On Gyroscopic Stabilization

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Jan Homeyer
CHAPTER 1

Introduction

We consider a system of the form

\[ M \ddot{x} + D \dot{x} + Kx = 0 \, , \quad x \in \mathbb{R}^n , \]  

(1.1)

with a positive definite mass matrix \( M \), a symmetric damping matrix \( D \) and a positive definite stiffness matrix \( K \). If the equilibrium in (1.1) is unstable, a small disturbance is enough to set the system in motion again. The motion of the system sustains itself, an effect which is called self-excitation or self-induced vibration. The reason behind this effect is the presence of negative damping, which results for example from dry friction, see [1], [5], [18]. Another example is the Van der Pol Oscillator (see [13], p. 9), which due to a non-constant damping coefficient, locally has negative damping. This example will be further examined below. Negative damping implies that the damping matrix \( D \) is indefinite or negative definite. Throughout our work, we assume \( D \) to be indefinite, and that system (1.1) possesses both stable and unstable modes and thus is unstable.

It is now the idea of gyroscopic stabilization to mix the modes of a system with indefinite damping such that the system is stabilized without introducing further dissipation. This is done by adding gyroscopic forces \( G \dot{x} \) with a suitable skew-symmetric matrix \( G \) to the left-hand side (see e.g. [16]).

**Definition 1.1.** We call \( G = -G^T \in \mathbb{R}^{n \times n} \) a gyroscopic stabilizer for the unstable system (1.1), if

\[ M \ddot{x} + (D + G) \dot{x} + K x = 0 \]

(1.2)

is asymptotically stable. In this case the system is gyroscopically stabilizable. We will even call \( G \) a gyroscopic stabilizer if

\[ M \ddot{x} + (D + \gamma G) \dot{x} + K x = 0 \]

for some \( \gamma \in \mathbb{R} \) is asymptotically stable, thus making the definition independent of the scaling of \( G \).
As mentioned above, an example for the occurrence of negative damping is given by the Van der Pol Oscillator (see [13], p. 9),

\[(1.3) \quad \ddot{x} - \mu(1 - x^2)\dot{x} + x = 0\]

with \(\mu > 0\). For \(x\) close to zero, the damping coefficient in (1.3) is negative, thus self-excitation occurs. The trajectories of the system tend towards a limit cycle (see Figure 1.1), since the damping coefficient becomes positive when \(x\) is sufficiently far away from zero. It follows that here self-excitation is a local effect depending on \(x\). We now stabilize system (1.3) by coupling it via a conservative, gyroscopic term with two damped harmonic oscillators. We put \(\mu = 3\) and consider the system

\[
\begin{bmatrix}
\ddot{x} \\
\ddot{y} \\
\ddot{z}
\end{bmatrix} + \begin{bmatrix}
-3(1 - x^2) & -g_3 & g_2 \\
g_3 & 2 & -g_1 \\
-g_2 & g_1 & 1.5
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} + \begin{bmatrix}
1 \\
\frac{1}{2} \\
1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \ddot{v} + (D(v) + G)\dot{v} + Kv = 0.
\]

(1.4)

For \(g_1 = g_2 = g_3 = 0\) the above system is still decoupled. Thus \(x\) tends towards the limit cycle, while \(y\) and \(z\) will tend to zero. By choosing \((g_1, g_2, g_3) = (4.5, 4.5, 4.5)\) and inspecting the eigenvalues, we see that the linearized system \(\ddot{v} + (D(0) + G)\dot{v} + Kv = 0\) is asymptotically stable. According to ([20], §29, IX), it is implied that the equilibrium of the nonlinear system is at least locally asymptotically stable.

In Figure 1.2, the solution of the coupled nonlinear system (1.4) with initial conditions \((x(0) \; \dot{x}(0) \; y(0) \; \dot{y}(0) \; z(0) \; \dot{z}(0)) = \frac{1}{10}(2 \; 1 \; 2 \; 1 \; 2 \; 1)\) is plotted versus time and in the phase space. In contrast to the non-coupled Van der Pol oscillator (1.3), the solution for \(z\) now tends to

Figure 1.1. Solution of (1.3) with initial conditions \(x(0) = 0.2\) and \(\dot{x}(0) = 0.1\) plotted versus time on the left; phase portrait on the right.
Figure 1.2. Solution for $x$ in (1.4) with initial conditions 
$(x(0) \dot{x}(0) \ y(0) \dot{y}(0) \ z(0) \dot{z}(0)) = \frac{1}{10}(2 1 2 1 2 1)$ plotted versus time on the left; phase portrait of $(x(t), \dot{x}(t))$ with $t \in [0, 100]$ on the right.

Figure 1.3 shows the solutions for $y$ and $z$ of (1.3). They behave very similar to the solution for $x$, the gyroscopic term aligns the behavior of $x, y, z$.

This example demonstrates that a gyroscopic stabilization may even be applied for certain nonlinear systems.

The problem we will analyze in our work is: under which circumstances does a gyroscopic stabilizer exists, and how can it be constructed?

Let $M$ be the unit matrix. A well-known necessary condition for gyroscopic stabilizability is that the traces of $D$ and $K^{-1}D$ are both positive, see [14] or [15]. In the recent paper [15], the authors ask whether this condition is also sufficient. In the case $n = 2$ they give an affirmative answer. For $n > 2$ gyroscopic stabilizability so far has only
been shown under additional conditions. We will use results on eigenvalue and eigenvector perturbation in order to derive a new sufficient condition for gyroscopic stabilizability. Our method is formulated as an inverse eigenvector problem, see chapter 2. We then show in chapter 3 how our method can be applied to systems of the form (1.1) in space dimension 2 and 3 and thereby show that the conditions $\text{tr } D, \text{tr } QDQ > 0$ are sufficient conditions for stabilizability. The two-dimensional case is known already and can be found in the literature, while the solution to the three-dimensional case so far was not known. An easily applicable construction method for a gyroscopic stabilizer $G$ is shown, and we derive a visualization of the set of $G$ in three dimensions for a given system of the form (1.1).

We also apply our construction method to space dimension 4 and 5. Actually, the construction in both cases is fairly similar. At least in dimension 4, we are again able to show the sufficiency of the conditions $\text{tr } D, \text{tr } QDQ > 0$ for gyroscopic stabilizability. Finally, we show how in some cases, the problem of constructing a gyroscopic stabilizer can be reduced to spaces of lower dimension.

Parts of chapters 2, 3 and 4 have already been published in a joint paper with Tobias Damm [7].
CHAPTER 2

Necessary and Sufficient Conditions

2.1. Necessary Condition

Since in the system
\[ M \ddot{x} + (D + G) \dot{x} + Kx = 0, \quad x \in \mathbb{R}^n \]  
we assume \( M > 0 \), there exists the unique positive definite square root of \( M \), thus we may define \( P > 0 \) with \( P^{-2} = M \). By multiplication from the left and with \( z = P^{-1}x \), (2.1) becomes
\[
PMPP^{-1} \ddot{x} + P(D + G)PP^{-1} \dot{x} + PKPP^{-1}x = 0,
\]
where \( I \) denotes the unit matrix, \( \hat{K} = \hat{K}^T > 0 \), \( \hat{G} = -\hat{G}^T \) and \( \hat{D} = \hat{D}^T \). Thus, without loss generality, from now on we assume that in (2.1) we have \( M = I \). Then, the second-order system can be written in first order form as
\[
\frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K & -D - G \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = A_G \begin{bmatrix} x \\ \dot{x} \end{bmatrix}.
\]
It is asymptotically stable if and only if \( \sigma(A_G) \subset \mathbb{C}_- \), which implies \( \text{tr} A_G < 0 \), i.e. \( \text{tr} D > 0 \). Moreover, since \( \sigma(A_G) \subset \mathbb{C}_- \) if and only if \( \sigma(A_G^{-1}) \subset \mathbb{C}_- \), where
\[
(2.2) \quad A_G^{-1} = \begin{bmatrix} -K^{-1}(D + G) & -K^{-1} \\ I & 0 \end{bmatrix},
\]
we conclude that also \( 0 < \text{tr} K^{-1}(D + G) = \text{tr} K^{-1}D \). These necessary criteria are well known (e.g. [14]), but their first appearance is difficult to track down. In the literature, [17] is often cited as the source, but actually, these conditions are not mentioned in the article.

We asked Prof. Dr. P. C. Müller about the origin of these criteria, which are well known to him for a long time, and he referred to Prof. Dr. K. Magnus. Müller was also so kind to provide us with his method to derive the conditions, which directly uses the characteristic polynomial \( p \)
2.2. AN INVERSE EIGENVECTOR PROBLEM

of (1.2). Consider
\[ p(\lambda) = \det(\lambda^2 I + \lambda(D + G) + K) = \lambda^m + \sum_{i=0}^{m-1} a_i \lambda^i, \]
where \( a_{m-1} = \text{tr} D \). For asymptotic stability, it is necessary that \( a_i > 0 \) for all \( i \), we immediately get the condition \( \text{tr} D > 0 \).

Define \( q \) via
\[ p(\lambda) = \lambda^n \det K \det(\mu^2 I + \mu K^{-1}(D + G)) = \lambda^n \det K q(\mu) \]
\[ = \lambda^n \det K (\mu^m + \sum_{i=0}^{m-1} b_i \mu^i), \]
where \( \mu = \frac{1}{\lambda} \). If \( p \) is asymptotically stable, then so is \( q \), where \( q \) is actually the characteristic polynomial of (2.2). The condition \( b_i > 0 \) then implies \( b_{m-1} = \text{tr}(K^{-1}D) > 0 \).

2.2. An Inverse Eigenvector Problem

To analyze sufficiency of the conditions \( \text{tr} D, \text{tr} K^{-1}D > 0 \) we first reformulate the gyroscopic stabilization problem as an inverse eigenvector problem. According to the spectral mapping theorem [12, Prop. A.1.16], the matrix \( A_G \) is asymptotically stable if and only if the matrix
\[ -A_G - A_G^{-1} = \begin{bmatrix} K^{-1}(D + G) & K^{-1} - I \\ K - I & D + G \end{bmatrix} \]
is positive stable (i.e. has all eigenvalues in \( \mathbb{C}_+ \)). Let \( Q \) denote the positive definite square root of \( K^{-1} \). A similarity transformation with \( T = \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \) brings \( -(A_G + A_G^{-1}) \) to the form
\[ -T^{-1}(A_G + A_G^{-1})T = \begin{bmatrix} Q(D + G)Q & Q - Q^{-1} \\ Q^{-1} - Q & D + G \end{bmatrix} \]
\[ = \begin{bmatrix} QGQ & 0 \\ 0 & G \end{bmatrix} + \begin{bmatrix} QDQ & 0 \\ 0 & Q^{-1} - Q \end{bmatrix}. \]
(2.3)

By a perturbation argument we can formulate a stabilizability criterion as an inverse eigenvector problem.

**Proposition 2.1.** Let \( \tau > 0 \) and \( D_\tau = D - \tau P \) for some positive definite \( P \).
For system (1.2) to be gyroscopically stabilizable it is sufficient that there exists a skew-symmetric matrix \( G = -G^T \) with the following properties:

(a) Both \( G \) and \( QGQ \) only have simple eigenvalues.
If \( v \) is an eigenvector of \( G \) then \( v^* D v \geq 0 \).

If \( w \) is an eigenvector of \( QGQ \) then \( w^* Q D w \geq 0 \).

**Proof.** Instead of (2.3) consider the matrix
\[
M_\varepsilon = \begin{bmatrix}
    QGQ & 0 \\
    0 & G
\end{bmatrix} + \varepsilon \begin{bmatrix}
    QDQ & Q - Q^{-1} \\
    Q^{-1} - Q & D
\end{bmatrix}.
\]

We show that for small \( \varepsilon > 0 \) this matrix is positive stable, which implies that \( \varepsilon^{-1} G \) is a gyroscopic stabilizer.

Note that all eigenvalues of \( M_\varepsilon \) are perturbations of the imaginary eigenvalues of \( M_0 \). We will show that for each eigenvalue \( \lambda_0 \in \sigma(M_0) = \sigma(G) \cup \sigma(QGQ) \subset i \mathbb{R} \) of multiplicity \( k \) the perturbed matrix \( M_\varepsilon \) has \( k \) eigenvalues with positive real part in a neighbourhood of \( \lambda_0 \).

(i) Assume \( \lambda_0 \in \sigma(G) \setminus \sigma(QGQ) \). Then \( G \) has an eigenvector \( v \in \mathbb{C}^n \), so that \( ||v|| = 1 \) and \( Gv = \lambda_0 v \). Condition (a) implies that \( \lambda_0 \) is a simple eigenvalue of \( M_0 \). A unit eigenvector of \( M_0 \) is given by \( v_0 = [0, v]^T \). For small \( \varepsilon > 0 \) a standard perturbation result (e.g. [19, Thm. IV 2.3]) gives that \( M_\varepsilon \) has a simple eigenvalue \( \lambda_\varepsilon = \lambda_0 + \varepsilon v^* D v + O(\varepsilon^2) \).

Since \( v^* D v > 0 \) by (b), we have \( \lambda_\varepsilon \in \mathbb{C}_+ \).

(ii) Assume \( \lambda_0 \in \sigma(QGQ) \setminus \sigma(G) \). For a corresponding unit eigenvector \( w_0 = [w, 0]^T \) of \( M_0 \), an analogous argument as in the first case shows that \( M_\varepsilon \) has a simple eigenvalue \( \lambda_\varepsilon = \lambda_0 + \varepsilon w^* Q D Q w + O(\varepsilon^2) \in \mathbb{C}_+ \).

(iii) Assume \( \lambda_0 \in \sigma(QGQ) \cap \sigma(G) \). Then \( \lambda_0 \) is a double eigenvalue of \( M_0 \). The corresponding two-dimensional invariant subspace is spanned by vectors \( v_0 \) and \( w_0 \) as in the first two cases. For small \( \varepsilon \geq 0 \) the perturbed matrix \( M_\varepsilon \) also has a two-dimensional invariant subspace, which depends smoothly on \( \varepsilon \) and coincides with \( \text{Span}\{v_0, w_0\} \) for \( \varepsilon = 0 \). The restriction of \( M_\varepsilon \) to this subspace has the representation (e.g. [19, Thm. V 2.8])
\[
[v_0, w_0]^* M_\varepsilon [v_0, w_0] + O(\varepsilon^2) = \lambda_0 I + \begin{bmatrix}
    v^* D v & v^* (Q - Q^{-1}) w \\
    w^* (Q - Q^{-1}) v & w^* Q D Q w
\end{bmatrix} + O(\varepsilon^2) .
\]

The \( 2 \times 2 \)-matrix in the previous term is positive stable, since it has positive trace and positive determinant. Thus \( M_\varepsilon \) has two positive stable eigenvalues (counting multiplicity) in a neighbourhood of \( \lambda_0 \).

\( \square \)
2.2. AN INVERSE EIGENVECTOR PROBLEM

If \( \frac{\text{tr} D}{n} \leq \frac{\text{tr} QDQ}{\text{tr} Q^2} \) we consider \( \tau = \frac{\text{tr} D}{n} \), such that we have

\[
\text{tr} D_\tau = 0, \quad \text{tr} QD_\tau Q \geq 0.
\]

Otherwise, if \( \frac{\text{tr} D}{n} > \frac{\text{tr} QDQ}{\text{tr} Q^2} \), we consider the system defined via \( A_\tau^{-1} \) in (2.2) as

\[
\frac{d}{dt} \begin{bmatrix} \dot{y} \\ y \end{bmatrix} = \begin{bmatrix} -K^{-1}(D + G) & -K^{-1} \\ I & 0 \end{bmatrix} \begin{bmatrix} \dot{y} \\ y \end{bmatrix},
\]

or equivalently

\[
\dot{y} + Q^2(D + G)y + Q^2y = 0.
\]

We put \( z = Q^{-1}y \) and multiply from the left with \( Q^{-1} \), then (2.4) becomes

\[
\dot{z} + Q(D + G)Q\dot{z} + Q^2z = 0.
\]

We set \( \tilde{D} = QDQ \), \( \tilde{Q} = Q^{-1} \) and \( \tilde{G} = QGQ \), i.e.

\[
\dot{z} + (\tilde{D} + \tilde{G})\dot{z} + \tilde{Q}^{-2}z = 0.
\]

System (2.5) is stable if and only if the original system (1.1) is stable.

Note that from \( \frac{\text{tr} D}{n} > \frac{\text{tr} QDQ}{\text{tr} Q^2} \) it follows that \( \text{tr} \tilde{Q}\tilde{D}\tilde{Q} > \frac{n\text{tr} \tilde{D}}{\text{tr} \tilde{Q}^{-2}} \).

For system (2.5) we get with \( \tilde{\tau} = \frac{\text{tr} \tilde{D}}{\text{tr} \tilde{Q}^{-2}} \) and \( P = \tilde{Q}^{-2} \):

\[
\text{tr}(\tilde{D} - \frac{\text{tr} \tilde{D}}{\text{tr} \tilde{Q}^{-2}} \tilde{Q}^{-2}) = 0, \quad \text{tr}(\tilde{Q}(\tilde{D} - \frac{\text{tr} \tilde{D}}{\text{tr} \tilde{Q}^{-2}} \tilde{Q}^{-2})\tilde{Q}) = \text{tr}(\tilde{Q}\tilde{D}\tilde{Q}) - \text{tr} \tilde{D} \frac{\text{tr} P}{\text{tr} \tilde{Q}^{-2}}
\]

\[
= \text{tr}(\tilde{Q}\tilde{D}\tilde{Q}) - \text{tr} \tilde{D} \frac{n}{\text{tr} \tilde{Q}^{-2}} \geq 0.
\]

Also, if \( \tilde{G} \) is such that \( \tilde{v}_i^*\tilde{D}_\tau \tilde{v}_i = 0 \) for all eigenvectors \( \tilde{v}_i \) of \( \tilde{G} \) and \( \tilde{w}_i^*\tilde{Q}\tilde{D}_\tau\tilde{Q}\tilde{w}_i \geq 0 \) for all eigenvectors \( \tilde{w}_i \) of \( \tilde{Q}\tilde{D}\tilde{Q} \), then for the skew-symmetric \( G = Q^{-1}\tilde{G}Q^{-1} \) we have \( v_i^*Dv_i \geq 0 \) for all eigenvectors \( v_i = \tilde{w}_i \) of \( G \) and \( w_i^*QDQw_i \geq 0 \) for all eigenvectors \( w_i = \tilde{v}_i \) of \( QDQ \).

Hence from now on we always consider \( \text{tr} D_\tau = 0 \) and \( \text{tr} QD_\tau Q \geq 0 \); for simplicity, we write again \( D, Q \) instead of \( \tilde{D}, \tilde{Q} \).

Throughout the following chapters, we will show that a solution to the following problem exists.

**Problem 2.2.** For symmetric matrices \( D, Q \in \mathbb{R}^{n \times n} \) satisfying \( Q \succ 0 \), \( \text{tr} D = 0 \) and \( \text{tr} QDQ \geq 0 \) find \( G = -G^T \) such that

(a) both \( G \) and \( QGQ \) only have simple eigenvalues,

(b) if \( v \) is an eigenvector of \( G \) then \( v^*Dv = 0 \),
if \( w \) is an eigenvector of \( QGQ \) then \( w^*QDQw \geq 0 \).

**Remark 2.3.** Consider the characteristic polynomial \( p \) of the system 
\[
\ddot{x} + (D + G)\dot{x} + Kx = 0.
\]
We have
\[
p(\lambda) = \det \left( \lambda^2 I + \lambda(D + G) + K \right)
\]
\[
= \det \left( \lambda^2 I + \lambda(D + G) + K \right)^T
\]
\[
= \det \left( \lambda^2 I + \lambda(D - G) + K \right).
\]
Thus it follows that \( G \) is a gyroscopic stabilizer if and only if \(-G\) is a gyroscopic stabilizer.

### 2.3. Traces And Indefinite Scalar Products

The conditions \( v^*Dv = 0, w^*QDQw \geq 0 \) from problem 2.2 can be considered in context of indefinite scalar products. Thus, we will in short give some background on the subject and introduce some notation that we will use in the subsequent chapters.

As before, by \( \text{tr} A \) we denote the trace of a square matrix. We will extend this notion now and define the trace of a matrix on a subspace. It is well-known that \( \text{tr} BC = \text{tr} CB \) if the product \( BC \) is a square matrix. Hence, if \( U = [u_1, \ldots, u_n] \in \mathbb{C}^{n \times n} \) is unitary then
\[
\text{tr} A = \text{tr} AUU^* = \text{tr} U^*AU = \sum_{j=1}^{n} u_j^*Au_j.
\]

More generally, if the matrix \( U = [u_1, \ldots, u_k] \in \mathbb{C}^{n \times k} \) has orthonormal columns, we write \( U = \text{Span}\{u_1, \ldots, u_k\} \) and \( P_U = UU^* \) for the orthogonal projection onto \( U \). Then \( P_UAP_U \) is the projection of \( A \) to \( U \) and
\[
\text{tr}_U A := \text{tr}(P_UAP_U) = \text{tr}(U^*AU) = \sum_{j=1}^{k} u_j^*Au_j
\]
is the trace of the projected matrix. It is important that \( \text{tr}_U \) depends continuously on \( U \), or, equivalently, on the orthogonal projector \( P_U \).

A matrix \( D = D^T \in \mathbb{R}^{n \times n} \) is positive definite, if \( \text{tr}_U D > 0 \) for all non-zero subspaces \( U \subset \mathbb{C}^n \). If \( D \) is indefinite, there exists a vector \( u \in \mathbb{R}^n \) with \( u^TDu = 0 \).

If \( G \) is skew-symmetric then \( \text{tr} G = 0 \). Moreover, if \( P = QQ^T \) is positive definite then also \( \text{tr} PG = \text{tr} Q^TGQ = 0 \).

Note that the eigenvalues of a skew-symmetric matrix \( G \) are either zero or complex conjugate pairs of purely imaginary numbers and a set of eigenvectors of a skew-symmetric matrix can be chosen as an orthonormal basis of \( \mathbb{C}^n \). The complex eigenvectors can be chosen as
conjugate pairs. If $v, w$ is any pair of normalized complex conjugate orthogonal vectors, we have

$$v^* Dv = w^* Dw = \frac{1}{2} \operatorname{tr}_{\text{Span}\{v, w\}} D$$

(2.6)

$$= \frac{1}{2} \operatorname{tr}_{\text{Span}\{b_1, b_2\}} D = \frac{1}{2} (b_1^* D b_1 + b_2^* D b_2)$$

for any orthonormal basis $\{b_1, b_2\}$ of $\text{Span}\{v, w\}$ and any matrix $D$ of suitable size, a fact we use frequently.

**Definition 2.4.** An indefinite scalar product $\langle \ldots \rangle_D$ induced by an indefinite matrix $D$ in the vector space $V = \mathbb{C}^n$ is a bilinear form with

(i) $\langle x, y \rangle_D = \langle y, x \rangle_D^*$ for all $x, y \in V$,

(ii) $\langle ax + by, z \rangle_D = a \langle x, z \rangle_D + b \langle y, z \rangle_D$ for all $x, y, z \in V$ and $a, b \in \mathbb{C}$.

If $D$ is singular and $x \in \ker(D)$, then we have $\langle x, y \rangle_D = 0$ for all $y \in \mathbb{C}^n$. In [11], a brief introduction on indefinite scalar products and Krein spaces can be found.

