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Vorwort

Das Tätigkeitsfeld des Fraunhofer-Instituts für Techno- und Wirtschaftsmathematik ITWM umfasst anwendungsnahe Grundlagenforschung, angewandte Forschung sowie Beratung und kundenspezifische Lösungen auf allen Gebieten, die für Techno- und Wirtschaftsmathematik bedeutsam sind.

In der Reihe »Berichte des Fraunhofer ITWM« soll die Arbeit des Instituts kontinuierlich einer interessierten Öffentlichkeit in Industrie, Wirtschaft und Wissenschaft vorgestellt werden. Durch die enge Verzahnung mit dem Fachbereich Mathematik der Universität Kaiserslautern sowie durch zahlreiche Kooperationen mit internationalen Institutionen und Hochschulen in den Bereichen Ausbildung und Forschung ist ein großes Potenzial für Forschungsberichte vorhanden. In die Berichtreihe sollen sowohl hervorragende Diplom- und Projektarbeiten und Dissertationen als auch Forschungsberichte der Institutsmitarbeiter und Institutsgäste zu aktuellen Fragen der Techno- und Wirtschaftsmathematik aufgenommen werden.

Darüber hinaus bietet die Reihe ein Forum für die Berichterstattung über die zahlreichen Kooperationsprojekte des Instituts mit Partnern aus Industrie und Wirtschaft.

Berichterstattung heißt hier Dokumentation des Transfers aktueller Ergebnisse aus mathematischer Forschungs- und Entwicklungsarbeit in industrielle Anwendungen und Softwareprodukte – und umgekehrt, denn Probleme der Praxis generieren neue interessante mathematische Fragestellungen.

Prof. Dr. Dieter Prätzel-Wolters
Institutsleiter

Kaiserslautern, im Juni 2001

A Generic Geometric Approach to Territory Design and Districting

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Abstract

Territory design and districting may be viewed as the problem of grouping small geographic areas into larger geographic clusters called territories in such a way that the latter are acceptable according to relevant planning criteria. The availability of GIS on computers and the growing interest in Geo-Marketing leads to an increasing importance of this area. Despite the wide range of applications for territory design problems, when taking a closer look at the models proposed in the literature, a lot of similarities can be noticed. Indeed, the models are many times very similar and can often be, more or less directly, carried over to other applications. Therefore, our aim is to provide a generic application-independent model and present efficient solution techniques. We introduce a basic model that covers aspects common to most applications. Moreover, we present a method for solving the general model which is based on ideas from the field of computational geometry. Theoretical as well as computational results underlining the efficiency of the new approach will be given. Finally, we show how to extend the model and solution algorithm to make it applicable for a broader range of applications and how to integrate the presented techniques into a GIS.

Subject classifications: Programming: Geometric, Programming: Heuristics, Information systems: Decision support systems

1 Introduction

Territory design and districting may be viewed as the problem of grouping small geographic areas, called basic areas, e.g., counties or zip code areas, into larger geographic clusters, called territories, in a way that the latter are acceptable according to relevant planning criteria. Two important criteria are balance and compactness. Balance describes the desire for territories that have approximately equal size, e.g., the same amount of workload, number of customers, or voting population. A territory is said to be geographically compact if it is round-shaped and undistorted. Compact territories usually reduce the sales persons unproductive travel time. Territory design problems (TDPs) are motivated by quite different applications ranging from political districting (Hess et al. (1965), George et al. (1997), Mehrotra et al. (1998), Bozkaya et al. (2003), Ricca et al. (2008)) over the design of territories for schools, social facilities, waste collection, or emergency services (Hanafi et al. (1999), D'Amico et al. (2002), Muyldermans et al. (2002), Perrier et al. (2006a), Perrier et al. (2006b)), to sales and service territory design (Fleischmann and Paraschis (1988), Drexl and Haase (1999), Blais et al. (2003), Fernández and Ríos-Mercado (2009)). See Williams (1995) and Kalcsics et al. (2005) for comprehensive overviews. For sales and service territories, well-planned decisions enable an efficient market penetration and lead to decreased costs and improved customer service, while for political districting an algorithmic approach protects against politically motivated manipulations during the districting process. As most applications have a strong spatial relation, it is obvious to integrate the algorithms into a Geographical Information System (GIS). Therewith, users can utilize the rich variety of maps, spatial databases, and geographical objects available in modern GIS.

Upon reviewing the literature, one can observe that only few papers consider the territory design problem independently from a specific practical background. Hence, the tendency to separate the model from the application and establish the model itself as a self-contained topic of research cannot be observed (Schröder (2001)). However, when taking a closer look at the proposed models, we observe that these models can often be, more or less directly, carried over to other applications. Therefore, we will introduce a generic application-independent model that covers criteria shared by most models in the literature. Typically, a decision making process does not follow a strict linear work flow but is rather an iterative and ongoing process of selecting appropriate planning parameters and data for the problem, computing a territory layout, and evaluating the solutions obtained. It is this interactive type of work that requires the fast generation of high quality solutions; especially for large-scale problems which are often encountered due to the availability of very detailed data in nowadays GIS.

Many different solution approaches for territory design problems have appeared in the literature. The first mathematical programming approach was proposed by Hess et al. (1965) who modeled the problem as a capacitated p-median facility location problem. Since then, several authors improved and modified this location–allocation procedure, see e.g. Fleischmann and Paraschis (1988); George et al. (1997). A second mathematical programming approach is based on set partitioning models, see e.g. Mehrotra et al. (1998); Nygreen (1988). In recent years a growing number of meta heuristics have been proposed for the TDP. Most notably, Simulated Annealing (D'Amico et al. (2002); Ricca and Simeone (2008)), Tabu Search (Blais et al. (2003); Bozkaya et al. (2003)), Genetic algorithms (Bergey et al. (2003); Forman and Yue (2003)), and GRASP (Fernández and Ríos-Mercado (2009)). Common to all these techniques is however, that they have an abstract view on the problem and completely neglect the inherent geographical nature of the problem. Therefore, our goal is to utilize the spatial information to develop a fast procedure that uses techniques from Computational Geometry and is suitable for an interactive use. Recently, Ricca et al. (2008) proposed to use weighted Voronoi diagrams. However, their preliminary computational tests indicate that the approach is not yet suitable for solving districting problems. In a slightly different context, Novaes et al. (2009) use Voronoi diagrams in association with continuous approximation models to solve location-districting problems.

The remainder of this paper is organized as follows. In the next section we introduce the basic model that covers criteria common to most applications. In Section 3 we give a sketch of the solution approach and derive theoretical results on the balance of the resulting territories. Afterward, we present the algorithm in detail (Section 4) and report computational results underlining its efficiency and the quality of the solutions obtained (Section 6). In Section 5 we show how to incorporate a broad range of extensions of the basic territory design model into the heuristic and how to integrate the methods into a Geographical Information System. The paper concludes with a summary and an outlook to future research.

2 A Basic Model for Territory Design

Since the early sixties, many authors have investigated territory design problems. Kalcsics et al. (2005) give an extensive overview of criteria and objectives encountered in literature. Despite the wide range of applications, most of them have the same basic premises, including the desire for compact, contiguous, and balanced territories. Therefore, we chose these criteria as the core of our generic model. Starting with a basic model has several advantages. Often, such a model already provides a sufficient approximation of the problem at hand (see Fleischmann and Paraschis (1988); George et al. (1997)). Moreover, this generic model can serve as a starting point for more complex models taking additional planning criteria into account, making it applicable to a much broader range of problems, see also Section 5. Finally, when providing algorithms for a general purpose GIS, one does not know the exact problem a user will have. Hence, modeling the most common aspects of the territory design problem allows a wide applicability of the algorithms. Next, we present the components of the basic model. Note that we provide rather informal descriptions here, as we first want to give a general idea of the model and the mathematical modeling of some of its components strongly depends on the chosen solution approach or the specific application, see Section 3.2 for more details.

Basic areas. A territory design problem comprises a set V of basic areas, also called sales coverage units. Let $M := |V|$. These basic areas are geographical objects in the plane: points (e.g., geo-coded addresses), lines (e.g., streets), or geographical areas (e.g., zip-code areas). In case of non-point objects, a basic area $i \in V$ is represented by a central point b_i, e.g., its geographical center. In what follows we assume, without loss of generality, that no more than two points b_i lie on a common line. Moreover, a quantifiable attribute $w_i \in I\!R_+$, called activity measure, is associated with each basic area $i \in V$. Typical examples are workload for servicing or visiting the customers within the area, estimated sales potential or number of inhabitants. For a subset $T \subset V$ of basic areas we define the activity measure of T as $w(T) = \sum_{i \in T} w_i$.

Number of territories. In the basic model we assume that the number of territories is given in advance and is denoted by p.

Complete assignment of basic areas. We require every basic area to be contained in exactly one territory, i.e., the territories define a partition of the set V of basic areas. Let $T_j \subset V$ denote the j-th territory, then $T_1 \cup \cdots \cup T_p = V$ and $T_j \cap T_k = \emptyset$, $\forall j \neq k$, $1 \leq j, k \leq p$.

Balance. We call a territory T perfectly balanced if its size $w(T)$ is equal to the average territory size $\mu = w(V)/p$. However, since perfectly balanced territories can usually not be achieved, a common way to measure balance is to compute the relative percentage deviation of the territory size from the average size. The larger this deviation is, the worse is the balance.

Contiguity. Unfortunately, a concise mathematical formulation of contiguity depends on the available data. If the basic areas are non-point objects, i.e., lines or polygons, we can easily derive neighborhood information and determine, whether a territory is connected or not. However, as we also have to take into account basic areas that represent point objects, we call a territory contiguous, if the convex hull of the (point representations of the) basic areas comprising the territory does not intersect the convex hull of the basic areas of another territory.

