The Multi Terminal q-FlowLoc Problem: A Heuristic*

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Abstract. In this paper the multi terminal q-FlowLoc problem (q-MT-FlowLoc) is introduced. FlowLoc problems combine two well-known modeling tools: (dynamic) network flows and locational analysis. Since the q-MT-FlowLoc problem is NP-hard we give a mixed integer programming formulation and propose a heuristic which obtains a feasible solution by calculating a maximum flow in a special graph H. If this flow is also a minimum cost flow, various versions of the heuristic can be obtained by the use of different cost functions. The quality of this solutions is compared.

Keywords: FlowLoc, (dynamic) network flows, heuristic, location theory

1 Introduction and Notations

FlowLoc problems combine two well studied modeling tools: network flow and locational analysis. Network flow models are often used to determine quickest flows or flow minimizing a given cost function (see [1] for an overview). Location theory on the other side is often used for finding "good" locations for facilities (see e.g. [5, 6, 10]). A field in which both problems occur is evacuation planning. A network flow represents people to be sent from a source to a sink. Depending on the objective function the overall evacuation time has to be minimized or the flow per time unit has to be maximized (see [2–4, 7–9]). On the other hand facilities (like first aid wards, fire engines, fish&chip shops, etc.) have to be placed. Although the placement of the facilities and the according reduction of the capacity of some edges have influence on the optimum flow, the two methods have only been considered in an integrated fashion by the research group of the authors. FlowLoc problems combine network flows and locational analysis to obtain results (e.g. lower bounds for evacuation times) taking more factors into account and hence yielding more realistic results.

In this paper we first introduce notations and definitions. Then we give an IP formulation. In Section 2 a graph H is introduced and it is shown that flows with value q in graph H represent the feasible solutions of the q-MT-FlowLoc problem. By adding

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different cost functions and calculating a minimum cost flow, different solutions can be obtained. These are compared in Section 3.

Let an undirected network $G = (V,E)$ with capacity function $u : E \to \mathbb{N}$, a set $\mathbb{P}$ of facilities and a size function $r : \mathbb{P} \to \mathbb{N}$ be given. Furthermore let $nol : E \to \mathbb{N}$ be a function assigning to each edge the maximum number of facilities that can be located there. Let $\mathbb{L} = \{e \in E : nol(e) > 0\} \subseteq E$ be the subset of edges on which facilities can be placed. The q-MT-FlowLoc problem asks for an optimal allocation of all facilities $p \in \mathbb{P}$ to edges $e \in \mathbb{L}$ (see also [7]). This means an allocation maximizing $\sum_{v<w} f_{vw}$, where f_{vw} is the flow value between vertex v and w, has to be found. Here facilities can only be placed on edges with capacity at least the size of the facility and the capacity u_e of edge e is reduced by the size of the largest facility $\tilde{p}$ placed on it to $u_e - r_{\tilde{p}}$. The number of facilities that has to be placed is denoted with $q = |\mathbb{P}|$ and the number of edges on which facilities can be placed with $L = |\mathbb{L}|$. The special case $q = 1$ is called 1-MT-FlowLoc problem or MT-FlowLoc single facility problem and is polynomial solvable (see [11]). For $q > 1$ the problem is NP-hard ([7]).

In IP 1 an integer programming formulation for the q-MT-FlowLoc problem is given. The objective function sums up the flow values of all vertex pairs. There are alternative objective functions (maximum difference, weighted sum) for the q-MT-FlowLoc problem but we will restrict ourselves to the sum objective function in this paper. This objective function can be used if it is not known in advance which vertex is source and which one sink.

IP 1 *q-MT-FlowLoc problem with sum objective function*

Variables

f_{vw}: *flow value of the flow with source v and sink w*

x_{ij}^{vw}: *flow on edge (i,j) of the flow from v to w*

y_{ep}: *indicator variable: equal to one, if facility p is placed on edge e, zero else*

Constants

nol_e: *maximal amount of facilities that can be placed on edge e*

r_p: *size of facility p*

u_e: *capacity of edge e*

$$\max \sum_{v<w \in V\times V} f_{vw} \tag{1}$$

$$s.t. \sum_{(i,v)\in E} x_{iw}^{vw} = f_{vw} \quad \forall (v,w) \in V \times V : v < w \tag{2}$$

$$\sum_{i:(i,j)\in E} x_{ij}^{vw} = 0 \quad \forall (v,w) \in V \times V : v < w, \forall i \in V \setminus \{v,w\} \tag{3}$$

$$\sum_{p\in\mathbb{P}} y_{ep} \leq nol_e \quad \forall e \in \mathbb{L} \tag{4}$$