**Definition 2.5.** A vector $x \in V$ is said to be

- $D$-positive if $\langle x, x \rangle_D > 0$,
- $D$-neutral or $D$-isotropic if $\langle x, x \rangle_D = 0$,
- $D$-negative if $\langle x, x \rangle_D < 0$.

Accordingly, we call a subspace $\mathcal{U} \subset V$ $D$-positive if $\langle x, x \rangle_D > 0$ for all $x \in \mathcal{U}$; a $D$-negative subspace is defined analogously. In addition, we call a subspace indefinite if it contains both positive and negative vectors.

For a finite-dimensional subspace $\mathcal{U} = \text{Span}\{u_1, \ldots, u_m\} \subseteq V$, we define the gramian

$$R_{\mathcal{U}} = \begin{pmatrix} \langle u_1, u_1 \rangle_D & \cdots & \langle u_1, u_m \rangle_D \\ \vdots & \ddots & \vdots \\ \langle u_m, u_1 \rangle_D & \cdots & \langle u_m, u_m \rangle_D \end{pmatrix}.$$  

Since $D$ is hermitian, the gramian $R_{\mathcal{U}}$ is hermitian as well.

**Lemma 2.6.** [11] Let $u_1, \ldots, u_m \in V$ be linearly independent. Then $\mathcal{U} = \text{Span}\{u_1, \ldots, u_m\}$ is a positive subspace of $V$ if and only if $R_{\mathcal{U}}$ is positive definite.

**Proof.** Since $u_1, \ldots, u_m$ are linearly independent, any $x \in \mathcal{U}$ has a unique representation $x = [u_1, \ldots, u_m] y$ with $y \in \mathbb{C}^m$. Since $\langle x, x \rangle_D = x^* D x = y^*[u_1, \ldots, u_m]^* D [u_1, \ldots, u_m] y > 0$, it follows that $R_{\mathcal{U}}$ is positive definite if and only if $\mathcal{U}$ is positive. \qed
The idea of the proof for indefinite and negative definite subspaces is similar. Note that an indefinite real subspace contains at least one nonzero real isotropic vector, a fact that we will use extensively.

We now recall a well-known fact about quadrics (see for example Prop. 14.3.1 in [3], [9]).

**Proposition 2.7.** Let $D \in \mathbb{R}^{n \times n}$ with $n \geq 3$ be a symmetric matrix. Then the set $\mathcal{M} = \{x \in \mathbb{R}^n \mid x^* Dx = 0\}$ is path-connected. Furthermore, the set $\mathcal{M}_1 = \{x \in \mathbb{R}^n \mid \|x\| = 1, x^* Dx = 0\}$ consists of at most two path-connected components.

Let $x, y \in \mathbb{R}^n$, $n \geq 3$, with $\|x\| = \|y\| = 1$ be given. Then by proposition 2.7 it follows that with some arbitrary $a, b \in \mathbb{R}^n$, $a < b$, there exists a continuous mapping $z : [a, b] \rightarrow \mathbb{R}^n$ with the properties $\|z(t)\| = 1$ for all $t$, $z(a) = x$ and either $z(b) = y$ or $z(b) = -y$.

For further use we define the numerical range of a symmetric matrix.

**Definition 2.8.** Let $D \in \mathbb{R}^{n \times n}$ be symmetric. Then the numerical range of $D$ is defined as 
$$\text{nr } D := \{x^* Dx \mid x \in \mathbb{R}^n, \|x\| = 1\}.$$  

**2.4. A Special Case**

Here we present a special case for which the existence of a gyroscopic stabilizer is already known, see for example [15]. We apply Problem 2.2 and show how to actually construct a gyroscopic stabilizer $G$.

**Proposition 2.9.** ([15]) Let $\text{tr } D = 0$ and $Q = cI$ with $c > 0$ be a multiple of the unit matrix. Then there exists a gyroscopic stabilizer $G$.

**Proof.** We write down the proof only for odd $n = 2m + 1$, the proof for even $n$ is analogous.

Since $Q = cI$, conditions (b) and (c) from problem 2.2 are identical. Thus, we just need to make sure that (b) is satisfied.

Given a symmetric matrix $D$ with $\text{tr } D = 0$, it is known ([8], [10]), that there exists a real orthogonal matrix $U = [u_1, \ldots, u_n]$ such that the diagonal elements of $U^* DU$ are all identically zero. Define 
$$G = U \text{ diag}(0, \rho_1 G_0, \ldots, \rho_m G_0) U^*,$$

$$G_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $\rho_i \neq 0$ for all $i$ and $\rho_i \neq \rho_j$ for $i \neq j$. Then $u_1$ is an eigenvector for the eigenvalue zero of $G$ and we have $u_1^* Du_1 = 0$ by construction.
Now consider the pair of eigenvalues $\pm i\rho_i$ and let $x, y$ be a pair of orthonormal eigenvectors for $\pm i\rho_i$. Then we have

$$\text{Span}\{x, y\} = \text{Span}\{u_{2i}, u_{2i+1}\}.$$ 

By construction,

$$\text{tr}_{\text{Span}\{u_{2i}, u_{2i+1}\}} D = u_{2i}^*D u_{2i} + u_{2i+1}^*D u_{2i+1} = 0,$$

thus it follows that with (2.6) that $x^*Dx = y^*Dy = 0$. □
CHAPTER 3

Space Dimensions Two and Three

3.1. Existence of $G$ in Space Dimension Two

The existence of a gyroscopic stabilizer in space dimension two is well known ([2]). Here we demonstrate that also our newly derived sufficient condition implies the existence of a gyroscopic stabilizer.

**Proposition 3.1.** Let $D, Q \in \mathbb{R}^{2\times2}$ with $Q > 0$ and $\text{tr} D = 0$. Then $G = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ solves problem 2.2.

**Proof.** Let $v_1, v_2$ be a pair of normalized eigenvectors of $G$. By (2.6), we have $v_1^* D v_1 = v_2^* D v_2 = \frac{1}{2} \text{tr} D = 0$. Analogously, let $w_1, w_2$ be a pair of normalized eigenvectors of $G$, then $w_1^* Q D Q w_1 = w_2^* Q D Q w_2 = \frac{1}{2} \text{tr} Q D Q > 0$. □

Since there is basically only one gyroscopic stabilizer $G$, the more interesting question is whether we can actually compute the coefficient $\gamma$, such that with $\text{tr} D, \text{tr} Q D Q > 0$, the system $\ddot{x} + (D + \gamma G) \dot{x} + K x = 0$ is stable. We use the approach from [2, p. 196 ff], where the Hurwitz criterion [12, Cor. 3.4.71] is applied. The result can also be found in [14].

**Lemma 3.2.** [12, Cor. 3.4.71], [14] Let $D, Q \in \mathbb{R}^{2\times2}$ be given with $\text{tr} D, \text{tr} Q D Q > 0$ and let $\ddot{x} + D \dot{x} + K x = 0$ with $K = Q^{-2}$ be unstable. Then $\ddot{x} + (D + \gamma G) \dot{x} + K x = 0$ with $G$ as in proposition 3.1 is stable if and only if

$$\gamma^2 > - \text{tr} K - \det D + \frac{\text{tr} D \det K}{\text{tr} K^{-1} D} + \frac{\text{tr} K^{-1} D}{\text{tr} D}.$$
3.2. EXISTENCE OF $G$ IN SPACE DIMENSION THREE

**Proof.** Let $p(\lambda) = \det(\lambda^2 I + \lambda(D + \gamma G) + Q^{-2})$. Direct computation shows that
\[
p(\lambda) = \lambda^4 + \text{tr} D \lambda^3 + (\text{tr} K + \det D + \gamma^2) \lambda^2 + \text{tr}(K^{-1} D)\lambda + \det K = \sum_{i=0}^{4} a_i \lambda^i.
\]
For stability, it is necessary and sufficient that all principal minors of the associated Hurwitz matrix $M(p)$ are positive. Here we have
\[
M(p) = \begin{pmatrix}
    a_3 & a_1 & 0 \\
    a_4 & a_2 & a_0 \\
    0 & a_3 & a_1
\end{pmatrix}.
\]
With the criterion of Liénard-Chipart [12, Th. 3.4.73] it is even necessary and sufficient for stability that $a_i > 0$ for all $i$ and the principal minors of odd order are positive, which in this case gives us the condition $\det M(p) = a_3(a_1a_2 - a_0a_3) - a_1^2a_4 > 0$.

By assumption, $a_4, a_3, a_1, a_0 > 0$, thus if $\det M(p) > 0$ then $a_2 > 0$. Here, we get
\[
\det M(p) = \text{tr} D(\text{tr} K^{-1} D \det K(\text{tr} K + \gamma^2 + \det D) - \det K \text{tr} D) - (\text{tr} K^{-1} D \det K)^2 > 0.
\]
Solving for $\gamma^2$ yields
\[
\gamma^2 > -\text{tr} K - \det D + \frac{\text{tr} D \det K}{\text{tr} K^{-1} D} + \frac{\text{tr} K^{-1} D}{\text{tr} D}.
\]

Note that if $\ddot{x} + D\dot{x} + Kx = 0$ is already stable, it follows that adding a gyroscopic term in $\mathbb{R}^2$ never destabilizes the system, since $G$ only plays a role in $a_2$ and $M(p)$, and both terms become larger by adding $G$. We can even conclude that in $\mathbb{R}^2$, the conditions of Proposition 2.1 are not only sufficient, but also necessary for gyroscopic stabilizability.

3.2. Existence of $G$ in Space Dimension Three

**Proposition 3.3.** Let $D,Q \in \mathbb{R}^{3 \times 3}$ with $Q > 0$ and $\text{tr} D = 0$. Choose $\omega \in \mathbb{R}\setminus\{0\}$ and an orthonormal basis $\{u_1, u_2, u_3\}$ of $\mathbb{R}^3$ so that $u_1^* Du_1 = 0$. Then
\[
G = \begin{bmatrix} u_1, u_2, u_3 \end{bmatrix} \begin{pmatrix}
    0 & 0 & 0 \\
    0 & 0 & \omega \\
    0 & -\omega & 0
\end{pmatrix} [u_1, u_2, u_3]^T
\]
solves Problem 2.2.
Proof. By construction, \( v_1 = u_1 \) is an eigenvector for the eigenvalue 0 of \( G \), and we have \( v_1^* D v_1 = 0 \). Since \( G \) is skew-symmetric, the eigenvectors are orthogonal and the eigenvectors \( v_2, v_3 \) for the imaginary eigenvalues can be assumed to be complex conjugate and normalized. From (2.6) it follows that it suffices to show that \( \text{tr} \text{Span}\{v_2, v_3\} D \geq 0 \). From \( \text{Span}\{v_2, v_3\} = \text{Span}\{u_2, u_3\} \) one has

\[
\text{tr} \text{Span}\{v_2, v_3\} D = \text{tr} \text{Span}\{u_2, u_3\} D = 0 .
\]

Since \( QGQQ^{-1} v_1 = 0 \), it follows that \( w_1 = \frac{Q^{-1} v_1}{\|Q^{-1} v_1\|} \) is an eigenvector for the eigenvalue 0 of \( QGQ \). Since \( w_1^* QDQ w_1 = 0 \), we have \( \text{tr} \text{Span} w_1 D = 0 \). Let \( w_2, w_3 \) denote the other eigenvectors of \( QDQ \). We use (2.6) again and get

\[
\text{tr} \text{Span}\{w_2, w_3\} QDQ = \text{tr} QDQ - \text{tr} w_1 QDQ = \text{tr} QDQ \geq 0 ,
\]

which completes the proof. \( \square \)

As seen in the previous section, in \( \mathbb{R}^2 \), any gyroscopic stabilizer satisfies the conditions provided by Proposition 2.1, basically there is just one. In \( \mathbb{R}^3 \), the situation is already different, there exist gyroscopic stabilizers whose eigenvectors do not satisfy the mentioned conditions, as the following example shows. But still, as shown with proposition

Example 3.4. Consider the system

\[
0 = \ddot{x} + \begin{pmatrix}
-4 & -2 & 3 \\
-2 & 8 & -12 \\
3 & -12 & 4
\end{pmatrix} \dot{x} + \begin{pmatrix}
5 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 5
\end{pmatrix} x .
\]

We have \( \text{tr} D = 8 \) and \( \text{tr} QDQ = 8 \), and the system is unstable, since the maximum of the real parts of the eigenvalues is 6.5803. Now, there exists a gyroscopic stabilizer

\[
G = \begin{pmatrix}
0 & -1 & -11 \\
1 & 0 & -26 \\
11 & 26 & 0
\end{pmatrix} ,
\]

such that the largest real part of the eigenvalues of \( \ddot{x} + (D + G) \dot{x} + Q^{-2} x = 0 \) is about \(-0.000032 \). But for the eigenvector \( v = (26, -11, 1) \) with eigenvalue zero, we have \( v^* D v = -168 < 0 \). Thus, the matrix \( G \) does not result from our construction in Proposition 3.3. This shows that in \( \mathbb{R}^3 \), not every gyroscopic stabilizer satisfies the conditions given by Problem 2.2.
3.3. The Set of Gyroscopic Stabilizers in $\mathbb{R}^3$

We first investigate the solutions of the *shifted* problem 2.2, which themselves are a subset of the gyroscopic stabilizers that satisfy the conditions of Proposition 2.1. Thus, at first we consider pairs $D, Q$ such that $\text{tr} \ D = 0, \text{tr} \ QDQ \geq 0$.

The construction of $G$ in Proposition 3.3 actually just depends on finding an isotropic vector with respect to $D$. Let

$$G = \begin{pmatrix}
0 & c & -b \\
-c & 0 & a \\
 b & -a & 0
\end{pmatrix}$$

be a skew symmetric matrix. Then $g = (a, b, c)^T$ is an eigenvector of $G$ for the eigenvalue zero. This shows that the specific choice of $\{u_2, u_3\}$ in Proposition 3.3 does not play any role in the construction of $G$. The actual form of $G$ is already completely determined just by $u_1$, but the question of how to scale $G$ remains open.

The question of how to scale $G$ with a suitable $\gamma$ such that the system actually becomes stable is still open, we just know that $\gamma$ exists. Therefore, here we consider $g$ to be normalized, and instead of speaking about the matrix $G$, we now just talk about the associated vector $g$.

For a given symmetric $D$ with $\text{tr} \ D = 0$, we define the sets

$$S_0 = \{g \mid g^*Dg = 0, \|g\| = 1\} ,$$
$$S_+ = \{g \mid g^*Dg > 0, \|g\| = 1\} ,$$
$$S_- = \{g \mid g^*Dg < 0, \|g\| = 1\} .$$

Throughout our work, a quadric $\{g \mid g^*Dg = 0\}$, where $D$ is symmetric, will be referred to as the cone generated by $D$. Clearly $S_0 \subset \{g \mid g^*Dg = 0\}$. If $g \in S_0$, then the associated $G$ satisfies the conditions of problem 2.2. The shape of the sets $S_0, S_+, S_-$ is determined by the eigenvalue structure of $D$. Let $v_-(D), v_0(D), v_+(D)$ denote the number of eigenvalues of $A$ counting algebraic multiplicities that are, respectively, negative, zero, and positive. Define the inertia of $D$ as $\text{In}(D) = (v_-(D) \ v_0(D) \ v_+(D))$.

We assume $D$ to be nonzero, then from $\text{tr} \ D = 0$ it follows that $\text{In}(D)$ takes either the form $(2,0,1)$ or $(1,1,1)$ or $(1,0,2)$.

The set $S_0$ is the intersection of the cone defined by $\{x \mid x^*Dx = 0\}$ and the unit sphere, and according to [9], the set $\{x \mid x^*Dx = 0\}$ is path-connected. The intersection with the unit sphere consists of one connected component in the case that $\text{In}(D) = (1,0,1)$, or two connected components if either $\text{In}(D) = (2,0,1)$ or $\text{In}(D) = (1,0,2)$.

Figure 3.1 shows the cone which is determined by the isotropic vectors
Figure 3.1. Intersection of unit sphere with cone \( S_0 \)

Figure 3.2. Sets \( S_+ \) (blue) and \( S_0 \) (red)
of the matrix $D = \text{diag}(1 \ 1 \ -2)$. Exactly one eigenvalue is negative and two are positive, thus each of the sets $S_-$ and $S_0$ splits into two connected components, while $S_+$ consists of exactly one connected component. Figure 3.2 shows the set $S_+$ visualized by blue dots, and the set $S_0$ in red; the latter is the intersection line of the cone in Figure 3.1 with the unit sphere.

In Figure 3.3, several possible shapes for $S_0$ depending on the given $D$ can be seen. The red set is for $D = \text{diag}(1 \ 1 \ -2)$, the green set for $D = \text{diag}(6 \ 1 \ -7)$ and the black set for $D = \text{diag}(1 \ 0 \ -1)$. While the first two sets in green and red are given by intersections of actual cones with the unit sphere, the black set arises from the degenerate case, where $S_0$ consists of exactly one connected component.

We now come to the original problem and consider pairs $D, Q$ with $\text{tr} \ D > 0, \text{tr} \ QDQ > 0$, as in Proposition 2.1. Consider the pair $D, Q$ with

$$D = \begin{pmatrix} 1.3 & 0 & 0 \\ 0 & 1.3 & 0 \\ 0 & 0 & -1.7 \end{pmatrix}, \quad Q = \begin{pmatrix} 7 & 3 & 0 \\ 3 & 4 & 2 \\ 0 & 2 & 6 \end{pmatrix}. $$

While in the situation $\text{tr} \ D = 0$ it was enough to identify the $g$ with $g^*Dg = 0$ in order to find the gyroscopic stabilizers, we now have
to consider the conditions $v^* Dv > 0$ for all eigenvectors of $G$ and $w^* QDQw > 0$ for all eigenvectors of $QGQ$ separately.

Consider a $g$ such that $0 < g^* Dg < \text{tr}(D)$. Let $v_2, v_3$ be a normalized pair of eigenvectors for the complex eigenvalues of the associated $G$. Then since $v_2^* Dv_2 = v_3^* Dv_3$ and $g^* Dg + v_2^* Dv_2 + v_3^* Dv_3 = \text{tr}(D)$ it follows that $0 < v_2^* Dv_2, v_3^* Dv_3 < \text{tr}(D)$. Figure 3.4 shows in blue the set of $g$ such that the conditions on the eigenvectors of the associated $G$ are satisfied. The boundaries of the blue set are determined by the cones defined by $x^* Dx = 0$ and $x^* (D - \text{tr}(D)I)x = 0$.

The two red lines show those $g$ for which $g^* (D - \frac{\text{tr}(D)}{3}I)g = 0$, thus they are the solutions to the shifted problem as formulated in 2.2 and coincide with the set $S_0$ as in Figure 3.2. By construction, the red set is completely contained in the blue set.

Now consider the eigenvectors of $QGQ$. As in Proposition 3.3, we know that $w_1 = \frac{Q^{-1}g}{\|Q^{-1}g\|}$ is an eigenvector for the eigenvalue zero of $QGQ$, and we denote normalized eigenvectors for the complex eigenvalues with $w_2, w_3$. Let $0 < w_1^* QDQw_1 = \frac{g^* Dg}{\|Q^{-1}g\|} < \text{tr}(QDQ)$. With $w_2^* QDQw_2 = w_3^* QDQw_3$ and

$$w_1^* QDQw_1 + w_2^* QDQw_2 + w_3^* QDQw_3 = \text{tr}(QDQ)$$
it follows that

$$0 < w_2^* Q D Q w_2, w_3^* Q D Q w_3 < \text{tr}(Q D Q),$$

satisfying the eigenvector conditions on $Q G Q$. In Figure 3.5, the set of the $g$ such that the eigenvector conditions for $Q G Q$ are satisfied, is shown in blue. The boundaries of the blue set are determined by the two cones defined by $x^* D x = 0$ and $x^* (Q D Q - \text{tr}(Q D Q) I) x = 0$. The red set is the same as in the previous figures and again completely contained in the blue set by construction.

Since the eigenvector conditions on $G$ and $Q G Q$ need to be satisfied simultaneously, the set of the gyroscopic stabilizers is given by the intersection of the sets in Figures 3.4 and 3.5. This intersection is the blue set in Figure 3.6, and it is non-empty, as can be seen by the fact that the red line, consisting of those $g$ for which $g^* (D - \frac{\text{tr} D}{3} I) g = 0$, is contained in the intersection. Altogether, in Figure 3.6 all $g$ such that the conditions in Proposition 2.1 are satisfied, are plotted in blue. This blue set is bounded by exactly three cones given by $g^* D g = 0$, $x^* (D - \text{tr}(D) I) x = 0$ and $x^* (Q D Q - \text{tr}(Q D Q) I) x = 0$. But Proposition 2.1 just provided a sufficient condition for the existence of a gyroscopic stabilizer $G$ (or $g$), thus one can assume that actually there are more gyroscopic stabilizers than we constructed so far.
Figure 3.6. $x$ with $0 < x^* Dx < \text{tr}(D)$ and $0 < \frac{x^* Dx}{\|Q^{-1}x\|^2} < \text{tr}(QDQ)$

Figure 3.7 shows again in blue the $g$ that result from Proposition 2.1 and in red the $g$ such that our sufficient conditions are violated but still the associated $G$ is a gyroscopic stabilizer. The red $g$ were found using Matlab: we directly computed the eigenvalues of the matrix

$$M = \begin{bmatrix} 0 & I \\ -K & -D - \gamma G \end{bmatrix}$$

with coefficients $\gamma$ in the range from zero to 2000; in the case the matrix was stable, the associated $g$ was plotted.
3.3. THE SET OF GYROSCOPIC STABILIZERS IN $\mathbb{R}^3$

Figure 3.7. $g$ resulting from Proposition 2.1 (blue), other stabilizing $g$ (red)
CHAPTER 4

Space Dimension Four

In the three-dimensional case, we exploited the fact that the skew-symmetric matrices $G, QGQ \in \mathbb{R}^{3 \times 3}$ both have a zero eigenvalue, and the corresponding eigenvectors are related via multiplication with $Q^{-1}$. Now we construct $G \in \mathbb{R}^{4 \times 4}$ with a double eigenvalue zero, allowing us to identify spaces containing eigenvectors of $QGQ$. Then we use a perturbation argument to move the zero eigenvalues along $i\mathbb{R}$.

4.1. Perturbation of Eigenvalues and Eigenvectors

**Proposition 4.1.** For some $\delta \in \mathbb{R}$ and an orthogonal matrix $Z = [z_1, z_2, z_3, z_4] \in \mathbb{R}^{4 \times 4}$ let

$$G_\delta = Z \begin{pmatrix} 0 & \delta & 0 & 0 \\ -\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} Z^T. \quad (4.1)$$

If for some $\tau > 0$ we have

(i) $\text{tr} \text{Span}\{z_1, z_2\} D_\tau \geq 0$ and $\text{tr} \text{Span}\{z_3, z_4\} D_\tau \geq 0$,

(ii) $\text{tr} Q^{-1} \text{Span}\{z_1, z_2\} QD_\tau Q \geq 0$ and $\text{tr} Q \text{Span}\{z_3, z_4\} QD_\tau Q \geq 0$,

then there exists $\delta \neq 0$ so that (a), (b), and (c) in Proposition 2.1 hold for $G_\delta$ and $\tau/2$.

