{x}), \quad a, d \in \mathbb{C}, \quad ad \neq 0 \tag{6}$$

Vice versa, if μ and ν satisfy (6) for $h(\tilde{x}) = \begin{pmatrix} a & b(\tilde{x}) \\ 0 & d \end{pmatrix}$ we have

$$f(\lambda,\tilde{x})h(\tilde{x}) = \begin{pmatrix} 1 & \mu(\lambda,\tilde{x}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b(\tilde{x}) \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & b(\tilde{x}) + d\mu(\lambda,\tilde{x}) \\ 0 & d \end{pmatrix} =$$
$$\begin{pmatrix} a & b(\lambda\tilde{x}) + a\nu(\lambda,\tilde{x}) \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & b(\lambda\tilde{x}) \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & \nu(\lambda,\tilde{x}) \\ 0 & 1 \end{pmatrix} = h(\lambda\tilde{x})f(\lambda,\tilde{x}).$$

This means that f and f' are equivalent. $\square$

3.3.2 Higher dimensional complex tori

One can also consider higher dimensional complex tori. Let $\Gamma \subset \mathbb{C}^g$ be a lattice,

$$\Gamma = \Gamma_1 \times \cdots \times \Gamma_g, \quad \Gamma_i = \mathbb{Z} + \mathbb{Z}\tau_i, \quad \operatorname{Im}\tau > 0.$$

Then as for one dimensional complex tori we obtain that $X = \mathbb{C}^g/\Gamma$ is a complex manifold. Clearly the map

$$\mathbb{C}^g \to \mathbb{C}^g/\Gamma = X, \quad x \mapsto [x]$$

is the universal covering of X. Since all vector bundles on C^g are trivial, we obtain a one-to-one correspondence between equivalence classes of r-dimensional factors of automorphy

$$f : \Gamma \times \mathbb{C}^g \to \mathrm{GL}_r(\mathbb{C})$$

and vector bundles of rank r on X.

Let $\Gamma = \mathbb{Z}^g + \Omega\mathbb{Z}^g$, where Ω is a symmetric complex $g \times g$ matrix with positive definite real part. Note that Ω is a generalization of τ from one dimensional case.

For any theta-characteristic $\xi = \Omega a + b$, where $a \in \mathbb{R}^g$, $b \in \mathbb{R}^g$ there is a holomorphic function $\theta_\xi : \mathbb{C}^g \to \mathbb{C}$ defined by

$$\theta_\xi(z) = \theta^a_b(z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp(\ \pi i (n+a)^t \Omega (n+a)\tau\ ) \exp(\ 2\pi i (n+a)^t \Omega (z+b)\ ),$$

which satisfies

$$\theta_\xi(\gamma + z) = \exp(2\pi i a^t \gamma - \pi i p^t \Omega p - 2\pi i p^t (z+\xi))\theta_\xi(z) = e_\xi(\gamma, z)\theta_\xi(z),$$

where $\gamma = \Omega p + q$ and $e_\xi(\gamma, z) = \exp(2\pi i a^t \gamma - \pi i p^t \Omega p - 2\pi i p^t (z+\xi))$. Since

$$e_\xi(\gamma_1 + \gamma_2, z) = e_\xi(\gamma_1, \gamma_2 + z) e_\xi(\gamma_2, z),$$

we conclude that $e_\xi(\gamma, z)$ is a factor of automorphy.

As above $\theta_\xi(z)$ defines a section of $E(e_\xi(\gamma, z))$.

For more detailed information on higher dimensional theta functions see [Mum].

3.4 Factors of automorphy depending only on the τ-direction of the lattice Γ

Here X is a complex torus, $X = \mathbb{C}/\Gamma$, $\Gamma = \mathbb{Z}\tau + \mathbb{Z}$, $\operatorname{Im}\tau > 0$. Denote $q = e^{2\pi i\tau}$. Consider the canonical projection

$$\mathrm{pr} : \mathbb{C}^* \to \mathbb{C}^*/ < q >, \quad u \to [u] = u < q > .$$

Clearly one can equip $\mathbb{C}^*/ < q >$ with the quotient topology. Therefore, there is a natural complex structure on $\mathbb{C}^*/ < q >$.

Consider the homomorphism

$$\mathbb{C} \overset{\exp}{\to} \mathbb{C}^* \overset{\mathrm{pr}}{\to} \mathbb{C}^*/ < q >, \quad z \mapsto e^{2\pi iz} \mapsto [e^{2\pi iz}].$$

It is clearly surjective. An element $z \in \mathbb{C}$ is in the kernel of this homomorphism if and only if $e^{2\pi iz} = q^k = e^{2\pi ik\tau}$ for some integer k. But this holds if and only if $z - k\tau \in \mathbb{Z}$ or, in other words, if $z \in \Gamma$. Therefore, the kernel of the map is exactly Γ, and we obtain an isomorphism of groups

$$iso : \mathbb{C}/\Gamma \to C^*/ < q > \simeq \mathbb{C}^*/\mathbb{Z}, \quad [z] \to [e^{2\pi iz}].$$

Since the diagram

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\mathrm{pr}"] \arrow[d, "\exp"'] & \mathbb{C}/\Gamma \arrow[d, leftrightarrow, "iso"] & \\ \mathbb{C}^* \arrow[r, "\mathrm{pr}"] & \mathbb{C}^*/\mathbb{Z} \arrow[r, equal] & \mathbb{C}^*/ < q > \end{tikzcd}$$

is commutative, we conclude that the complex structure on $\mathbb{C}^*/ < q >$ inherited from $\mathbb{C}/\Gamma$ by the isomorphism iso coincides with the natural complex structure on $\mathbb{C}^*/ < q >$. Therefore, iso is an isomorphism of complex manifolds. Thus complex tori can be represented as $\mathbb{C}^*/ < q >$, where $q = e^{2\pi i\tau}$, $\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$.

