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On Mathematical Aspects of a Combined Inversion of Gravity and Normal Mode Variations by a Spline Method

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On Mathematical Aspects of a Combined Inversion of Gravity and Normal Mode Variations by a Spline Method

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This paper provides a brief overview of two linear inverse problems concerned with the determination of the Earth's interior: inverse gravimetry and normal mode tomography. Moreover, a vector spline method is proposed for a combined solution of both problems. This method uses localised basis functions, which are based on reproducing kernels, and is related to approaches which have been successfully applied to the inverse gravimetric problem and the seismic traveltime tomography separately.

Key Words: inverse gravimetric problem, gravimetry, Newton potential, harmonic density, normal modes, splitting function, mass density, body wave velocity, reproducing kernel, ill-posed, regularisation, spline.

AMS(2000) Classification: 45B05, 45Q05, 46E22, 65F22, 65D07, 86A22.

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1 Introduction

There exist essentially two types of data which reveal information of the Earth's interior and surface: gravitational data and seismic data. Whereas the mass density ϱ, the compressional velocity α, and the shear velocity β are important rheological constants for the description of a material, the gravitational field is only linked to the mass density (via Newton's Law of Gravitation). Moreover, the so-called inverse gravimetric problem of calculating the mass density out of gravitational data involves a severe problem of a non-uniqueness of the solution (see [Michel and Fokas, 2008] for further details). More precisely, only the harmonic projection of the mass density can be reconstructed from gravity data. On the other hand, modern satellite missions such as CHAMP, GRACE, and GOCE provide models of the gravitational field in an unprecedented precision. Therefore, the use of gravitational data should be an essential component of a modelling of the Earth's structure. This is in particular the case for the modelling of the upper-most layer of the Earth, since the maximum principle for harmonic functions implies that the obtained harmonic density anomalies will show the strongest variations at the surface, which corresponds to the real density anomalies.

Seismic data that are connected to the named material parameters can be subdivided into travel time data of earthquake waves (in this case: body waves) and anomalies of the spectrum of free oscillations (which are also called normal modes) after major earthquakes. In the first case, the travel times are linked to the velocities of the compressional and the shear waves. In the linearised problem, the relation between the travel time and the slowness, which is the reciprocal of the velocity, is linear, which means that the dependence of the travel time on the velocity remains non-linear. There exist formulas which enable the calculation of the mass density out of the velocities. However, these are strongly simplified.

The analysis of the normal modes, however, gives information on all three unknown material parameters, where the associated equation is linear, as it is the case for the inverse gravimetric problem. For this reason, the inclusion of normal mode tomography in the inverse gravimetric problem appears to be a good choice to reduce the non-uniqueness problem of gravity inversion. Thus, this paper focusses on the combination of these two linear inverse problems. Its purpose is to provide a theoretical basis for the combination of gravitational data and spectral anomalies of free oscillations to calculate an approximate model of the three parameters α, β, and ϱ of the Earth. More precisely, the task is the determination of variations with respect to a radially symmetric model such as PREM [Dziewonski and Anderson, 1981, 1984]. For this purpose, the relevant parts of the known theory are summarised (to provide a reference for future research) and a numerical method is proposed.

Whereas there already exist several methods for solving these inverse problems (see e.g. [Ishii and Tromp, 2004, Kuo and Romanowicz, 2002, Li et al., 1991, Masters et al., 2000, Resovsky and Ritzwoller, 1999] for the normal mode tomography, [Michel and Fokas, 2008] for an overview of methods for the inverse gravimetric problem and [Ishii and Tromp, 2001] for a combined inversion of normal modes and free-air gravity anomaly), the aim of this paper is to develop the theory of a method which involves localised basis functions based