The idea behind the above statement is as follows. The matrix $G_0$ is constructed such that after a small perturbation $\delta$ of the double zero eigenvalue of $G$, the conditions (a), (b) and (c) in Proposition 2.1 are satisfied, which is ensured by the assumptions (i) and (ii). Then, by changing the value of $\delta$ by a tiny amount, the space associated with the zero eigenvalues of $QG_\delta Q$ is changed a bit as well, but still the conditions posed on the eigenvectors of $QG_\delta Q$ remain satisfied as long as the change in $\delta$ remains sufficiently small.

**Proof.** By continuity of eigenvalues, it is clear that (a) in Proposition 2.1 holds for small $|\delta| \neq 0$.

Using (2.6) and the structure of $G_\delta$ we conclude that assumption (b) is equivalent to $\text{tr} \text{Span}\{z_1, z_2\} D_\tau \geq 0$ and $\text{tr} \text{Span}\{z_3, z_4\} D_\tau \geq 0$ for all
4.2. Construction Of $G$ Via Eigenvectors Of $Q$

$\delta \in \mathbb{R} \setminus \{0\}$.

To verify (c), note that $QG_\delta Q$ has two conjugate pairs of imaginary eigenvalues, which we denote by $\pm \lambda_\delta$ and $\pm \mu_\delta$. These depend continuously on $\delta$ (where $\lambda_0 = 0$). The same is true (e.g. [19]) for the invariant subspaces

$$V_\lambda(\delta) := \text{Ker} \left( (QG_\delta Q)^2 + |\lambda_\delta|^2 I \right),
V_\mu(\delta) := \text{Ker} \left( (QG_\delta Q)^2 + |\mu_\delta|^2 I \right).$$

By assumption for $\delta = 0$ and $\eta = \tau$ we have with $D = D_\tau + \tau P$:

$$\text{tr}_{V_\lambda(\delta)} QDQ \geq \eta \text{tr}_{V_\lambda(\delta)} PQ > 0,$n$$
$$\text{tr}_{V_\mu(\delta)} QDQ \geq \eta \text{tr}_{V_\mu(\delta)} PQ > 0.$$

By continuity, the same holds for $\eta = \tau/2$ and sufficiently small $\delta$. Together with (2.6) this completes the proof.

Thus, we can relax the conditions in Problem 2.2 slightly and reformulate it, such that it relates to two-dimensional spaces containing pairs of complex conjugate eigenvectors instead of relating directly to eigenvectors.

**Problem 4.2.** For symmetric matrices $D, Q \in \mathbb{R}^{4 \times 4}$ satisfying $Q > 0$, $\text{tr} \ D = 0$ and $\text{tr} \ QDQ \geq 0$, find $G_0$ as in (4.1) such that $\text{tr}_{\text{Ker} \ G_0} D = 0$ and $\text{tr} \ QDQ \geq \text{tr}_{\text{Ker} \ QG_0 Q} QDQ \geq 0$.

Let $\{u_1, u_2, u_3, u_4\}$ denote an orthonormal set of eigenvectors of $Q$ with corresponding eigenvalues $\lambda_k > 0$. We consider the numbers $u_i^* Du_i$. Since $\text{tr} \ D = 0$ we either have $u_i^* Du_i = 0$ for all $i$ or some of these numbers are positive and some are negative. In the following propositions, we make a complete distinction between all possible cases.

### 4.2. Construction Of $G$ Via Eigenvectors Of $Q$

**Proposition 4.3.** Assume that for some ordering of the $u_i$ the spaces $U_{12} = \text{Span} \{u_1, u_2\}$ and $U_{34} = \text{Span} \{u_3, u_4\}$ are both $D$-indefinite or $D$-singular. Then there exists a skew-symmetric $G_0$ solving Problem 4.2.

The idea of the following proof is to construct a two-dimensional space via identifying a basis consisting of $D$-neutral vectors. Note that a $D$-neutral basis does not imply that each vector in the space is $D$-neutral. The fact that $U_{12}$ and $U_{34}$ are orthogonal and remain orthogonal under multiplication with $Q^{-1}$ is used.

**Proof.** By our assumptions on $U_{12}$ and $U_{34}$, there exist normalized vectors $z_1 \in U_{12}$ and $z_2 \in U_{34}$ with $\langle z_1, z_1 \rangle_D = \langle z_2, z_2 \rangle_D = 0$. Let $Z = [z_1, z_2, z_3, z_4] \in \mathbb{R}^{4 \times 4}$ be orthogonal and define $G_0$ as in (4.1).
Then \( \text{Span}\{z_1, z_2\} = \text{Ker} G_0 \) and \( \text{tr}_{\text{Ker} G_0} D = 0 \).

Again by construction, \( \{Q^{-1}z_1, Q^{-1}z_2\} \) is an orthogonal basis of the space \( \text{Ker} QG_0Q \) and
\[
\langle Q^{-1}z_1, Q^{-1}z_2 \rangle_{QDQ} = \langle Q^{-1}z_2, Q^{-1}z_2 \rangle_{QDQ} = \langle z_1, z_1 \rangle_D = \langle z_2, z_2 \rangle_D = 0.
\]

Hence \( \text{tr}_{\text{Ker} QG_0Q} QDQ = 0 \), i.e. \( G_0 \) solves Problem 4.2. \( \square \)

Note, that the assumptions of Prop. 4.3 may only fail, if three of the numbers \( u_i^* Du_i \) are positive and one is negative, or vice versa.

**Proposition 4.4.** Assume that \( u_i^* Du_i < 0 \) for exactly one fixed \( i \in \{1, 2, 3, 4\} \) and \( u_k^* Du_k > 0 \) for all \( k \neq i \).

Assume further that
\[
K_{mn} = \begin{pmatrix}
  u_m^* Du_m & u_m^* Du_n \\
  u_n^* Du_m & u_n^* Du_n
\end{pmatrix}
\]
be nonnegative definite for any choice of distinct \( m, n \in \{1, 2, 3, 4\} \backslash \{i\} \).

Then there exists a skew-symmetric \( G_0 \) solving Problem 4.2.

**Remark 4.5.** We can fix \( j, m, n \) arbitrarily provided that we have \( \{i, j, m, n\} = \{1, 2, 3, 4\} \). Thus we may assume \( \lambda_j, \lambda_m, \lambda_n \) to be ordered arbitrarily. For any such choice,
\[
K_{ij} = \begin{pmatrix}
  u_i^* Du_i & u_i^* Du_j \\
  u_j^* Du_i & u_j^* Du_j
\end{pmatrix}
\]
necessarily is indefinite. If \( K_{mn} \) was not nonnegative definite, then it would be indefinite or singular, and we could apply Prop. 4.3.

**Proof.** (of Proposition 4.4) We denote the eigenvalues of \( K_{ij} \) by \( \mu_i \), \( \mu_j \) and those of \( K_{mn} \) by \( \mu_m \) and \( \mu_n \), where in accordance with our assumptions \( \mu_i < 0 \), \( \mu_j > 0 \), and \( \mu_n \geq \mu_m \geq 0 \). Since \( \text{tr} D = \mu_i + \mu_j + \mu_m + \mu_n = 0 \), we have \( \mu_i \leq -\mu_m - \mu_n \).

Thus \( [\mu_j, \mu_i] \supset [-\mu_m, -\mu_n] \), i.e. (e.g. [4, Ex. I.2.9])
\[
\{x_i^* K_{ij} x_1 \mid \|x_1\| = 1\} \supset \{-x_2^* K_{mn} x_2 \mid \|x_2\| = 1\}.
\]

Thus, for each normalized \( z_2 \in \text{Span}\{u_m, u_n\} \) there is a normalized \( z_1 = z_1(z_2) \in \text{Span}\{u_i, u_j\} \) so that
\[
z_1^* Dz_1 = -z_2^* Dz_2. \tag{4.2}
\]

We can choose \( z_1 = f(\alpha) = \cos(\alpha) \hat{u}_i + \sin(\alpha) \hat{u}_j \) with \( \alpha \in [0, \pi/2] \), where \( \hat{u}_i, \hat{u}_j \) are orthonormal and \( [\hat{u}_i \hat{u}_j]^* D[\hat{u}_i \hat{u}_j] = \text{diag}(\mu_i, \mu_j) \). Then the mapping \( g : \alpha \mapsto f(\alpha)^* Df(\alpha) \) is continuous and strictly monotonically increasing and therefore continuously invertible. Since the mapping \( z_2 \mapsto z_2^* Dz_2 \) is also continuous, we can assume the mapping \( z_2 \mapsto z_1(z_2) = z_1(g^{-1}(z_2^* Dz_2)) \) to be continuous.
4.2. CONSTRUCTION OF $G$ VIA EIGENVECTORS OF $Q$

We now consider three different cases.

(i) Assume $\lambda_i = \max_k \lambda_k$. Since

$$0 \leq \text{tr } QDQ = \sum_{k=1}^{4} u_k^* QDQ u_k$$

$$= \sum_{k=1}^{4} \lambda_k^2 u_k^* D u_k \leq \lambda_i^2 \sum_{k=1}^{4} u_k^* D u_k = 0,$$

it follows that $\lambda_i = \lambda_k$ for all $k$, i.e. $Q = \lambda_k I$. But this case is solved by Proposition 2.9.

(ii) Let $\min_k \lambda_k < \lambda_i < \max_k \lambda_k$ and assume without loss of generality that $\lambda_m \leq \lambda_i, \lambda_j \leq \lambda_n$. Then

$$\lambda_m^{-1} = \|Q^{-1} u_m\| \geq \|Q^{-1} z_1(u_m)\|,$$

$$\lambda_n^{-1} = \|Q^{-1} u_n\| \leq \|Q^{-1} z_1(u_n)\|.$$  

By the mean value theorem there exists a normalized $z_2 = \cos(\beta) u_m + \sin(\beta) u_n \in \text{Span}\{u_m, u_n\}$ so that

$$\|Q^{-1} z_2\| = \|Q^{-1} z_1(z_2)\|.  \tag{4.3}$$

We extend $z_1 = z_1(z_2)$ and $z_2$ to an orthogonal matrix $Z = [z_1, \ldots, z_4]$ and define $G_0$ as in (4.1).

Then $\text{Span}\{z_1, z_2\} = \text{Ker } G_0$ and $\text{tr}_{\text{Ker } G_0} D = 0$. Moreover, we have that $\{Q^{-1} z_1, Q^{-1} z_2\}$ is an orthogonal basis of $\text{Ker } QG_0 Q$ and (using (4.2) and (4.3)) we have

$$\text{tr}_{\text{Ker } QG_0 Q} QDQ = \frac{z_1^* D z_1}{\|Q^{-1} z_1\|^2} + \frac{z_2^* D z_2}{\|Q^{-1} z_2\|^2} = 0.$$

Hence $G_0$ solves Problem 4.2.

(iii) Let $\lambda_i = \min_k \lambda_k$ and assume $\lambda_m \geq \lambda_j$. Let $z_2 = u_m$ and $z_1 = z_1(z_2) \in \text{Span}\{u_i, u_j\}$. Then

$$\lambda_i^{-2} \geq \|Q^{-1} z_1\|^2 \geq \lambda_j^{-2} \geq \lambda_m^{-2} = \|Q^{-1} z_2\|^2.$$  

With $G_0$ again as in (4.1), we have $\text{tr}_{\text{Ker } G_0} D = 0$ and

$$\text{tr}_{\text{Ker } QG_0 Q} QDQ = \frac{z_1^* D z_1}{\|Q^{-1} z_1\|^2} + \frac{z_2^* D z_2}{\|Q^{-1} z_2\|^2} \geq 0,$$

because $-z_1^* D z_1 = z_2^* D z_2 = u_m^* D u_m$. On the other hand

$$\text{tr}_{\text{Ker } QG_0 Q} QDQ = \frac{z_1^* D z_1}{\|Q^{-1} z_1\|^2} + \frac{z_2^* D z_2}{\|Q^{-1} z_2\|^2}$$

$$\leq u_m^* D u_m (\lambda_m^2 - \lambda_i^2)$$

$$\leq \sum_{k=1}^{4} \lambda_k^2 u_k^* D u_k = \text{tr } QDQ.$$
Again $G_0$ solves Problem 4.2. 

Finally we consider the case where three of the numbers $u_i^*Du_i$ are negative and one is positive.

**Proposition 4.6.** Assume that $u_i^*Du_i > 0$ for exactly one fixed $i \in \{1, 2, 3, 4\}$ and $u_k^*Du_k < 0$ for all $k \neq i$. Assume further that

$$K_{mn} = \begin{pmatrix} u_m^*Du_m & u_m^*Du_n \\ u_n^*Du_m & u_n^*Du_n \end{pmatrix}$$

be nonpositive definite for any choice of distinct $m, n \in \{1, 2, 3, 4\}\setminus\{i\}$. Then there is a skew-symmetric $G_0$ solving Problem 4.2.

The proof is essentially the same as in Proposition 4.4.

**Proof.** We denote the eigenvalues of $K_{ij}$ by $\mu_i, \mu_j$ and those of $K_{mn}$ by $\mu_m$ and $\mu_n$, where in accordance with our assumptions $\mu_i > 0$, $\mu_j < 0$, and $\mu_n \leq \mu_m \leq 0$. Since $\text{tr} D = \mu_i + \mu_j + \mu_m + \mu_n = 0$, we have

$$\mu_i \geq -\mu_m - \mu_n.$$ 

Thus $[\mu_j, \mu_i] \supset [-\mu_m, -\mu_n]$, i.e. (e.g. [4, Ex. I.2.9])

$$\{x_1^*K_{ij}x_1 \mid \|x_1\| = 1\} \supset \{-x_2^*K_{mn}x_2 \mid \|x_2\| = 1\}.$$ 

Thus, for each normalized $z_2 \in \text{Span}\{u_m, u_n\}$ there is a normalized $z_1 = z_1(z_2) \in \text{Span}\{u_i, u_j\}$ so that

$$z_1^*Dz_1 = -z_2^*Dz_2.$$ 

We can choose $z_1 = f(\alpha) = \cos(\alpha)\tilde{u}_i + \sin(\alpha)\tilde{u}_j$ with $\alpha \in [0, \pi/2]$, where $\tilde{u}_i, \tilde{u}_j$ are orthonormal and $[\tilde{u}_i, \tilde{u}_j]^*D[\tilde{u}_i, \tilde{u}_j] = \text{diag}(\mu_i, \mu_j)$, where $\mu_i > 0 > \mu_j$. Then the mapping $g : \alpha \mapsto f(\alpha)^*Df(\alpha)$ is continuous and strictly monotonically decreasing on $[0, \pi/2]$ and therefore continuously invertible. Since the mapping $z_2 \mapsto z_2^*Dz_2$ is also continuous, we can assume the mapping $z_2 \mapsto z_1(z_2) = z_1(g^{-1}(z_2^*Dz_2))$ to be continuous.

We now consider three different cases.

(i) Assume $\lambda_i = \min_k \lambda_k$. Since

$$0 \leq \text{tr} QDQ = \sum_{k=1}^4 u_k^*QDQu_k = \sum_{k=1}^4 \lambda_k^2 u_k^*Du_k \leq \lambda_i^2 \sum_{k=1}^4 u_k^*Du_k = 0,$$

it follows that $\lambda_i = \lambda_k$ for all $k$, i.e. $Q = \lambda_i I$. But this case is solved by Proposition 2.9.
(ii) Let min \( \lambda_k \) \( \lambda_k < \lambda_i < \max \lambda_k \) and assume without loss of generality that \( \lambda_m = \min \lambda_i, \lambda_j = \max \lambda_i \). Then
\[
\lambda_m^{-1} = \|Q^{-1} u_m\| \geq \|Q^{-1} z_1(u_m)\|, \\
\lambda_n^{-1} = \|Q^{-1} u_n\| \leq \|Q^{-1} z_1(u_n)\|.
\]
By the mean value theorem there exists a normalized \( z_2 = \cos(\beta)u_m + \sin(\beta)u_n \in \text{Span}\{u_m, u_n\} \) so that
\[
(4.5) \quad \|Q^{-1} z_2\| = \|Q^{-1} z_1(u_n)\|.
\]
We extend \( z_1 = z_1(z_2) \) and \( z_2 \) to an orthogonal matrix \( Z = [z_1, \ldots, z_4] \) and define \( G_0 \) as in (4.1).
Then we have \( \text{Span}\{z_1, z_2\} = \ker G_0 \) and \( \text{tr}_{\ker G_0} D = 0 \). Moreover, \( \{Q^{-1} z_1, Q^{-1} z_2\} \) is an orthogonal basis of \( \ker QG_0Q \) and (using (4.4) and (4.5)) we have
\[
\text{tr}_{\ker QG_0Q} QDQ = \frac{z_1^* D z_1}{\|Q^{-1} z_1\|^2} + \frac{z_2^* D z_2}{\|Q^{-1} z_2\|^2} = 0.
\]
Hence \( G_0 \) solves Problem 4.2.
(iii) Let \( \lambda_i = \max \lambda_k \) and assume \( \lambda_m = \lambda_j \).
Let \( z_2 = u_m \) and \( z_1 = z_1(z_2) \in \text{Span}\{u_i, u_j\} \). Then
\[
\lambda_i^{-1} \leq \|Q^{-1} z_1\|^2 \leq \lambda_j^{-1} \leq \lambda_m^{-2} = \|Q^{-1} z_2\|^2.
\]
With \( G_0 \) again as in (4.1), we have \( \text{tr}_{\ker G_0} D = 0 \) and
\[
\text{tr}_{\ker QG_0Q} QDQ = \frac{z_1^* D z_1}{\|Q^{-1} z_1\|^2} + \frac{z_2^* D z_2}{\|Q^{-1} z_2\|^2} \geq 0,
\]
because \(-z_1^* D z_1 = z_2^* D z_2 = u_m^* D u_m\). On the other hand
\[
\text{tr}_{\ker QG_0Q} QDQ \leq \frac{z_1^* D z_1}{\|Q^{-1} z_1\|^2} + \frac{z_2^* D z_2}{\|Q^{-1} z_2\|^2} \\
\leq u_m^* D u_m (\lambda_m^2 - \lambda_i^2) \\
= \lambda_m^2 u_m^* D u_m - \lambda_i^2 u_m^* D u_m \\
= \lambda_m^2 u_m^* D u_m + \lambda_i^2 (u_i^* D u_i + u_j^* D u_j + u_n^* D u_n) \\
\leq \sum_{k=1}^4 \lambda_i^2 u_k^* D u_k = \text{tr} QDQ.
\]
Again \( G_0 \) solves Problem 4.2.
\]
Since all appearing cases have been considered, we have the following result.

**Theorem 4.7.** Let \( D \) and \( Q > 0 \) be in \( \mathbb{R}^{4 \times 4} \) with \( \text{tr} D, \text{tr} QDQ > 0 \). Then there exists a gyroscopic stabilizer.
4.3. Examples

Here, we provide examples for some of the cases discussed in the previous section and show how to do an explicit construction. Since $Q$ is positive definite and therefore diagonalizable, for simplicity we directly take examples where $Q$ is diagonal. All computations are performed in Matlab.

**Example 4.8.** We construct an example as in Proposition 4.3. Consider the system

$$
\ddot{x} + D\dot{x} + Q^{-2}x = 0,
$$

where

$$
D = \begin{pmatrix}
1 & -3 & 0 & -2 \\
-3 & -2 & -1 & -3 \\
0 & -1 & -3 & 3 \\
-2 & -3 & 3 & 8
\end{pmatrix}, \quad Q = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 4
\end{pmatrix}.
$$

The system is unstable since there is an eigenvalue with positive real part $4.7740$. We have

$$
\frac{\text{tr}D}{n} = 1 < \frac{94}{30} = \frac{\text{tr}QDQ}{\text{tr}Q^2},
$$

thus we take $\tau = 1$ and $P = I$ according to Proposition 2.1 and get

$$
D_\tau = \begin{pmatrix}
0 & -3 & 0 & -2 \\
-3 & -3 & -1 & -3 \\
0 & -1 & -4 & 3 \\
-2 & -3 & 3 & 7
\end{pmatrix}.
$$

Now $\text{Span}\{e_1, e_2\}$ and $\text{Span}\{e_3, e_4\}$ are both either $D$-indefinite or singular, thus in each of them there exists a $D$-isotropic vector.

In $\text{Span}\{e_1, e_2\}$, a normalized isotropic vector can be taken as $z_1 = e_1$.

We diagonalize $K_{34} = [e_3\ e_4]^*D[e_3\ e_4] = U_{34}V_{34}U_{34}^*$ and get

$$
U_{34} = \begin{pmatrix}
-0.9690 & 0.2471 \\
0.2471 & 0.9690
\end{pmatrix}, \quad V_{34} = \begin{pmatrix}
-4.7650 & 0 & 0 \\
0 & 0 & 7.7650
\end{pmatrix},
$$

thus an isotropic vector is given by

$$
\tilde{z}_2 = U\begin{pmatrix}
\sqrt{7.7650} \\
-\sqrt{4.7650}
\end{pmatrix}, \quad z_2 = [e_3\ e_4]_{\|z_2\|} = [e_3\ e_4]\begin{pmatrix}
0 \\
0 \\
-0.9152 \\
-0.4030
\end{pmatrix}.
$$

We complete $z_1, z_2$ to an orthonormal basis and according to Proposition 4.1 we put

$$
G_\delta = Z\begin{pmatrix}
0 & \delta & 0 & 0 \\
-\delta & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}Z^T,
$$
4.3. EXAMPLES

Figure 4.1. Plot of the real parts of the eigenvalues of \( \ddot{x} + (D + \alpha G_{0.0001}) \dot{x} + K x = 0 \) with \( \alpha \) on the horizontal axis

where the matrix \( Z \) is chosen as

\[
Z = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & -0.9152 & 0 & -0.4030 \\
0 & -0.4030 & 0 & 0.9152
\end{pmatrix}.
\]

According to our construction, we just know that \( \alpha G_\delta \) stabilizes our system for small \( \delta \) and large \( \alpha \). We put \( \delta = 0.0001 \) and plot the real parts of the eigenvalues of the system versus \( \alpha \). Via Matlab we find that the system is stable for all \( \alpha \delta > 11.7564 \) as can be seen in Figure 4.1. While our construction is independent of the scaling or norm of \( Q \), it should be noted that the actual \( \alpha_\delta \) does depend on it.

Figure 4.2 shows the dependance of \( \alpha \) on \( \delta \): the green area signifies the parameter combinations \( (\alpha, \delta) \), for which the system is unstable, the blue area signifies stability. Our construction just predicted that small \( \delta \) and large \( \alpha \) yield stability, but the actual stability domain (the blue area) appears to be a lot larger.

Also interesting is the fact that the stability domain as computed via MATLAB is not symmetric around the horizontal axis, which means that replacing \( \delta \) by \( -\delta \) might alter the stability properties of the system, a fact which cannot be deduced from our construction. Note that \( G_\delta \neq -G_{-\delta} \), since only in one eigenvalue block the sign is changed.
Figure 4.2. The blue area indicates parameter combinations \((\alpha, \delta)\) that result in stability of \(\ddot{x} + (D + \alpha G_\delta) \dot{x} + Kx = 0\), the green area indicates instability.