So for any complex torus $X = \mathbb{C}^*/ < q >$ we have a natural surjective holomorphic map

$$\mathbb{C}^* \to \mathbb{C}^*/ < q >= X, \quad u \to [u].$$

This map is moreover a covering of X. Consider the group $\mathbb{Z}$. It acts holomorphically on $X = \mathbb{C}^*$:

$$\mathbb{Z} \times \mathbb{C}^* \to \mathbb{C}^*, \quad (n, u) \mapsto q^n u.$$

Moreover, since $\mathrm{pr}(q^n u) = \mathrm{pr}(u)$, $\mathbb{Z}$ is naturally identified with a subgroup in the group of deck transformations $\mathrm{Deck}(X/\mathbb{C}^*)$. It is easy to see that $\mathbb{Z}$ satisfies the property $(\mathbf{T})$. We obtain that there is a one-to-one correspondence between classes of isomorphism of vector bundles over X and classes of equivalence of factors of automorphy

$$f : \mathbb{Z} \times \mathbb{C} \to \mathrm{GL}_r(\mathbb{C}).$$

Consider the following action of Γ on $\mathbb{C}^*$:

$$\Gamma \times \mathbb{C}^* \to \mathbb{C}^*; \quad (\lambda, u) \mapsto \lambda u =: e^{2\pi i \lambda} u$$

Let $A : \Gamma \times \mathbb{C}^* \to \mathrm{GL}_r(\mathbb{C})$ be a holomorphic function satisfying

$$A(\lambda + \lambda', u) = A(\lambda, \lambda' u) A(\lambda', u) \tag{$*$}$$

for all $\lambda, \lambda' \in \Gamma$. We call such functions $\mathbb{C}^*$-factors of automorphy. Consider the map

$$\mathrm{id}_\Gamma \times \exp : \Gamma \times \mathbb{C} \to \Gamma \times \mathbb{C}^*, \quad (\lambda, x) \to (\lambda, e^{2\pi i x})$$

Then the function

$$f_A = A \circ (\mathrm{id}_\Gamma \times \exp) : \Gamma \times \mathbb{C} \to \mathrm{GL}_r(\mathbb{C})$$

is an r-dimensional factor of automorphy, because

$$\begin{aligned} f_A(\lambda + \lambda', x) = A(\lambda + \lambda', e^{2\pi i x}) = A(\lambda, e^{2\pi i \lambda'} e^{2\pi i x}) A(\lambda', e^{2\pi i x}) = \\ A(\lambda, e^{2\pi i (\lambda' + x)}) A(\lambda', e^{2\pi i x}) = f_A(\lambda, \lambda' + x) f_A(\lambda', x). \end{aligned}$$

So, factors of automorphy on $\mathbb{C}^*$ define factors of automorphy on $\mathbb{C}$.

We restrict ourselves to factors of automorphy $f : \Gamma \times \mathbb{C} \to \mathrm{GL}_r(\mathbb{C})$ with the property

$$f(m\tau + n, x) = f(m\tau, x), \quad m, n \in \mathbb{Z}. \tag{7}$$

It follows from this property that $f(n, x) = f(0, x) = \mathrm{id}_{\mathbb{C}^r}$. Therefore,

$$f(\lambda + k, x) = f(\lambda, k + x) f(k, x) = f(\lambda, k + x) \text{ for all } \lambda \in \Gamma,\ k \in \mathbb{Z}$$

and it is possible to define the function

$$A_f : \Gamma \times \mathbb{C}^* \to \mathrm{GL}_r(\mathbb{C}), \quad (\lambda, e^{2\pi i x}) \mapsto f(\lambda, x),$$

which is well-defined because from $e^{2\pi i x_1} = e^{2\pi i x_2}$ follows $x_1 = x_2 + k$ for some $k \in \mathbb{Z}$ and $f(\lambda, x_1) = f(\lambda, x_2 + k) = f(\lambda, x_2)$.

Consider A with the property $A(m\tau + n, u) = A(m\tau, u) =: A(m, u)$. Then clearly $f_A(m\tau + n, u) = f_A(m\tau, u)$. So for any $\mathbb{C}^*$-factor of automorphy $A : \Gamma \times \mathbb{C}^* \to \mathrm{GL}_r(\mathbb{C})$ with the property $A(m\tau + n, u) = A(m\tau, u)$ one obtains the factor of automorphy f_A satisfying (7). We proved the following

Theorem 3.13. *Factors of automorphy $f : \Gamma \times \mathbb{C} \to \mathrm{GL}_r(\mathbb{C})$ with the property (7) are in a one-to-one correspondence with $\mathbb{C}^*$-factors of automorphy with property $A(m\tau + n, u) = A(m\tau, u)$.*

Now we want to translate the conditions for factors of automorphy with the property (7) to be equivalent in the language of $\mathbb{C}^*$-factors of automorphy with the same property.

Theorem 3.14. *Let f, f' be r-factors of automorphy with the property (7). Then $f \sim f'$ if and only if there exists a holomorphic function $B : \mathbb{C}^* \to \mathrm{GL}_r(\mathbb{C})$ such that*

$$A_f(m, u)B(u) = B(q^m u)A_{f'}(m, u))$$

for $q := e^{2\pi i\tau}$, where $A(m, u) := A(m\tau, u)$. In this case we also say A_f is equivalent to $A_{f'}$ and write $A_f \sim A_{f'}$.

Proof. Let $f \sim f'$. By definition it means that there exists a holomorphic function $h : \mathbb{C} \to \mathrm{GL}_r(\mathbb{C})$ such that $f(\lambda, x)h(x) = h(\lambda x)f'(\lambda, x)$. Therefore, from $f(n, x)h(x) = h(n + x)f'(n, x)$ and $f(n, x) = f'(n, x) = \mathrm{id}_{\mathbb{C}^r}$ it follows $h(x) = h(n + x)$ for all $n \in \mathbb{Z}$. Therefore, the function

$$\begin{aligned} B : \mathbb{C}^* &\to \mathrm{GL}_r(\mathbb{C}) \\ e^{2\pi ix} &\mapsto h(x) \end{aligned}$$

is well-defined. We have

$$\begin{aligned} A_f(m, e^{2\pi ix})B(e^{2\pi ix}) =& f(m\tau, x)h(x) = h(m\tau + x)f'(m\tau, x) = \\ & B(e^{2\pi i(m\tau+x)})f'(m, e^{2\pi ix}) = B(q^m e^{2\pi ix})A_{f'}(m, e^{2\pi ix}). \end{aligned}$$

Vice versa, let B be such that $A_f(m, u)B(u) = B(q^m A_{f'}(m, u))$. Define $h = B \circ \exp$. We obtain

$$\begin{aligned} f(m\tau + n, x)h(x) = & A_f(m\tau + n, e^{2\pi ix})B(e^{2\pi ix}) = \\ & B(q^m e^{2\pi ix})A_{f'}(m\tau + n, e^{2\pi ix}) = B(e^{2\pi i(m\tau+x)})A_{f'}(m\tau + n, e^{2\pi ix}) = \\ & B(e^{2\pi i(m\tau+n+x)})A_{f'}(m\tau + n, e^{2\pi ix}) = h(m\tau + n + x)f'(m\tau + n, x). \end{aligned}$$

□

Remark. The last two theorems allow us to embed the set $Z^1(\mathbb{Z}, r)$ of factors of automorphy $\mathbb{Z} \times X \to \mathrm{GL}_r(\mathbb{C})$ to the set $Z^1(\Gamma, r)$. The embedding is

$$\Psi : Z^1(\mathbb{Z}, r) \to Z^1(\Gamma, r), \quad f \mapsto g, \quad g(n\tau + m, x) := f(n, x).$$

Two factors of automorphy from $Z^1(\mathbb{Z}, r)$ are equivalent if and only if their images under Ψ are equivalent in $Z^1(\Gamma, r)$. That is why it is enough to consider only factors of automorphy

$$\Gamma \times \mathbb{C} \to \mathrm{GL}_r(\mathbb{C})$$

satisfying (7).