Compactness. A territory is said to be geographically compact if it is somewhat round-shaped and undistorted. Although being a very intuitive concept, a rigorous definition of compactness does not exist. Compactness can be evaluated using relative measures, like the Roeck and Schwartzberg tests, or absolute measures, e.g., the (weighted) moment of inertia (see Young (1988)). For our solution approach we will derive a measure based on convex hulls to achieve compact territories.

Objective. The objective can be informally described as follows: Partition all basic areas V into a number of p territories that are balanced, contiguous, compact, and non-overlapping.

3 Theoretical Results

In the following we present the principle ideas of our approach, which are based on methods from computational geometry and utilize the underlying geographical information of the problem. Although this type of approach has already been mentioned in the literature, no details were given (Forrest (1964)). The idea is to recursively subdivide the problem geometrically using lines into smaller and smaller subproblems, until an elemental level is reached where we can efficiently solve the TDP. The solutions to these problems then directly yield a solution for the original problem. Hence, the basic operation is to divide a subset $B \subseteq V$ of the basic areas, i.e., points, into two "halves" B_l and B_r by placing a line within this set of points. B_l (B_r) are then defined as the set of points, i.e., the set of basic areas, located "left" ("right") of the line. By this, we partition the territory design problem for B into two disjoint subproblems, one for B_l and one for B_r. These subproblems are then solved independently from one another, again by dividing each of them using a line. See Figure 1 for an example. This iterative partitioning gives the heuristic its name: *successive dichotomies*. The partitioning is thereby achieved by means of so–called *line partitions*. Note that one could consider more general methods to geometrically

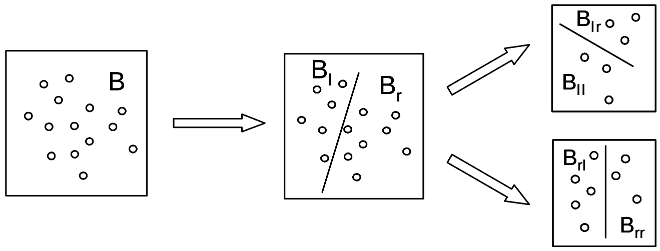

Figure 1: A recursive partition of a set of points B into four disjoint subsets.

define a partition, e.g., using curves. However, as we will see in following, using lines has several (computational) advantages. But before, we will state some basic definitions.

3.1 Definitions

An undirected graph $G = (V, E)$ consists of a set of nodes $V = \{v_1, \ldots, v_n\}$ and a set of edges $E = \{e_1, \ldots, e_m\}$ connecting the nodes. A path $P(v_i, v_j)$ between two nodes v_i and v_j is an alternating sequence of nodes and edges connecting v_i and v_j where all nodes and edges are distinct. The length $\ell(P)$ of a path $P(v_i, v_j)$ is the number of edges of the path. A tree $T = (V, E)$ is an undirected graph which is connected, i.e., there exists a path between any pair of nodes, and acyclic, i.e., there are no closed paths. Let $v^0 \in V$ be a distinguished node of the tree, the so-called root. Let $v_i \in V$, $v_i \neq v^0$, be a node and v_j be adjacent to v_i. v_j is called son or child of v_i, if v_i is on the unique path connecting v_j with v^0; moreover, v_i is called parent or father of v_j. A tree is called binary, if a node is either a leaf or has exactly two sons. We say that a node v_i is at depth or level l of the tree, if the length of the unique path from v_i to the root has length l: $lev(v_i) = \ell(P(v_i, v^0))$. The height l^{max} of a rooted tree is the greatest depth of a node of the tree, i.e., $l^{max} = \max_{v_i \in V} lev(v_i)$.

3.2 Line Partitions

First, we will formally define line partitions and afterwards discuss properties of line partitions regarding balance, contiguity, and compactness. We denote a line $L = L(z, \alpha)$ in the plane by a footpoint $z = (x_z, y_z) \in I\!R^2$ and an angle $\alpha \in [0, 2\pi)$ of the line with the positive x-axis, that is $L(z, \alpha) = \{(x, y) \in I\!R^2 \mid 0 = mx - y + a\}$, where $m = \tan\alpha$ and $a = y_z - x_z \tan\alpha$ (we set $\tan\alpha := 0$ for $\alpha \in \{0, 180\}$). Every line $L((x_z, y_z), \alpha)$ in $I\!R^2$ divides the plane into two halfspaces. Let $H^\diamond(z, \alpha) := \{(x, y) \in I\!R^2 \bigm| mx - y + a \diamond 0\}$, where $\diamond \in \{<, \leq, =, \geq, >\}$.

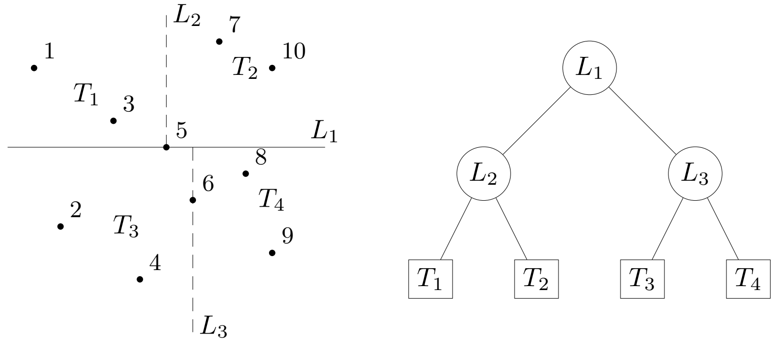

Figure 2: Partition of a set of basic areas and the corresponding partition tree.

A *partition problem* $PP = (B,\, q)$ is defined as a subset of basic areas $B \subseteq V$ and an integer $1 \leq q \leq p$ denoting the number of territories B has to be partitioned into. PP is called trivial if $q = 1$, as in this case B already defines a territory. As there are exponentially many possible partitions of the set B, we restrict ourselves to a special class of partitions, defined as follows.

Definition 3.1 *A* line partition $LP = (B_l,\, B_r,\, q_l,\, q_r)$ of a partition problem $PP = (B,\, q)$ *is defined by two sets* $B_l,\, B_r \subset B$*, a line* $L(z,\, \alpha)$ *in* $\mathbb{R}^2$ *such that* $B_l = B \cap H^{\leq}(z,\, \alpha)$ *and* $B_r = B \cap H^{>}(z,\, \alpha)$*, and two numbers* $1 \leq q_l,\, q_r \leq p$ *with* $q_l + q_r = q$*.*

Observe that $B_l \cup B_r = B$ and $B_l \cap B_r = \emptyset$. We say that L generates or induces the line partition LP. Note that the number of partitions of B induced by a line is bounded by $|B|^2$, as we assumed that no more than two points lie on a common line. We call $PP_l := (B_l,\, q_l)$ and $PP_r := (B_r,\, q_r)$ the left and right subproblem of PP, and PP the father of PP_l and PP_r. As a nontrivial partition problem generates two new subproblems, the recursive partitioning resembles a binary tree, called *partition tree*. The root of the tree is the problem $(V,\, p)$ we start with, each interior node represents a nontrivial partition problem, and the leaves correspond to territories $(T_i,\, 1)$.

Example 3.1

Consider the set of basic areas $B := \{1,\, 2,\, \ldots,\, 10\}$ depicted in Figure 2 with the following weights: $w = (4, 2, 4, 3, 5, 7, 6, 5, 6, 8)$. Let $q = 4$. The figure depicts a partition of the point-set B into four territories. First, we partition the point-set B using the horizontal line L_1 into the left (upper) and right subproblem $PP_l = (\{1, 3, 5, 7, 10\},\, 2)$ and $PP_r = (\{2, 4, 6, 8, 9\},\, 2)$. Afterward, PP_l is further subdivided by L_2 into two territories $T_1 = (\{1, 3, 5\},\, 1)$ and $T_2 = (\{7, 10\},\, 1)$; analogously, for PP_r. This structure is resembled in the partition tree, where we only give the corresponding partitioning line in an interior node.

In Section 2 we introduced, rather informally, the components of the basic model. Some of them require further specification. This will be done next. Note that we do not obtain the final territory layout till the very end of the method (territories correspond to leafs of the partition tree); hence, evaluating the criteria only for the resulting territories would be too late. Therefore, given a problem that needs further subdivision, we try to choose a line partition such that the two resulting subproblems are again disjoint, balanced, contiguous, and compact.

Balance of a Line Partition

Given a partition problem $PP = (B, q)$, following the territory design literature, we measure its balance by computing the relative percentage deviation of the territory size from the average size $\mu = w(V)/p$

$$bal(PP) \;=\; bal(B, q) \;:=\; \frac{|w(B) \;-\; q\,\mu|}{q\,\mu}\,.$$

The larger this value is, the worse is the balance of the territory. The balance $bal(TL)$ of a territory layout $TL = \{T_1, \ldots, T_p\}$ is defined as the maximal balance of one of the territories T_j of the layout, that is $bal(TL) := \max_{j=1,\ldots,p} bal(T_j)$. In the following, our aim is to derive an upper bound on $bal(TL)$. The balance of a line partition $LP = (B_l, B_r, q_l, q_r)$ is defined by means of the resulting subproblems $PP_l = (B_l, q_l)$ and $PP_r = (B_r, q_r)$: $bal(LP) := \max\big\{bal(PP_l), bal(PP_r)\big\}$. Next, we will show that the balance of at least one of the two subproblems PP_l and PP_r will be worse than the balance of PP.