**Example 4.9.** Here we will do a construction as in Proposition 4.4 (iii). Consider the system \(\ddot{x} + D\dot{x} + Q^{-2}x = 0\), where

\[
D = \begin{pmatrix}
-13 & -1 & 2 & 2 \\
-1 & 9 & 2 & 5 \\
2 & 2 & 2 & 1 \\
2 & 5 & 1 & 6
\end{pmatrix}, \quad Q = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 4
\end{pmatrix}.
\]

The largest real part of the eigenvalues of the system is 13.5055, so the system is unstable. With \(\frac{\text{tr} D}{n} = 1 < \frac{137}{30} = \frac{\text{tr} QDQ}{\text{tr} Q^2}\) we put \(\tau = 1\) and \(P = I\) according to Proposition 2.1 and get

\[
D_\tau = \begin{pmatrix}
-14 & -1 & 2 & 2 \\
-1 & 8 & 2 & 5 \\
2 & 2 & 1 & 1 \\
2 & 5 & 1 & 5
\end{pmatrix}.
\]

The space \(\text{Span}\{e_1, e_j\}\) is \(D\)-indefinite for any \(j = 2, \ldots, 4\) while for any \(p, q \neq 1\) the space \(\text{Span}\{e_p, e_q\}\) is \(D\)-positive definite. In particular, \(e_1^* D e_1\) is negative and \(\lambda_1\) is the smallest eigenvalue of \(Q\), thus we are
4.3. EXAMPLES

in the case (iii) of Proposition 4.4.
We fix, according to the construction in the proposition, \(i = 1\) and \(j = 2\) and put \(z_2 = e_4\). A normalized \(z_1 = z_1(z_2)\) such that \(z_1^* D z_1 + z_2^* D z_2 = 0\) can be constructed as follows. Consider a normalized vector \(y \in \mathbb{R}^{2 \times 2}\) such that

\[
y^* K_{12} y = -z_2^* D z_2.
\]

Then

\[
y^*(K_{12} + z_2^* D z_2 I)y = 0,
\]
so as in the example before we need an isotropic vector of the matrix \(T = K_{12} + z_2^* D z_2 I\). We diagonalize \(T = UV^*\) and obtain

\[
U = \begin{pmatrix}
-0.9990 & -0.0453 \\
-0.0453 & 0.9990 \\
\end{pmatrix}, \quad V = \begin{pmatrix}
-9.0454 \\
13.0454 \\
\end{pmatrix},
\]

thus an isotropic vector of \(T\) is given by

\[
\tilde{z}_1 = U \begin{pmatrix} \sqrt{13.0454} \\ -\sqrt{9.0454} \end{pmatrix}, \quad z_1 = [e_1 \ e_2] \frac{\tilde{z}_1}{\|\tilde{z}_1\|} = [e_1 \ e_2] \begin{pmatrix} -0.7387 \\ -0.6741 \end{pmatrix}.
\]

In fact, we have \(z_2^* D z_2 = 5 = -z_1^* D z_1\). We complete \(z_1, z_2\) to an orthonormal basis and according to Proposition 4.1 we define

\[
Z = \begin{pmatrix}
0 & -0.7387 & 0 & 0.6741 \\
0 & -0.6741 & 0 & -0.7387 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
G_\delta = Z \begin{pmatrix}
0 & \delta & 0 & 0 \\
-\delta & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
\end{pmatrix} Z^T.
\]

With \(\delta = 0.0001\), the system \(\ddot{x} + (D + \alpha G_\delta) \dot{x} + Q^{-2}x = 0\) is stable for \(\alpha \geq 69752\). Note that in this case, the coefficient \(\alpha\) needed to stabilize the system is a lot larger than in the previous example, the eigenvalues of the system \(\ddot{x} + (D + \alpha G_{0.0001}) \dot{x} + K x = 0\) are less sensitive with respect to the parameter \(\alpha\) as can be seen in Figure 4.3. One of the reasons of this effect is, as already mentioned, the scaling of \(Q\).

Figure 4.4 shows again the dependance of \(\alpha\) on \(\delta\), where the green area signifies the parameter combinations \((\alpha, \delta)\), for which the system is unstable, the blue area signifies stability. It can be seen that the higher \(\alpha\) gets, the smaller the absolute value of \(\delta\) can be chosen, such
that the pair $\alpha, \delta$ yields stability. In contrast to the previous example, the value $\delta = 0$ does not seem to be admissible.

**Example 4.10.** Here, we construct a stabilizer for the case of Proposition 4.4 (ii). The main difficulty is the construction of the mapping $z_2 \mapsto z_1(z_2)$. Consider the system $\ddot{x} + D\dot{x} + Q^{-2}x = 0$, where

$$D = \begin{pmatrix} 9 & -1 & 2 & 5 \\ -1 & -13 & 2 & 2 \\ 2 & 2 & 2 & 1 \\ 5 & 2 & 1 & 6 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

The largest real part of the eigenvalues of the system is 13.5055, the system is unstable. With $\frac{\text{tr} D}{\tau} = 1 < \frac{137}{30} = \frac{\text{tr} QDQ}{\text{tr} Q^2}$ we put $\tau = 1$ and $P = I$ in Proposition 2.1 and get

$$D_\tau = \begin{pmatrix} 8 & -1 & 2 & 5 \\ -1 & -14 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 5 & 2 & 1 & 5 \end{pmatrix}.$$

Any space $\text{Span}\{e_2, e_j\}$ with $j \neq 2$ is $D$-indefinite, while the spaces $\text{Span}\{e_1, e_3\}, \text{Span}\{e_1, e_4\}, \text{Span}\{e_3, e_4\}$ are each $D$-positive definite, thus we are in the situation of Proposition 4.4 (ii).
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Figure 4.4. The blue area indicates parameter combinations $(\alpha, \delta)$ that result in stability of $\ddot{x} + (D + \alpha G_\delta)\dot{x} + Kx = 0$, the green area indicates instability.

We put $(i,j,m,n) = (2,3,1,4)$ according to Proposition 4.4, and

$$K_{ij} = [e_i e_j]^* D [e_i e_j] = \begin{pmatrix} -14 & 2 \\ 2 & 1 \end{pmatrix},$$

$$K_{mn} = [e_m e_n]^* D [e_m e_n] = \begin{pmatrix} 8 & 5 \\ 5 & 5 \end{pmatrix}.$$

Let the eigenvalues of $K_{ij}$ be $\mu_i, \mu_j$ with $\mu_i \leq 0 \leq \mu_j$ and those of $K_{mn}$ be $\mu_m, \mu_n > 0$. Clearly there exists a pair of orthonormal vectors $x_i, x_j \in \text{Span}\{u_i, u_j\}$ and $x_m, x_n \in \text{Span}\{u_m, u_n\}$ such that with $X_{ij} = [x_i, x_j]$ and $X_{mn} = [x_m, x_n]$ we have

$$\text{diag}(\mu_i, \mu_j) = \text{diag}(14.2621, 1.2621) = X_{ij}^T D X_{ij},$$

$$\text{diag}(\mu_m, \mu_n) = \text{diag}(1.2798, 11.7202) = X_{mn}^T D X_{mn}.$$

Since $K_{ij}$ is indefinite, we can assume without loss of generality

$$\mu_i \leq u_i^* Du_i < 0 \leq u_j^* Du_j \leq \mu_j.$$

From $\text{tr } D = \sum_{k=1}^{4} u_k^* D u_k = \sum_{k=1}^{4} \mu_k = 0$ we conclude
\[
\mu_i \leq - (\mu_n + \mu_m) \leq 0 \leq \mu_j ,
\]
(4.6)

Therefore, for each normalized $z_2 \in \text{Span}\{u_m, u_n\}$ there exists a normalized $z_1 \in \text{Span}\{u_i, u_j\}$ such that
\[
z_1^* D z_1 = - z_2^* D z_2 .
\]

Let $z_2(\beta) = \cos(\beta) x_m + \sin(\beta) x_n$ for $\beta \in [0, 2\pi)$ and define $z_1(\alpha_\beta) = \cos(\alpha_\beta) x_i + \sin(\alpha_\beta) x_j$. Now we construct a continuous mapping $\beta \mapsto \alpha_\beta$ that satisfies
\[
z_1(\alpha_\beta)^* D z_1(\alpha_\beta) = - z_2(\beta)^* D z_2(\beta)
\]
for all $\beta$. We write (4.7) as
\[
0 = \cos^2(\alpha_\beta) \mu_i + \sin^2(\alpha_\beta) \mu_j + \cos^2(\beta) \mu_m + \sin^2(\beta) \mu_n .
\]

With $\sin^2(\alpha_\beta) = 1 - \cos^2(\alpha_\beta)$ we get
\[
0 = \cos^2(\alpha_\beta) (\mu_i - \mu_j) + \mu_j + \cos^2(\beta) \mu_m + \sin^2(\beta) \mu_n .
\]

and finally
\[
\cos^2(\alpha_\beta) = \frac{\mu_j + \cos^2(\beta) \mu_m + \sin^2(\beta) \mu_n}{\mu_j - \mu_i} ,
\]

where the denominator is nonzero by construction. Also, the real square root of the above fraction exists and is less or equal 1, since from (4.6) it follows that
\[
\frac{\mu_j + \cos^2(\beta) \mu_m + \sin^2(\beta) \mu_n}{\mu_j - \mu_i} \leq \frac{\mu_j + \cos^2(\beta) \mu_m + \sin^2(\beta) \mu_n}{\mu_j + \mu_m + \mu_n} \leq \frac{\mu_j + \mu_m + \mu_n}{\mu_j + \mu_m + \mu_n} \leq 1 ,
\]

and we also have
\[
\frac{\mu_j + \cos^2(\beta) \mu_m + \sin^2(\beta) \mu_n}{\mu_j - \mu_i} \geq 0 ,
\]

since both numerator and denominator are nonnegative. Therefore,
\[
\alpha(\beta) = \arccos \left( \frac{\mu_j + \cos^2(\beta) \mu_m + \sin^2(\beta) \mu_n}{\mu_j - \mu_i} \right)^{\frac{1}{2}}
\]
is well defined and satisfies (4.7). In our case, it takes the form
\[
\alpha(\beta) = \arccos \left( \frac{1.2621 + \cos^2(\beta) 1.2798 + \sin^2(\beta) 11.7202}{1.2621 + 14.2621} \right)^{\frac{1}{2}} .
\]
With
\[
  z_1(\alpha_\beta) = \cos(\alpha_\beta)x_i + \sin(\alpha_\beta)x_j
\]
\[
  z_2(\beta) = \cos(\beta)x_m + \sin(\beta)x_n
\]

we define the mapping
\[
f : [0, 2\pi) \to \mathbb{R} , \quad f(\beta) = \frac{z_1(\alpha_\beta)^*Dz_1(\alpha_\beta)}{\|Q^{-1}z_1(\alpha_\beta)\|^2} + \frac{z_2(\beta)^*Dz_2(\beta)}{\|Q^{-1}z_2(\beta)\|^2},
\]

which now according to Proposition 4.4 has at least one zero. Figure 4.5 shows that actually there are several zeros we can choose from, we take \( \beta = 3.4547 \).

Figure 4.5. Plot of the function \( f \) on \([0 2\pi)\)

With \( \delta = 0.0001 \), the system \( \ddot{x} + (D + \alpha G_\delta)\dot{x} + Q^{-2}x = 0 \) is stable for \( \alpha \geq 25595 \) as can also be seen in figure 4.6.
Figure 4.6. Plot of the real parts of the eigenvalues of $\ddot{x} + (D + \alpha G_{0.0001}) \dot{x} + Kx = 0$ with $\alpha$ on the horizontal axis.
CHAPTER 5

Space Dimension Five and Higher

The approach we take here is similar to our approach in the four dimensional case. Again the fact that the eigenspace for the eigenvalue zero of $G$ results in an easy to identify eigenspace for the eigenvalue zero of $QGQ$ is used.

**Proposition 5.1.** For any $\delta \in \mathbb{R}$ and an orthogonal matrix $Z = [z_1, z_2, z_3, z_4, z_5] \in \mathbb{R}^{5 \times 5}$ set

$$G_\delta = Z \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta & 0 & 0 \\ 0 & -\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} Z^T.$$  \hspace{1cm} (5.1)

If for $\tau > 0$ with $\text{tr} \, D_\tau = 0$, $\text{tr} \, QD_\tau Q \geq 0$ we have

(i) $\text{tr}_{\text{Span}\{z_1\}} D_\tau = \text{tr}_{\text{Span}\{z_2,z_3\}} D_\tau = \text{tr}_{\text{Span}\{z_4,z_5\}} D_\tau = 0$,

(ii) $\text{tr}_{Q^{-1}\text{Span}\{z_1,z_2,z_3\}} QD_\tau Q \geq \text{tr}_{Q^{-1}\text{Span}\{z_1\}} QD_\tau Q = 0$,

then there exists $\delta \neq 0$ such that (a), (b), and (c) in Proposition 2.1 hold for $G_\delta$ and $\tau/2$.

The idea behind the above proposition is as follows. The matrix $G_0$ is constructed such that after a small perturbation $\delta$, the triple eigenvalue zero of $G_0$ is split into a pair of imaginary eigenvalues $\pm i\delta$ and a single eigenvalue zero, such that the corresponding eigenvectors of $G_\delta$ and $QG_\delta Q$ satisfy the conditions (a), (b) and (c) in Proposition 2.1 for sufficiently small $\delta$.

**Proof.** By continuity of eigenvalues, it is clear that (a) in Proposition 2.1 holds for small $|\delta| \neq 0$.

Using (2.6) and the structure of $G_\delta$ we conclude that assumption (b) is equivalent to $\text{tr}_{\text{Span}\{z_1\}} D_\tau \geq 0$, $\text{tr}_{\text{Span}\{z_2,z_3\}} D_\tau \geq 0$ and $\text{tr}_{\text{Span}\{z_4,z_5\}} D_\tau \geq 0$ for all $\delta \in \mathbb{R} \setminus \{0\}$.

To verify (c), first observe that for $\delta \neq 0$ the matrix $QG_\delta Q$ has a zero eigenvalue and two conjugate pairs of imaginary eigenvalues, which
we denote by $\pm \lambda_\delta$ and $\pm \mu_\delta$. These depend continuously on $\delta$ (where $\lambda_0 = 0$). The same is true (e.g. [19]) for the invariant subspaces
\[
V_{0,\lambda}(\delta) := \text{Ker} \left( QG_\delta Q \left( (QG_\delta Q)^2 + |\lambda_\delta|^2 I \right) \right),
\[
V_{\mu}(\delta) := \text{Ker} \left( (QG_\delta Q)^2 + |\mu_\delta|^2 I \right).
\]
Also, independently of $\delta$, $Q^{-1}z_{1}$ is always an eigenvector for the eigenvalue zero of $QG_\delta Q$. By construction we have
\[
\text{tr}_{Q^{-1}\text{Span}\{z_1\}} QD_\tau Q = 0.
\]
We define for any $\delta$ the orthogonal complement $V_\lambda(\delta)$ of $Q^{-1}\text{Span}\{z_1\}$ in $V_{0,\lambda}(\delta)$ and get then $V_\lambda(\delta) \oplus Q^{-1}\text{Span}\{z_1\} = V_{0,\lambda}(\delta)$.

By assumption for $\delta = 0$ and $\eta = \tau$ we have with $D = D_\tau + \tau P$:
\[
\begin{align*}
\text{tr}_{Q^{-1}\text{Span}\{z_1\}} QDQ &= \eta \text{tr}_{Q^{-1}\text{Span}\{z_1\}} PQP > 0, \\
\text{tr}_{V_{0,\lambda}(\delta)} QDQ &\geq \eta \text{tr}_{V_{0,\lambda}(\delta)} PQP > 0, \\
\text{tr}_{V_{\mu}(\delta)} QDQ &\geq \eta \text{tr}_{V_{\mu}(\delta)} PQP > 0.
\end{align*}
\]

Now, since $Q^{-1}\text{Span}\{z_1\} \subset V_{0,\lambda}(\delta)$ and $PQP$ is positive definite, it follows that
\[
\text{tr}_{V_{0,\lambda}(\delta)} QDQ \geq \eta \text{tr}_{V_{0,\lambda}(\delta)} PQP > \eta \text{tr}_{V_{\mu}(\delta)} PQP > 0,
\]
implying that
\[
\text{tr}_{V_{\lambda}(\delta)} QDQ \geq \eta \text{tr}_{V_{\lambda}(\delta)} PQP = \eta \left( \text{tr}_{V_{0,\lambda}(\delta)} PQP - \text{tr}_{V_{\mu}(\delta)} PQP \right) > 0.
\]
We altogether have
\[
\begin{align*}
\text{tr}_{Q^{-1}\text{Span}\{z_1\}} QDQ &= \eta \text{tr}_{Q^{-1}\text{Span}\{z_1\}} PQP > 0, \\
\text{tr}_{V_{\lambda}(\delta)} QDQ &\geq \eta \text{tr}_{V_{\lambda}(\delta)} PQP > 0, \\
\text{tr}_{V_{\mu}(\delta)} QDQ &\geq \eta \text{tr}_{V_{\mu}(\delta)} PQP > 0.
\end{align*}
\]

By continuity, the same holds for $\eta = \tau/2$ and sufficiently small $\delta$. Together with (2.6) this completes the proof. \hfill \Box

Thus, as in the four dimensional case, we can relax the conditions in Problem 2.2 and reformulate it, such that it relates to two-dimensional spaces containing pairs of complex conjugate eigenvectors instead of relating directly to eigenvectors.

**Problem 5.2.** For symmetric matrices $D, Q \in \mathbb{R}^{5 \times 5}$ with $Q > 0$, $\text{tr} D = 0$ and $\text{tr} QDQ \geq 0$, find $G_0$ as in (5.1) so that
\[
\begin{align*}
(i) \quad \text{tr}_{\text{Span}\{z_1\}} D &= \text{tr}_{\text{Span}\{z_2, z_3\}} D = \text{tr}_{\text{Span}\{z_4, z_5\}} D = 0, \\
(ii) \quad \text{tr}_{Q^{-1}\text{Span}\{z_1, z_2, z_3\}} QDQ &\geq 0, \\
(iii) \quad \text{tr}_{Q\text{Span}\{z_4, z_5\}} QDQ &\geq 0.
\end{align*}
\]

The conditions in Problem 5.2 can be weakened even further:
**Lemma 5.3.** Let $\text{tr} \ D = 0$ and $\text{tr} \ QDQ \geq 0$ and suppose that one of the following conditions holds.

(i) There exists a two-dimensional space $\text{Span}\{z_4, z_5\}$ with 

$$\text{tr}_{\text{Span}\{z_4, z_5\}} \ D = 0$$

and

$$0 \leq \text{tr}_Q \text{Span}\{z_4, z_5\} \ QDQ \leq \text{tr} \ QDQ .$$

(ii) There exists a three-dimensional space $\text{Span}\{z_1, z_2, z_3\}$ such that we have $\text{tr}_{\text{Span}\{z_1, z_2, z_3\}} \ D = 0$ and 

$$0 \leq \text{tr}_{Q^{-1} \text{Span}\{z_1, z_2, z_3\}} \ QDQ \leq \text{tr} \ QDQ .$$

(iii) There exists a two-dimensional space $\text{Span}\{z_4, z_5\}$ with 

$$\text{tr}_{\text{Span}\{z_4, z_5\}} \ D = 0$$

and

$$0 \leq \text{tr}_Q \text{Span}\{z_4, z_5\} \ QDQ$$

and a three-dimensional space $\text{Span}\{z_1, z_2, z_3\}$ orthogonal on $\text{Span}\{z_4, z_5\}$ with $\text{tr}_{\text{Span}\{z_1, z_2, z_3\}} \ D$ and 

$$0 \leq \text{tr}_{Q^{-1} \text{Span}\{z_1, z_2, z_3\}} \ QDQ .$$

Then there exists a solution to Problem 5.2.

**Proof.** We prove (i) exemplarily. The other cases can be handled analogously.

Without loss of generality, we assume $z_1, \ldots, z_5$ to be an orthonormal basis of $\mathbb{R}^5$. The condition $\text{tr}_{\text{Span}\{z_4, z_5\}} \ D = 0$ implies $\text{tr}_{\text{Span}\{z_1, z_2, z_3\}} \ D = 0$, which implies the existence of a $D$-isotropic vector in $\text{Span}\{z_1, z_2, z_3\}$. By a change of the basis, we can assume this $D$-isotropic vector to be $z_1$. Thus, (i) in Problem 5.2 holds. Also,

$$0 \leq \text{tr}_Q \text{Span}\{z_4, z_5\} \ QDQ \leq \text{tr} \ QDQ$$

implies

$$0 \leq \text{tr}_{Q^{-1} \text{Span}\{z_1, z_2, z_3\}} \ QDQ \leq \text{tr} \ QDQ ,$$

so (ii) and (iii) in Problem 5.2 hold.

In the subsequent sections, we will construct spaces according to the conditions given by Lemma 5.3.

**Proposition 5.4.** Let $\text{tr} \ D = 0$ and $\text{tr} \ QDQ \geq 0$ and let $u_1, \ldots, u_5$ and $\lambda_1, \ldots, \lambda_5$ be a set of normalized eigenvectors and corresponding eigenvalues of $Q$ in no particular order.

Assume that $u_5^* Du_5 = 0$ Then there exists a gyroscopic stabilizer.
PROOF. We define
\[ \hat{D} = [u_1 \ u_2 \ u_3 \ u_4]^* D [u_1 \ u_2 \ u_3 \ u_4], \]
\[ \hat{Q} = [u_1 \ u_2 \ u_3 \ u_4]^* Q [u_1 \ u_2 \ u_3 \ u_4]. \]

It directly follows \( \text{tr} \hat{D} = 0, \text{tr} \hat{Q} \hat{D} \hat{Q} = \text{tr} Q D Q \geq 0. \) From section 4 it follows that there exists a twodimensional subspace \( \mathcal{Z} \subset \mathbb{R}^4 \) such that \( \text{tr}_\mathcal{Z} \hat{D} = 0 \) and \( 0 \leq \text{tr} \hat{Q}^{-1} \hat{D} \hat{Q} \leq \text{tr} \hat{D} \hat{Q} \). Let \( z_1, z_2 \in \mathbb{R}^4 \) be an orthonormal basis of \( \mathcal{Z} \) and define \( \mathcal{Y} \) as the span of the columns of \([u_1 \ u_2 \ u_3 \ u_4] [z_1 \ z_2] = [y_1 \ y_2] \). Then by construction \( y_1, y_2, u_5 \) are orthonormal and \( \text{tr}_{\text{Span} \{y_1, y_2, u_5\}} \hat{D} = 0 \).

Since \( Q^{-1} u_5 \perp Q^{-1} y_1, y_2 \), it also follows
\[ \text{tr} \hat{Q}^{-1} \text{Span} \{y_1, y_2, u_5\} \hat{D} \hat{Q} = \text{tr} \hat{Q}^{-1} \text{Span} \{y_1, y_2\} \hat{D} \hat{Q} + \text{tr} \hat{Q}^{-1} \text{Span} \{u_5\} \hat{D} \hat{Q} = \text{tr} \hat{Q}^{-1} \text{Span} \{y_1, y_2\} \hat{D} \hat{Q}, \]
which now implies \( 0 \leq \text{tr} \hat{Q}^{-1} \text{Span} \{y_1, y_2, u_5\} Q D Q \leq \text{tr} Q D Q \). Thus, according to lemma 5.3 (iii), there exists a gyroscopic stabilizer.

The above proposition shows that in the case \( u_i^* D u_i = 0 \) for some eigenvector \( u_i \) of \( Q \) the construction of a gyroscopic stabilizer can be reduced to the four-dimensional case. Thus from here on, if not mentioned otherwise, we consider \( u_i^* D u_i \neq 0 \) for all eigenvectors.