on reproducing kernels. Localised functions for the inverse gravimetry problems have been developed and successfully applied for the last ten years. In this context, wavelet-based multiscale methods (see [Freeden and Michel, 2004, Michel, 1998, 1999, 2002a,b, 2005a, Michel and Fokas, 2008]) were constructed as well as spline methods (see [Fengler et al., 2006, Michel and Wolf, 2008]). The latter use reproducing kernel Hilbert spaces and are more flexible with respect to the use of different types of data and are a further development of the theory of spherical splines [Freeden, 1981a,b, 1999]. A similar spline method on a three-dimensional ball was developed in [Amirbekyan, 2006, Amirbekyan and Michel, 2008] for the seismic (body wave) traveltime tomography. The success of these methods motivates the development of a corresponding method for the problem discussed in this paper. A new difficulty which occurs here is the fact that three unknown functions have to be calculated at the same time. This will be solved by enhancing the known approach to the vectorial case, where the underlying kernels can be constructed differently to adapt, for example, their localisation to the data situation.
This paper is organised as follows: In Section 2 two known orthonormal basis systems for square-integrable functions on a three-dimensional ball together with the system of spherical harmonics are recapitulated and the used notations are introduced. In Section 3 theoretical aspects of the involved inverse problems are summarised. In particular, the representation of the available data as the values of functionals applied to the unknown functions is investigated, since such formulae are essential for the construction of the spline method later on. In Section 4 the theory of vectorial splines is introduced. For this purpose, scalar Sobolev-like reproducing kernel Hilbert spaces on the three-dimensional ball are constructed in analogy to [Amirbekyan, 2006, Amirbekyan and Michel, 2008]. The associated reproducing kernels are used to propose a method for the calculation of a solution to the combined inverse problem. This solution is located in a model space which involves the previously derived functionals representing the available data. In Section 5 the details of the application of the spline method to the involved inverse problems (in particular also their combination) are discussed and explicit formulae for the implementation of the method are derived. Finally, Section 6 provides conclusions and an outlook.

2 Notation

The letters $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{R}$ denote the set of positive integers, non-negative integers and real numbers, respectively. $\mathrm{C}^{(k)}(D)$, $D \subset \mathbb{R}^m$, $k \in \mathbb{N}_0 \cup \{\infty\}$, $m \in \mathbb{N}$, denotes the set of all functions $f : D \to \mathbb{R}$ which are k-times continuously differentiable. The space $\mathrm{L}^2(D)$, $D \subset \mathbb{R}^m$ (Lebesgue) measurable, $m \in \mathbb{N}$, of all equivalence classes of almost everywhere identical square-integrable functions $f : D \to \mathbb{R}$ equipped with the scalar product

$$\langle F, G\rangle_{\mathrm{L}^2(D)} := \int_D F(x)\, G(x)\, \mathrm{d}x\,, \quad F, G \in \mathrm{L}^2(D)\ ,$$

is a Hilbert space. $\mathcal{L}(X, \mathbb{R}^m)$, $m \in \mathbb{N}$, denotes the space of all bounded linear functionals $\mathcal{F} : X \to \mathbb{R}^m$, where X is a normed space.
This paper deals with a spherical model of the Earth denoted by $\mathcal{B} := \{x \in \mathbb{R}^3 \big|\, |x| \leq a\}$,

its boundary is denoted by $\partial\mathcal{B}$. $\Omega := \{x \in \mathbb{R}^3 \big| \, |x| = 1\}$ is the unit sphere.
$\{Y_{n,j}\}_{n\in\mathbb{N}_0,\, j\in\{1,\ldots,2n+1\}}$ denotes a complete $\mathrm{L}^2(\Omega)$-orthonormal system of scalar spherical harmonics (see e.g. [Freeden et al., 1998]). Moreover, the Jacobi polynomials on $[-1,1]$ are represented by $\{P_m^{(\alpha,\beta)}\}_{m\in\mathbb{N}_0}$ (see e.g. [Szegö, 1939]).
Two complete orthonormal systems in $\mathrm{L}^2(\mathcal{B})$ are given by (see e.g. [Ballani et al., 1993, Dufour, 1977, Michel, 2005b, Tscherning, 1996])

$$\begin{aligned} G^{\mathrm{I}}_{m,n,j}(x) &:= \sqrt{\frac{4m+2n+3}{a^3}}\, P_m^{(0,n+1/2)}\left(2\,\frac{|x|^2}{a^2}-1\right)\left(\frac{|x|}{a}\right)^n Y_{n,j}\left(\frac{x}{|x|}\right), \\ G^{\mathrm{II}}_{m,n,j}(x) &:= \begin{cases} \sqrt{\frac{2m+3}{a^3}}\, P_m^{(0,2)}\left(2\,\frac{|x|}{a}-1\right) Y_{n,j}\left(\frac{x}{|x|}\right), & \text{if } x \neq 0 \\ 1\,, & \text{if } x = 0\,, \end{cases} \end{aligned}$$

$m,n \in \mathbb{N}_0$, $j = 1,\ldots,2n+1$, $x \in \mathcal{B}$. If simply $G_{m,n,j}$ is written in this paper, then system I as well as system II could be chosen. However, this choice has to be fixed throughout the whole theory.