Corollary 3.15. *A factor of automorphy f with property (7) is trivial if and only if $A_f(m,u) = B(q^m u)B(u)^{-1}$ for some holomorphic function $B : \mathbb{C}^* \to \mathrm{GL}_r(\mathbb{C})$.*

Theorem 3.16. *Let A be a $\mathbb{C}^*$-factor of automorphy. $A(m,u)$ is uniquely determined by $A(u) := A(1,u)$.*

$$A(m,u) = A(q^{m-1}u) \dots A(qu)A(u), \quad m > 0 \tag{8}$$
$$A(-m,u) = A(q^{-m}u)^{-1} \dots A(q^{-1}u)^{-1}, \quad m > 0. \tag{9}$$

$A(m,u)$ is equivalent to $A'(m,u)$ if and only if

$$A(u)B(u) = B(qu)A'(u) \tag{10}$$

for some holomorphic function $B : \mathbb{C}^* \to \mathrm{GL}_r(\mathbb{C})$. In particular $A(m,u)$ is trivial iff $A(u) = B(qu)B(u)^{-1}$.

Proof. Since $A(1,u) = A(u)$ the first formula holds for $m = 1$. Therefore,

$$A(m+1,u) = A(1,q^m u)A(m,u) = A(q^m)A(m,u)$$

and we prove the first formula by induction.

Now $\mathrm{id} = A(0,u) = A(m-m,u) = A(m,q^{-m}u)A(-m,u)$ and hence

$$\begin{aligned} A(-m,u) =& A(m,q^{-m}u)^{-1} = (A(q^{m-1}q^{-m}u) \dots A(qq^{-m}u)A(q^{-m}u))^{-1} = \\ & A(-m,u) = A(q^{-m}u)^{-1} \dots A(q^{-1}u)^{-1} \end{aligned}$$

which proves the second formula.

If $A(m,u) \sim A'(m,u)$ then clearly (10) holds.

Vice versa, suppose $A(u)B(u) = B(qu)A'(u)$. Then

$$\begin{aligned} & A(m,u)B(u) = A(q^{m-1}u) \dots A(qu)A(u)B(u) = \\ & A(q^{m-1}u) \dots A(qu)B(qu)A'(u) = \dots = B(q^m u)A'(q^{m-1}u) \dots A'(qu)A'(u) = \\ & B(q^m u)A'(m,u) \end{aligned}$$

for $m > 0$.

Since $A(-m,u) = A(m,q^{-m}u)^{-1}$ we have

$$\begin{aligned}A(-m,u)B(u) =& A(m,q^{-m}u)^{-1}B(u) = (B(u)^{-1}A(m,q^{-m}u))^{-1} = \\ & (B(u)^{-1}A(m,q^{-m}u)B(q^{-m}u)B(q^{-m}u)^{-1})^{-1} = \\ & (B(u)^{-1}B(q^{m}q^{-m}u)A'(m,q^{-m}u)B(q^{-m}u)^{-1})^{-1} = \\ & (B(u)^{-1}B(u)A'(m,q^{-m}u)B(q^{-m}u)^{-1})^{-1} = \\ & B(q^{-m}u)A'(m,q^{-m}u)^{-1} = B(q^{-m}u)A'(-m,u)\end{aligned}$$

which proves the statement. □

Theorem 3.17. *Let $A : \mathbb{C}^* \to \mathrm{GL}_n(\mathbb{C})$, $B : \mathbb{C}^* \to \mathrm{GL}_m(\mathbb{C})$ be two holomorphic maps. Then $E(A) \otimes E(B) \simeq E(A \otimes B)$.*

Proof. By theorem 3.7 we have

$$E(A) \otimes E(B) \simeq E(A(n,u)) \otimes E(B(n,u)) \simeq E(A(n,u) \otimes B(n,u)).$$

Since $A(1,u) \otimes B(1,u) = A(u) \otimes B(u)$, we obtain $E(A) \otimes E(B) \simeq E(A \otimes B)$. □

3.5 Classification of vector bundles over a complex torus

Here we work with factors of automorphy depending only on τ, i. e., with holomorphic functions $\mathbb{C}^* \to GL_r(\mathbb{C})$.

3.5.1 Vector bundles of degree zero

We return to extensions of the type $0 \to I_1 \to E \to I_1 \to 0$, where I_1 denotes the trivial vector bundle of rank 1.

Theorem 3.11 can be rewritten as follows.

Theorem 3.18. *A function* $A(u) = \begin{pmatrix} 1 & a(u) \\ 0 & 1 \end{pmatrix}$ *defines the trivial bundle if and only if* $a(u) = b(qu) - b(u)$ *for some holomorphic function* $b : \mathbb{C}^* \to \mathbb{C}$.

Corollary 3.19. $A(u) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *defines a non-trivial vector bundle.*

Proof. Suppose A defines the trivial bundle. Then $1 = b(qu) - b(u)$ for some holomorphic function $b : \mathbb{C}^* \to \mathbb{C}$. Considering the Laurent series expansion $\sum_{-\infty}^{+\infty} b_k u^k$ of b we obtain $1 = b_0 - b_0 = 0$ which shows that our assumption was false. $\square$

Let $a : \mathbb{C}^* \to \mathbb{C}$ be a holomorphic function such that $A_2(U) = \begin{pmatrix} 1 & a(u) \\ 0 & 1 \end{pmatrix}$ defines non-trivial bundle, i. e., by Theorem 3.18, there exists no holomorphic function $b : \mathbb{C}^* \to \mathbb{C}$ such that

$$a(u) = b(qu) - b(u).$$

Let F_2 be the bundle defined by A_2. Then by Theorem 3.5 there exists an exact sequence

$$0 \to I_1 \to F_2 \to I_1 \to 0.$$

For $n \geqslant 3$ we define $A_n : \mathbb{C}^* \to \mathrm{GL}_n(\mathbb{C})$,

$$A_n = \begin{pmatrix} 1 & a & & \\ & \ddots & \ddots & \\ & & 1 & a \\ & & & 1 \end{pmatrix},$$

where empty entries stay for zeros.

Let F_n be the bundle defined by A_n. By (3.5) one sees that A_n defines the extension

$$0 \to I_1 \to F_n \to F_{n-1} \to 0.$$

Theorem 3.20. *F_n is not the trivial bundle. The extension*

$$0 \to I_1 \to F_n \to F_{n-1} \to 0.$$

is non-trivial for all $n \geqslant 2$.