### 5.1. One \( D \)-negative Eigenvector of \( Q \)

In this section, we will prove the analogue of Proposition 4.4. Thus we consider the values \( u_i^* D u_i \) for the eigenvectors \( u_i \) of \( Q \). We assume that exactly one of the numbers \( u_i^* D u_i, i = 1, \ldots, 5 \) is negative and all others are positive. Additionally, we assume the two-dimensional spaces \( \text{Span} \{u_j, u_k\} \) with \( j, k \neq i \) to be positive definite, analogously to Proposition 4.4.

Note that, if \( u_i = u_5 \) is an eigenvector for the largest eigenvalue of \( Q \), it follows that \( Q = c I \) is a multiple of the unit matrix, see the proof of Proposition 4.4. This case is already solved in Proposition 2.9. Thus in this section, we only need to consider the cases \( i = 1, \ldots, 4 \).

Let the eigenvectors \( u_1, \ldots, u_5 \) of \( Q \) be ordered such that \( \lambda_1 \leq \cdots \leq \lambda_5 \).

In this section, we then consider the cases

(i) \( u_1^* D u_1 < 0, u_j^* D u_j > 0 \) for all \( j \neq 1 \)

(ii) \( u_2^* D u_2 < 0, [u_j \ u_k]^* D [u_j \ u_k] > 0 \) for all distinct \( j, k \neq 2 \)

(iii) \( u_3^* D u_3 < 0, [u_j \ u_k]^* D [u_j \ u_k] > 0 \) for all distinct \( j, k \neq 3 \)

(iv) \( u_4^* D u_4 < 0, [u_j \ u_k]^* D [u_j \ u_k] > 0 \) for all distinct \( j, k \neq 4 \).
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It should be noted, that case (i) is more general than the others since 
$[u_j u_k]^* D [u_j u_k] > 0$ implies $u_j^* D u_j, u_k^* D u_k > 0$.

**Proposition 5.5.** Let $\text{tr} D = 0$ and $\text{tr} QDQ \geq 0$ and let $u_1, \ldots, u_5$ 
and $\lambda_1 \leq \cdots \leq \lambda_5$ be a set of normalized eigenvectors and corresponding 
eigenvalues of $Q$. 
Assume that $u_1^* D u_1 < 0$ and $u_j^* D u_j > 0$ for all $j \neq 1$. Then there 
exists a gyroscopic stabilizer.

**Proof.** In the proof, we first show the existence of a 3-dimensional 
space $\mathcal{Y}_a$ with 
(a) $\text{tr}_{\mathcal{Y}_a} D = 0$ and $0 \leq \text{tr}_{Q^{-1} \text{Span} \mathcal{Y}_a} QDQ$, 
(b) $0 \leq \text{tr}_{Q \mathcal{Y}_b} QDQ$, where $\mathcal{Y}_b = \mathcal{Y}_a$.

Then according to Lemma 5.3 (iii), there exists a gyroscopic stabilizer. 
(a) From $\text{tr} D = \sum_{k=1}^5 u_k^* D u_k = 0$ it follows that 
$$u_1^* D u_1 < -(u_3^* D u_3 + u_4^* D u_4) < 0 < u_2^* D u_2.$$ 

Then the number $-(u_3^* D u_3 + u_4^* D u_4)$ is in the numerical range of the 
gramian given by $[u_1 u_2]^* D [u_1 u_2]$, so there exists a normalized $y_1 = 
a u_1 + b u_2$ with $y_1^* D y_1 + u_3^* D u_3 + u_4^* D u_4 = 0$.

We define $\mathcal{Y}_a = \text{Span} \{y_1, u_3, u_4\}$. By construction, we have $\text{tr}_{\mathcal{Y}_a} D = 0$.

Since $y_1, u_3, u_4$ remain orthogonal under multiplication with $Q$, we can compute 
$$\text{tr}_{Q^{-1} \mathcal{Y}_a} QDQ = \frac{(Q^{-1} y_1)^* Q D Q (Q^{-1} y_1)}{\|Q^{-1} y_1\|^2} + \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4$$ 
$$= \frac{y_1^* D y_1}{\|Q^{-1} y_1\|^2} + \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4.$$ 

From the definition of $y_1 = a u_1 + b u_2$ with $a, b \in \mathbb{R}$ and $a^2 + b^2 = 1$ we get 
$$\frac{1}{\|Q^{-1} y_1\|^2} \in [\lambda_1^2, \lambda_2^2].$$ 

It follows with $y_1^* D y_1 < 0$ and an estimation for the denominator in 
equation (5.2) that 
$$\text{tr}_{Q^{-1} \mathcal{Y}_a} QDQ \geq \lambda_1^2 y_1^* D y_1 + \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4$$ 
$$= \lambda_2^2 (-u_3^* D u_3 - u_4^* D u_4) + \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4$$ 
$$\geq -\lambda_3^2 u_3^* D u_3 - \lambda_4^2 u_4^* D u_4 + \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 = 0.$$
With $\mathcal{Y}_b = \mathcal{Y}_a^\perp$ and $\text{tr}_{\mathcal{Y}_a} D = 0$, it follows that $\text{tr}_{\mathcal{Y}_b} D = 0$. With $Q\mathcal{Y}_b \perp Q^{-1}\mathcal{Y}_a$ we get
\[
\text{tr}_{Q\mathcal{Y}_b} QDQ = \text{tr} QDQ - \text{tr}_{Q^{-1}\mathcal{Y}_a} QDQ
\]
\[
= \sum_{k=1}^{5} \lambda_k^2 u_k^* D u_k - \frac{y_1^* D y_1}{\|Q^{-1}y_1\|^2} - \lambda_3^2 u_3^* D u_3 - \lambda_4^2 u_4^* D u_4
\]
\[
\geq \lambda_1^2 u_1^* D u_1 + \lambda_2^2 u_2^* D u_2 + \lambda_5^2 u_5^* D u_5
\]
\[
+ \lambda_1^2 (u_3^* D u_3 + u_4^* D u_4)
\]
\[
\geq \lambda_1^2 \sum_{k=1}^{5} u_k^* D u_k = 0,
\]
which shows that we fulfill the requirements of lemma 5.3 (iii).

**Proposition 5.6.** Let $\text{tr} D = 0$ and $\text{tr} QDQ \geq 0$, and let $u_1, \ldots, u_5$ and $\lambda_1 \leq \cdots \leq \lambda_5$ be a set of normalized eigenvectors and corresponding eigenvalues of $Q$. Assume that $u_2^* D u_2 < 0$ and let
\[
K_{jk} = [u_j u_k] D [u_j u_k]
\]
be positive definite for each pair $j, k \in \{1, 3, 4, 5\}$. Then there exists a gyroscopic stabilizer.

**Proof.** The proof is divided into the following steps.
(a) We show the existence of a two-dimensional space $\mathcal{A}$ with
\[
\text{tr}_{\mathcal{A}} D = 0, \quad \text{tr}_{Q\mathcal{A}} QDQ \leq \text{tr} QDQ.
\]
(b) A second two-dimensional space $\mathcal{B}$ is constructed with
\[
\text{tr}_{\mathcal{B}} D = 0, \quad \text{tr}_{Q\mathcal{B}} QDQ \geq 0.
\]
(c) The existence of a continuous mapping $\mathcal{Y}$, whose image is in the set $\mathcal{P}$ of two-dimensional subspaces of $\mathbb{R}^5$, with the properties
\[
\mathcal{Y} : [0, \frac{\pi}{2}] \mapsto \mathcal{P}, \quad t \mapsto \mathcal{Y}(t), \quad \dim \mathcal{Y}(t) = 2 \text{ for all } t
\]
and the existence of a $t_0 \in [0, \frac{\pi}{2}]$ with $\text{tr}_{Q\mathcal{Y}(t_0)} QDQ = 0$ is shown. Then from lemma 5.3 (i), the existence of a gyroscopic stabilizer follows.
Note that the specific parameterization of $\mathcal{Y}$ with $t \in [0, \frac{\pi}{2}]$ is completely arbitrary. It should also be mentioned that in general it holds that $(Q^{-1}\mathcal{Y}) = Q^{-1}(\mathcal{Y}^\perp)$, which is why we introduced the notation with $\mathcal{Y}_b$, $\mathcal{Y}_a$, and avoided $\mathcal{Y}_b^\perp$.
(a) From $u_2^* D u_2 < 0$ and $u_j^* D u_j > 0$ for all $j \neq 2$ and $0 = \text{tr} D = \sum_{i=1}^{5} u_i^* D u_i$ it follows that there exists a normalized $x \in \text{Span}\{u_1, u_2\}$
with \( x^* D x = -u_5^* D u_5 \). We then have with \( \mathcal{A} = \text{Span}\{x, u_5\} \) and \( x^\perp \) such that \( \{x, x^\perp\} \) is an orthonormal basis of \( \text{Span}\{u_1, u_2\} \):

\[
\text{tr}_\mathcal{A} D = 0 ,
\]

\[
\text{tr}_{\mathcal{A}} Q D Q = \lambda_5^2 u_5^* D u_5 + \frac{(Q x)^* Q D Q (Q x)}{\|Q x\|^2}
\]

\[
= \lambda_5^2 u_5^* D u_5 + \text{tr}_{\text{Span}\{u_1, u_2\}} Q D Q - \frac{(Q^{-1} x^\perp)^* Q D Q (Q^{-1} x^\perp)}{\|Q^{-1} x^\perp\|^2}
\]

\[
= \lambda_5^2 u_5^* D u_5 + \lambda_1^2 u_1^* D u_1 + \lambda_2^2 u_2^* D u_2 - \frac{(x^\perp)^* D x^\perp}{\|Q^{-1} x^\perp\|^2} .
\]

Again from \( \text{tr} D = 0 \) and \( x^* D x = -u_5^* D u_5 \) it follows that

\( (x^\perp)^* D x^\perp = -(u_3^* D u_3 + u_4^* D u_4) \)

and thus

\[
\text{tr}_{\mathcal{A}} Q D Q = \lambda_5^2 u_5^* D u_5 + \lambda_1^2 u_1^* D u_1 + \lambda_2^2 u_2^* D u_2 + \frac{u_3^* D u_3 + u_4^* D u_4}{\|Q^{-1} x^\perp\|^2} .
\]

Now we have \( \frac{1}{\|Q^{-1} x^\perp\|^2} \in [\lambda_1^2, \lambda_5^2] \) and therefore

\[
\text{tr}_{\mathcal{A}} Q D Q \leq \text{tr} Q D Q .
\]

(b) On the other hand, there exists a normalized \( y \in \text{Span}\{u_2, u_4\} \) with \( y^* D y = -u_5^* D u_5 \). With \( \mathcal{B} = \text{Span}\{y, u_5\} \) and \( (y^\perp)^* D y^\perp = -(u_1^* D u_1 + u_3^* D u_3) \) we get

\[
\text{tr}_{\mathcal{B}} Q D Q = \lambda_5^2 u_5^* D u_5 + \frac{(Q y)^* Q D Q (Q y)}{\|Q y\|^2} = \lambda_5^2 u_5^* D u_5 + \frac{u_1^* D u_1 + u_3^* D u_3}{\|Q^{-1} y^\perp\|^2} .
\]

\[
\geq \lambda_5^2 u_5^* D u_5 + \lambda_2^2 u_2^* D u_4 + \lambda_3^2 u_3^* D u_4 + \lambda_4^2 (u_1^* D u_1 + u_3^* D u_3)
\]

\[
\geq \lambda_5^2 \sum_{i=1}^5 u_i^* D u_i = 0 .
\]

(c) We consider the space \( \text{Span}\{u_1, u_2, u_4\} \). We already showed in (a) and (b) that \(-u_5^* D u_5\) is in the numerical range of

\[
K_{124} = [u_1, u_2, u_4]^* D [u_1, u_2, u_4] .
\]

This implies that 0 is in the numerical range of \((K_{124} + u_5^* D u_5 I_3)\).

By construction of \( x \) and \( y \), it follows that \( x = [u_1, u_2, u_4] z_x \) for some
normalized \( z \in \mathbb{R}^3 \) with \( z^*(K_{124} + u_5^*D u_5 I_3)z = 0 \), and we have \( y = [u_1, u_2, u_4]z_y \). Now by Proposition 2.7 there exists a continuous mapping \( z : [0, \frac{\pi}{2}] \mapsto \mathbb{R}^5 \) with \( \|z(t)\| = 1 \), \( z(0) = z_x \) and either \( z(\frac{\pi}{2}) = z_y \) or \( z(\frac{\pi}{2}) = -z_y \). We define \( \mathcal{Y}(t) = \text{Span}\{[u_1, u_2, u_4]z(t), u_5]\} \) for all \( t \). Note that \( \mathcal{Y}(\frac{\pi}{2}) = \text{Span}\{y, u_5\} \) and that \( \mathcal{Y} \) is continuous by definition. As seen above in (a) and (b), we have

\[
\text{tr}_{Q\mathcal{Y}(0)} QDQ = \text{tr}_{Q\mathcal{Y}(\frac{\pi}{2})} QDQ \leq \text{tr} QDQ
\]

and

\[
\text{tr}_{Q\mathcal{Y}(\frac{\pi}{2})} QDQ = \text{tr}_{Q\mathcal{Y}(\frac{\pi}{2})} QDQ \geq 0 .
\]

If already either

\[
0 \leq \text{tr}_{Q\mathcal{Y}(0)} QDQ \leq \text{tr} QDQ
\]

or

\[
0 \leq \text{tr}_{Q\mathcal{Y}(\frac{\pi}{2})} QDQ \leq \text{tr} QDQ ,
\]

we have found a space as in lemma 5.3 (ii), and the proposition is shown. Otherwise, we have

\[
\text{tr}_{Q\mathcal{Y}(0)} QDQ \leq 0 ,
\]

\[
\text{tr}_{Q\mathcal{Y}(\frac{\pi}{2})} QDQ \geq \text{tr} QDQ .
\]

But then, by continuity of the trace and the intermediate value theorem, for any value \( c \in [\text{tr}_{Q\mathcal{Y}(0)} QDQ, \text{tr}_{Q\mathcal{Y}(\frac{\pi}{2})} QDQ] \) there exists a \( t_0 \in [0, \frac{\pi}{2}] \) such that \( c = \text{tr}_{Q\mathcal{Y}(t_0)} QDQ \). Choose \( c = \frac{\text{tr} QDQ}{2} \) and \( t_0 \) accordingly. Then

\[
0 \leq \text{tr}_{Q\mathcal{Y}(t_0)} QDQ \leq \text{tr} QDQ ,
\]

as in Lemma 5.3 (i).

\[\square\]

**Proposition 5.7.** Let \( \text{tr} D = 0 \) and \( \text{tr} QDQ \geq 0 \) and let \( u_1, \ldots, u_5 \) and \( \lambda_1 \leq \cdots \leq \lambda_5 \) be a set of normalized eigenvectors and corresponding eigenvalues of \( Q \). Assume that \( u_3^*D u_3 < 0 \) and let

\[
K_{jk} = [u_j u_k]^*D[u_j u_k]
\]

be positive definite for each pair \( j, k \in \{1, 2, 4, 5\} \). Then there exists a gyroscopic stabilizer.

**Proof.** The strategy for the proof is as follows. (a) We show the existence of a space \( \mathcal{Y}_a \) with

\[
\text{tr}_{\mathcal{Y}_a} D = 0 , \quad \text{tr}_{Q\mathcal{Y}_a} QDQ \geq 0 ,
\]
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(b) and the existence of a second space $\mathcal{Y}_b$ with
\[ \text{tr}_{\mathcal{Y}_b} D = 0, \quad \text{tr}_{Q\mathcal{Y}_b} QDQ \leq 0. \]

(c) We construct a continuous mapping $\mathcal{Y}$, whose image is in the set $\mathcal{P}$ of two-dimensional subspaces of $\mathbb{R}^5$, with the properties
\[ \mathcal{Y} : [0, \pi] \mapsto \mathcal{P}, \quad t \mapsto \mathcal{Y}(t), \quad \dim \mathcal{Y}(t) = 2 \text{ for all } t \]
\[ (5.4) \quad \mathcal{Y}(0) = \mathcal{Y}_a, \quad \mathcal{Y}(\pi) = \mathcal{Y}_b, \quad \text{tr}_{\mathcal{Y}(t)} D = 0 \text{ for all } t, \]
and show the existence of a $t_0 \in [0, \pi]$ with $\text{tr}_{Q\mathcal{Y}(t_0)} QDQ = 0$.

(a) From our assumptions it follows that $u_5^*Du_3 < -u_5^*Du_5 < u_4^*Du_4$, thus there exists a normalized $x \in \text{Span}\{u_3, u_4\}$ with $x^*Dx = -u_5^*Du_5$.
We then have with $\mathcal{Y}_a = \text{Span}\{x, u_5\}$ and an $x^\perp$ such that $x, x^\perp$ is an orthonormal basis of $\text{Span}\{u_3, u_4\}$:
\[ \text{tr}_{Q\mathcal{Y}_a} QDQ = \lambda_3^2u_3^*Du_5 + \frac{(Qx)^*QDQ(Qx)}{\|Qx\|^2} \]
\[ = \lambda_3^2u_3^*Du_5 + \text{tr}_{\text{Span}\{u_3, u_4\}} QDQ \]
\[ - \frac{(Q^{-1}x^\perp)^*QDQ(Q^{-1}x^\perp)}{\|Q^{-1}x^\perp\|^2} \]
\[ = \lambda_3^2u_3^*Du_5 + \lambda_3^2u_3^*Du_3 + \lambda_4^2u_4^*Du_4 - \frac{(x^\perp)^*Dx^\perp}{\|Q^{-1}x^\perp\|^2}. \]

From $\text{tr} \ D = 0$ and
\[ u_3^*Du_3 + u_4^*Du_4 = x^*Dx + (x^\perp)^*Dx^\perp = -u_5^*Du_5 + (x^\perp)^*Dx^\perp \]
it follows that
\[ (x^\perp)^*Dx^\perp = u_3^*Du_3 + u_4^*Du_4 + u_5^*Du_5 \]
and thus
\[ \text{tr}_{Q\mathcal{Y}_a} QDQ = \lambda_3^2u_3^*Du_3 + \lambda_4^2u_4^*Du_4 + \lambda_5^2u_5^*Du_5 \]
\[ - \frac{u_3^*Du_3 + u_4^*Du_4 + u_5^*Du_5}{\|Q^{-1}x^\perp\|^2}. \]

We have $\frac{1}{\|Q^{-1}x^\perp\|^2} \in [\lambda_3^2, \lambda_4^2]$ since $x^\perp$ is a normalized element of $\text{Span}\{u_3, u_4\}$, therefore we get with $-(u_3^*Du_3 + u_4^*Du_4 + u_5^*Du_5) > 0$:
\[ \text{tr}_{Q\mathcal{Y}_a} QDQ \geq \lambda_3^2u_3^*Du_3 + \lambda_4^2u_4^*Du_4 + \lambda_5^2u_5^*Du_5 \]
\[ - \lambda_3^2(u_3^*Du_3 + u_4^*Du_4 + u_5^*Du_5) \]
\[ = (\lambda_3^2 - \lambda_3^2)u_5^*Du_5 + (\lambda_4^2 - \lambda_3^2)u_4^*Du_4 \geq 0. \]

(b) On the other hand, there exists a normalized $y \in \text{Span}\{u_2, u_3\}$ with $y^*Dy = -u_1^*Du_1$. With $\mathcal{Y}_b = \text{Span}\{y, u_1\}$ and $y^\perp$ such that $\{y, y^\perp\}$ is
an orthonormal basis of $\text{Span}\{u_2, u_3\}$, we get

$$
\text{tr}_{Q_y} Q D Q = \lambda_1^2 u_1^* D u_1 + \frac{(Q y)^* Q D Q (Q y)}{\|Q y\|^2}
$$

$$
= \lambda_1^2 u_1^* D u_1 + \text{tr}_{\text{Span}\{u_2, u_3\}} Q D Q \left(\frac{(Q^{-1} y^\perp)^* Q D Q (Q^{-1} y^\perp)}{\|Q^{-1} y^\perp\|^2}\right)
$$

$$
= \lambda_1^2 u_1^* D u_1 + \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3 - \frac{(y^\perp)^* D y^\perp}{\|Q^{-1} y^\perp\|^2}.
$$

From $\text{tr} D = 0$ and

$$
u_2^* D u_2 + u_3^* D u_3 = y^* D y + (y^\perp)^* D y^\perp = -u_1^* D u_1 + (y^\perp)^* D y^\perp
$$

it follows that

$$(y^\perp)^* D y^\perp = u_1^* D u_1 + u_2^* D u_2 + u_3^* D u_3
$$

and

$$
\text{tr}_{Q_y} Q D Q = \lambda_1^2 u_1^* D u_1 + \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3 - \frac{u_1^* D u_1 + u_2^* D u_2 + u_3^* D u_3}{\|Q^{-1} y^\perp\|^2}.
$$

We have $\frac{1}{\|Q^{-1} y^\perp\|^2} \in [\lambda_2^2, \lambda_3^2]$ since $y^\perp$ is a normalized element of $\text{Span}\{u_2, u_3\}$, therefore, since $-(u_1^* D u_1 + u_2^* D u_2 + u_3^* D u_3) > 0$,

$$
\text{tr}_{Q_y} Q D Q \leq \lambda_1^2 u_1^* D u_1 + \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3 - \lambda_2^2 (u_1^* D u_1 + u_2^* D u_2 + u_3^* D u_3)
$$

$$
= (\lambda_1^2 - \lambda_2^2) u_1^* D u_1 + (\lambda_2^2 - \lambda_3^2) u_2^* D u_2 \leq 0.
$$

(c) From $u_3^* D u_3 < -u_5^* D u_5 < u_4^* D u_4$, $u_2^* D u_2$ it follows that $-u_5^* D u_5$ is in the numerical range of

$$
K_{234} = [u_2, u_3, u_4]^* D [u_2, u_3, u_4],
$$

in particular there exists $x \in \text{Span}\{u_3, u_4\}$ with $x^* D x = -u_5^* D u_5$ as in (a), and there exists $w \in \text{Span}\{u_2, u_3\}$ with $w^* D w = -u_5^* D u_5$. If we put $x = [u_2, u_3, u_4] z_x$ and $w = [u_2, u_3, u_4] z_w$, then both $z_x, z_w$ are zeros of the quadratic form given by $(K_{234} + u_5^* D u_5 I_3)$.

By Proposition 2.7 there is a continuous $z = z(t)$ with $z(0) = z_x$ and either $z(\frac{\pi}{2}) = z_w$ or $z(\frac{\pi}{2}) = -z_w$. We put $\mathcal{Y}(t) = \text{Span}\{[u_2, u_3, u_4] z(t), u_5\}$ for all $t \in [0, \frac{\pi}{2}]$. Note that $\mathcal{Y}(\frac{\pi}{2}) = \text{Span}\{w, u_5\}$, and $\mathcal{Y}$ is continuous by definition.