Proposition 3.1 *Let $PP = (B, q)$ be a partition problem and $LP = (B_l, B_r, q_l, q_r)$ be a line partition of PP. Then, $bal(PP) \leq \max\big\{bal(PP_l), bal(PP_r)\big\}$.*

Proof. Assume, w.l.o.g., that $w(B) \geq q\mu$. Let $\delta := w(B) - q\mu \geq 0$. Moreover, define $\delta_l := w(B_l) - q_l\mu \in I\!R$ and $\delta_r := w(B_r) - q_r\mu \in I\!R$. Then, $\delta_l + \delta_r = \delta = \delta\frac{q_l}{q} + \delta\frac{q_r}{q}$, since $w(B) = w(B_l) + w(B_r)$ and $q = q_l + q_r$. We distinguish two cases:

1. $\delta_l \geq \delta\, q_l/q$. Since $\delta \geq 0$, we have $\delta_l \geq 0$ and
$$bal(B_l, q_l) \;=\; \frac{|w(B_l) - q_l\mu|}{q_l\mu} \;=\; \frac{\delta_l}{q_l\mu} \;\geq\; \frac{\delta\frac{q_l}{q}}{q_l\mu} \;=\; \frac{\delta}{q\mu} \;=\; bal(B, q)\,.$$
2. $\delta_l < \delta\, q_l/q$. Then, $\delta_r - \delta\frac{q_r}{q} \;=\; \delta\frac{q_l}{q} - \delta_l > 0$. Therefore, $\delta_r > \delta\frac{q_r}{q} \geq 0$ and we can repeat the argument in 1. to show that $bal(B_r, q_r) > bal(B, q)$.

Hence, the result follows. □

As the balance deteriorates for at least one of the sons with every line partition, we should choose a well balanced partition for each problem. Let a partition problem (B, q) and numbers $1 \leq q_l, q_r \leq q$ with $q_l + q_r = q$ be given. For a given angle α, we are looking for a line $L(\cdot, \alpha)$ inducing a partition of (B, q) such that the resulting two subproblems (B_l, q_l) and (B_r, q_r) are again well balanced. Let us assume for the moment that $\alpha = \pi/2$, i.e., we consider separating lines parallel to the $y-$axis. First, we sort the points in B by non-decreasing $x-$coordinate. Obviously, every possible partition along a line parallel to the $y-$axis divides this sorted sequence into a left and a right part. Thus, there are $O(|B|)$ (nontrivial) partitions, as we assumed that no more than two points lie on a common line. If α is different from $\pi/2$, the same idea applies after rotating the coordinate system so that the line through the origin with angle α becomes the $y-$axis. Denote the sorted sequence of points of B as $a_1, a_2, \ldots, a_n$, $n = |B|$.

Every possible, nontrivial line partition of (B, q) with respect to this angle is given by $LP(k) = (B_l^k, B_r^k, q_l, q_r)$, $k = 1, \ldots, n-1$, where $B_l^k := \{a_1, \ldots, a_k\}$ and $B_r^k := \{a_{k+1}, \ldots, a_n\}$. Next, we determine the partition LP^* for which the maximal relative deviation of the two resulting subproblems from the average territory size with respect to (B, q) is minimal, i.e., the index k^* minimizing

$$\min_{k=1,\ldots,n-1} \max \left\{ \frac{|w(B_l^k) - q_l\mu'|}{q_l\mu'}, \frac{|w(B_r^k) - q_r\mu'|}{q_r\mu'} \right\} \tag{1}$$

where $\mu' := w(B)/q$. The index k^* can be computed as follows. First, we determine an index k' such that $w(B_l^{k'}) < q_l\mu'$ and $w(B_l^{k'+1}) \geq q_l\mu'$. Let $w^{k'+1}$ be the weight of the point $a_{k'+1}$. Then, k^* is given by

$$k^* := \begin{cases} k' & \text{if } q_l\mu' - w(B_l^{k'}) \leq \frac{1}{2}w^{k'+1} \\ k'+1 & \text{otherwise} \end{cases} \tag{2}$$

The correctness of this construction is verified in the following proposition.

Proposition 3.2 *Let a partition problem (B, q), $q_1, q_2 \geq 0$ with $q_1 + q_2 = q$, an angle α, and the corresponding sorted sequence $a_1, a_2, \ldots, a_n$ of points of B be given. Moreover, let k^* be defined as in* (2)*. Then, k^* minimizes* (1)*.*

Proof. Let k' be defined as above. We start showing that for k^* minimizing (1) in fact $k^* \in \{k', k'+1\}$. First, let $k < k'$. Then, $|w(B_l^k) - q_l\mu'| = q_l\mu' - w(B_l^k) > q_l\mu' - w(B_l^{k'}) = |w(B_l^{k'}) - q_l\mu'|$ and, as $|w(B_r^j) - q_r\mu'| = |w(B) - w(B_l^j) - q\mu' + q_l\mu'| = |w(B_l^j) - q_l\mu'|$, $j =$

$1, \ldots, n-1$, also $|w(B_r^k) - q_r\mu'| > |w(B_r^{k'}) - q_r\mu'|$. Analogously, for $k > k'+1$ we obtain that the deviation for $k'+1$ is always smaller than the one for k. Hence, $k^* \in \{k',\, k'+1\}$.

Now, first assume that $q_l\mu' - w(B_l^{k'}) \leq \frac{1}{2}w^{k'+1}$. Then, $w(B_l^{k'+1}) - q_l\mu' \geq \frac{1}{2}w^{k'+1}$. Therefore, $|w(B_l^{k'}) - q_l\mu'| = q_l\mu' - w(B_l^{k'}) \leq w(B_l^{k'+1}) - q_l\mu' = |w(B_l^{k'+1}) - q_l\mu'|$, and $|w(B_r^{k'}) - q_r\mu'| = |w(B_l^{k'}) - q_l\mu'| \leq |w(B_l^{k'+1}) - q_l\mu'| = |w(B_r^{k'+1}) - q_r\mu'|$. Hence, $k^* = k'$ minimizes (1). Using similar arguments, we obtain for $q_l\mu' - w(B_l^{k'}) > \frac{1}{2}w^{k'+1}$ that $k^* = k'+1$ minimizes (1). □

Note that, in principle, we may have $k^* < q_l$ or $n - k^* < q_r$. This, however, would mean that there exists a basic area i with $w_i > \mu'$; a situation which is very unlikely in practice as this basic area then already comprises a territory in itself and could be removed from V a priori.

Example 3.1 (cont.)

For $\alpha = 0$, i.e., a horizontal line, the sorted sequence of points is given by $\{7, 1, 10,\ 3, 5, 8, 6, 2, 9, 4\}$, see Figure 2. Let $q = 4$ and $q_l = q_r = 2$. Hence, $\mu = \mu' = 12.5$. Then, $k' = 4$ as $w(B_l^4) = 22 < 25$ and $w(B_l^5) = 27 \geq 25$. As $w^5 = w_5 = 5$, we have $q_l\mu' - w(B_l^4) = 3 > 2.5 = \frac{w^5}{2}$. Hence, $k^* = 5$ and $LP^* = (\{1, 3, 5, 7, 10\},\ \{2, 4, 6, 8, 9\},\ 2,\ 2)$. Note that $bal(LP^*) = 2/25 < 3/25 = bal(LP(k'))$. ⌟

In Proposition 3.1 we have seen that the balance of any line partition of a problem PP is always worse than the balance of PP itself. The question that now arises is: How worse can it get if we use the best balanced line partition? The answer is given in the following proposition.

Proposition 3.3 *Let $(B,\, q)$ be a partition problem, $q_1, q_2 \geq 0$ with $q_1 + q_2 = q$, and $LP(k^*)$ be a line partition of PP for a given angle α, where k^* is defined as in* (2). *Then, for $w_B^{max} := \max_{i \in B} w_i$,*

$$bal(B_l^{k^*},\, B_r^{k^*},\, q_l,\, q_r) \;\leq\; bal(B,\, q) \;+\; \frac{w_B^{max}}{2\,\min\{q_l, q_r\}\,\mu}\,.$$

Proof. First, consider $PP_l = (B_l^{k^*},\, q_l)$. Denote $B_l^* = B_l^{k^*}$ and $\mu' = w(B)/q$. From (2) and the proof of Proposition 3.2 follows $|w(B_l^*) - q_l\mu'| \leq \frac{w_B^{max}}{2}$. Then,

$$\begin{aligned} bal(B_l^*,\, q_l) &= \frac{|w(B_l^*) - q_l\mu' + q_l\mu' - q_l\mu|}{q_l\mu} \leq \frac{|w(B_l^*) - q_l\mu'| + |q_l\mu' - q_l\mu|}{q_l\mu} \\ &\leq \frac{w_B^{max}}{2q_l\mu} + \frac{|q\mu' - q\mu|}{q\mu} = \frac{w_B^{max}}{2q_l\mu} + bal(B,\, q)\,. \end{aligned}$$

Analogously, we obtain $bal(B_r^{k^*},\, q_r) \leq \frac{w_B^{max}}{2q_r\mu} + bal(B,\, q)$, and the result follows. □

As we will see in the next example, this bound can be tight.

Example 3.1 (cont.)

For $q = 4$, $q_l = q_r = 2$, and k^* as defined as in (2), we get the following upper bound for any line partition $LP(k^*)$ of $(B, 4)$: $bal(B_l, B_r, 2, 2) \le bal(B, 4) + \frac{8}{2 \cdot 2 \cdot 12.5} = \frac{4}{25}$. For the partition $LP(k^*)$ for $\alpha = 0$, the actual balance is $bal(LP(k^*)) = 2/25$. Let now $\alpha = \pi/4$, i.e., we use the first main diagonal. Hence, the sorted sequence is $\{1, 3, 7, 2, 5, 10, 6, 4, 8, 9\}$. We obtain $k^* = k' = 5$ and $bal(LP(k^*)) = \max\left\{\frac{|21-25|}{25}, \frac{|29-25|}{25}\right\} = \frac{4}{25}$, i.e., the upper bound is tight.