3 Theoretical Aspects of the Involved Inverse Problems

3.1 The Inverse Gravimetric Problem

When searching the mass density distribution of the Earth, it appears to be a quite natural choice to look at gravity data. Indeed, Newton's Law of Gravitation

$$V(y) = \gamma \int_{\mathcal{B}} \frac{\varrho(x)}{|x-y|}\,\mathrm{d}x, \quad y \in \mathbb{R}^3 \setminus \mathcal{B}, \tag{1}$$

links the gravitational potential V to the mass density distribution ϱ, where γ is the gravitational constant. It is well-known that V is harmonic outside the gravitating body and regular at infinity (provided that ϱ satisfies some appropriate conditions, see, e.g., [Mikhlin, 1970, p. 230]) such that it can be expanded in terms of outer harmonics

$$V(y) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} V_{n,j}\left(\frac{a}{|y|}\right)^{n+1} Y_{n,j}\left(\frac{y}{|y|}\right), \quad |y| > a, \tag{2}$$

where

$$V_{n,j} = \int_{\partial\mathcal{B}} V(y)\,\frac{1}{a}\,Y_{n,j}\left(\frac{y}{|y|}\right)\mathrm{d}\omega(y). \tag{3}$$

Note that the series in (2) is uniformly convergent for all y with $|y| > a+\varepsilon$, $\varepsilon > 0$ fixed, and L^2-convergent on the surface $\partial\mathcal{B}$. The assumption $\varrho \in \mathrm{L}^2(\mathcal{B})$ justifies the ansatz

$$\varrho(x) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} \varrho_{n,j}(|x|) Y_{n,j}\left(\frac{x}{|x|}\right), \quad x \in \mathcal{B},$$

where the series converges with respect to $\|\cdot\|_{\mathrm{L}^2(\mathcal{B})}$. It is well-known that the Fredholm integral equation of the first kind given by (1) is not uniquely solvable (see [Michel and Fokas, 2008] for further details and historical references).

Theorem 1 *Provided that ϱ and V satisfy the conditions stated above, the following identity holds true*

$$\int_0^a r^{n+2} \varrho_{n,j}(r)\,\mathrm{d}r = \frac{2n+1}{4\pi} \cdot a^{n+1} \cdot V_{n,j}. \tag{4}$$

Obviously, the choice of the radially dependent functions $\varrho_{n,j}$ is not unique. Numerous different constraints were discussed in the literature for the purpose of uniqueness, where none of them yielded a satisfactory physical interpretation (see [Michel and Fokas, 2008] for a survey). From the point of view of functional analysis, there exists a decomposition

$$\mathrm{L}^2(\mathcal{B}) = \left\{ F \in \mathrm{C}^{(2)}(\mathcal{B}) \,\middle|\, \Delta F = 0 \right\} \oplus \left\{ F \in \mathrm{C}^{(2)}(\mathcal{B}) \,\middle|\, \Delta F = 0 \right\}^{\perp_{\mathrm{L}^2(\mathcal{B})}}.$$

It is known that the Fredholm integral operator T corresponding to (1) has the following null-space

$$\ker T = \left\{ F \in \mathrm{C}^{(2)}(\mathcal{B}) \,\middle|\, \Delta F = 0 \right\}^{\perp_{\mathrm{L}^2(\mathcal{B})}}.$$

Hence, only the harmonic projection of ϱ is uniquely determined by V whereas the orthogonal complement (which is called an anharmonic function) is arbitrary. This is convincingly demonstrated by the well-known result of an expansion of ϱ in terms of the basis of type I.

Theorem 2 *Let $V : \overline{\mathbb{R}^3 \backslash \mathcal{B}} \to \mathbb{R}$ given by (2) satisfy the conditions*

- $V|_{\partial\mathcal{B}} \in \mathrm{L}^2(\partial\mathcal{B})$,
- $\sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} V_{n,j}^2\, n^3 < +\infty$.