We will now extend the map $\mathcal{Y}$ and use an argument analogous to the
first part of the proof of Proposition 4.4. By assumption, \( \text{Span}\{u_1, u_5\} \) is \( D \)-positive definite, thus the numerical range of 
\[
K_{15} = [u_1, u_5]^*D[u_1, u_5]
\]
is completely contained in the interval \([0, u_1^*Du_1 + u_5^*Du_5]\), we write
\[
\text{nr}(K_{15}) \subseteq [0, u_1^*Du_1 + u_5^*Du_5].
\]
Since we have
\[
u_3^*Du_3 \leq -(u_5^*Du_5 + u_1^*Du_1) \leq u_2^*Du_2,
\]
it follows that \( \text{nr}(-K_{15}) \subseteq \text{nr}(K_{23}) \). Thus for each normalized element \( v_2 \in \text{Span}\{u_1, u_5\} \) there exists a normalized element \( v_1 = v_1(v_2) \in \text{Span}\{u_2, u_3\} \) so that \( v_1^*Dv_1 = -v_2^*Dv_2 \). We can choose \( v_1 = f(\alpha) = \cos(\alpha)\tilde{u}_2 + \sin(\alpha)\tilde{u}_3 \) with \( \alpha \in [0, \frac{\pi}{2}] \) and \( \tilde{u}_2, \tilde{u}_3 \) are orthonormal such that 
\[
[\tilde{u}_2\tilde{u}_3]^*D[\tilde{u}_2\tilde{u}_3] = \text{diag}(\mu_2, \mu_3)
\]
where \( \mu_3 < 0 < \mu_2 \). Then the mapping \( g: \alpha \mapsto f(\alpha)^*Df(\alpha) \) is continuous and strictly monotonically decreasing and therefore continuously invertible. Since the mapping \( v_2 \mapsto v_2^*Dv_2 \) is also continuous, the mapping \( v_2 \mapsto v_1(v_2) = v_1(g^{-1}(v_2^*Dv_2)) \) is continuous as well.

We set \( v_2(t) := \cos(t)u_5 + \sin(t)u_1 \) and \( v_1(t) := v_1(v_2(t)) \). Then the space \( \mathcal{Y}(t) = \text{Span}\{v_1(t), v_2(t)\} \) is continuous on \([\frac{\pi}{2}, \pi]\) with \( \mathcal{Y}(\frac{\pi}{2}) = \text{Span}\{w, u_5\} \) and \( \mathcal{Y}(\pi) = \text{Span}\{y, u_1\} \), where \( y \) is as in (b).

Altogether \( \mathcal{Y} \) is continuous on the complete interval \([0, \pi]\) and
\[
\begin{align*}
\text{tr}_{\mathcal{Y}(0)} QDQ & \geq 0, \\
\text{tr}_{\mathcal{Y}(\pi)} QDQ & \leq 0.
\end{align*}
\]
The intermediate value theorem implies the existence of \( t_0 \in [0, \pi] \) with
\[
\text{tr}_{\mathcal{Y}(t_0)} QDQ = 0.
\]
Thus by Lemma 5.3 (i), there exists a gyroscopic stabilizer. \( \square \)

**Proposition 5.8.** Let \( \text{tr} \ D = 0 \) and \( \text{tr} \ QDQ \geq 0 \) and let \( u_1, \ldots, u_5 \) and \( \lambda_1 \leq \cdots \leq \lambda_5 \) be a set of normalized eigenvectors and corresponding eigenvalues of \( Q \).

Assume that \( u_4^*Du_4 < 0 \) and let 
\[
K_{jk} = [u_ju_k]^*D[u_ju_k]
\]
be positive definite for each pair \( j, k \in \{1, 2, 3, 5\} \).

Then there exists a gyroscopic stabilizer.

**Proof.** The strategy is basically the same as in the previous proof.

(a) We show the existence of a space \( \mathcal{Y}_a \) with
\[
\text{try}_a \ D = 0, \quad \text{tr}_{\mathcal{Y}_a} QDQ \geq \text{tr} \ QDQ
\]
(b) then the existence of a second space $\mathcal{Y}_b$ with
\[ \text{tr}_{\mathcal{Y}_b} D = 0, \quad \text{tr}_{Q_{\mathcal{Y}_b}} QDQ \leq 0. \]
(c) We construct a continuous mapping $\mathcal{Y}$, whose image is in the set $\mathcal{P}$ of two-dimensional subspaces of $\mathbb{R}^5$, with the properties
\[ \mathcal{Y} : [0, \frac{\pi}{2}] \mapsto \mathcal{P}, \quad t \mapsto \mathcal{Y}(t), \quad \dim \mathcal{Y}(t) = 2 \text{ for all } t \]
(5.5) \[ \mathcal{Y}(0) = \mathcal{Y}_a, \quad \mathcal{Y}(\frac{\pi}{2}) = \mathcal{Y}_b, \quad \text{tr}_{\mathcal{Y}(t)} D = 0 \text{ for all } t. \]
Then the existence of a $t_0 \in [0, \frac{\pi}{2}]$ with $\text{tr}_{Q_{\mathcal{Y}(t_0)}} QDQ = 0$ follows.
(a) From our assumptions it follows that there exists a normalized $x \in \text{Span}\{u_3, u_4\}$ with $x^* Dx = -u_5^* Du_5$. We then have with $\mathcal{Y}_a = \text{Span}\{x, u_3\}$ and $x^\perp$ such that $\{x, x^\perp\}$ is an orthonormal basis of the space $\text{Span}\{u_3, u_4\}$:
\[ \text{tr}_{Q_{\mathcal{Y}_a}} QDQ = \lambda_3^2 u_3^* Du_5 + \frac{(Qx)^* QDQ(Qx)}{\|Qx\|^2} \]
\[ = \lambda_3^2 u_3^* Du_5 + \text{tr}_{\text{Span}\{u_3, u_4\}} QDQ \]
\[ - \frac{(Q^{-1}x^\perp)^* QDQ(Q^{-1}x^\perp)}{\|Q^{-1}x^\perp\|^2} \]
\[ = \lambda_3^2 u_3^* Du_5 + \lambda_4^2 u_4^* Du_3 + \lambda_4^2 u_4^* Du_4 - \frac{(x^\perp)^* Dx^\perp}{\|Q^{-1}x^\perp\|^2}. \]
From $\text{tr} D = 0$ and
\[ u_3^* Du_3 + u_4^* Du_4 = x^* Dx + (x^\perp)^* Dx^\perp = -u_5^* Du_5 + (x^\perp)^* Dx^\perp \]
it follows that
\[ (x^\perp)^* Dx^\perp = u_3^* Du_3 + u_4^* Du_4 + u_5^* Du_5 = -(u_1^* Du_1 + u_2^* Du_2) \]
and thus
\[ \text{tr}_{Q_{\mathcal{Y}_a}} QDQ = \lambda_3^2 u_3^* Du_3 + \lambda_4^2 u_4^* Du_4 + \lambda_4^2 u_4^* Du_5 + \frac{u_1^* Du_1 + u_2^* Du_2}{\|Q^{-1}x^\perp\|^2}. \]
We have $\frac{1}{\|Q^{-1}x^\perp\|^2} \in [\lambda_3^2, \lambda_4^2]$, since $x^\perp$ is a normalized element of $\text{Span}\{u_3, u_4\}$, therefore
\[ \text{tr}_{Q_{\mathcal{Y}_a}} QDQ \geq \lambda_3^2 u_3^* Du_3 + \lambda_4^2 u_4^* Du_4 + \lambda_4^2 u_4^* Du_5 + \lambda_3^2 (u_1^* Du_1 + u_2^* Du_2) \]
\[ \geq \sum_{i=1}^5 \lambda_i^2 u_i^* Du_i = \text{tr} QDQ \geq 0. \]
(b) On the other hand, there exists a normalized $y \in \text{Span}\{u_3, u_4\}$ with $y^* Dy = -u_2^* Du_2$. With $\mathcal{Y}_b = \text{Span}\{y, u_2\}$ and $y^\perp$ such that $y, y^\perp$ is an
orthonormal basis of $\text{Span}\{u_3, u_4\}$ we get

$$\text{tr}_{Qy_b} QDQ = \lambda_2^2 u_2^* Du_2 + \frac{(Qy)^* QDQ(Qy)}{\|Qy\|^2}$$

$$= \lambda_2^2 u_2^* Du_2 + \text{tr}_{\text{Span}\{u_3, u_4\}} QDQ$$

$$- \frac{(Q^{-1}y^\perp)^* QDQ(Q^{-1}y^\perp)}{\|Q^{-1}y^\perp\|^2}$$

$$= \lambda_2^2 u_2^* Du_2 + \lambda_3^2 u_3^* Du_3 + \lambda_4^2 u_4^* Du_4 - \frac{(y^\perp)^* Dy^\perp}{\|Q^{-1}y^\perp\|^2}.$$ 

From $\text{tr} D = 0$ and

$$u_3^* Du_3 + u_4^* Du_4 = y^* Dy + (y^\perp)^* Dy^\perp = -u_2^* Du_2 + (y^\perp)^* Dy^\perp$$

it follows that

$$(y^\perp)^* Dy^\perp = u_2^* Du_2 + u_3^* Du_3 + u_4^* Du_4$$

and

$$\text{tr}_{Qy_b} QDQ = \lambda_2^2 u_2^* Du_2 + \lambda_3^2 u_3^* Du_3 + \lambda_4^2 u_4^* Du_4$$

$$- \frac{u_2^* Du_2 + u_3^* Du_3 + u_4^* Du_4}{\|Q^{-1}y^\perp\|^2}.$$ 

We have $\frac{1}{\|Q^{-1}y^\perp\|^2} \in [\lambda_3^2, \lambda_4^2]$ since $y^\perp$ is a normalized element of $\text{Span}\{u_3, u_4\}$, therefore, since $-(u_2^* Du_2 + u_3^* Du_3 + u_4^* Du_4) > 0$,

$$\text{tr}_{Qy_b} QDQ \leq \lambda_2^2 u_2^* Du_2 + \lambda_3^2 u_3^* Du_3 + \lambda_4^2 u_4^* Du_4$$

$$- \lambda_2^2 (u_2^* Du_2 + u_3^* Du_3 + u_4^* Du_4)$$

$$= \left(\lambda_2^2 - \lambda_3^2\right) u_2^* Du_2 + \left(\lambda_2^2 - \lambda_4^2\right) u_3^* Du_3 \leq 0.$$ 

(c) We will now construct the map $\mathcal{Y}$ exactly as in the proof of Proposition 5.7. By assumption, $\text{Span}\{u_2, u_3\}$ is $D$-positive definite, thus we have $\text{nr}(-K_{25}) \subseteq \text{nr}(K_{34})$. It follows that for each normalized element $v_2 \in \text{Span}\{u_2, u_3\}$ there exists a normalized element $v_1 = v_1(v_2) \in \text{Span}\{u_3, u_4\}$ so that $v_1^* Dv_1 = -v_2^* Dv_2$. We can choose $v_1 = f(\alpha) = \cos(\alpha)\tilde{u}_3 + \sin(\alpha)\tilde{u}_4$ with $\alpha \in [0, \frac{\pi}{2}]$ and $\tilde{u}_3, \tilde{u}_4$ are orthonormal such that $[\tilde{u}_3 \tilde{u}_4]^* D[\tilde{u}_3 \tilde{u}_4] = \text{diag}(\mu_3, \mu_4)$ where $\mu_4 < 0 < \mu_3$. Then the mapping $g : \alpha \mapsto f(\alpha)^* Df(\alpha)$ is continuous and strictly monotonically decreasing and therefore continuously invertible. Since the mapping $v_2 \mapsto v_2^* Dv_2$ is also continuous, we can assume the mapping

$$v_2 \mapsto v_1(v_2) = v_1(g^{-1}(v_2^* Dv_2))$$
to be continuous.
In particular we can assume, after choosing the parameter $t$ accordingly, that $\mathcal{Y}(t) = \text{Span}\{v_1(t), v_2(t)\}$ is continuous on $[0, \frac{\pi}{2}]$ with $\mathcal{Y}(0) = \text{Span}\{x, u_3\}$ and $\mathcal{Y}(\frac{\pi}{2}) = \text{Span}\{y, u_2\}$ where $x, y$ are as constructed in (a) and (b).
Now, $\text{tr}_{Q\mathcal{Y}(0)} QDQ \geq \text{tr} QDQ \geq 0$ and $\text{tr}_{Q\mathcal{Y}(\frac{\pi}{2})} QDQ \leq 0$. By continuity of $\mathcal{Y}$, the intermediate value theorem implies the existence of $t_0 \in [0, \frac{\pi}{2}]$ with
$$\text{tr}_{Q\mathcal{Y}(t_0)} QDQ = 0.$$ Then our proposition follows from Lemma 5.3 (i). \qed

\section*{5.2. One $D$-positive Eigenvector of $Q$}

Here we prove the analogue of Proposition 4.6 with an approach as in section 5.1. Thus, we consider the following cases:

(i) $u_5^* Du_5 > 0$  $u_j^* Du_j < 0$ \quad for all $j \neq 5$
(ii) $u_4^* Du_4 > 0$  $[u_j u_k]^* D[u_j u_k] < 0$ \quad for all distinct $j, k \neq 4$
(iii) $u_3^* Du_3 > 0$  $[u_j u_k]^* D[u_j u_k] < 0$ \quad for all distinct $j, k \neq 3$
(iv) $u_2^* Du_2 > 0$  $[u_j u_k]^* D[u_j u_k] < 0$ \quad for all distinct $j, k \neq 2$.

As was the case in section 5.1, case (i) is more general than the others since $[u_j u_k]^* D[u_j u_k] < 0$ implies $u_j^* Du_j, u_k^* Du_k < 0$.

**Proposition 5.9.** Let $\text{tr} D = 0$ and $\text{tr} QDQ \geq 0$ and let $u_1, \ldots, u_5$ and $\lambda_1 \leq \cdots \leq \lambda_5$ be a set of normalized eigenvectors and corresponding eigenvalues of $Q$.
Assume that $u_5^* Du_5 > 0$ and $u_j^* Du_j < 0$ for all $j \neq 5$. Then there exists a gyroscopic stabilizer.

**Proof.** We construct a 3-dimensional space $\mathcal{Y}_a$ with
(a) $\text{tr}_{\mathcal{Y}_a} D = 0$ and $0 \leq \text{tr} Q_{\mathcal{Y}_a} QDQ$,
(b) $0 \leq \text{tr} Q_{\mathcal{Y}_b} QDQ$, where $\mathcal{Y}_b = \mathcal{Y}_a^\perp$.
Then according to lemma 5.3 (iii), there exists a gyroscopic stabilizer.
(a) From $\text{tr} D = \sum_{k=1}^5 u_k^* Du_k = 0$ it follows that
$$u_5^* Du_5 \geq -(u_2^* Du_2 + u_3^* Du_3) \geq 0 \geq u_4^* Du_4.$$ This implies that there exists a normalized $y_1 \in \text{Span}\{u_4, u_5\}$ with
$$y_1^* Dy_1 + u_2^* Du_2 + u_3^* Du_3 = 0.$$ We define $\mathcal{Y}_a = \text{Span}\{y_1, u_2, u_3\}$. By construction, we have
$$\text{tr}_{\mathcal{Y}_a} D = 0.$$
Also by construction, \( y_1, u_3, u_4 \) remain orthogonal under multiplication with \( Q \) or \( Q^{-1} \), thus we can compute

\[
\text{tr}_{Q^{-1}Y_a} QDQ = \lambda_2^2 u_2^* Du_2 + \lambda_3^2 u_3^* Du_3 + \frac{(Q^{-1}y_1)^*QDQ(Q^{-1}y_1)}{\|Q^{-1}y_1\|^2}.
\]

(5.6)

\[
= \lambda_2^2 u_2^* Du_2 + \lambda_3^2 u_3^* Du_3 + \frac{y_1^* Dy_1}{\|Q^{-1}y_1\|^2}.
\]

Now, since \( y_1 \in \text{Span}\{u_4, u_5\} \), we have

\[
\frac{1}{\|Q^{-1}y_1\|^2} \in [\lambda_4^2, \lambda_5^2].
\]

It follows with \( y_1^* Dy_1 > 0 \) and an estimation for the denominator of (5.6) that

\[
\text{tr}_{Q^{-1}Y_a} QDQ \geq \lambda_2^2 u_2^* Du_2 + \lambda_3^2 u_3^* Du_3 + \lambda_1^2 (-u_2^* Du_2 - u_3^* Du_3) \equiv \lambda_2^2 u_2^* Du_2 + \lambda_3^2 u_3^* Du_3 \geq 0.
\]

(b) We define \( Y_b = Y_a^\perp \). From \( \text{tr}_{Y_a} D = 0 \) and \( \text{tr} D = 0 \) it follows that \( \text{tr}_{Y_b} D = 0 \). We get

\[
\text{tr}_{QY_b} QDQ = \text{tr} QDQ - \text{tr}_{Q^{-1}Y_a} QDQ
\]

\[
= \sum_{k=1}^{5} \lambda_k^2 u_k^* Du_k - \frac{y_1^* Dy_1}{\|Q^{-1}y_1\|^2} - \lambda_2^2 u_2^* Du_2 - \lambda_3^2 u_3^* Du_3
\]

\[
\geq \lambda_2^2 u_1^* Du_1 + \lambda_3^2 u_4^* Du_4 + \lambda_5^2 u_5^* Du_5
\]

\[
+ \lambda_2^2 (u_2^* Du_2 + u_3^* Du_3)
\]

\[
\geq \lambda_2^2 \sum_{k=1}^{5} u_k^* Du_k = 0,
\]

which shows that we fulfill the requirements of Lemma 5.3 (iii). Thus there exists a gyroscopic stabilizer. \( \square \)

**Proposition 5.10.** Let \( \text{tr} D = 0 \) and \( \text{tr} QDQ \geq 0 \) and let \( u_1, \ldots, u_5 \) and \( \lambda_1 \leq \cdots \leq \lambda_5 \) be a set of normalized eigenvectors and corresponding eigenvalues of \( Q \). Assume that \( u_4^* Du_4 > 0 \) and let let

\[
K_{jk} = [u_j u_k]^* D[u_j u_k]
\]

be negative definite for each pair \( j, k \in \{1, 2, 3, 5\} \). Then there exists a gyroscopic stabilizer.
Proof. The proof is divided into the following steps.
(a) We show the existence of a two-dimensional space \( A \) with
\[
\text{tr}_A D = 0, \quad \text{tr}_{QA} QDQ \leq \text{tr} QDQ.
\]
(b) A second two-dimensional space \( B \) is constructed with
\[
\text{tr}_B D = 0, \quad \text{tr}_{QB} QDQ \geq 0.
\]
(c) The existence of a continuous mapping \( Y \), whose image is in the set \( \mathcal{P} \) of two-dimensional subspaces of \( \mathbb{R}^5 \), with the properties
\[
Y : [0, \frac{\pi}{2}] \rightarrow \mathcal{P}, \quad t \mapsto Y(t), \quad \dim Y(t) = 2 \text{ for all } t, \quad (5.7) \quad Y(0) = A, \quad Y(\frac{\pi}{2}) = B, \quad \text{tr}_{Y(t)} D = 0 \text{ for all } t,
\]
and the existence of a \( t_0 \in [0, \frac{\pi}{2}] \) with \( \text{tr}_{Y(t_0)} QDQ = 0 \) is shown. Then our proposition follows from Lemma 5.3 (i).

(a) From \( u_i^* Du_4 > 0 \) and \( u_i^* Du_4 < 0 \) for all \( j \neq 4 \) and \( 0 = \text{tr} D = \sum_{i=1}^{5} u_i^* Du_i \) it follows that there exists a normalized \( x \in \text{Span}\{u_4, u_5\} \) with \( x^* Dx = -u_1^* Du_1 \). We then have with \( A = \text{Span}\{x, u_1\} \) and \( x^\perp \) such that \( \{x, x^\perp\} \) is an orthonormal basis of \( \text{Span}\{u_4, u_5\} \):
\[
\text{tr}_{QA} QDQ = \lambda_1^2 u_1^* Du_1 + \frac{(Qx)^* QDQ(Qx)}{\|Qx\|^2} = \lambda_1^2 u_1^* Du_1 + \text{tr}_{\text{Span}\{u_4, u_5\}} QDQ - \frac{(Q^{-1}x^\perp)^* QDQ(Q^{-1}x^\perp)}{\|Q^{-1}x^\perp\|^2} = \lambda_1^2 u_1^* Du_1 + \lambda_2^2 u_4^* Du_4 + \lambda_3^2 u_5^* Du_5 - \frac{(x^\perp)^* Dx^\perp}{\|Q^{-1}x^\perp\|^2}.
\]
Again from \( \text{tr} D = 0 \) and \( x^* Dx = -u_1^* Du_1 \) it follows that
\[
(x^\perp)^* Dx^\perp = -(u_2^* Du_2 + u_3^* Du_4)
\]
and thus
\[
\text{tr}_{QA} QDQ = \lambda_1^2 u_1^* Du_1 + \lambda_2^2 u_4^* Du_4 + \lambda_3^2 u_5^* Du_5 + \frac{u_2^* Du_2 + u_3^* Du_3}{\|Q^{-1}x^\perp\|^2}.
\]
Now we have \( \frac{1}{\|Q^{-1}x^\perp\|^2} \in [\lambda_4^2, \lambda_5^2] \) and \( u_2^* Du_2 + u_3^* Du_3 < 0 \), therefore
\[
\text{tr}_{QA} QDQ \leq \lambda_1^2 u_1^* Du_1 + \lambda_2^2 u_4^* Du_4 + \lambda_3^2 u_5^* Du_5 + \lambda_4^2 (u_2^* Du_2 + u_3^* Du_3) \leq \sum_{i=1}^{5} \lambda_i^2 u_i^* Du_i = \text{tr} QDQ.
\]
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(b) On the other hand, there exists a normalized $y \in \text{Span}\{u_2, u_4\}$ with $y^*Dy + u_1^*Du_1 = 0$. With $B = \text{Span}\{y, u_1\}$ and $(y^{})^*Dy^{} = -(u_3^*Du_3 + u_5^*Du_5)$ we get

$$\text{tr}_{QB} QDQ = \lambda_1^2 u_1^*Du_1 + \frac{(Qy)^*QDQ(Qy)}{\|Qy\|^2}$$

$$= \lambda_1^2 u_1^*Du_1 + \lambda_2^2 u_2^*Du_2 + \lambda_4^2 u_4^*Du_4 - \frac{(y^{})^*Dy^{} - \|Q^{-1}y^{}\|^2}{\|Q^{-1}y^{}\|^2}$$

$$= \lambda_1^2 u_1^*Du_1 + \lambda_2^2 u_2^*Du_2 + \lambda_4^2 u_4^*Du_4 + \frac{u_3^*Du_3 + u_5^*Du_5}{\|Q^{-1}y^{}\|^2}$$

$$\geq \lambda_1^2 u_1^*Du_1 + \lambda_2^2 u_2^*Du_2 + \lambda_4^2 u_4^*Du_4 + \lambda_5^2(u_3^*Du_3 + u_5^*Du_5)$$

$$\geq \lambda_2^2 \sum_{i=1}^5 u_i^*Du_i = 0.$$  

(c) We consider the space $\text{Span}\{u_2, u_4, u_5\}$. We already showed in (a) and (b) that $-u_1^*Du_1$ is in the numerical range of $K_{245} = [u_2, u_4, u_5]^*D[u_2, u_4, u_5]$. This implies that 0 is in the numerical range of $(K_{245} + u_1^*Du_1I_3)$. By construction of $x$ and $y$, it follows that $x = [u_2, u_4, u_5]z_x$ for some normalized $z_x \in \mathbb{R}^3$ with $z_x^*(K_{245} + u_1^*Du_1I_3)z_x = 0$ and we have $y = [u_2, u_4, u_5]z_y$. Now by Proposition 2.7 there exists a continuous mapping $z : [0, \pi/2] \mapsto \mathbb{R}^5$ with $\|z(t)\| = 1$, $z(0) = z_x$ and either $z(\pi/2) = z_y$ or $z(\pi/2) = -z_y$. We put

$$\mathcal{Y}(t) = \text{Span}\{[u_2, u_4, u_5]z(t), u_1\}$$

for all $t$. Note that $\mathcal{Y}(\pi/2) = \text{Span}\{y, u_1\}$ and $\mathcal{Y}$ is continuous by definition. As seen above in (a) and (b), we have

$$\text{tr}_{Q\mathcal{Y}(0)} QDQ = \text{tr}_{Q\mathcal{A}} QDQ \leq \text{tr} QDQ$$

and

$$\text{tr}_{Q\mathcal{Y}(\pi/2)} QDQ = \text{tr}_{Qg} QDQ \geq 0.$$  

If we already have either

$$0 \leq \text{tr}_{Q\mathcal{Y}(0)} QDQ \leq \text{tr} QDQ$$

or

$$0 \leq \text{tr}_{Q\mathcal{Y}(\pi/2)} QDQ \leq \text{tr} QDQ,$$
we have found a space as proposed.
Otherwise, we have
\[
\text{tr}_{QY(0)} QDQ \leq 0, \\
\text{tr}_{QY(\pi/2)} QDQ \geq \text{tr} QDQ.
\]
But then, by continuity of the trace and the intermediate value theorem, for any value \(c \in [\text{tr}_{QY(0)} QDQ, \text{tr}_{QY(\pi/2)} QDQ]\) there exists a \(t_0 \in [0, \pi/2]\) such that \(c = \text{tr}_{QY(t_0)} QDQ\). Choose \(c = \frac{\text{tr} QDQ}{2}\) and \(t_0\) accordingly. Then
\[
0 \leq \text{tr}_{QY(t_0)} QDQ \leq \text{tr} QDQ.
\]
Then our proposition follows from Lemma 5.3 (i).

\[ \square \]

**Proposition 5.11.** Let \(\text{tr} D = 0\) and \(\text{tr} QDQ \geq 0\) and let \(u_1, \ldots, u_5\) and \(\lambda_1 \leq \cdots \leq \lambda_5\) be a set of normalized eigenvectors and corresponding eigenvalues of \(Q\). Assume that \(u_3^*Du_3 > 0\) and let
\[
K_{jk} = [u_j u_k]^* D [u_j u_k]
\]
be negative definite for each pair \(j, k \in \{1, 2, 4, 5\}\). Then there exists a gyroscopic stabilizer.