From Proposition 3.3, we can derive two straightforward, but important consequences. First, the upper bound is independent of the line direction α. Secondly, if we use the best line partition LP^* for a problem PP, the balance of the two subproblems is at most $w^{max}/(2\min\{q_l, q_r\}\,\mu)$ worse than the balance of PP. As we try to obtain well balanced territories, we should choose values q_l^* and q_r^* such that this term is as small as possible. Therefore, the best values for q_l^* and q_r^* are given by

$$\begin{aligned} &q_l^* = q_r^* = \tfrac{q}{2} && \text{if } q \text{ is even} \\ &q_l^* = \tfrac{q-1}{2} \text{ and } q_r^* = \tfrac{q+1}{2} && \text{if } q \text{ is odd}. \end{aligned} \tag{3}$$

From the proof of Proposition 3.3 we can directly derive the following result.

Corollary 3.1 *Let $PP = (B, q)$ and $PP' = (B', q')$ be two subproblems of (V, p), where PP is the father problem of PP' and PP' generated by a line partition $LP(k^*)$ of PP, where k^* is defined as in (2). Then, $bal(PP') \le bal(PP) + \frac{w_B^{max}}{2\,q'\,\mu}$.*

As this bound can be applied recursively, we can derive an upper bound for the balance of the final territory layout, as we show next. Recall that in the partition tree for a problem (V, p), the root corresponds to the initial problem, $PP^0 = (V, p)$, and the leafs to territories T_j. A node on an intermediate level i of the tree stands for a partition problem PP^i whose subdivision yields two subproblems PP_l^{i+1} and PP_r^{i+1} at depth $i+1$. First, we consider $p = 2^s$, $s \le \lfloor \log_2 n \rfloor$. Choosing q_l and q_r according to (3), we always have $q_l = q_r = q/2$. Thus, all leafs v are at the highest level, $lev(v) = l^{max}$; moreover, $l^{max} = s$, i.e., s is the height of the tree.

Theorem 3.4 *Let $PP^0 = (V, p)$ be the initial problem with $p = 2^s$, $s \ge 1$. If we always choose values q_l and q_r according to (3), and a line partition $LP(k^*)$ where k^* is defined as in (2), then*

$$bal(TL) = \max_{j=1,\dots,p} bal(T_j, 1) \le \frac{w^{max}}{\mu},$$

where the T_j are the point-sets of the leafs, i.e., the final territories, TL the territory layout $\{(T_1, 1), \ldots, (T_p, 1)\}$, and $w^{max} := \max_{i \in V} w_i$.

Proof. Let $PP^i = (B, q)$ be a partition problem at level $0 \leq i \leq s$ of the binary tree. Then, $q = p/2^i = 2^{s-i}$. Moreover, let PP^{i-1} be the father problem of PP^i and PP^i be generated by a line partition $LP(k^*)$ of PP^{i-1} where k^* is defined as in (2). From $w^{max} \geq w_B^{max}$ and Corollary 3.1, we obtain

$$\begin{aligned} bal(PP^i) &\leq bal(PP^{i-1}) + \frac{w^{max}}{2\,q\,\mu} = bal(PP^{i-1}) + \frac{w^{max}}{2^{s-i+1}\,\mu} \\ &\leq bal(PP^{i-2}) + \frac{w^{max}}{2^{s-i+2}\,\mu} + \frac{w^{max}}{2^{s-i+1}\,\mu} \leq \ldots \\ &\leq bal(PP^0) + \frac{w^{max}}{\mu}\left(\frac{1}{2^s} + \ldots + \frac{1}{2^{s-i+2}} + \frac{1}{2^{s-i+1}}\right). \end{aligned}$$

Hence, for $i = s$, we get

$$bal(PP^s) \leq bal(V, p) + \frac{w^{max}}{\mu}\left(\frac{1}{2^s} + \ldots + \frac{1}{4} + \frac{1}{2}\right) \leq \frac{w^{max}}{\mu}$$

since $bal(V, p) = 0$. The result follows, as $PP^s = (T_j, 1)$, for $j \in \{1, \ldots, p\}$. □

Example 3.1 (cont.)

For the problem $PP = (B, 4)$, we obtain as upper bound for the deviation of the final territories: $bal(T_j, 1) \leq \frac{8}{12.5} = 0.64$. For the partition depicted in Figure 2 on page 7, we obtain $\max_{j=1,\ldots,4} bal(T_j, 1) = 0.12$. ⌟

From Theorem 3.4 follows that the size of the final territories deviates at most w^{max} from the average size μ. Observe, that for a given set of points V, the number of territories has a strong impact on the upper bound and, as we will see in Section 6, also on the actual balance of the territories. Unfortunately, for $2^s < p < 2^{s+1}$, $s \leq \lfloor \log_2 n \rfloor - 1$, we obtain a weaker bound.

Theorem 3.5 *Let (V, p) be the initial problem with $2^s < p < 2^{s+1}$, $s \geq 1$. If we always choose values q_l and q_r according to (3), and a line partition $LP(k^*)$ where k^* is defined as in (2), then*

$$bal(TL) = \max_{j=1,\ldots,p} bal(T_j, 1) \leq 2\,\frac{w^{max}}{\mu},$$

where T_j are the point-sets of the leafs, i.e., the final territories, TL is the territory layout $\{(T_1, 1), \ldots, (T_p, 1)\}$, and $w^{max} := \max_{i \in V} w_i$.

Proof. Let $PP^i = (B^i, q^i)$ be a partition problem on level $0 \leq i \leq l^{max}$ of the partition tree. For p being a power of two, we could directly determine the value of q. Here, this is not possible (so easily), but we can derive a lower bound on q^i as follows. If we always choose q_l and q_r according to (3), the smallest possible value of q^i is obtained if we have a sequence of partition problems $PP^j = (B^j, q^j)$, $1 \leq j \leq i-1$, such that always $q^j = (q^{j-1} - 1)/2$. Hence, for $j = 1$: $q^1 = (q^0 - 1)/2 = (p-1)/2$, for $j = 2$: $q^2 = (q^1 - 1)/2 = (p-3)/4$, for $j = 3$: $q^3 = (q^2 - 1)/2 = (p-7)/8$, and so on. Consequently, $q^j = (p+1)/2^j - 1$. Therefore,

$$q^i \;\geq\; \frac{p+1}{2^i} - 1 \;>\; \frac{2^s}{2^i} - 1 \;=\; 2^{s-i} - 1 \quad\Rightarrow\quad q^i \;\geq\; 2^{s-i} \qquad (*)$$

Using this lower bound for q^i, we now derive a bound on $bal(TL)$. Let PP^{i-1} be the father problem of PP^i and PP^i be generated by a line partition $LP(k^*)$ of PP^{i-1} where k^* is defined as in (2). Using $(*)$ and applying Corollary 3.1 recursively, we obtain, analogously to Theorem 3.4, that

$$\begin{aligned} bal(PP^i) \;&\leq\; bal(PP^{i-1}) \;+\; \frac{w^{max}}{2\,q^i\,\mu} \;\leq\; bal(PP^{i-1}) \;+\; \frac{w^{max}}{2^{s-i+1}\,\mu} \\ &\leq\; bal(PP^0) \;+\; \frac{w^{max}}{\mu}\left(\frac{1}{2^s} + \ldots + \frac{1}{2^{s-i+2}} + \frac{1}{2^{s-i+1}}\right). \end{aligned}$$

Unfortunately, the height, l^{max}, of the tree is now $s+1$: $l^{max} \geq s+1$, as $p > 2^s$; on the other hand, to show that $l^{max} \leq s+1$, we assume that there exists a leaf at level $s+2$. Let $PP^{s+1} = (B^{s+1}, q^{s+1})$ be the father of the problem in the leaf. Then, $q^{s+1} \geq 2$. If we always choose q_l and q_r according to (3), then, for a partition problem PP^i at level i and its son PP^{i+1}, we have $q^i \geq 2\,q^{i+1} - 1$. For a son, PP^{i+2}, of PP^{i+1}, we get $q^i \geq 4\,q^{i+2} - 3$. Recursively applying this argument, we obtain for a descendant PP^{i+j} of PP^i at level $i+j$ that $q^i \geq 2^j\,(q^{i+j} - 1) + 1$. Hence, for $i = 0$ and $j = s+1$ we have $p = q^0 \geq 2^{s+1}\,(q^{s+1} - 1) + 1 \geq 2^{s+1} + 1 > p$, which leads to a contradiction. Therefore, $l^{max} = s+1$ and

$$bal(PP^{s+1}) \;\leq\; bal(PP^0) \;+\; \frac{w^{max}}{\mu}\left(\frac{1}{2^s} + \ldots + \frac{1}{4} + \frac{1}{2} + 1\right) \;\leq\; 2\,\frac{w^{max}}{\mu},$$

which concludes the proof, as $bal(PP^i) \leq w^{max}/\mu$ for $i \leq s$. □

3.3 Contiguity of a Line Partition

We call a subset $B \subset V$ of points contiguous with respect to V, if no point in $V \setminus B$ is contained in the convex hull, $ch(B)$, of the set B. If B is divided into two subsets, B_l and B_r, using a line, then the convex hulls of the two subsets will be disjoint. Hence, this criterion is always fulfilled for line partitions as well as partition problems.

3.4 Compactness of a Line Partition

For reasons of computational efficiency we decided against using one of the manifold explicit compactness measures, but rather evaluate compactness indirectly. The measure we propose is based on the following reasoning. Let a set B and a line L that partitions B into two subsets B_l and B_r be given. The segment of L that lies "within" B will contribute to the total length of the borders of B_l and B_r and therefore likely also to the territory borders in the final layout. If we try to make this segment short, we can hope to end up with a small total border length and therefore with a compact layout. Hence, we do not measure the compactness of subproblems (or territories) but the compactness of line partitions.