Then the unique solution $\varrho \in \mathrm{C}^{(2)}(\mathcal{B})$ of (1) subject to the constraint $\Delta\varrho = 0$ in $\operatorname{int}\mathcal{B}$ satisfies

$$\begin{aligned} \varrho(x) &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \varrho_{n,j}\, G^{\mathrm{I}}_{0,n,j}(x), \quad x \in \mathcal{B}, \\ \varrho_{n,j} &= \frac{2n+1}{4\pi} \sqrt{\frac{2n+3}{a}}\, V_{n,j}, \end{aligned}$$

where the series converges with respect to $\mathrm{L}^2(\mathcal{B})$.

If V is given at a position y_k above the surface, i.e. if $|y_k| =: \sigma > a$ (which is the case for airborne and spaceborne data), then the exponential ill-posedness of the involved downward continuation problem comes into play. More precisely, we can expand V at the sphere $\sigma\Omega$ as follows:

$$\begin{aligned} V(y) &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} V_{n,j} \left(\frac{a}{|y|}\right)^{n+1} Y_{n,j}\left(\frac{y}{|y|}\right) \\ &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} V_{n,j} \left(\frac{a}{\sigma}\right)^{n+1} \sigma \cdot \frac{1}{\sigma} Y_{n,j}\left(\frac{y}{\sigma}\right), \quad y \in \sigma\Omega, \end{aligned}$$

i.e.,

$$\left\langle V|_{\sigma\Omega}\,, \frac{1}{\sigma}\,Y_{n,j}\left(\frac{\cdot}{\sigma}\right)\right\rangle_{\mathrm{L}^2(\sigma\Omega)} \;=\; \frac{a^{n+1}}{\sigma^n}\,V_{n,j}.$$

The exponential ill-posedness now becomes obvious in the identity

$$\varrho_{n,j} \;=\; \frac{2n+1}{4\pi}\sqrt{\frac{2n+3}{a^3}}\left(\frac{\sigma}{a}\right)^n\left\langle V|_{\sigma\Omega}\,, \frac{1}{\sigma}\,Y_{n,j}\left(\frac{\cdot}{\sigma}\right)\right\rangle_{\mathrm{L}^2(\sigma\Omega)},$$

since $\sigma > a$. Note that the functional mapping $\varrho \in \mathrm{L}^2(\mathcal{B})$ to $V(y_k) \in \mathbb{R}$ is, consequently, represented by

$$\mathcal{F}^k\varrho := \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}\left(\frac{a}{|y_k|}\right)^n\frac{4\pi}{2n+1}\sqrt{\frac{a^3}{2n+3}}\;\left\langle\varrho, G^{\mathrm{I}}_{0,n,j}\right\rangle_{\mathrm{L}^2(\mathcal{B})}\;\frac{1}{|y_k|}\,Y_{n,j}\left(\frac{y_k}{|y_k|}\right). \tag{5}$$

This formula can be extended to the case where first-order or second-order normal or tangential derivatives of V are given, which is relevant for satellite missions such as CHAMP, GRACE, and GOCE. The corresponding formulae were derived in [Fengler et al., 2006] and [Michel, 2005a].

The expansion of ϱ in terms of the basis of type I has the advantage that it yields an obvious characterisation of the non-uniqueness. Since, on the other hand, the use of the basis of type II may have computational advantages due to the separability of the reproducing kernels (see below) into the radial and the angular part, we present here a new result which gives an explicit formula for the harmonic projection of ϱ in terms of the basis of type II. The proof uses the fact that the harmonic solution is also the solution of minimal $\mathrm{L}^2(\mathcal{B})$-norm, which is a consequence of basic functional analytic arguments.

Theorem 3 *The solution in Theorem 2 can also be represented by*

$$\begin{aligned}
\varrho(x) \;&=\; \sum_{n=0}^{\infty}\frac{2n+1}{4\pi a^2}\left(\sum_{l=0}^{n}(2l+3)\left(\frac{n!(n+2)!}{(n-l)!(n+3+l)!}\right)^2\right)^{-1} \\
&\qquad\times\sum_{m=0}^{n}(2m+3)\frac{n!(n+2)!}{(n-m)!(n+3+m)!}P_m^{(0,2)}\left(2\frac{|x|}{a}-1\right) \\
&\qquad\times\sum_{j=1}^{2n+1}V_{n,j}\,Y_{n,j}\left(\frac{x}{|x|}\right) \\
&=\; \sum_{m=0}^{n}\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}\frac{2n+1}{4\pi}\left(\sum_{l=0}^{n}(2l+3)\left(\frac{n!(n+2)!}{(n-l)!(n+3+l)!}\right)^2\right)^{-1} \\
&\qquad\qquad\times\sqrt{\frac{2m+3}{a}}\;\frac{n!(n+2)!}{(n-m)!(n+3+m)!}\,V_{n,j}\,G^{\mathrm{II}}_{m,n,j}(x),
\end{aligned} \tag{6}$$

where also this series converges with respect to $\mathrm{L}^2(\mathcal{B})$.