**Proof.** The strategy for the proof is as follows.
(a) We show the existence of a two-dimensional space \(\mathcal{Y}_a\) with
\[
\text{tr}_{\mathcal{Y}_a} D = 0, \quad \text{tr}_{Q\mathcal{Y}_a} QDQ \geq 0.
\]
(b) Then we show the existence of a second two-dimensional space \(\mathcal{Y}_b\) with
\[
\text{tr}_{\mathcal{Y}_b} D = 0, \quad \text{tr}_{Q\mathcal{Y}_b} QDQ \leq 0.
\]
(c) We construct a continuous mapping \(\mathcal{Y}\), whose image is in the set \(\mathcal{P}\) of two-dimensional subspaces of \(\mathbb{R}^5\), with the properties
\[
\mathcal{Y} : [0, \pi] \mapsto \mathcal{P}, \quad t \mapsto \mathcal{Y}(t), \quad \dim \mathcal{Y}(t) = 2 \text{ for all } t,
\]

(5.8) \(\mathcal{Y}(0) = \mathcal{Y}_a, \quad \mathcal{Y}(\pi) = \mathcal{Y}_b, \quad \text{tr}_{\mathcal{Y}(t)} D = 0 \text{ for all } t.
\]
We show the existence of a \(t_0 \in [0, \pi]\) with \(\text{tr}_{Q\mathcal{Y}(t_0)} QDQ = 0\), then from Lemma 5.3 (i), the existence of a gyroscopic stabilizer follows.
(a) From our assumptions it follows that \(u_3^*Du_3 > -u_1^*Du_1 > u_2^*Du_2\), thus there exists a normalized \(x \in \text{Span}\{u_2, u_3\}\) with \(x^*Dx = -u_1^*Du_1\).
We then have with \(\mathcal{Y}_a = \text{Span}\{x, u_1\}\) and \(x^\perp\) such that \(x, x^\perp\) is an
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orthonormal basis of \( \text{Span}\{u_2, u_3\} \):

\[
\text{tr}_{Qy_a} QDQ = \lambda_1^2 u_1^* D u_1 + \frac{(Qx)^* QDQ(Qx)}{\|Qx\|^2}
\]

\[
= \lambda_1^2 u_1^* D u_1 + \text{tr}_{\text{Span}\{u_2, u_3\}} QDQ - (Q^{-1} x^+)^* QDQ(Q^{-1} x^+) \|Q^{-1} x^+\|^2
\]

\[
= \lambda_1^2 u_1^* D u_1 + \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3 - \frac{(x^+)^* D x^+}{\|Q^{-1} x^+\|^2}.
\]

From \( \text{tr} D = 0 \)

\[
u_2^* D u_2 + u_3^* D u_3 = x^* D x + (x^+)^* D x^+ = -u_1^* D u_1 + (x^+)^* D x^+
\]

it follows that

\[
(x^+)^* D x^+ = u_1^* D u_1 + u_2^* D u_2 + u_3^* D u_3
\]

and thus

\[
\text{tr}_{Qy_a} QDQ = \lambda_1^2 u_1^* D u_1 + \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3
\]

\[
- \frac{u_1^* D u_1 + u_2^* D u_2 + u_3^* D u_3}{\|Q^{-1} x^+\|^2}.
\]

We have \( \frac{1}{\|Q^{-1} x^+\|^2} \in [\lambda_2^2, \lambda_3^2] \) since \( x^+ \) is a normalized element of \( \text{Span}\{u_2, u_3\} \), therefore, since \(- (u_1^* D u_1 + u_2^* D u_2 + u_3^* D u_3) < 0, \)

\[
\text{tr}_{Qy_a} QDQ \geq \lambda_1^2 u_1^* D u_1 + \lambda_2^2 u_2^* D u_2 + \lambda_3^2 u_3^* D u_3
\]

\[
- \lambda_2^2 (u_1^* D u_1 + u_2^* D u_2 + u_3^* D u_3)
\]

\[
= (\lambda_1^2 - \lambda_2^2) u_1^* D u_1 + (\lambda_2^2 - \lambda_3^2) u_2^* D u_2 \geq 0.
\]

(b) On the other hand, there exists a normalized \( y \in \text{Span}\{u_3, u_4\} \) with \( y^* D y = -u_5^* D u_5 \). With \( y_a = \text{Span}\{y, u_4\} \) and \( y^\perp \) such that \( y, y^\perp \) is an orthonormal basis of \( \text{Span}\{u_3, u_4\} \) we obtain

\[
\text{tr}_{Qy_b} QDQ = \lambda_3^2 u_5^* D u_5 + \frac{(Qy)^* QDQ(Qy)}{\|Qy\|^2}
\]

\[
= \lambda_3^2 u_5^* D u_5 + \text{tr}_{\text{Span}\{u_3, u_4\}} QDQ - (Q^{-1} y^\perp)^* QDQ(Q^{-1} y^\perp) \|Q^{-1} y^\perp\|^2
\]

\[
= \lambda_3^2 u_5^* D u_5 + \lambda_4^2 u_4^* D u_4 + \lambda_5^2 u_5^* D u_5 - \frac{(y^\perp)^* D y^\perp}{\|Q^{-1} y^\perp\|^2}.
\]

From \( \text{tr} D = 0 \)

\[
u_3^* D u_3 + u_4^* D u_4 = y^* D y + (y^\perp)^* D y^\perp = -u_5^* D u_5 + (y^\perp)^* D y^\perp
\]
it follows that
\[ (y^\perp)^* D y^\perp = u_3^* D u_3 + u_4^* D u_4 + u_5^* D u_5 \]
and
\[ \text{tr}_{QYb} QDQ = \frac{\lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 + \lambda_5^2 u_5^* D u_5 - u_3^* D u_3 + u_4^* D u_4 + u_5^* D u_5}{\|Q^{-1}y^\perp\|^2} . \]
We have \( \frac{1}{\|Q^{-1}y^\perp\|^2} \in [\lambda_3^2, \lambda_4^2] \) since \( y^\perp \) is a normalized element of \( \text{Span}\{u_3, u_4\} \), therefore, since \( -(u_3^* D u_3 + u_4^* D u_4 + u_5^* D u_5) < 0 \),
\[ \text{tr}_{QYb} QDQ \leq \lambda_3^2 u_3^* D u_3 + \lambda_4^2 u_4^* D u_4 + \lambda_5^2 u_5^* D u_5 - \lambda_3^2 (u_3^* D u_3 + u_4^* D u_4 + u_5^* D u_5) \]
\[ = (\lambda_3^2 - \lambda_3^2) u_3^* D u_4 + (\lambda_3^2 - \lambda_3^2) u_5^* D u_5 \leq 0 . \]
(c) From \( u_3^* D u_3 \geq -u_1^* D u_1 \geq u_2^* D u_2 \) it follows that \(-u_1^* D u_1\) is in the numerical range of 
\[ K_{234} = [u_2, u_3, u_4]^* D[u_2, u_3, u_4] , \]
in particular there exists \( x \in \text{Span}\{u_2, u_3\} \) with \( x^* D x = -u_1^* D u_1 \) as in part (a) and there exists also \( w \in \text{Span}\{u_3, u_4\} \) with \( w^* D w = -u_1^* D u_1 \). If we put \( x = [u_2, u_3, u_4]z_x \) and \( w = [u_2, u_3, u_4]z_w \), then both \( z_x, z_w \) are zeros of the quadratic form given by \( (K_{234} + u_1^* D u_1 I_3) \).

Now by Proposition 2.7 there exists a continuous mapping \( z : [0, \frac{\pi}{2}] \mapsto \mathbb{R}^5 \) with \( \|z(t)\| = 1 \), \( z(0) = z_x \) and either \( z(\frac{\pi}{2}) = z_w \) or \( z(\frac{\pi}{2}) = -z_w \). We put \( Y(t) = \text{Span}\{[u_2, u_3, u_4]z(t), u_1\} \) for all \( t \in [0, \frac{\pi}{2}] \). Note that \( Y(\frac{\pi}{2}) = \text{Span}\{w, u_1\} \), and \( Y \) is continuous by definition.

We will now extend the map \( Y \) and use an argument analogous to the first part of the proof of Proposition 4.3. By assumption, \( \text{Span}\{u_1, u_5\} \) is \( D \)-negative definite, thus the numerical range of 
\[ K_{15} = [u_1, u_5]^* D[u_1, u_5] \]
is completely contained in the interval \([u_1^* D u_1 + u_5^* D u_5, 0]\), we write \( \text{nr}(K_{15}) \subseteq [u_1^* D u_1 + u_5^* D u_5, 0] \). Since we have 
\[ u_3^* D u_3 \geq -(u_5^* D u_5 + u_1^* D u_1) \geq 0 \geq u_4^* D u_4 \]
it follows that \( \text{nr}(-K_{15}) \subseteq \text{nr}(K_{34}) \). Thus for each normalized element \( v_2 \in \text{Span}\{u_4, u_5\} \) there exists a normalized element \( v_1 = v_1(v_2) \in \text{Span}\{u_3, u_4\} \) such that \( v_1^* D v_1 = -v_2^* D v_2 \). We can choose \( v_1 = f(\alpha) = \cos(\alpha)\tilde{u}_3 + \sin(\alpha)\tilde{u}_4 \) with \( \alpha \in [0, \frac{\pi}{2}] \) and \( \tilde{u}_3, \tilde{u}_4 \) are orthonormal such that \( [\tilde{u}_3\tilde{u}_4]^* D[\tilde{u}_3\tilde{u}_4] = \text{diag}(\mu_3, \mu_4) \) where \( \mu_3 > 0 \geq \mu_2 \). Then the mapping \( g : \alpha \mapsto f(\alpha)^* D f(\alpha) \) is continuous and strictly monotonically
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decreasing and therefore continuously invertible. Since the mapping
$v_2 \mapsto v_2^* D v_2$ is also continuous, we can assume the mapping
\[ v_2 \mapsto v_1(v_2) = v_1(g^{-1}(v_2^* D v_2)) \]
to be continuous.

In particular we can assume, after choosing the parameter $t$ accordingly,
that $\mathcal{Y}(t) = \text{Span}\{v_1(t), v_2(t)\}$ is continuous on $[\frac{\pi}{2}, \pi]$ with
$\mathcal{Y}(\frac{\pi}{2}) = \text{Span}\{w, u_1\}$ and $\mathcal{Y}(\pi) = \text{Span}\{y, u_5\}$ where $y$ is as constructed in (b).

Altogether $\mathcal{Y}$ is continuous on the complete interval $[0, \pi]$ and
\[
\text{tr}_{Q \mathcal{Y}(t) \mathcal{Y}(t) \mathcal{Y}(t)} DQ \geq 0, \\
\text{tr}_{Q \mathcal{Y}(\pi) \mathcal{Y}(\pi) \mathcal{Y}(\pi)} DQ \leq 0.
\]
The intermediate value theorem implies the existence of $t_0 \in [0, \pi]$ with
\[
\text{tr}_{Q \mathcal{Y}(t_0) \mathcal{Y}(t_0) \mathcal{Y}(t_0)} DQ = 0.
\]
Thus by Lemma 5.3 (i), there exists a gyroscopic stabilizer.

**Proposition 5.12.** Let $\text{tr} D = 0$ and $\text{tr} QDQ \geq 0$ and let $u_1, \ldots, u_5$
and $\lambda_1 \leq \cdots \leq \lambda_5$ be a set of normalized eigenvectors and corresponding eigenvalues of $Q$. Assume that $u_2^* D u_2 > 0$ and let
\[
K_{jk} = [u_j \ u_k]^* D [u_j \ u_k]
\]
be negative definite for each pair $j, k \in \{1, 3, 4, 5\}$.

Then there exists a gyroscopic stabilizer.

**Proof.** The strategy is basically the same as in the previous proof.

(a) We show the existence of a space $\mathcal{Y}_a$ with
\[
\text{tr}_{\mathcal{Y}_a} D = 0, \quad \text{tr}_{Q \mathcal{Y}_a \mathcal{Y}_a} DQ \geq \text{tr} QDQ,
\]
(b) then the existence of a second space $\mathcal{Y}_b$ with
\[
\text{tr}_{\mathcal{Y}_b} D = 0, \quad \text{tr}_{Q \mathcal{Y}_b \mathcal{Y}_b} DQ \leq 0.
\]
(c) We construct a continuous mapping $\mathcal{Y}$, whose image is in the set $\mathcal{P}$
of two-dimensional subspaces of $\mathbb{R}^5$, with the properties
\[
\mathcal{Y} : [0, \frac{\pi}{2}] \mapsto \mathcal{P}, \quad t \mapsto \mathcal{Y}(t), \quad \dim \mathcal{Y}(t) = 2 \text{ for all } t
\]
\[
(5.9) \quad \mathcal{Y}(0) = \mathcal{Y}_a, \quad \mathcal{Y}(\frac{\pi}{2}) = \mathcal{Y}_b, \quad \text{tr}_{\mathcal{Y}(t) \mathcal{Y}(t) \mathcal{Y}(t)} D = 0 \text{ for all } t.
\]
Then the existence of a $t_0 \in [0, \frac{\pi}{2}]$ with $\text{tr}_{Q \mathcal{Y}(t_0) \mathcal{Y}(t_0) \mathcal{Y}(t_0)} DQ = 0$ follows, and
from Remark 5.3 (i), the existence of a gyroscopic stabilizer follows.

(a) By assumption there exists a normalized $x \in \text{Span}\{u_2, u_3\}$ with
$x^*Dx = -u_1^*Du_1$. Then we have with $\mathcal{V}_a = \text{Span}\{x, u_1\}$ and $x^\perp$ such that $\{x, x^\perp\}$ is an orthonormal basis of the space $\text{Span}\{u_2, u_3\}$:

$$\text{tr}_{\mathcal{V}_a} QDQ = \lambda_1^2 u_1^*Du_1 + \frac{(Qx)^*QDQ(Qx)}{\|Qx\|^2}$$

$$= \lambda_1^2 u_1^*Du_1 + \text{tr}_{\text{Span}\{u_2, u_3\}} QDQ$$

$$- \frac{(Q^{-1}x^\perp)^*QDQ(Q^{-1}x^\perp)}{\|Q^{-1}x^\perp\|^2}$$

$$= \lambda_1^2 u_1^*Du_1 + \lambda_2^2 u_2^*Du_2 + \lambda_3^2 u_3^*Du_3 - \frac{(x^\perp)^*Dx^\perp}{\|Q^{-1}x^\perp\|^2}.$$ 

From $\text{tr}D = 0$ and $u_2^*Du_2 + u_3^*Du_3 = x^*Dx + (x^\perp)^*Dx^\perp = -u_1^*Du_1 + (x^\perp)^*Dx^\perp$ it follows that $$(x^\perp)^*Dx^\perp = u_1^*Du_1 + u_2^*Du_2 + u_3^*Du_3 = -(u_4^*Du_4 + u_5^*Du_5)$$

and thus

$$\text{tr}_{\mathcal{V}_a} QDQ = \lambda_1^2 u_1^*Du_1 + \lambda_2^2 u_2^*Du_2 + \lambda_3^2 u_3^*Du_3 + \frac{u_4^*Du_4 + u_5^*Du_5}{\|Q^{-1}x^\perp\|^2}.$$ 

We have $\frac{1}{\|Q^{-1}x^\perp\|^2} \in [\lambda_2^2, \lambda_3^2]$ since $x^\perp$ is a normalized element of $\text{Span}\{u_2, u_3\}$, and with $u_4^*Du_4 + u_5^*Du_5 < 0$ we get

$$\text{tr}_{\mathcal{V}_a} QDQ \geq \lambda_1^2 u_1^*Du_1 + \lambda_2^2 u_2^*Du_2 + \lambda_3^2 u_3^*Du_3 + \lambda_4^2 (u_4^*Du_4 + u_5^*Du_5)$$

$$\geq \sum_{i=1}^5 \lambda_i^2 u_i^*Du_i = \text{tr}QDQ \geq 0.$$ 

(b) On the other hand there exists a normalized $y \in \text{Span}\{u_2, u_3\}$ with $y^*Dy = -u_1^*Du_1$. With $\mathcal{V}_a = \text{Span}\{y, u_1\}$ and $y^\perp$ such that $y, y^\perp$ is an orthonormal basis of $\text{Span}\{u_2, u_3\}$ we get

$$\text{tr}_{\mathcal{V}_a} QDQ = \lambda_1^2 u_1^*Du_1 + \frac{(Qy)^*QDQ(Qy)}{\|Qy\|^2}$$

$$= \lambda_1^2 u_1^*Du_1 + \text{tr}_{\text{Span}\{u_2, u_3\}} QDQ$$

$$- \frac{(Q^{-1}y^\perp)^*QDQ(Q^{-1}y^\perp)}{\|Q^{-1}y^\perp\|^2}$$

$$= \lambda_2^2 u_2^*Du_2 + \lambda_3^2 u_3^*Du_3 + \lambda_4^2 u_4^*Du_4 - \frac{(y^\perp)^*Dy^\perp}{\|Q^{-1}y^\perp\|^2}.$$ 

From $\text{tr}D = 0$ and $u_2^*Du_2 + u_3^*Du_3 = y^*Dy + (y^\perp)^*Dy^\perp = -u_1^*Du_1 + (y^\perp)^*Dy^\perp$
it follows that\[ (y^\perp)^* D y^\perp = u_2^* Du_2 + u_3^* Du_3 + u_4^* Du_4\]
and
\[
\text{tr}_{Qy_0} QDQ = \lambda_2^2 u_2^* Du_2 + \lambda_3^2 u_3^* Du_3 + \lambda_4^2 u_4^* Du_4 \\
= \frac{u_2^* Du_2 + u_3^* Du_3 + u_4^* Du_4}{\|Q^{-1} y^\perp\|^2}.
\]
We have \[\frac{1}{\|Q^{-1} y^\perp\|^2} \in [\lambda_2^2, \lambda_3^2]\] since \(y^\perp\) is a normalized element of \(\text{Span}\{u_2, u_3\}\), therefore, since \(-(u_2^* Du_2 + u_3^* Du_3 + u_4^* Du_4) < 0,\)
\[
\text{tr}_{Qy_0} QDQ \leq \lambda_2^2 u_2^* Du_2 + \lambda_3^2 u_3^* Du_3 + \lambda_4^2 u_4^* Du_4 \\
= (\lambda_2^2 - \lambda_3^2) u_3^* Du_3 + (\lambda_4^2 - \lambda_3^2) u_4^* Du_4 \leq 0.
\]
(c) We will now construct the map \(Y\) as in the proof of Proposition 5.7. By assumption, \(\text{Span}\{u_1, u_4\}\) is \(D\)-negative definite, thus we have \(\text{nr}(-K_{14}) \subseteq \text{nr}(K_{23})\). It follows that for each normalized element \(v_2 \in \text{Span}\{u_1, u_4\}\) there exists a normalized element \(v_1 = v_1(v_2) \in \text{Span}\{u_2, u_3\}\) so that \(v_1^* D v_1 = -v_2^* D v_2\). We can choose \(v_1 = f(\alpha) = \cos(\alpha) \tilde{u}_2 + \sin(\alpha) \tilde{u}_3\) with \(\alpha \in [0, \pi]\) and \(\tilde{u}_2, \tilde{u}_3\) are orthonormal such that \([\tilde{u}_2 \tilde{u}_3]^* D [\tilde{u}_2 \tilde{u}_3] = \text{diag}(\mu_2, \mu_3)\) where \(\mu_2 > 0 \geq \mu_3\). Then the mapping \(g : \alpha \mapsto \tilde{f}(\alpha)^* D \tilde{f}(\alpha)\) is continuous and strictly monotonically decreasing and therefore continuously invertible. Since the mapping \(v_2 \mapsto v_2^* D v_2\) is also continuous, we can assume the mapping\[ v_2 \mapsto v_1(v_2) = v_1(g^{-1}(v_2^* D v_2))\]
to be continuous.
In particular we can assume, after choosing the parameter \(t\) accordingly, that \(Y(t) = \text{Span}\{v_1(t), v_2(t)\}\) is continuous on \([0, \pi]\) with \(Y(0) = \text{Span}\{x, u_1\}\) and \(Y(\pi) = \text{Span}\{y, u_4\}\) where \(x, y\) are as constructed in (a) and (b).
Now, \(\text{tr}_{QY(0)} QDQ \geq \text{tr} QDQ \geq 0\) and \(\text{tr}_{QY(\pi/2)} QDQ \leq 0\). By continuity of \(Y\), the intermediate value theorem implies the existence of \(t_0 \in [0, \pi/2]\) with \(\text{try}(t_0) D = 0\) and \(\text{tr}_{QY(t_0)} QDQ = 0\).
Then our proposition follows from Lemma 5.3 (i). \(\square\)
5.3. Two D-indefinite Subspaces

Proposition 4.3
In section 5.1 we were considered all cases where one of the \( u_j^* D u_i \)
was negative and the two-dimensional spaces \( \text{Span}\{u_j, u_k\} \) with \( j, k \neq i \) were each positive definite, in section 5.2 equivalently. Thus the case where we have two indefinite spaces each spanned by a pair of eigenvectors remains to be investigated.

So far we were not able to give a complete proof of the existence of a gyroscopic stabilizer in this case. Thus in this section we will present a construction that at least partly solves of our problem of constructing \( G \).