As B is a discrete set of points, we measure the length of the intersection of $L(z, \alpha)$ with the convex hull. By convexity, L intersects $ch(B)$ in at most two points c_1 and c_2. Note that possibly $c_1 = c_2$. The Euclidian distance between c_1 and c_2 defines the length of the segment and is a measure of the compactness of the line partition: $cp(LP) := l_2(c_1, c_2)$.

4 The Successive Dichotomies Heuristic

In this section, we present the successive dichotomies heuristic for solving our basic territory design model. A brief outline of the heuristic has already been given in Kalcsics et al. (2005). The heuristic explores the partition tree with nodes corresponding to partition problems and terminates when all leaves are generated. Two questions need to be answered:

- How do we perform the partitioning of a problem into subproblems?
- How do we explore the partition tree?

Before we answer these questions, we make the following assumption. We assume that a lower bound L and an upper bound U for the activity measure of a territory are given. A partition problem (B, q) is called feasible if $L \leq w(B)/q \leq U$. For example, L and U can be calculated from a maximally allowed deviation $\tau > 0$ from the average size by $L = (1 - \tau)\,\mu$ and $U = (1 + \tau)\,\mu$. Then (B, q) is feasible, if $bal(B, q) \leq \tau$.

4.1 Partitioning a Problem

Let a partition problem (B, q) be given. If $q > 1$, we have to make two decisions:

1. Select numbers $q_l, q_r \geq 1$ with $q_l + q_r = q$.

2. Select a line partition $LP = (B_l, B_r)$ to split B into two subsets B_l and B_r.

Concerning 1., we choose q_l and q_r according to (3) to reduce the imbalance of the subproblems. As we restricted ourselves to line partitions, we have a quadratic number of possible partitions. Unfortunately, this is still too much for large scale problems. Therefore, just those partitions of B are considered that are generated by a limited number K of line directions. Although this seems to be rather restrictive, we found that it still produces very good results, see Section 6. Next, we will show how to generate and rank line partitions in terms of balance and compactness.

Generating and Ranking Line Partitions

Let now a partition problem (B, q) and the numbers q_l and q_r be given. Moreover, denote K the number of line directions to be considered. We consider the angles $\alpha_i = i\,\frac{\pi}{K}$ for $i = 0, 1, \ldots, K-1$. We are looking for a line inducing a partition of B such that the resulting two subproblems are balanced, compact, contiguous, and non-overlapping. As the last two criteria are fulfilled by definition of a line partition, we only have to consider the first two. Therefore, we compute for every angle α_i, $i = 0, \ldots K-1$, the sorted sequence $a_1, a_2, \ldots, a_n$ of points of B. Afterward, we determine the line partition $LP(k^*)$, where k^* is defined as in (2). Note that we discard the partition for k^* if it is infeasible, i.e., if $k^* < q_l$ or $n - k^* < q_r$, or if $w(B_l)/q_l$ or $w(B_r)/q_r$ is not in $[L, U]$. We repeat this process for all given directions α_i and, if q is odd, for both combinations of q_l and q_r. All feasible line partitions are stored in a list $\mathcal{FLP}$. Note that this list contains at most $2K$ elements.

Among the feasible partitions in $\mathcal{FLP}$, we then choose the most balanced and compact one and implement it. To rank all partitions, we evaluate them in terms of balance and compactness. Since compactness is measured as an absolute value, we divide the compactness measure of each line partition by the maximal compactness value cp^{max} of a partition: $cp^{max} := \max\{cp(B_l, B_r) \mid LP \in \mathcal{FLP}\}$. Analogously, we also scale the balance values by the maximal balance, bal^{max}, of a line partition in $\mathcal{FLP}$. The ranking value of a line partition LP is a convex combination of the scaled balance and compactness measure

$$rk(LP) := \beta\,\frac{bal(LP)}{bal^{max}} + (1-\beta)\,\frac{cp(B_l, B_r)}{cp^{max}}\,, \tag{4}$$

where β is the weighting factor for the two criteria. The smaller the ranking value is, the better the partition. Finally, we sort the partitions in nondecreasing order of their ranking value. The partition to be implemented is then given by

$$LP^* := \operatorname*{argmin}_{LP \in \mathcal{FLP}} rk(LP)\,.$$

We summarize the steps in Algorithm 4.1.

***Algorithm 4.1**: Compute and rank all feasible line partitions of a problem*

Input: *Partition problem* (B, q)*; number of line directions* K*; bounds* L *and* U*.*

Output: $\mathcal{FLP}$ *sorted in nondecreasing order of their ranking value.*

1 **if** q *is even* **then** *set* $Q := \{(\frac{q}{2}, \frac{q}{2})\}$.
  else *set* $Q := \{(\frac{q-1}{2}, \frac{q+1}{2}), (\frac{q+1}{2}, \frac{q-1}{2})\}$.
  Set $\mathcal{FLP} := \emptyset$.

2 **for** $i = 0, \ldots, K-1$ **do**
  Let $\alpha_i := i\,\frac{\pi}{K}$ *and determine the sorted sequence* $a_1, \ldots, a_n$ *w.r.t.* α_i*.*
  forall $(q_l, q_r) \in Q$ **do**
    Compute k^**, where* k^* *is defined as in* (2)*.*
    if $LP(k^*)$ *is feasible* **then** *set* $\mathcal{FLP} := \mathcal{FLP} \cup \{LP(k^*)\}$.
  end
end

3 *Sort the partitions in* $\mathcal{FLP}$ *in nondecreasing order of their ranking value.*

4 **return** $\mathcal{FLP}$.

Complexity of Algorithm 4.1

The sorted sequence $a_1, \ldots, a_n$ of points can be computed in $O(|B| \log |B|)$ time and the partition $LP(k^*)$ in $O(|B|)$. As $|Q| \leq 2$, the complexity of Step 2 to generate all feasible partitions is $O(K\,|B| \log |B|)$. To determine the ranking value of a partition in Step 3, we first compute the boundary of the convex hull of B and then we intersect the line with the boundary. These two steps can be done in $O(|B| \log |B|)$ and $O(|B|)$ time, respectively, see Klein (1997). As the ranking of the line partitions requires $O(K \log K)$ time, the overall complexity of the algorithm is $O(K|B| \log |B| + K \log K)$.

4.2 Exploring the Partition Tree

In the last section we explained how we generate and rank line partitions. The straightforward "greedy" approach to choose just the best partition according to this ranking is, however, often not sufficient. Even though we only consider feasible and well balanced partitions for a certain line direction, there is no guarantee that we do not run into an infeasible subproblem further down in the partition tree. Therefore, we incorporate a backtracking mechanism into the heuristic that allows to revisit a partition problem at a higher level to revise the subdivision made there, and choose the next best line partition and continue with this partition.

The search encounters at least $2p-1$ nodes until it terminates. However, due to backtracking operations, the number of nodes examined can be much larger. Especially proving infeasibility of the problem requires to examine all feasible partitions for all problems; in general, this number is exponential in K and p. Therefore, it is necessary to limit the search. As it is usually better to report some result, even an infeasible one, instead of no result, we decrease L and increase U by some amount after a given number $NodeMax$ of nodes has been examined, and thus enlarge the number of feasible partitions. However, we do not restart the heuristic, so the relaxed bounds apply only to newly generated nodes of the search tree. This relaxation is repeated a few times, if necessary. If the heuristic still does not terminate after a given number $RelMax$ of relaxations, we finally set $L = 0$ and $U = \infty$. Afterward, the algorithm performs no more backtracking and terminates quickly. In our tests, $NodeMax = 10p$ and $RelMax = 3$ proved to be suitable values. An outline of the procedure is given in Algorithm 4.2. (The list PP stores the yet untreated partition problems.)

Complexity of Algorithm 4.2

The most time consuming operation is to compute and rank all feasible partitions of a node. Using Algorithm 4.1, this can be done in $O(K|B|\log|B|+K\log K)$ time for a partition problem (B, q), where K is the number of different line directions. To determine the overall complexity, we distinguish two cases:

1. $L = 0$ and $U = \infty$:

   The complexity of the algorithm is $O(\log p\, KM \log M \;+\; p\, K \log K)$, where $M = |V|$. To see this, denote $PP_1^i, \ldots, PP_s^i$, $s \leq 2^i$, the partition problems on level i of the partition tree. Consequently, the point-sets B_k^i, $1 \leq k \leq s$, of these problems are pairwise disjoint. Hence, the effort to compute the feasible partitions of all nodes on level i and determine their ranking value is $O(K|B_1^i|\log|B_1^i| + \ldots + K|B_s^i|\log|B_s^i|) = O(KM\log M)$. As the partition tree has $O(\log p)$ levels (see Theorem 3.5) and at most $2p-1$ nodes, the result follows (we have to sort the line partitions for each node).

Algorithm 4.2: *The successive dichotomies heuristic for the basic model*

Input: *Set of basic areas V with activity measures w_i, $i \in V$; number of territories p; parameters τ, K, β, $NodeMax$, and $RelMax$.*

Output: *Territory layout $TL = \{T_1, \ldots, T_p\}$.*

1 **Initialization**

Set $L := (1 - \tau)\,\mu$, $U := (1 + \tau)\,\mu$, $NodeCtr := 0$, and $RelCtr := 0$.

Set $PP := \{v^0 = (V, p)\}$ and compute and rank all feasible partitions of v^0.