$$\sum_{e\in\mathbb{L}} y_{ep} = 1 \quad \forall p \in \mathbb{P} \tag{5}$$

$$x_e^{vw} + x_e^{wv} = 0 \quad \forall (v,w) \in V \times V : v < w, \forall e \in E \tag{6}$$

$$x_e^{vw} + r_p \cdot y_{ep} \leq u_e \quad \forall e \in \mathbb{L}, (v,w) \in V \times V : v < w, p \in \mathbb{P} \tag{7}$$

$$x_e^{vw} \leq u_e \quad \forall e \in E \setminus \mathbb{L}, (v,w) \in V \times V : v < w \tag{8}$$

$$y_{ijp} = y_{jip} \quad \forall (i,j) \in \mathbb{L}, p \in \mathbb{P} \tag{9}$$

$$y_{ep} \in \mathbb{B} \quad \forall e \in \mathbb{L}, p \in \mathbb{P} \tag{10}$$

2 Heuristic

In this section a heuristic is introduced that obtains feasible solutions by solving a maximum flow problem in a special network. This network represents all possible allocation of the facilities and all maximum flows correspond to feasible solutions for the q-MT-FlowLoc problem if the flow value is equal to q and vice versa. The main advantage of this heuristic is, that it always finds a feasible solution if one exists and that by adding a cost function to the edges in the network and calculating a minimum cost flow, it is possible to adapt the heuristic to different needs like objective functions or special graph classes.

Definition 1. *Given the network $G = (V,E)$ and the facilities $\mathbb{P}$ with size $r : \mathbb{P} \to \mathbb{N}$ of a (multi terminal) q-FlowLoc problem, the* q-FlowLoc feasible solution network *H with vertex set V_H and edge set E_H is defined as follows:*

$$V_H = \{s\} \cup \{t\} \cup \{p_i : 1 \leq i \leq L\} \cup \{e_j : 1 \leq j \leq L\}$$
$$E_H = \underbrace{\{(s,p_i)\ \forall 1 \leq i \leq q\}}_{E_H^0} \cup \underbrace{\{(e_j,t)\ \forall 1 \leq j \leq L\}}_{E_H^1} \cup \underbrace{\{(p_i,e_j)\ \forall i,j \text{ with } r(p_i) \leq u(e_j)\}}_{E_H^2}$$

$$u(e) = \begin{cases} 1 & e \in E_H^0 \cup E_H^1 \\ nol(e_j) & e \in E_H^2 \text{ and } e = (p_i,e_j) \end{cases}$$

Using network H all feasible solutions for the q-MT-FlowLoc problem can be determined:

Theorem 1. *There exists a feasible solution for the (multi terminal) q-FlowLoc problem in G if and only if there exists a maximum s-t-flow in H with flow value q. Furthermore there is a one-to-one correspondence between the feasible solutions of the FlowLoc problem in G and the flows with flow value q in H.*

Proof. Let $l : \mathbb{P} \to \mathbb{L}$ be a feasible allocation of the facilities to the edges in G. Then define flow $x : E_H \to \mathbb{N}$ as follows:

$$x(e) = \begin{cases} 1 & e \in E_H^0 \text{ or } E_H^2 \text{ and } l(p_i) = e_j \\ \left|\{p \in \mathbb{P} : l(p) = e_j\}\right| & e = (e_j,t) \in E_H^1 \\ 0 & \text{else} \end{cases}$$

For x the capacity constraints are fulfilled since the allocation is feasible and hence the number of facilities placed on edge e_j is less or equal to $nol(e_j)$. Furthermore the flow conservation constraints hold for the vertices $v_i \in V_H$ since every facility has to be placed and for the vertices $e_j \in V_H$ since every facility has to be placed on exactly one edge.

On the other hand let x be a flow with value q in H, then define $l : \mathbb{P} \to \mathbb{L}$ as follows: $l(p_i) = e_j$, if $x(p_i, e_j) = 1$. This edge exists and is unique since the flow has flow value q and the flow conservation constraint is fulfilled for vertex v_i. On each edge $e_j \in \mathbb{L}$ at most $nol(e_j)$ facilities are placed because the capacity on edge $(e_j, t) \in E_H$ is equal to $nol(e_j)$. Facilities are only placed on edges having large enough capacity because of the definition of the edge set E_H. □

An immediate consequence of Theorem 1 is the following generic heuristic to find a feasible solution for the q-MT-FlowLoc problem.