Proof. Suppose F_n is trivial. Then $A_n(u)B(u) = B(qu)$ for some $B = (b_{ij})_{ij^n}$. In particular it means $b_{ni}(u) = b_{ni}(qu)$ for $i = \overline{1,n}$. Let $b_{ni} = \sum\limits_{-\infty}^{+\infty} b_k^{(ni)} u^k$ be the expansion of $b_n i$ in Laurent series. Then $b_{ni}(u) = b_{ni}(qu)$ implies $b_k^{(ni)} = q^k b_k^{(ni)}$ for all k.

Note that $|q| < 1$ because $\tau = \xi + i\eta$, $\eta > 0$ and

$$|q| = |e^{2\pi i \tau}| = |e^{2\pi i(\xi + i\eta)}| = |e^{2\pi i \xi} e^{-2\pi\eta}| = e^{-2\pi\eta} < 1.$$

Therefore, $b_k^{(ni)} = 0$ for $k \neq 0$ and we conclude that b_{ni} should be constant functions.

We also have

$$b_{n-1i}(u) + b_{ni} a(n) = b_{n-1i}(qu).$$

Since at least one of b_{ni} is not equal to zero because of invertibility of B, we obtain

$$a(u) = \frac{1}{b_{ni}} (b_{n-1i}(qu) - b_{n-1i}(u))$$

for some i, which contradicts the choice of a. Therefore, F_n is not trivial.

Assume now, that for some $n > 2$ the extension

$$0 \to I_1 \to F_n \to F_{n-1} \to 0$$

is trivial(for $n = 2$ it is not trivial since F_2 is not a trivial vector bundle). This means $A_n \sim \begin{pmatrix} 1 & 0 \\ 0 & A_{n-1} \end{pmatrix}$, i. e., there exists a holomorphic function $B : \mathbb{C}^* \to \mathrm{GL}_n(\mathbb{C})$, $B = (b_{ij})_{i,j}^n$ such that

$$A_n(u)B(u) = B(qu) \begin{pmatrix} 1 & 0 \\ 0 & A_{n-1} \end{pmatrix}.$$

Considering the elements of the first and second columns we obtain for the first column

$$\begin{aligned} &b_{n1}(u) = b_{n1}(qu), \\ &b_{i1}(u) + b_{i+11}(u)a(u) = b_{i1}(qu), \quad i < n \end{aligned}$$

and for the second column

$$\begin{aligned} &b_{n2}(u) = b_{n2}(qu), \\ &b_{i2}(u) + b_{i+12}(u)a(u) = b_{i2}(qu), \quad i < n. \end{aligned}$$

For the first column as above considering Laurent series we have that b_{n1} should be a constant function. If $b_{n1} \neq 0$ it follows

$$a(u) = \frac{1}{b_{n1}}(b_{n-11}(qu) - b_{n-11}(u)),$$

which contradicts the choice of a. Therefore, $b_{n1} = 0$ and $b_{n-11}(qu) = b_{n-11}(u)$, in other words b_{n-11} is a constant function. Proceeding by induction one obtains that b_{11} is a constant function and $b_{i1} = 0$ for $i > 1$.

For the second column absolutely analogously we obtain a similar result: b_{12} is constant, $b_{i2} = 0$ for $i > 1$. This contradicts the invertibility of $B(u)$ and proves the statement. □

Corollary 3.21. *The vector bundle F_n is the only vector bundle of rank n and degree 0 that has non-trivial sections.*

Proof. This follows from Theorem 2.10. □

So we have that the vector bundles $F_n = E(A_n)$ are exactly F_n's defined by Atiyah in [At].

Remark. Note that constant matrices A and B having the same Jordan normal form are equivalent. This is clear because $A = SBS^{-1}$ for some constant invertible matrix S, which means that A and B are equivalent.

Consider an upper triangular matrix $B = (b_{ij})_1^n$ of the following type:

$$b_{ii} = 1, \quad b_{ii+1} \neq 0. \tag{11}$$

It is easy to see that this matrix is equivalent to the upper triangular matrix A,

$$a_{ii} = a_{ii+1} = 1, \quad a_{ij} = 0, \quad j \neq i+1, \quad j \neq i. \tag{12}$$

In fact, these matrices have the same characteristic polynomial $(t-1)^n$ and the dimension of the eigenspace corresponding to the eigenvalue 1 is equal to 1 for both matrices. Therefore, A and B have the same Jordan form. By Remark above we obtain that A and B are equivalent. We proved the following:

Lemma 3.22. *A matrix satisfying (11) is equivalent to the matrix defined by (12). Moreover, two matrices of the type (11) are equivalent, i. e., they define two isomorphic vector bundles.*

Theorem 3.23. *$F_n \simeq S^{n-1}(F_2)$.*

Proof. We know that F_2 is defined by the constant matrix $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We know by Theorem 3.6 that $S^n(F_2)$ is defined by $S^n(A_2)$. We calculate $S^n(f_2)$ for $n \in \mathbb{N}_0$. Since f_2 is a constant matrix, $S^n(f_2)$ is also a constant matrix defining a map $S^n(\mathbb{C}^2) \to S^n(\mathbb{C}^2)$. Let e_1, e_2 be the standard basis of $\mathbb{C}^2$, then $S^n(\mathbb{C})$ has a basis

$$\{e_1^k e_2^{n-k} |\ k = n, n-1, \ldots, 0\}.$$

Since $A_2(e_1) = e_1$ and $A_2(e_2) = e_1 + e_2$, we conclude that $e_1^k e_2^{n-k}$ is mapped to

$$A_2(e_1)^k A_2(e_1)^{n-k} = e_1^k (e_1 + e_2)^{n-k} = e_1^k \sum_{i=0}^{n-k} \binom{n-k}{i} e_1^{n-k-i} e_2^i = \sum_{i=0}^{n-k} \binom{n-k}{i} e_1^{n-i} e_2^i.$$

Therefore,

$$S^n(A_2) = \begin{pmatrix} 1 & 1 & 1 & \ldots & \binom{n}{0} \\ & 1 & 2 & \ldots & \binom{n}{1} \\ & & 1 & \ldots & \binom{n}{2} \\ & & & \ddots & \vdots \\ & & & & \binom{n}{n} \end{pmatrix},$$

where empty entries stay for zero. In other words, the columns of $S^n(A_2)$ are columns of binomial coefficients. By Lemma 3.22 we conclude that $S^n(A_2)$ is equivalent to A_{n+1}. This proves the statement of the theorem. □

Let E be a 2-dimensional vector bundle over a topological space X. Then there exists an isomorphism

$$S^p(E) \otimes S^q(E) \simeq S^{p+q}(E) \oplus (\det E \otimes S^{p-1}(E) \otimes S^{q-1}(E)).$$

This is the Clebsch-Gordan formula. If $\det E$ is the trivial line bundle, then we have $S^p(E) \otimes S^q(E) \simeq S^{p+q}(E) \oplus S^{p-1}(E) \otimes S^{q-1}(E)$, and by iterating one gets

$$S^p(E) \otimes S^q(E) \simeq S^{p+q}(E) \oplus S^{p+q-2}(E) \oplus \cdots \oplus S^{p-q}(E), \quad p \geqslant q. \tag{13}$$