2 **while** $PP \neq \emptyset$ **do**

- *Let $v = (B_v, q_v) \in PP$ be a partition problem. Set $NodeCtr := NodeCtr + 1$.*
- **if** $q_v = 1$ **then** *set $TL := TL \cup \{v\}$, $PP := PP \setminus \{v\}$, and* **continue** *with Step 2.*
- **if** *there are no more feasible partitions left for v* **then**
  - */* Backtrack */*
  - **if** $v = v^0$ *is the root node* **then**
    - **if** $RelCtr \geq RelMax$ **then** *set $L := 0$ and $U := \infty$.*
    - **else** *set $L := L - (U - L)/2$, $U := U + (U - L)/2$, and $RelCtr := RelCtr + 1$.*
    - *Compute and rank again all feasible partitions of v^0.*
  - **else**
    - *set $PP := PP \cup \{v_f\}$ for the father v_f of v and delete all descendants of v_f from PP.*
  - **end**
- **else**
  - */* Partition */*
  - *Implement the highest ranked partition creating two new nodes $v_l = (B_{v_l}, q_{v_l})$ and $v_r = (B_{v_r}, q_{v_r})$. Compute and rank all feasible partitions of v_l and v_r. Set $PP := PP \setminus \{v\} \cup \{v_l, v_r\}$.*
- **end**
- **if** $NodeCtr = NodeMax$ **then**
  - **if** $RelCtr \geq RelMax$ **then** *set $L := 0$ and $U := \infty$.*
  - **else** *set $L := L/2$, $U := 2U$, and $RelCtr := RelCtr + 1$.*
  - *Set $NodeCtr := 0$.*
- **end**

end

3 **return** TL.

2. $L > 0$ and $U < \infty$:

   The complexity depends now on the actual number of nodes explored in the search for a feasible territory plan. If we choose $NodeMax = 10p$ and $RelMax = 3$, then the maximal number of nodes examined is linear in p and we get as complexity $O(p\,K(M \log M + \log K))$.

Observe that the heuristic is subquadratic in p, K, and M.

In Figure 3, we present an example of two sales territory layouts of German zip-code areas (marked as “x”) into 70 territories created by applying the above heuristic. Two different sets of line directions were used: one with 2, see the left-hand side image, and the other with 16 directions, see the right-hand side picture. (For details on the quality of the solutions and the running times, we refer to Section 6.)

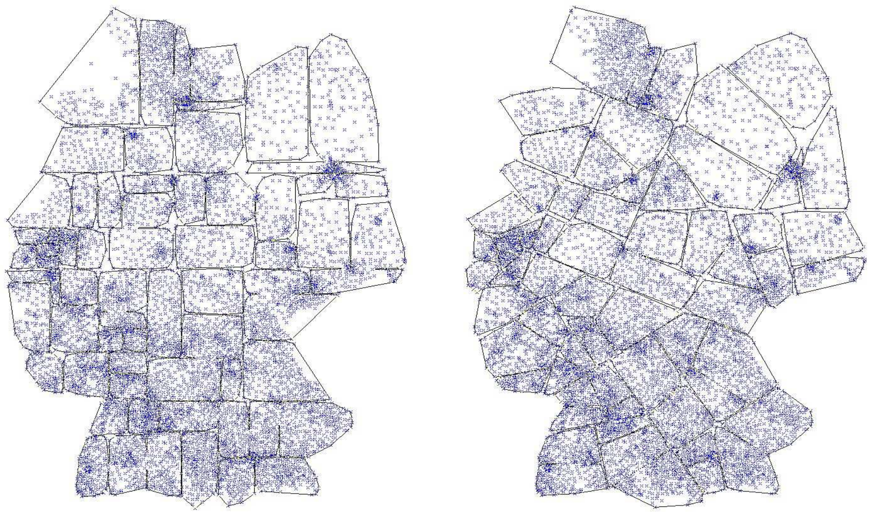

Figure 3: Two territory layouts based on German zip-code areas.

5 Extensions and Planning Scenarios

Several characteristics of territory design problems often encountered in practice are not covered by the basic model. In the following we distinguish between two different types that differ in the way they are included into a solution approach for the basic TDP. The former, simply called extensions, require a modification of the solution algorithm for the basic model itself and, consequently, their implementation strongly depends on the chosen solution method. Whereas the latter, called planning scenarios, leave the algorithm itself unchanged and embed it into a larger framework, calling it (repeatedly) with appropriate parameters.

First, we will discuss how to incorporate the extensions into the heuristic. Although we do this for each extension separately, they can easily be combined. Afterward, we show how to implement the two planning scenarios.

5.1 Extensions

5.1.1 Several activity measures

Often, more than one activity measure has to be considered in the planning process. Let R be the number of different activity measures. For $1 \leq r \leq R$, denote w_i^r the r-th activity measure of basic area $i \in V$. To incorporate multiple activity measures into the partition process, we aggregate them using weighting factors γ^r, $1 \leq r \leq R$: $\bar{w}_i = \sum_{r=1}^{R} \gamma^r w_i^r$. Using these aggregate weights we then determine a well balanced line partition.

5.1.2 Neighborhood information

Assume that we are given neighborhood information about the basic areas that is stored in the so-called neighborhood graph $\mathcal{NG} = (\mathcal{V}, \mathcal{E})$. In this graph, every basic area $i \in V$ corresponds to a node $v_i \in V$ and two nodes v_i and v_j are connected by an edge, if and only if the respective basic areas i and j are neighboring. We call a territory contiguous, if the basic areas comprising the territory induce a connected subgraph in $\mathcal{NG}$. Given a partition problem $PP = (B, q)$, we denote $G(B)$ the subgraph of $\mathcal{NG}$ induced by the set of basic areas B. Moreover, if $G(B)$ is not connected, we denote $G_1, \ldots, G_s$, $s \geq 2$, the connected components of $G(B)$, and B_i the underlying set of basic areas of G_i. Then, $B_i \subset B$ and $B_1 \cup \ldots \cup B_s = B$. A partition problem $PP = (B, q)$ is (not) contiguous if $G(B)$ is (dis)connected, and a line partition is called contiguous if the resulting two subproblems PP_l and PP_r are contiguous.

Assume first that PP is not contiguous. In this case, instead of subdividing PP along a line, we partition the problem based on its connected components into s disjoint problems $PP_1 = (B_1, q_1), \ldots, PP_s = (B_s, q_s)$. Whereas the partition of the set B is induced by the connected components, values for the q_i are not so obvious to determine. Here, we use the following approach. We determine values for the q_i such that the maximal balance of one of the resulting subproblems $PP_1, \ldots, PP_s$ is as small as possible. That is, we want to solve the problem

$$\min_{\substack{q_1+\ldots+q_s=q \\ q_i \geq 1,\, i=1,\ldots,s}} \; \max_{i=1,\ldots,s} \; bal(B_i, q_i) \tag{5}$$

As each connected component has to yield at least one territory, we enforce $q_i \geq 1$. Note that this problem is well-defined only if $s \leq q$, i.e., if the number of connected components is not larger than the number of territories PP has to be partitioned into. Fortunately, problem (5) can be solved optimally using dynamic programming with a worst case time complexity of $O(|B| + s\,p^2)$.

(The stages correspond to the connected components and the states to the remaining number of territories that have to be assigned to connected components.)

Now, let the partition problem PP be contiguous. Moreover, let α be an angle and $a_1, \ldots, a_n$ the corresponding sorted sequence of points of B. For the basic model, we added, starting with a_1, iteratively basic areas to the left problem, PP_l, until both subproblems were well balanced. Now, we start again with a_1. But then we add the next basic area in the sorted sequence to left subproblem only if it is adjacent to a basic area already in PP_l. In this way, we try to obtain a contiguous left problem. If, however, we can not add enough adjacent basic areas to the left subproblem, we have to add a basic area that is not adjacent to PP_l. Nevertheless, if we continue adding basic areas, PP_l might get contiguous again. This approach, however, not necessarily yields a contiguous right subproblem PP_r. To assure this, we could, in principle, add the next basic area to the left subproblem only if it is adjacent to a basic area already contained in PP_l and if it does not lead to a non-contiguous right subproblem (PP_r comprises the basic areas not yet assigned to PP_l). However, this is in general too restrictive as a non-contiguous right problem at an intermediate stage of the process may still result in PP_r being contiguous at the end of the process. Therefore, we check the contiguity of PP_r only after we have found a well balanced partition. If the left or right subproblem is not contiguous for the final line partition, we add the value 1 to the ranking value of the partition. By doing this, we assure that we first use contiguous line partitions for the subdivision of PP. Only if no feasible and contiguous partitions remain, we fall back to non-contiguous ones.

5.2 Planning Scenarios

5.2.1 Unknown number of territories

In some applications, the number of territories may not be known in advance. Instead, an upper or lower bound, or both, on the size of the territories is given and the region under consideration has to be partitioned into an appropriate number of territories such that these bounds are not violated. Consider a company providing on-site service for their products in a certain region. Due to the limited working time of the service staff, the size of each territory is bounded from above, e.g., in terms of the expected number of service calls an employee can handle in one month. The task is to partition all basic areas into an appropriate number of territories, each attended to by a single service person, such that the size of each service district does not exceed the threshold value. Hence, we are given an upper bound UB or a lower bound LB, or both, and we require the size of each territory to be within the interval $[LB, UB]$.

However, specifying just a lower or upper bound is usually not sufficient. Assume, that all basic areas have (almost) identical activity measures and we are given only an upper bound. Then, we always have the trivial but optimal solution where each territory consists of a single basic area. Therefore, we need to extend the problem formulation. Coming back to the example, the service company obviously wants to cover the region with as few territories as possible. This leads to the revised problem formulation: "Partition all basic areas into a minimal (maximal) number of territories such that they satisfy the planning criteria of balance, compactness, and contiguity and their size does not exceed (fall below) the upper (lower) bound." As both cases are analogous, we only discuss the case of an upper bound in the following.