Heuristic 1 for the MTFLMFP with the sum objective function

Require: undirected graph $G = (V, E)$, capacities $u : E \to \mathbb{N}$, set of possible locations $\mathbb{L} \subseteq E$, set of facilities $\mathbb{P}$ with size $r(p)$, maximal number $nol(e)$ of facilities that can be placed on edge $e \in \mathbb{L}$
Ensure: allocation $l : \mathbb{P} \to \mathbb{L}$
1: construct the q-FlowLoc feasible solution network H
2: calculate a maximum s-t-flow x in H
3: construct allocation $l : \mathbb{P} \to \mathbb{L}$ from x (see proof of Theorem 1)
4: **return** l

The quality of the solution computed in Heuristic 1 can be influenced by computing special maximum s-t-flows in 2. For instance a minimum cost s-t-flows sending q flow units can be calculated. For the cost function there are several choices, some of which are listed in Table 1. Any combination of the cost functions for edges (p_i, e_j) and (e_j, t)

	a	b	c	d	e
$c(p_i, e_j)$	$-u(e_j)$	$-u(e_j) + r(p_i)$	$-u(e_j)nol(e_j)$	$-nol(e_j)(u(e_j) - r(p_i))$	0
	i	ii	iii	iv	
$c(e_j, t)$	1	$-nol(e_j)$	$-u(e_j)$	0	

Table 1. Cost functions for the edges in E_H

can be chosen. The idea of the cost functions is, that the larger the capacity of an edge, the larger the amount of facilities that can be placed on it and the smaller the facility itself, the smaller is the influence on the maximum flow values. It is not necessary to assign cost to edges (s, p_i) because every flow with flow value q has to use these edges, so in all cost combinations $c(s, p_i) = 0$.

3 Comparison of the cost functions

To indicate the influence of the cost selection, we list in Table 2 the performance of the heuristic for randomly generated test graphs (using random graph generator of BGL

[12]) with $n = 30$ vertices and an edge density of 40%. A maximum of $nol_{\max} = 2$ facilities can be placed on a single edge. We tested all combinations of the cost functions given in Table 1. The number q of facilities and the ratio $L\%$ of edges in $\mathbb{L}$ to edges in E is given in the top row of Table 2. The entries in the table give the quotient of the heuristic solution and the optimal solution value computed by IP 1. All cost combinations for the tested heuristics yield solutions close to the optimum value. No heuristic achieved outstanding results for all tested parameter combinations. The "right" choice of the cost function depends on the observed problem and parameter setting. The advantage of graph H is that it is easy to construct and all feasible solutions are represented. Furthermore graph H is independent of the objective function of the FlowLoc problem and hence suitable for many problems. The cost function can be chosen corresponding to the objective function of the FlowLoc problem and the parameters of graph G. The special structure and the small size of H are further advantages.

$q, L\%$	8,40	16,40	5,100	10,100
(a,i)	0.8698	0.8943	0.9017	0.9104
(a,ii)	0.8698	0.8958	0.9041	0.9124
$(a,iii$	0.8737	0.8943	0.9005	0.9025
(a,iv)	0.8698	0.8943	0.9017	0.9104
(b,i)	0.8662	0.8785	0.9024	0.9061
(b,ii)	0.8662	0.8808	0.9047	0.9171
(b,iii)	0.8753	0.8785	0.9010	0.8995
(b,iv)	0.8662	0.8785	0.9024	0.9061
(c,i)	0.8712	0.8985	0.9017	0.9180
(c,ii)	0.8712	0.8985	0.9014	0.9169
(c,iii)	0.8777	0.8985	0.9005	0.9143
(c,iv)	0.8712	0.8985	0.9017	0.9180
(d,i)	0.8712	0.8985	0.9017	0.9180
(d,ii)	0.8712	0.8985	0.9041	0.9169
(d,iii)	0.8777	0.8985	0.9005	0.9143
(d,iv)	0.8712	0.8985	0.9017	0.9180
(e,i)	0.8730	0.8875	0.8989	0.9117
(e,ii)	0.8745	0.8967	0.9004	0.9122
(e,iii)	0.8775	0.8951	0.9051	0.9052
(e,iv)	0.8730	0.8875	0.8989	0.9117

Table 2. Comparison of Heuristics 1-3 and the different cost function combinations

4 Conclusion

The q-MT-FlowLoc problem is NP-hard and thus optimum solutions cannot be computed in polynomial time (unless P=NP). The q-FlowLoc feasible solution network incorporates all feasible solutions of the q-FlowLoc problem into the s-t-flows with flow value equal to q. By the use of different cost functions it is possible to obtain solutions for the q-MT-FlowLoc problem with objective value near the optimum. The time needed to calculate a maximum flow algorithm or a minimum cost flow in the small graph and the corresponding allocation of the facilities is negligible compared to solving the IP formulation to obtain a optimum solution. With the help of Heuristic 1 it is possible to find feasible solutions with reasonable objective values in short time also for large instances. The goal is now to quantify the performance of the heuristic by using the structure of the IP.

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