Theorem 3.24. *$F_p \otimes F_q \simeq F_{p+q-1} \oplus F_{p+q-3} \oplus \cdots \oplus F_{p-q+1}$ for $p \geqslant q$.*

Proof. Using Theorem 3.23 and (13) we obtain

$$\begin{aligned} F_p \otimes F_q \simeq & S^{p-1}(F_2) \otimes S^{q-1}(F_2) \simeq S^{p+q-2}(F_2) \oplus S^{p+q-4}(F_2) \oplus \cdots \oplus S^{p-q}(F_2) \simeq \\ & F_{p+q-1} \oplus F_{p+q-3} \oplus \cdots \oplus F_{p-q+1}. \end{aligned}$$

□

Remark. The possibility of proving the last theorem using Theorem 3.23 is exactly what Atiyah states in remark (1) after Theorem 9.[10]

We have already given (Corollary 3.21) a description of vector bundles of degree zero with non-trivial sections. We give now a description of all vector bundles of degree zero.

Consider the function $\varphi_0(z) = \exp(-\pi i\tau - 2\pi iz) = q^{-1/2}u^{-1} = \varphi(u)$, where $u = e^{2\pi iz}$. It defines the factor of automorphy

$$e_0(p\tau + q, z) = \exp(-\pi ip^2\tau - 2\pi izp) = q^{-\frac{p^2}{2}}u^{-p}$$

corresponding to the theta-characteristic $\xi = 0$.

Theorem 3.25. $\deg E(\varphi_0) = 1$, *where as above* $\varphi_0(z) = \exp(-\pi i\tau - 2\pi iz) = q^{-1/2}u^{-1} = \varphi(u)$.

Proof. Follows from Theorem 3.9 for $\xi = 0$. □

Theorem 3.26. *Let* $L' \in \mathcal{E}(1, d)$. *Then there exists* $x \in X$ *such that* $L' \simeq t_x^*E(\varphi_0) \otimes E(\varphi_0)^{d-1}$.

Proof. Since $E(\varphi_0)^d$ has degree d, we obtain that there exists $\tilde{L} \in \mathcal{E}(1, 0)$ such that $L' \simeq E(\varphi_0)^d \otimes \tilde{L}$. We also know that $\tilde{L} \simeq t_x^*E(\varphi_0) \otimes E(\varphi_0)^{-1}$ for some $x \in X$. Combining these one obtains

$$L' \simeq E(\varphi_0)^d \otimes t_x^*E(\varphi_0) \otimes E(\varphi_0)^{-1} \simeq t_x^*E(\varphi_0) \otimes E(\varphi_0)^{d-1}.$$

□

Theorem 3.27. *The map*

$$\mathbb{C}^*/ < q > \to \mathrm{Pic}^0(X), \quad \bar{a} \mapsto E(a).$$

is well-defined and is an isomorphism of groups.

Proof. Let $\varphi_0(z) = \exp(-\pi i\tau - 2\pi iz)$ as above. For $x \in X$ consider $t_x^*E(\varphi_0)$, where the map

$$t_x : X \to X, \quad y \mapsto y + x$$

is the translation by x. Let $\xi \in \mathbb{C}$ be a representative of x. Clearly, $t_x^*E(\varphi_0)$ is defined by

$$\varphi_{0\xi}(z) = t_\xi\varphi_0(z) = \varphi_0(z + \xi) = \exp(-\pi i\tau - 2\pi iz - 2\pi i\xi) = \varphi_0(z)\exp(-2\pi i\xi).$$

[10] see [At], page 439.

(Note that if η is another representative of x, then φ_{0_ξ} and φ_{0_η} are equivalent.) Therefore, the bundle $t_x^*E(\varphi_0) \otimes E(\varphi_0)^{-1}$ is defined by

$$(\varphi_{0_\xi}{\varphi_0}^{-1})(z) = \varphi_0(z)\exp(-2\pi i\xi){\varphi_0}^{-1}(z) = \exp(-2\pi i\xi).$$

Since for any $L \in \mathcal{E}(1,0)$ there exists $x \in X$ such that $L \simeq t_x^*E(\varphi_0) \otimes E(\varphi_0)^{-1}$, we obtain $L \simeq E(a)$ for $a = \exp(-2\pi i\xi) \in \mathbb{C}^*$, where $\xi \in \mathbb{C}$ is a representative of x. We proved that any line bundle of degree zero is defined by a constant function $a \in \mathbb{C}^*$.

Vice versa, let $L = E(a)$ for $a \in \mathbb{C}^*$. Clearly, there exists $\xi \in \mathbb{C}$ such that $a = \exp(-2\pi i\xi)$. Therefore,

$$L \simeq E(a) \simeq L(\varphi_{0_\xi}{\varphi_0}^{-1}) \simeq t_x^*E(\varphi_0) \otimes E(\varphi_0)^{-1},$$

where x is the class of ξ in X, which implies that $E(a)$ has degree zero. So we obtained that the line bundles of degree zero are exactly the line bundles defined by constant functions.

We have the map

$$\phi : \mathbb{C}^* \to \mathrm{Pic}^0(X), \quad a \mapsto E(a),$$

which is surjective. By Theorem 3.17 it is moreover a homomorphism of groups. We are looking now for the kernel of this map.

Suppose $E(a)$ is a trivial bundle. Then there exists a holomorphic function $f : \mathbb{C}^* \to \mathbb{C}^*$ such that $f(qu) = af(u)$. Let $f = \sum f_\nu a^\nu$ be the Laurent series expansion of f. Then from $f(qu) = af(u)$ one obtains

$$af_\nu = f_\nu q^\nu \text{ for all } \nu \in \mathbb{Z}.$$

Therefore, $f_\nu(a - q^\nu) = 0$ for all $\nu \in \mathbb{Z}$.

Since $f \not\equiv 0$, we obtain that there exists $\nu \in \mathbb{Z}$ with $f_\nu \neq 0$. Hence $a = q^\nu$ for some $\nu \in \mathbb{Z}$.