Let \( \text{tr} \ D = 0 \) and \( \text{tr} \ Q D Q \geq 0 \) and let \( u_1, \ldots, u_5 \) and \( \lambda_1, \ldots, \lambda_5 \) be a set of normalized eigenvectors and corresponding eigenvalues of \( Q \), given in no particular order. Assume that with \( \{i, j, k, m, n\} = \{1, 2, 3, 4, 5\} \) the matrices

\[
K_{jk} = [u_j u_k]^* D [u_j u_k], \quad K_{mn} = [u_m u_n]^* D [u_m u_n]
\]

are each indefinite or singular.

The goal is to construct a three-dimensional space such that the conditions of Lemma 5.3 (ii) are satisfied. Since \( K_{jk}, K_{mn} \) are indefinite or singular, there exist normalized \( z_1 \in \text{Span}\{u_j, u_k\} \) and \( z_2 \in \text{Span}\{u_m, u_n\} \) with \( z_1^* D z_1 = z_2^* D z_2 = 0 \). By construction, \( Q^{-1} z_1 \perp Q^{-1} z_2 \), so we can also compute

\[
\text{tr}_{Q^{-1}\text{Span}\{z_1, z_2\}} Q D Q \frac{(Q^{-1} z_1)^* Q D Q \ast Q^{-1} z_1}{\|Q^{-1} z_1\|^2} + \frac{(Q^{-1} z_2)^* Q D Q \ast Q^{-1} z_2}{\|Q^{-1} z_2\|^2} = 0.
\]

We will extend \( Q^{-1}\text{Span}\{z_1, z_2\} \) to a three-dimensional space, therefore we analyze the value \( \text{tr}_{Q^{-1}\text{Span}\{z_1, z_2, z_3\}} Q D Q \) for a given \( z_3 \) orthogonal on \( z_1, z_2 \). Let \( v_3 \) be orthonormal on \( z_1 \) in \( \text{Span}\{u_j, u_k\} \) and \( v_4 \) orthonormal on \( z_2 \) in \( \text{Span}\{u_m, u_n\} \). Then by construction, \( v_3, v_4, u_i \) are pairwise orthogonal and remain so under multiplication with \( Q \) or \( Q^{-1} \).

Now \( \{Q_{v_3} \|Q v_3\|^{-1}, Q^{-1}_{z_3} \|Q^{-1} z_3\|^{-1}\} \) is an orthonormal basis of

\[
\text{Span}\{u_j, u_k\} = Q^{-1}\text{Span}\{u_j, u_k\} = Q \text{Span}\{u_j, u_k\},
\]

and \( \{Q_{v_4} \|Q v_4\|^{-1}, Q^{-1}_{z_2} \|Q^{-1} z_2\|^{-1}\} \) is an orthonormal basis of

\[
\text{Span}\{u_m, u_n\} = Q^{-1}\text{Span}\{u_m, u_n\} = Q \text{Span}\{u_m, u_n\}.
\]

Let \( z_3 \) be a normalized vector in \( \text{Span}\{v_3, v_4, u_i\} \). Then

\[
\{Q^{-1}_{z_1} \|Q^{-1} z_1\|^{-1}, Q^{-1}_{z_2} \|Q^{-1} z_2\|^{-1}, PQ^{-1}_{z_3} \|Q^{-1} z_3\|^{-1}\}
\]
is an orthonormal basis of $Q^{-1}\text{Span}\{z_1, z_2, z_3\}$, where $P$ denotes the projection onto the space $Q\text{Span}\{v_3, v_4, u_i\}$:

\[
P = \frac{Qv_3 v_3^* P}{\|Qv_3\|^2} + \frac{Qv_4 v_4^* Q}{\|Qv_4\|^2} + \frac{Qu_i u_i^* Q}{\|Qu_i\|^2} = \frac{Qv_3 v_3^* P}{\|Qv_3\|^2} + \frac{Qv_4 v_4^* Q}{\|Qv_4\|^2} + u_i u_i^*,
\]

\[
PQ^{-1} = \frac{Qv_3 v_3^*}{\|Qv_3\|^2} + \frac{Qv_4 v_4^*}{\|Qv_4\|^2} + \frac{u_i u_i^*}{\lambda_i}.
\]

With \(z_3 = av_3 + bv_4 + cu_i\) we get

\[
PQ^{-1}(av_3 + bv_4 + cu_i) = a \frac{Qv_3}{\|Qv_3\|^2} + b \frac{Qv_4}{\|Qv_4\|^2} + c \frac{u_i}{\lambda_i}
\]

\[
= \begin{bmatrix} Qv_3 \\ [Qv_3]^2 \\ Qv_4 \\ [Qv_4]^2 \\ u_i \\ \lambda_i \\ a \\ b \\ c \end{bmatrix},
\]

and

\[
\|PQ^{-1}(av_3 + bv_4 + cu_i)\|^2 = \frac{a^2}{\|Qv_3\|^2} + \frac{b^2}{\|Qv_4\|^2} + \frac{c^2}{\lambda_i^2}.
\]

Now we have

\[
\text{tr}_{Q^{-1}\text{Span}\{z_1, z_2, z_3\}} QDQ
\]

\[
= \text{tr}_{Q^{-1}\text{Span}\{z_1, z_2\}} QDQ + (\frac{PQ^{-1}z_3}{\|PQ^{-1}z_3\|})^* QDQ(\frac{PQ^{-1}z_3}{\|PQ^{-1}z_3\|})
\]

\[
= \frac{z_3^* PQDPQ^{-1} z_3}{\|PQ^{-1}z_3\|^2}.
\]

We put $F = Q^{-1}PQDPQ^{-1}$ and get

\[
F = \begin{pmatrix} \frac{v_3^* Q^2 DQ v_3}{\|Qv_3\|^2} & \frac{v_4^* Q^2 DQ v_4}{\|Qv_4\|^2} & \frac{u_i^* Q^2 DQ u_i}{\|Qu_i\|^2} \\ \frac{v_3^* Q^2 DQ v_3}{\|Qv_3\|^2} & \frac{v_4^* Q^2 DQ v_4}{\|Qv_4\|^2} & \frac{u_i^* Q^2 DQ u_i}{\|Qu_i\|^2} \\ \frac{v_3^* Q^2 DQ v_3}{\|Qv_3\|^2} & \frac{v_4^* Q^2 DQ v_4}{\|Qv_4\|^2} & \frac{u_i^* DQ u_i}{\|Qu_i\|^2} \end{pmatrix}
\]

and

\[
Q_F = (PQ^{-1}[\frac{Qv_3}{\|Qv_3\|^2}\frac{Qv_4}{\|Qv_4\|^2}\frac{u_i}{\lambda_i}])^* PQ^{-1}[\frac{Qv_3}{\|Qv_3\|^2}\frac{Qv_4}{\|Qv_4\|^2}\frac{u_i}{\lambda_i}]
\]

\[
= \begin{pmatrix} 1 \\ [Qv_3]^{-2} \\ [Qv_4]^{-2} \\ \frac{1}{\lambda_i^2} \end{pmatrix},
\]
this allows us to write with $\|v\| = 1$ and $v = (a \ b \ c)$:

$$\text{tr}_{Q^{-1} \text{Span}\{z_1,z_2,z_3\}} QDQ = \frac{v^*Fv}{v^*Q_Fv}$$

and $\text{tr} \ F_0 = \text{tr} \ QDQ$. We also define

$$\tilde{D} = [v_3 \ v_4 \ u_i]^*D[v_3 \ v_4 \ u_i] .$$

In this formulation, what we need is a normalized $v$ such that

$$0 = v^*\tilde{D}v ,$$

$$0 \leq \frac{v^*Fv}{v^*Q_Fv} \leq \text{tr} \ QDQ .$$

(5.10)

Now either the quadratic forms given by $F$ and $\tilde{D}$ have a nontrivial common zero $v_0$, then we put $z_3 = [v_3 \ v_4 \ u_i]v_0$ and have

$$\text{tr}_{\text{Span}\{z_1,z_2,z_3\}} D = 0 ,$$

$$\text{tr}_{Q^{-1} \text{Span}\{z_1,z_2,z_3\}} QDQ = 0 ,$$

and our statement is shown.

Or there exist according to [9] numbers $g,h \in \mathbb{R}$ such that the matrix given by $g\tilde{D} + hF$ is positive definite, which just means that the sign of $v^*Fv$ is invariant on the isotropic vectors of $\tilde{D}$. In that case, one needs to establish that a normalized $v$ as in (5.10) can be found, which so far we were not able to prove.

Our approach here might be too restrictive since with Lemma 5.3, we already demand a very specific form for a gyroscopic stabilizer with a pair of eigenvalues close to zero.

But still, for given pairs $D,Q$, we were always able to find a gyroscopic stabilizer $G$ in $\mathbb{R}^5$ by doing a random search in Matlab. A counterexample in $\mathbb{R}^5$, where $\text{tr} \ D, \text{tr} \ QDQ > 0$ and no gyroscopic stabilizer exists, has not yet been found. We therefore formulate our missing case in $\mathbb{R}^5$ as an open problem.

**Problem 5.13.** Let $\text{tr} \ D = 0$ and $\text{tr} \ QDQ \geq 0$ and let $u_1, \ldots, u_5$ be a set of eigenvectors of $Q$. For $\{i,j,k,m,n\} = \{1,2,3,4,5\}$ in no particular order, let the matrices

$$K_{jk} = [u_j \ u_k]^*D[u_j \ u_k] , \quad K_{mn} = [u_m \ u_n]^*D[u_m \ u_n]$$

be indefinite. Show that there exists a gyroscopic stabilizer.
5.4. Example

In this section, we will demonstrate the construction of a gyroscopic stabilizer in $\mathbb{R}^5$ as in Proposition 5.7. The construction in each case in the first two sections of this chapter is very similar. The difficulty lies in constructing a function $Y$.

**Example 5.14.** Consider the unstable system $\ddot{x} + D \dot{x} + Q^{-2}x = 0$ with

$$
D = \begin{pmatrix}
10 & 4 & -8 & -3 & -8 \\
4 & 6 & -1 & -2 & -2 \\
-8 & -1 & -25 & 12 & 3 \\
-3 & -2 & 12 & 3 & 1 \\
-8 & -2 & 3 & 1 & 11
\end{pmatrix}, \quad Q = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 5
\end{pmatrix}.
$$

With $\frac{\text{tr}D}{n} = 1 < \frac{132}{55} = \frac{\text{tr}QQ\text{tr}Q}{\text{tr}Q^2}$ we put $\tau = 1$ and $P = I$ in Proposition 2.1 and get

$$
D_\tau = \begin{pmatrix}
9 & 4 & -8 & -3 & -8 \\
4 & 5 & -1 & -2 & -2 \\
-8 & -1 & -26 & 12 & 3 \\
-3 & -2 & 12 & 2 & 1 \\
-8 & -2 & 3 & 1 & 10
\end{pmatrix}.
$$

Any space $\text{Span}\{e_3, e_j\}$ with $j \neq 3$ is $D$-indefinite, while the space $\text{Span}\{e_1, e_2, e_4, e_5\}$ is $D$-positive definite, thus we are in the situation of 5.7.

The proposed spaces $Y_a$ and $Y_b$ can be chosen as

$$
Y_a = \text{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ -0.3679 \\ 0.9299 \\ 0 \end{pmatrix}, e_5 \right\}, \quad Y_b = \text{Span}\left\{ \begin{pmatrix} 0 \\ 0.7184 \\ -0.6956 \\ 0 \end{pmatrix}, e_1 \right\}.
$$

Note that even if we stick to the proposed construction, the spaces $Y_a, Y_b$ are not unique, since in $\text{Span}\{e_3, e_4\}$ there are in general at least two linearly independent normalized vectors $x, y$ with $x^*Dx = y^*Dy = -e_5^*D_\tau e_5$. For $Y_b$ holds the same.

For $t \in [0, \frac{\pi}{2}]$, we put $u(t) = \cos(t)u_4 + \sin(t)u_2$. The proposed normalized $x(t)$ with $x(t)^*Dx(t) + e_5^*D_\tau e_5 = 0$ can then be constructed as follows. Define

$$
K(t) = [e_3 \ u(t)]*D[e_3 \ u(t)] + e_5^*D_\tau e_5I.
$$
Let $U(t)$ be a matrix whose columns are an orthonormal basis of eigenvectors of $K(t)$ and $V(t)$ a diagonal matrix with corresponding eigenvalues of $K(t)$ on the diagonal. As the components of $K(t)$ are continuous, so are the eigenvectors and eigenvalues. Thus $U(t), V(t)$ can also be chosen to depend continuously on $t$. Now choose an isotropic vector $\hat{y}$ of $V(t)$, see example 4.8. Since isotropy now only depends on the eigenvalues on the diagonal of $V(t)$, we can choose $\hat{y}$ such that it depends continuously on $t$. Then $y(t) = U(t)(1 1)^T$ is isotropic for $K(t)$. In particular, it also depends continuously on $t$. Then for

$$x(t) = [e_3 \ u(t)]y(t)$$

we have $x(t)^* Dx(t) = -e_5^* D e_5$ by construction.

For $t \in [\frac{\pi}{2}, \pi]$, we define $\hat{u}(t) = \sin(t)e_5 - \cos(t)u_1$ and

$$\hat{K}(t) = [e_3 \ e_2]^* D[e_3 \ e_2] + \hat{u}(t)^* D \hat{u}(t) I .$$

As above, we construct $y(t)$ as an isotropic vector of $\hat{K}(t)$ and put $x(t) = [e_3 \ e_2]y(t)$ for $t \in [\frac{\pi}{2}, \pi]$. With

$$\mathcal{Y}(t) = \begin{cases} \text{Span}\{x(t), u_5\} , & t \in [0, \frac{\pi}{2}] , \\ \text{Span}\{x(t), \hat{u}(t)\} , & t \in [\frac{\pi}{2}, \pi] . \end{cases}$$

We now have $\text{tr}\mathcal{Y}(t) D = 0$ for all $t \in [0, \pi]$. Via Matlab we get

$$\text{tr}_{\mathcal{Y}(0)} Q D Q = 181.9306$$
$$\text{tr}_{\mathcal{Y}(\pi)} Q D Q = -137.7026 ,$$

so none of the above two spaces satisfies the desired trace condition

$$0 < \text{tr}_{\mathcal{Y}(0)} Q D Q < \text{tr} Q D Q ,$$

but there exists an intermediate value such that our condition is satisfied.
5.4. EXAMPLE

In figure 5.1, we see a plot of \( f(t) = \text{tr}_{Q^{-1}Y(t)} QDQ \), the horizontal red lines show the values 0 and \( \text{tr} QDQ \). By construction, there is now an area where \( f(t) \) is in between 0 and \( \text{tr} QDQ \). We pick \( t_0 = 1.2563 \) and get \( f(1.2563) = 62.8646 \) and

\[
\mathcal{Y}(t_0) = \text{Span}\left\{ \begin{pmatrix} 0 \\ 0.7559 \\ -0.6067 \\ 0.2459 \\ 0 \end{pmatrix} \right\}.
\]

We now choose a normalized \( D \)-isotropic vector \( z_1 \) in \( \mathcal{Y}(t_0)^\perp \), complete \( z_1 \) to an orthonormal basis \( \{z_1, z_2, z_3\} \) of \( \mathcal{Y}(t_0)^\perp \) and put

\[
Z = \begin{bmatrix} z_1 & z_2 & z_3 & x(t_0) & u_5 \end{bmatrix}
\begin{pmatrix}
-0.5314 & -0.8170 & -0.2238 & 0 & 0 \\
0.3367 & -0.0663 & -0.5575 & 0.7559 & 0 \\
0.1075 & 0.1423 & -0.7746 & -0.6067 & 0 \\
-0.7698 & 0.5549 & -0.1975 & 0.2459 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

By construction,

\[
G = Z \text{ diag} \left( 0, \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) Z^* 
\]

now satisfies our eigenvector conditions. Via Matlab we find that the system is stabilized by \( \alpha G \) with \( \alpha \geq 296097 \).
5.5. A Reduction Method

In some cases, the problem of constructing a gyroscopic stabilizer \( G \) as in Proposition 2.1 can be reduced to lower space dimensions. The basic idea is similar to case 3.1 in [15].

**Proposition 5.15.** Let \( \text{tr} D, \text{tr} QDQ > 0 \) and let \( \mathcal{B} = \{u_1, \ldots, u_n\} \) be a basis of \( \mathbb{R}^n \) consisting of normalized eigenvectors of \( Q \). Suppose there is a partition
\[
\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 = \{u_1, \ldots, u_t\} + \{u_{t+1}, \ldots, u_n\}
\]
such that \( t \) is even and
\[
\text{tr} \text{Span} \mathcal{B}_1 D > 0, \quad \text{tr} \text{Span} \mathcal{B}_2 D > 0, \quad \text{tr} \text{Span} \mathcal{B}_1 QDQ > 0, \quad \text{tr} \text{Span} \mathcal{B}_2 QDQ > 0.
\]
With \( B_1 = [u_1, \ldots, u_t], B_2 = [u_{t+1}, \ldots, u_n] \) define \( D_1 = B_1^* DB_1, Q_1 = B_1^* QB_1 \) and \( D_2 = B_2^* DB_2, Q_2 = B_2^* QB_2 \).

Let \( G_1 \) be a gyroscopic stabilizer such that
\[
v_1^* D_1 v_1 > 0, \quad w_1^* Q_1 D_1 Q_1 w_1 > 0
\]
for all eigenvectors \( v_1 \) of \( G_1 \) and all eigenvectors \( w_1 \) of \( Q_1 G_1 Q_1 \), for \( G_2 \) equivalently. Then
\[
G = [B_1 \ B_2] \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} [B_1 \ B_2]^* 
\]
is a gyroscopic stabilizer for the pair \( D, Q \).

The assumption that at least one of \( G_1, G_2 \) is of even dimension is necessary to ensure that the eigenvalues of \( G \) are simple. If both \( G_1 \) and \( G_2 \) were of odd dimension, then both would have a zero eigenvalue, resulting in a double eigenvalue zero in \( G \).

**Proof.** By scaling \( G_2 \), we can assume that \( \sigma(G_1) \cap \sigma(G_2) = \emptyset \), thus no multiple eigenvalues appear in \( G \).

Suppose \( v_1 \) is an eigenvector of \( G_1 \), then \( v = B_1 v_1 \) is an eigenvector of \( G \). We get
\[
v^* Dv = v_1^* B_1^* DB_1 v_1 = v_1^* D_1 v_1 > 0.
\]
Now let \( w_1 \) be an eigenvector of \( Q_1 G_1 Q_1 \), then \( w = B_1 w_1 \) is an eigenvector of \( QGQ \). We get, using that \( \text{im}(QB_1) = \text{Span} \mathcal{B}_1 \),
\[
w^* QDQ w = w_1^* B_1^* QDQ B_1 w_1 = w_1^* B_1^* Q B_1^* DB_1 B_1^* Q B_1 w_1 = w_1^* Q_1 D_1 Q_1 w_1 > 0.
\]
The same holds for the eigenvectors of \( G_2 \) and \( Q_2 G_2 Q_2 \). \( \square \)
5.6. CONCLUSION

In case 3.1 in [15] the existence of a gyroscopic stabilizer $G$ is shown under the condition that $v^*Dv > 0$ for every eigenvector of the eigenproblem $(\lambda^2 I + K)v = 0$. Clearly $(\lambda^2 I + K)v = 0$ if and only if $v$ is an eigenvector of $Q$.

If $v^*Dv > 0$ for every eigenvector of $Q$, then the conditions of Proposition 5.15 are met for any partition of the set of eigenvectors $B$. Thus $G$ can be defined for odd $n = 2m + 1$ as

$$G = [v_1 \ldots v_n] \text{diag}(0, \rho_1 G_0, \ldots, \rho_m G_0)[v_1 \ldots v_n]^*,$$

$$G_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where the $\rho_i \in \mathbb{R}\setminus\{0\}$ are pairwise distinct. For even $n$, the construction is equivalent.

Finally, we present a condition that guarantees that every $G$ with $D$-isotropic eigenvectors is a gyroscopic stabilizer.

**Proposition 5.16.** Let $\text{tr } D = 0$ and $\text{tr } QDQ \geq 0$ and let $\mu_1 \leq \cdots \leq \mu_n$ be the eigenvalues of $QDQ$. If $\mu_1 < 0 < \mu_2$ and $|\mu_1| \leq |\mu_2|$, then for any $G$ with $v^*Dv = 0$ for all eigenvectors $v$ of $G$ and pairwise distinct eigenvalues also satisfies $w^*QDQw \geq 0$ for all eigenvectors $w$ of $QGQ$.

**Proof.** By assumption, condition (a) and (b) of Problem 2.2 are satisfied, thus we just have to verify (c).

If the space dimension $n$ is odd, then there is an an eigenvalue zero of $G$ with eigenvector $v_0$. By assumption $v_0^*Dv_0 = 0$, which implies that for the eigenvector $w_0 = Q^{-1}v_0$ for the eigenvalue zero of $QGQ$ we have $w_0^*QDQw_0 = 0$ as well.

Now let $w_k, w_l$ be the eigenvectors for a complex conjugate pair of eigenvalues of $QGQ$. By assumption on the eigenvalues of $QDQ$, it follows that

$$\text{tr}_W QDQ \geq \mu_1 + \mu_2 \geq 0$$

for any two-dimensional subspace $W$. In particular,

$$\text{tr}_{\text{Span}\{w_k, w_l\}} QDQ \geq 0$$

for every pair of eigenvectors $w_k, w_l$, implying $w^*QDQw \geq 0$ for all eigenvectors $w$ of $QDQ$. Thus (c) in Problem 2.2 holds. \qed

5.6. Conclusion

In our work we were able to devise a new sufficient condition for gyroscopic stabilizability. This condition allowed us to show the necessity of $\text{tr } D > 0$ and $\text{tr } QDQ > 0$ for the existence of $G$ in the space dimensions 3 and 4. We were able to develop a construction method for $G$
in these dimensions and by means of several examples showed how our construction can be applied. The case \( n = 5 \) was not solved completely in our work. For \( n \geq 5 \) the construction of a gyroscopic stabilizer so far remains an open problem. Clearly it is desirable to develop an inductive algorithm for the construction of gyroscopic stabilizers that would allow us to reduce the construction problem to lower space dimensions. We were not yet able to devise a method to achieve that. The difficulty in generalizing our approaches is mainly the fact that in general orthogonality of vectors and/or spaces is destroyed once they are multiplied with \( Q \) or \( Q^{-1} \). That was the reason we relied so much on eigenvectors of \( Q \) in our construction because for them orthogonality was preserved. For further research we suggest to analyze for the shifted problem with \( \text{tr} \ D = 0 \) the elements of the set 

\[
\mathcal{A} = \{ T \in O(n) \mid \text{the diagonal of } T^T DT \text{ consists of zeros} \},
\]

where \( O(n) \subset \mathbb{R}^{n \times n} \) denotes the orthogonal group. For \( n \geq 3 \), this set appears to become decomposed into a finite number of path-connected components. Then by using continuity arguments it might to be possible to find a \( T \in \mathcal{A} \) that qualifies for the construction of \( G \) via

\[
G = T \ \text{diag}(0, \rho_1 G_0, \ldots, \rho_n G_0) T^* , \ \rho_i \neq \rho_j \text{ for } i \neq j ,
\]

\[
G_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .
\]

This approach involves differential geometry and topology and for the moment is out of reach for us.
Bibliography


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