Let $LB = 0$ and $UB > w^{max}$ (otherwise, the problem is infeasible). The smaller p is, the larger is the average territory size $\mu = w(V)/p$ and therefore also the actual size of the territories. Unfortunately, we cannot directly compute a minimal value for p due to the discrete structure of the problem and the fact that the problem is usually solved using a heuristic method. However, a good lower bound on p is given by $\lceil w(V)/UB \rceil$ and we can perform a binary or interval search to determine a minimal value for which a feasible layout still exists. Note, however, that this value is not necessarily optimal, as it may happen that the heuristic finds a feasible layout for a value q but not for $q + 1$, although the average territory size for $q + 1$ is smaller than for q.

5.2.2 Incomplete assignment

In the basic model we completely partition the basic areas into territories, i.e., all basic areas are assigned to a district. However, often not all basic areas have to be (or can be) partitioned into territories due to certain planning restrictions. For example, if a company can not afford to employ more than a given number of sales persons and each of them can only develop a certain maximal total market potential. Therefore, if the total market potential of the region is larger, a complete partition of all basic areas will not make sense; some will stay unassigned and will not be attended to by a representative of the company. Moreover, the sales territories should not only be designed such that their total market potential is below the maximal potential, but also as close as possible to this maximal potential to provide for a fair living for the sales staff. In the following, we assume that we are only given an upper bound. The situation just with a lower bound or a lower as well as an upper bound is analogous.

Let $LB = 0$ and $UB \leq \mu$, $\mu = w(V)/p$. (If $UB > \mu$, then the bound does not pose a restriction.) Hence, it will be impossible to subdivide all basic areas into territories. As UB is not just a mere upper bound that has to be fulfilled but rather a target value for the size of the territories, we obtain the following, revised problem formulation: "Assign basic areas to a given number of territories such that the latter satisfy the planning criteria of balance, compactness and contiguity and their size is as close as possible but not above the upper bound."

A straightforward approach is to ignore the upper bound in a first step and compute a complete partition of the basic areas into territories using some heuristic for the basic model. Then, in a second step, we prune territories T for which $w(T) > UB$ by iteratively removing basic areas from the territory, e.g., starting with the ones farthest away from the center of the territory, until the size of T is within the upper bound. Unfortunately, if UB is much smaller than μ this pruning often leads to rather dispersed territories. Moreover, the resulting territories are usually fairly uneven in terms of their geographical extent. That is, some territories cover a small region while others span a large area. The reason is, that territories are typically not centered around concentrations of basic areas with high activity measures. Coming back to the above example, this would mean that some sales persons have to travel very far to attend to their customers, e.g., in rural areas, whereas the territories of others are concentrated in a much smaller region, e.g., within a city. Obviously, this is undesirable. Hence, the geographical extent of all sales territories should be as small as possible, i.e., the territories should be located in areas where a high market potential is concentrated in a relatively small region.

To achieve this, we use a different approach. The idea is to partition the set V not into p territories but into a larger number, p', of territories. By increasing this number, the average territory size μ decreases and, for p' sufficiently large, finally is below the upper bound UB. Hence, if we solve the problem with this new number of territories using an algorithm for the basic model, we obtain a layout where the size of each territory is below the upper bound. Then, we select a set of p territories from this layout whose geographical extent is as small as possible. What remains to be discussed is how to determine an appropriate value for p'. This is done using the same approach as for the first scenario with an unknown number of territories, i.e., we assume that only the upper bound UB is given but not the number of territories. Thereby, we obtain a value p' for which a feasible layout exists and where the size of the territories is close to UB. Then, we choose the p territories with the smallest geographical extent.

6 Computational Results

We tested the heuristic on problems with 100 up to 1000 basic areas in steps of 100. For each number of basic areas, five instances were generated using real-world data obtained from the GIS *ArcView*: basic areas correspond to German zip-code areas and the activity measure equals the number of inhabitants. Therefore, we have in total 50 different instances. In addition, we created one instance containing all 8270 German zip-code areas to illustrate the efficiency of the heuristic for large-scale problems. The number of territories p was determined based on a parameter Q specifying the average number of basic areas per territory, i.e., $p = M/Q$. For Q, we chose values of 10, 20, 30, 40, and 50. For the number of line directions we choose five different values: $K \in \{2, 4, 8, 16, 32\}$. The maximal allowed deviation τ from the average territory size μ was set to 5%. Finally, the weighting factor β required to determine the ranking value of a line partition equals 0.5 and for $NodeMax$ and $RelMax$ we chose $10p$ and 3.

The heuristic was implemented in *C++* and the results were obtained on a Pentium 4, 2.6 GHz with 512 MB Ram. For each problem instance, the solution time and the quality of the resulting territories in terms of balance and compactness were obtained. These values were then averaged over all instances with the same set of parameters (i.e., number of basic areas, territories, line directions, etc.). First, we will analyze the behavior of the algorithm in terms of running times, balance, and compactness with respect to different numbers of line directions before we discuss the influence of the parameter β. For a comparison of the successive dichotomies heuristic with other methods for the basic model we refer to Kalcsics et al. (2005).

Varying the Number of Line Directions and the Tolerance

First, we will present results concerning the balance of the territory layouts.

Balance

In Table 1, we report the average and maximal balance for $K \in \{2, 4, 8, 16, 32\}$. For $Q = 10, \ldots, 50$, each entry in the left hand side (right hand side) part of the table is the average (maximal) balance over all 50 problem instances. Note that most values are considerably less than the 5%-tolerance. However, for $Q = 10$ and $K = 2, 4$, the algorithm could not always find a territory layout within the 5%-tolerance. In these instances, the heuristic performed several backtracking operations and was forced to relax the upper and lower bound, L and U.

We note that the balance improves for larger numbers of basic areas per territory. This is to

	Average Balance					Max Balance				
$Q \setminus K$	2	4	8	16	32	2	4	8	16	32
10	6.3	4.7	4.3	4.0	4.0	29.0	15.0	4.9	5.0	4.9
20	3.9	2.9	2.2	1.7	1.6	4.9	4.9	4.7	3.9	3.9
30	3.1	1.8	1.3	1.0	0.9	4.9	4.1	3.3	2.4	4.0
40	2.2	1.4	1.0	0.6	0.6	4.6	3.9	2.4	1.4	1.6
50	1.6	0.9	0.7	0.4	0.3	4.0	2.1	1.7	1.0	1.1

Table 1: Average and maximal balance in % for $\tau = 0.05$.

be expected from a theoretical as well as a practical point of view. Theoretically, as the maximal balance of a territory is bounded from above by $2w^{max}/\mu$, and w^{max} is fix and $\mu = w(V)/p$ increases, if we increase Q. From a practical point, the more basic areas we have per territory, the less likely will larger than average areas lead to unbalanced partitions. Moreover, also with increasing K the average balance improves due to an increased number of options for choosing a well balanced (and compact) line partition. As the improvement from $K = 16$ to $K = 32$ is negligible, it is unlikely that larger values of K will further improve the results.

In Table 2 we report the average balance of the resulting territories for $K = 16$, including the instance with all German zip-code areas. The entries are averaged over the different values for Q. We observe that the balance worsens with an increasing number of basic areas. This, however, can be expected as the number of territories also increases and, consequently, also the height of the partition tree. However, the imbalance is still acceptable.

Summing up the results, even for a small number of basic areas per territory, we obtain well balanced solutions for values of $K \geq 8$.

Running Times

In Table 3, we report the average running times in seconds, which are computed again over all 50 problem instances. As expected, the running times increase for an increasing number of angles and territories. However, the running times are still negligibly small. Another major factor influencing the execution times is the problem size. In Table 4 we give results depending

M	100	200	300	400	500	600	700	800	900	1000	8270
Balance	0.8	1.2	1.5	1.3	1.4	1.6	1.9	1.8	1.9	1.8	2.8

Table 2: Average balance in % for different numbers of basic areas for $K = 16$.

$Q \setminus K$	2	4	8	16	32
10	0.03	0.04	0.08	0.14	0.24
20	0.02	0.03	0.05	0.09	0.17
30	0.02	0.03	0.04	0.07	0.13
40	0.02	0.03	0.04	0.07	0.12
50	0.02	0.03	0.04	0.06	0.10

Table 3: Running times in seconds.

on the number of basic areas for $K = 32$. Each entry is the mean over the five instances and $Q = 10, \ldots, 50$ (except for the last column). As expected, the execution times increase proportional to the number of basic areas. However, for up to 1000 areas they are still below one second and for the large example below five seconds.

Compactness

Finally, we will compare the territory layouts resulting from different parameter settings in terms of compactness. As we measured compactness for efficiency reasons only indirectly in our algorithm, we use for the computational results the weighted moment of inertia to measure the compactness of a territory $T_j \subset V$, as proposed by Hess et al. (1965). This is the weighted sum of the squared Euclidean distances from the center of gravity, c_j, of the territory to the basic areas of T_j: $\overline{cp}(T_j) = \sum_{i \in T_j} w_i \, l_2^2(c_j, b_i)$. The smaller the moment of inertia is, the more compact the district is. The compactness of a territory layout TL is the sum of the weighted moments of inertia of the territories comprising the layout $\overline{cp}(TL) = \sum_{T_j \in TL} \overline{cp}(T_j)$. As this is an absolute measure, to compare two solutions we determine the relative percentage deviation of the compactness of the territory plans for two different values K_1 and K_2 as

$$\text{deviation} = \frac{\overline{cp}(TL_1) - \overline{cp}(TL_2)}{\overline{cp}(TL_2)} * 100\% \, ,$$

where TL_i is the layout obtained using K_i. Hence, for a positive (negative) deviation, layout TL_1 is less (more) compact than TL_2.