Vice versa, if $a = q^\nu$, for $f(u) = u^\nu$ we get

$$f(qu) = q^\nu u^\nu = af(u).$$

This means that $E(a)$ is the trivial bundle, which proves $\mathrm{Ker}\,\phi =< q >$. We obtain the required isomorphism

$$\mathbb{C}^*/ < q > \to \mathrm{Pic}^0(X), \quad \bar{a} \mapsto E(a).$$

$\square$

Theorem 3.28. *For any $F \in \mathcal{E}(r,0)$ there exists a unique $\bar{a} \in \mathbb{C}^*/ < q >$ such that $F \simeq E(A_r(a))$, where*

$$A_r(a) = \begin{pmatrix} a & 1 & & \\ & \ddots & \ddots & \\ & & a & 1 \\ & & & a \end{pmatrix}.$$

Proof. By Theorem 2.10 $F \simeq F_r \otimes L$ for a unique $L \in \mathcal{E}(1,0)$. Since $F_r \simeq E(A_r)$ and $L \simeq E(a)$ for a unique $\bar{a} \in \mathbb{C}^*/ < q >$ we get $F \simeq E(A_r \otimes a)$. So F is defined by the matrix

$$\begin{pmatrix} a & a & & \\ & \ddots & \ddots & \\ & & a & a \\ & & & a \end{pmatrix},$$

where empty entries stay for zeros. It is easy to see that the Jordan normal form of this matrix is

$$\begin{pmatrix} a & 1 & & \\ & \ddots & \ddots & \\ & & a & 1 \\ & & & a \end{pmatrix}.$$

This proves the statement of the theorem. $\square$

3.5.2 Vector bundles of arbitrary degree

Denote by $E_\tau = \mathbb{C}/\Gamma_\tau$, where $\Gamma_\tau = \mathbb{Z}\tau + \mathbb{Z}$. Consider the r-covering

$$\pi_r : E_{r\tau} \to E_\tau, \quad [x] \mapsto [x].$$

Theorem 3.29. *Let F be a vector bundle of rank n on E_τ defined by $A(u) = A(1,u) = A(\tau,u)$. Then $\pi_r^*(F)$ is defined by*

$$\tilde{A}(r\tau, u) = \tilde{A}(u) = \tilde{A}(1,u) := A(r\tau, u) = A(q^{r-1}u)\dots A(qu)A(u).$$

Proof. Consider the following commutative diagram.

$$\begin{tikzcd} & \mathbb{C} \arrow[dl, "p_{r\tau}"'] \arrow[dr, "p_\tau"] & \\ E_{r\tau} \arrow[rr, "\pi_r"] & & E_\tau \end{tikzcd}$$

Consider the map

$$\begin{aligned} \mathbb{C}\times\mathbb{C}^n/\tilde{A} = E(\tilde{A}) \to \pi_r^*(E(A)) = E_{r\tau}\times_{E_\tau} E(A) = \{([z]_{r\tau},\ [z,v]_\tau) \in E_{r\tau}\times E(A)\} \\ [z,v]_{r\tau} \mapsto ([z]_{r\tau},\ [z,v]_\tau). \end{aligned}$$

It is clearly bijective. It remains to prove that it is biholomorphic. From the construction of $E(A)$ and $E(\tilde{A})$ it follows that the diagram

$$\begin{tikzcd} E(\tilde{A}) \arrow[dr] & & \\ & \pi_r^*(E(A)) \arrow[r] \arrow[d] & E(A) \arrow[d] \\ & E_{r\tau} \arrow[r] & E_\tau \end{tikzcd}$$

locally looks as follows

$$\begin{tikzcd} U\times\mathbb{C}^n \arrow[dr] & & \\ & \Delta(U\times U)\times\mathbb{C}^n \arrow[r] \arrow[d] & U\times\mathbb{C}^n, \arrow[d] \\ & U \arrow[r, equal] & U \end{tikzcd} \qquad \begin{tikzcd} (z,v) \arrow[dr, mapsto] & & \\ & ((z,z),v) \arrow[r, mapsto] \arrow[d, mapsto] & (z,v). \arrow[d, mapsto] \\ & z \arrow[r, mapsto] & z \end{tikzcd}$$

This proves the statement.

$\square$

Theorem 3.30. *Let F be a vector bundle of rank n on $E_{r\tau}$ defined by $\tilde{A}(u) = \tilde{A}(r\tau, u)$. Then $\pi_{r*}(F)$ is defined by* $A(u) = \begin{pmatrix} 0 & I_{(r-1)n} \\ \tilde{A}(u) & 0 \end{pmatrix}$.

Proof. Consider the following commutative diagram.

$$\begin{tikzcd} & \mathbb{C} \arrow[dl, "p_{r\tau}"'] \arrow[dr, "p_\tau"] & \\ E_{r\tau} \arrow[rr, "\pi_r"] & & E_\tau \end{tikzcd}$$

Let $z \in \mathbb{C}$. Consider $y = p_{r\tau}(z) \in E_{r\tau}$ and $x = p_\tau(z) = \pi_r p_{r\tau}(z) \in E_\tau$.

Choose a point $b \in \mathbb{C}$ such that $z \in V_b$, where V_b is the standard parallelogram at point b. Clearly $x \in U_b = p_r(V_b)$ and we have the isomorphism $\varphi_b : U_b \to V_b$ with $\varphi_b(x) = z$.

Consider $\pi_r^{-1}(U_b) = W_b \bigsqcup \cdots \bigsqcup W_{b+(r-1)\tau}$, where $y \in W_b$ and $\pi_r|_{W_{b+i\tau}} : W_{b+i\tau} \to U_b$ is an isomorphism for each $0 \leqslant i < r$.

We have

$$\begin{aligned} \pi_{r*}(\mathcal{E}(\tilde{A}))(U_b) =& \mathcal{E}(\tilde{A})(\pi_r^{-1}(U_b)) = \mathcal{E}(\tilde{A})(W_b \bigsqcup \cdots \bigsqcup W_{b+(r-1)\tau}) = \\ & \mathcal{E}(\tilde{A})(W_b) \oplus \cdots \oplus \mathcal{E}(\tilde{A})(W_{b+(r-1)\tau}), \end{aligned}$$

where $\mathcal{E}(\tilde{A})$ is the sheaf of sections of $E(\tilde{A})$.

Choose $a \in \mathbb{C}$ such that $z \notin V_a$, $z \in V_{a+\tau}$. We have $\varphi_a(x) = z + \tau$. As above, $\pi_r^{-1}(U_a) = W_a \bigsqcup \cdots \bigsqcup W_{a+(r-1)\tau}$ and

$$\begin{aligned} \pi_{r*}(\mathcal{E}(\tilde{A}))(U_a) =& \mathcal{E}(\tilde{A})(\pi_r^{-1}(U_a)) = \mathcal{E}(\tilde{A})(W_a \bigsqcup \cdots \bigsqcup W_{a+(r-1)\tau}) = \\ & \mathcal{E}(\tilde{A})(W_a) \oplus \cdots \oplus \mathcal{E}(\tilde{A})(W_{a+(r-1)\tau}). \end{aligned}$$

Since $g_{ab}(x) = A(\varphi_a(x) - \varphi_b(x), \varphi_b(x))$, we obtain

$$g_{ab}(x) = A(\varphi_a(x) - \varphi_b(x), \varphi_b(x)) = A(z + \tau - z, z) = A(\tau, z).$$

Therefore, to obtain $A(\tau, z)$ it is enough to compute $g_{ab}(x)$.

Note that $\pi_{r*}(\mathcal{E}(\tilde{A}))_x = \mathcal{E}(\tilde{A})_y \oplus \cdots \oplus \mathcal{E}(\tilde{A})_{y+(r-1)\tau}$. Note also that g_{ab} is a map from $\pi_{r*}(\mathcal{E}(\tilde{A}))(U_b) = \mathcal{E}(\tilde{A})(W_b) \oplus \cdots \oplus \mathcal{E}(\tilde{A})(W_{b+(r-1)\tau})$ to $\pi_{r*}(\mathcal{E}(\tilde{A}))(U_a) = \mathcal{E}(\tilde{A})(W_a) \oplus \cdots \oplus \mathcal{E}(\tilde{A})(W_{a+(r-1)\tau})$.

One easily sees that $y \in W_b$, $y \in W_{a+(r-1)\tau}$ and $y + i\tau \in W_{b+i\tau}$, $y + i\tau \in W_{a+(i-1)\tau}$ for $0 < i < r$. Therefore,

$$g_{ab}(x) = \begin{pmatrix} 0 & \tilde{g}_{a\ b+\tau}(y+\tau) & & \\ \vdots & & \ddots & \\ 0 & & & \tilde{g}_{a+(r-2)\tau\ b+(r-a)\tau}(y+(r-1)\tau) \\ \tilde{g}_{a+(r-1)\tau\ b}(y) & 0 & \dots & 0 \end{pmatrix}.$$

It remains to compute the entries of this matrix. Since

$\tilde{g}_{a+(r-1)\tau\ b}(y) = \tilde{A}(\tilde{\varphi}_{a+(r-1)\tau}(y) - \tilde{\varphi}_b(y), \tilde{\varphi}_b(y)) = \tilde{A}(z + r\tau - z, z) = \tilde{A}(r\tau, z)$ and $\tilde{g}_{a+(i-1)\tau\ b+i\tau}(y + i\tau) = \tilde{A}(\tilde{\varphi}_{a+(i-1)\tau}(y + i\tau) - \tilde{\varphi}_{b+i\tau}(y + i\tau), \tilde{\varphi}_{b+i\tau}(y + i\tau)) =$
$\tilde{A}(z + i\tau - (z + i\tau) = \tilde{A}(0, z + i\tau) = I_n$,

one obtains

$$g_{ab}(x) = \begin{pmatrix} 0 & I_n & & \\ \vdots & & \ddots & \\ 0 & & & I_n \\ \tilde{A}(z) & 0 & \dots & 0 \end{pmatrix}.$$

Therefore, $A(z) = \begin{pmatrix} 0 & I_n & & \\ \vdots & & \ddots & \\ 0 & & & I_n \\ \tilde{A}(z) & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_{(r-1)n} \\ \tilde{A}(u) & 0 \end{pmatrix}$. This proves the statement. □

Lemma 3.31. *Let $A_i \in \mathrm{GL}_n(\mathbb{R})$, $i = 1, \dots, n$. Then*

$$\prod_{i=1}^{r} \begin{pmatrix} 0 & I_{(r-1)n} \\ A_i & 0 \end{pmatrix} = \mathrm{diag}(A_r, \dots, A_1)$$

Proof. Straightforward calculation. □

From Theorem 3.29 and Theorem 3.30 one obtains the following:

Corollary 3.32. *Let $E(A)$ be a vector bundle of rank n on $E_{r\tau}$, where $A : \mathbb{C}^* \to \mathrm{GL}_n(\mathbb{C}V)$ is a holomorphic function. Then $\pi_r^* \pi_{r*} E(A)$ is defined by*

$$\mathrm{diag}(A(q^{r-1}u), \dots, A(qu), A(u)).$$

In other words $\pi_r^* \pi_{r*} E(A)$ is isomorphic to the direct sum

$$\bigoplus_{i=0}^{r-1} E(A(q^i u)).$$

Proof. We know that $\pi_r^* \pi_{r*} E(A)$ is defined by $B(r, u)$, where

$$B(1, u) = \begin{pmatrix} 0 & I_{(r-1)n} \\ A & 0 \end{pmatrix}.$$

Therefore, using Lemma 3.31, one obtains

$$B(r,u) = \begin{pmatrix} 0 & I_{(r-1)n} \\ A(q^{r-1}u) & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & I_{(r-1)n} \\ A(qu) & 0 \end{pmatrix} \begin{pmatrix} 0 & I_{(r-1)n} \\ A(u) & 0 \end{pmatrix} = \mathrm{diag}(A(q^{r-1}u), \dots, A(qu), A(u)).$$

□

Corollary 3.33. *Let $L \in \mathcal{E}(r,0)$, then $\pi_r^*\pi_{r*}L = \bigoplus_1^r L$.*

Proof. Clear, since $L = E(A)$ for a constant matrix A by Theorem 3.28. □

Note that for a covering $\pi_r : E_{r\tau} \to E_\tau$ the group of deck transformations $\mathrm{Deck}(E_{r\tau}/E_\tau)$ can be identified with the kernel $\mathrm{Ker}(\pi_r)$. But $\mathrm{Ker}\,\pi_r$ is cyclic and equals $\{1, [q], \dots [q]^{r-1}\}$, where $[q]$ is a class of $q = e^{2\pi i\tau}$ in $E_{r\tau}$. Clearly

$$[q]^*(E(A(u))) = E(A(qu)).$$

Therefore, we get one more corollary.

Corollary 3.34. *Let ϵ be a generator of* $\mathrm{Deck}(E_{r\tau}/E_\tau)$. *Then for a vector bundle E on $E_{r\tau}$ we have*

$$\pi_r^*\pi_{r*}E = E \oplus \epsilon^*E \oplus \cdots \oplus (\epsilon^{r-1})^*E.$$

To proceed we need the following result from [Oda](Theorem 1.2, (i)):

Theorem. *Let $\varphi : Y \to X$ be an isogeny of g-dimensional abelian varieties over a field k, and let L be a line bundle on Y such that the restriction of the map*

$$\Lambda(L) : Y \to \mathrm{Pic}^0(Y), \quad y \mapsto t_y^*L \otimes L^{-1},$$

to the kernel of φ is an isomorphism. Then $\mathrm{End}(\varphi_*L) = k$ *and φ_*L is an indecomposable vector bundle on X.*

Theorem 3.35. *Let $L \in \mathcal{E}(1,d)$ and let $(r,d) = 1$. Then $\pi_{r*}(L) \in \mathcal{E}(r,d)$.*

Proof. It is clear that $\pi_{r*}L$ has rank r and degree d. It remains to prove that $\pi_{r*}L$ is indecomposable.