$\tau \setminus M$	100	200	300	400	500	600	700	800	900	1000	8270
0.05	0.02	0.05	0.06	0.10	0.13	0.16	0.20	0.22	0.28	0.31	4.91

Table 4: Running times in seconds for $K = 32$.

$Q \setminus K$	2 / 4	4 / 8	8 / 16	16 / 32
10	7.5	4.1	-1.2	-0.2
20	6.6	3.3	2.8	1.1
30	7.1	4.0	0.6	2.7
40	6.1	4.1	1.2	0.3
50	1.9	6.5	1.5	1.1

Table 5: Pairwise comparison of the average compactness.

The average relative percentage deviations for the pairwise comparisons are given in Table 5. For example, for $K_1 = 2$, the average compactness is 7.5% worse compared to $K_2 = 4$. For larger values of K this difference reduces more and more. That is, the territory layouts generated using 8, 16, and 32 different line directions are, more or less, equally compact. We observe that for $Q = 10$ and $K = 16, 32$, the deviation is negative, i.e., the compactness of, at least, some territory layouts is worse than for the previous number of angles. Hence, there is not necessarily a monotone improvement of the compactness for increasing values of K. Note that there is no significant difference between different values of Q.

Summing up the results, $K = 16$ or $K = 32$ seem to be suitable values to provide stable and high quality results, even for a small number of basic areas per territory.

Trade off between balance and compactness

Next, we will discuss the influence of the parameter β on the quality of the solutions. Recall that β was the weighting factor between balance and compactness for computing the ranking value of a partition LP. For the comparisons, we choose $K = 16$ and $\beta \in \{0.25, 0.33, 0.5, 0.66, 0.75\}$. The results obtained in terms of balance are reported in Table 6. As expected, the average balance improves for an increasing β. But to the same extent as the balance improves, the compactness deteriorates, as we can see in right hand side of the table. However, for larger values, i.e., a decreased emphasis on the compactness of the territories, the compactness deteriorates disproportionately to the improvement of the balance, which is almost uniform for increasing values of β. Consequently, from a certain point on, we have to pay more in terms of compactness for a certain improvement of the balance. We finally decided to use $\beta = 0.5$. As the average running times are identical for varying values of β, namely 0.09 seconds, we do not list them here.

	Balance					Compactness			
$Q \setminus \beta$	0.25	0.33	0.5	0.66	0.75	0.33/0.25	0.5/0.33	0.66/0.5	0.75/0.66
10	4.1	4.5	4.0	3.9	3.5	1.0	1.3	3.6	5.5
20	2.7	2.3	1.7	1.3	1.1	-0.3	3.0	4.9	2.0
30	1.7	1.4	1.0	0.7	0.7	0.3	1.9	2.1	1.1
40	1.2	1.0	0.6	0.5	0.4	0.1	2.1	1.2	1.8
50	0.6	0.5	0.4	0.3	0.3	0.7	1.0	2.3	4.3
Avg	2.1	1.9	1.5	1.4	1.2	0.4	1.8	2.8	2.9

Table 6: Balance and Compactness in % for varying values of Q and β.

7 Integration into GIS

Enhanced with the extensions discussed in Section 5, the successive dichotomies heuristic can be integrated into a Geographic Information System (GIS). The user benefits from this integration in several ways. First, he can access the manifold of maps and data available in GIS. Moreover, GIS are common tools in geo-marketing and the user has access to all GIS functionality to work on his planning data and the resulting territories. Secondly, the seamless integration of territory design heuristics allows the user to access these methods without being an expert in Operations Research. After the computations performed by the heuristics in the background are finished, an immediate visualization of the results in the GIS allows the user to examine the proposed solution. Then, he has the option to manually adjust the solution or to change the planning parameters and start a new run of the optimization engine. It is this interactive type of work with the heuristics that requires the fast generation of solutions, already mentioned. The technical side of the integration into the GIS is sketched in Figure 4. While the user interaction and data management is all done within the GIS, the optimization engine is an external, underlying component. In line with the distinction between different extensions of the basic model discussed in Section 5, we distinguish two layers in the optimization engine. The lower layer contains the implementation of the successive dichotomies heuristic and the extensions detailed in Section 5.1. An intermediate layer contains the so-called scenario manager. This layer selects and combines the algorithms in the heuristics layer that are suited to produce an answer to the user's planning problem. This layer comprises, among others, the planning scenarios discussed in Section 5.2.

The various heuristics, embedded in the above methodology, are the algorithmic base of a commercial software product for geo-marketing called *BusinessManager*. The BusinessManager

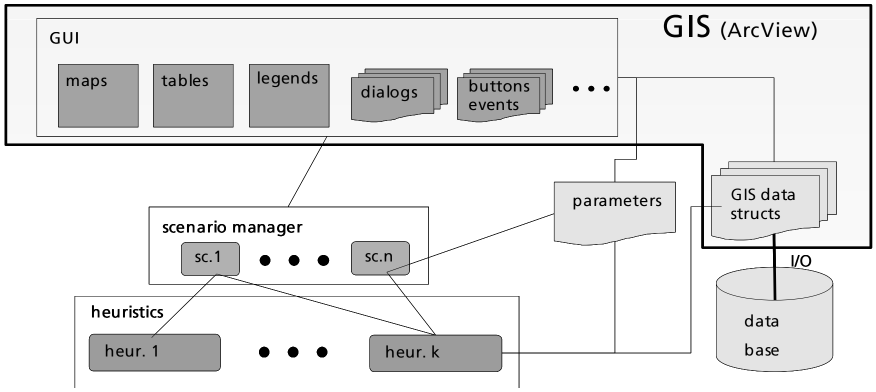

Figure 4: Integration of the heuristics into ArcView GIS.

is an extension of ESRI's ArcView GIS and has been developed by *geomer GmbH* (www.geomer.de) together with *Fraunhofer ITWM* (www.itwm.fraunhofer.de). The interface is integrated with the GIS so the user can access data from arbitrary shape files. Figure 5 shows a screenshot of the BusinessManager software.

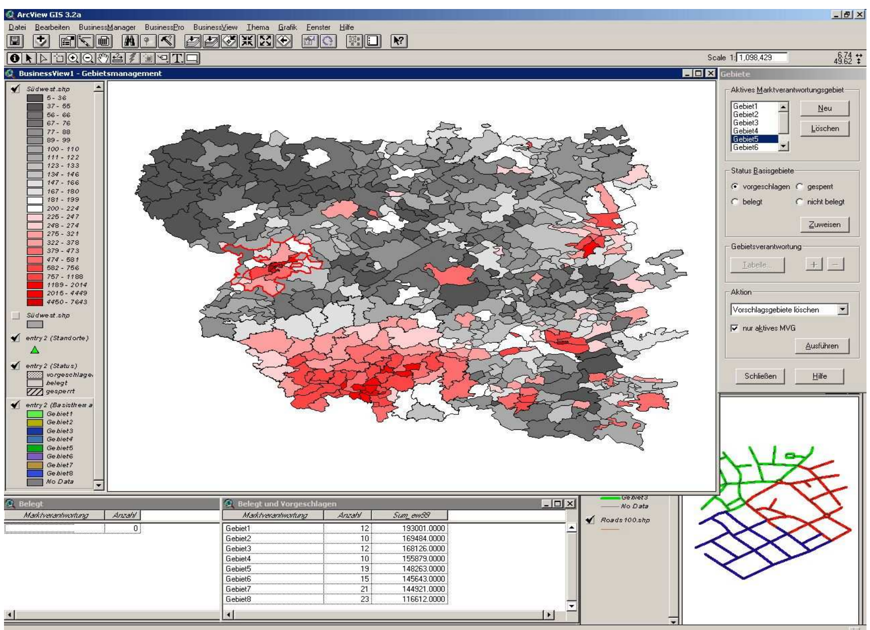

Figure 5: Screenshot of the BusinessManager software.

8 Conclusions

In this treatise we presented a generic approach to the territory design problem. Based on the basic model, its extensions, and the planning scenarios, we introduced a general framework adequately supporting decision makers to solve a large variety of applications in an interactive environment. As nowadays GISs provide the user with very detailed and comprehensive data, fast and flexibel algorithms are needed to solve large-scale practical problems. To this end, we presented a new algorithm that covers common criteria encountered in a manifold of applications. This method is based on techniques from Computational Geometry and utilizes the underlying geographical information of the problem. Although being a construction heuristic, our computational analysis shows that the algorithm provides very good results in almost negligible running times even for large problem instances with over 8000 basic areas. Therefore, the heuristic is suitable for a stand-alone, operational use in an interactive planning tool.

Apart from the computational tests, we also presented a theoretical analysis of the quality of the solutions obtained in terms of balance, which is usually the most important design criterion. By bounding the worsening of the balance of a partition problem compared to the balance of its father problem, we could derive an upper bound on the maximal balance of the final territories. Moreover, to solve different planning scenarios, we developed a new framework that repeatedly calls the algorithm for the basic model with varying numbers of territories until a satisfactory result has been obtained. As most applications for territory design problems have a strong spatial relation, we showed how to integrate them into a Geographical Information System.

There are still several open topics to work on. One open question is whether we obtain better results if we use a direct measure for the compactness of a line partition instead of the implicit one currently used? And in which time? Moreover, a computational study to fathom the applicability of the solution approaches for the extensions and planning scenarios of the basic model has to be done. The fast and efficient algorithm for the TDP gives rise to a promising decomposition heuristic for multifacility location problems. The idea is to partition the problem into a certain number of territories and then solve in each territory a location problem with a now reduced number of demand points and facilities. By doing so, we can likely solve the smaller problems to optimality, even for more elaborate location models. An open questions is, for example, how “far” should we decompose the problem? That is, what is an appropriate number of territories? Moreover, the approach to use the heuristic for demand point aggregation deserves further investigation.

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