We have the isogeny $\pi_r : E_{r\tau} \to E_r$. Since $Y = E_{r\tau}$ is a complex torus (elliptic curve), $Y \simeq \mathrm{Pic}^0(Y)$ with the identification $y \leftrightarrow t_y^*E(\varphi_0) \otimes E(\varphi_0)^{-1}$. We know that $L = E(\varphi_0)^d \otimes \tilde{L}$ for some $\tilde{L} = E(a) \in \mathcal{E}(1,0)$, $a \in \mathbb{C}^*$. Since $t_y^*(\tilde{L}) = t_y^*(E(a)) = E(a) = \tilde{L}$, as in the proof of Theorem 3.27 one gets

$$\begin{aligned}\Lambda(L)(y) =& t_y^*(L) \otimes L^{-1} = t_y^*(E(\varphi_0)^d \otimes \tilde{L}) \otimes (E(\varphi_0)^d \otimes \tilde{L})^{-1} = \\ & t_y^*(E(\varphi_0)^d) \otimes t_y^*(\tilde{L}) \otimes E(\varphi_0)^{-d} \otimes \tilde{L}^{-1} = t_y^*(E(\varphi_0)^d) \otimes E(\varphi_0)^{-d} = \\ & t_y^*(E(\varphi_0^d)(z)) \otimes E(\varphi_0^{-d}) = E(\varphi_0^d(z+\eta)) \otimes E(\varphi_0^{-d}) = \\ & E(\varphi_0^d(z+\eta)\varphi_0^{-d}(z)) = E(\exp(-2\pi i d\eta)) = t_{dy}^*(E(\varphi_0)) \otimes E(\varphi_0)^{-1},\end{aligned}$$

where $\eta \in \mathbb{C}$ is a representative of y. This means that the map $\Lambda(L)$ corresponds to the map

$$d_Y : E_{r\tau} \to E_{r\tau}, \quad y \mapsto dy.$$

Since $\operatorname{Ker} \pi_r$ is isomorphic to $\mathbb{Z}/r\mathbb{Z}$, we conclude that the restriction of d_Y to $\operatorname{Ker} \pi_r$ is an isomorphism if and only if $(r,d) = 1$. Therefore, using Theorem mentioned above, we prove the required statement. □

Now we are able to prove the following main theorem:

Theorem 3.36. *(i) Every indecomposable vector bundle $F \in \mathcal{E}_{E_\tau}(r,d)$ is of the form $\pi_{r'*}(L' \otimes F_h)$, where $(r,d) = h$, $r = r'h$, $d = d'h$, $L' \in \mathcal{E}_{E_{r'\tau}}(1,d')$.*

(ii) Every vector bundle of the form $\pi_{r'*}(L' \otimes F_h)$, where L' and r' are as above, is an element of $\mathcal{E}_{E_\tau}(r,d)$.

Proof. [11] (i) By Theorem 2.18 we obtain $F \simeq E_A(r,d) \otimes L$ for some line bundle $L \in \mathcal{E}(1,0)$. By Theorem 2.16 we have $E_A(r,d) \simeq E_A(r',d') \otimes F_h$, hence $F \simeq E_A(r',d') \otimes F_h \otimes L$.

Consider any line bundle $\tilde{L} \in \mathcal{E}_{E_{r'\tau}}(1,d')$. Since by Theorem 3.35 $\pi_{r'*}(\tilde{L}) \in \mathcal{E}(r',d')$, it follows from Theorem 2.18 that there exists a line bundle L'' such that $E_A(r',d') \otimes L \simeq \pi_{r'*}(\tilde{L}) \otimes L''$.

Using the projection formula, we get

$$F \simeq \pi_{r'*}(\tilde{L}) \otimes L'' \otimes F_h \simeq \pi_{r'*}(\tilde{L} \otimes \pi^*_{r'}(L'') \otimes \pi^*_{r'}(F_h)) \simeq \pi_{r'*}(L' \otimes \pi^*_{r'}(F_h))$$

for $L' = \tilde{L} \otimes \pi^*_{r'}(L'')$.

Since F_h is defined by a constant matrix we obtain by Theorem 3.29 that $\pi^*_{r'}(F_h)$ is defined by $f_h^{r'}$, which is has the same Jordan normal form as f_h. Therefore, $\pi^*_{r'}(F_h) \simeq F_h$ and finally one gets $F \simeq \pi_{r'*}(L' \otimes F_h)$.

(ii) Consider $F = \pi_{r'*}(L' \otimes F_h)$. As above $F_h = \pi^*_{r'}(F_h)$. Using the projection formula we get

$$F = \pi_{r'*}(L' \otimes F_h) = \pi_{r'*}(L' \otimes \pi^*_{r'}(F_h)) = \pi_{r'*}(L') \otimes F_h.$$

By Theorem 3.35 $\pi_{r'*}(L')$ is an element from $\mathcal{E}_{E_\tau}(r',d')$. Therefore, $\pi_{r'*}(L') = E_A(r',d') \otimes L$ for some line bundle $L \in \mathcal{E}_{E_\tau}(1,0)$. Finally we obtain

$$F = \pi_{r'*}(L') \otimes F_h = E_A(r',d') \otimes L \otimes F_h = E_A(r'h,d'h) \otimes L = E_A(r,d) \otimes L,$$

which means that F is an element of $\mathcal{E}_{E_\tau}(r,d)$. □

[11] The proof of this theorem uses the ideas from lectures presented by Bernd Kreußler at University of Kaiserslautern.

Remark. Since any line bundle of degree d' is of the form $t_x^*E(\varphi_0)\otimes E(\varphi_0)^{d'-1}$, Theorem 3.36(i) takes exactly the form of Proposition 1 from the paper of Polishchuk and Zaslow.

Any line bundle of degree d' over $E_{r\tau}$ is of the form $E(a)\otimes E(\varphi^{d'})$, where $a\in\mathbb{C}^*$. Therefore, $L'\otimes F_h = E(a)\otimes E(\varphi_0^{d'})\otimes E(A_h) = E(\varphi_0^{d'}A_h(a))$. Using Theorem 3.30 we obtain the following:

Theorem 3.37. *Indecomposable vector bundles of rank r and degree d on E_τ are exactly those defined by the matrices*

$$\begin{pmatrix} 0 & I_{(r'-1)h} \\ \varphi_0^{d'}A_h(a) & 0 \end{pmatrix},$$

where $(r,d)=h$, $r'=r/h$, $d'=d/h$, $\varphi_0(u)=q^{-\frac{\tau}{2}}u^{-1}$, $q=e^{2\pi i\tau}$, $a\in\mathbb{C}^*$, and

$$A_h(a)=\begin{pmatrix} a & 1 & & \\ & \ddots & \ddots & \\ & & a & 1 \\ & & & a \end{pmatrix}\in \mathrm{GL}_h(\mathbb{C}).$$

Note that if $d=0$, we get $h=r$, $r'=1$, and $d'=0$. In this case the statement of Theorem 3.37 is exactly Theorem 3.28.

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