Proof. The use of the basis of type II involves the representation of the radially dependent spherical harmonics coefficients of ϱ as follows

$$\varrho_{n,j}(r) = \sum_{m=0}^{\infty} \varrho_{m,n,j}\, P_m^{(0,2)}\left(2\frac{r}{a}-1\right), \quad r \in [0,a]. \tag{7}$$

Within this proof, the notation $\mathrm{L}^2_{(0,2)}$ will be used for the Hilbert space with the following weighted L^2-inner product

$$\langle F, G\rangle_{\mathrm{L}^2_{(0,2)}} := \int_{-1}^{1} (1+t)^2 F(t)G(t)\,\mathrm{d}t.$$

Note that the series in (7) is well-defined with respect to $\mathrm{L}^2_{(0,2)}$ (with $r = (1+t)a/2$). Since the harmonic solution is also the unique solution which minimises

$$\|\varrho\|^2_{\mathrm{L}^2(\mathcal{B})} = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} \int_0^a r^2\, \varrho_{n,j}(r)^2\,\mathrm{d}r,$$

the problem of finding this density is equivalent to the following family of minimisation problems:

$$\begin{aligned} &\text{minimise} && \int_0^a r^2\left(\varrho_{n,j}(r)\right)^2\,\mathrm{d}r,\\ &\text{subject to} && \int_0^a \varrho_{n,j}(r) r^{n+2}\,\mathrm{d}r = \frac{2n+1}{4\pi}\cdot a^{n+1}\cdot V_{n,j}, \end{aligned}$$

where (4) has been used here. Equation (7) now yields

$$\begin{aligned} \int_0^a \varrho_{n,j}(r) r^{n+2}\,\mathrm{d}r &= \int_{-1}^{1} \varrho_{n,j}((1+t)a/2)\left(\frac{a(1+t)}{2}\right)^{n+2}\frac{a}{2}\,\mathrm{d}t\\ &= \sum_{m=0}^{\infty} \varrho_{m,n,j}\left(\frac{a}{2}\right)^{n+3}\int_{-1}^{1} P_m^{(0,2)}(t)(1+t)^n(1+t)^2\,\mathrm{d}t. \end{aligned}$$

Since $P_m^{(0,2)}$ is $\mathrm{L}^2_{(0,2)}$-orthogonal to all polynomials of degree $< m$, the series can be reduced to a finite summation:

$$\begin{aligned} &\int_0^a \varrho_{n,j}(r) r^{n+2}\,\mathrm{d}r\\ &= \left(\frac{a}{2}\right)^{n+3}\sum_{m=0}^{n} \varrho_{m,n,j}\int_{-1}^{1} P_m^{(0,2)}(t)(1+t)^n(1+t)^2\,\mathrm{d}t. \end{aligned}$$

By using the Rodrigues formula (see, e.g., [Szegö, 1939, p. 67])

$$(1+t)^2 P_m^{(0,2)}(t) = \frac{(-1)^m}{2^m m!}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m\left((1-t)^m(1+t)^{m+2}\right) \tag{8}$$

the occurring integral can be calculated via integration by parts

$$
\begin{aligned}
&\int_{-1}^{1} P_m^{(0,2)}(t)(1+t)^n(1+t)^2\,\mathrm{d}t \\
&= \frac{(-1)^m}{2^m m!}\int_{-1}^{1}(1+t)^n\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m\left((1-t)^m(1+t)^{m+2}\right)\,\mathrm{d}t \\
&= \frac{(-1)^m}{2^m m!}(-1)^m\,\frac{n!}{(n-m)!}\int_{-1}^{1}(1+t)^{n-m}(1-t)^m(1+t)^{m+2}\,\mathrm{d}t \\
&= \frac{n!}{2^m m!(n-m)!}\int_{-1}^{1}(1+t)^{n+2}(1-t)^m\,\mathrm{d}t \\
&= \frac{n!}{2^m m!(n-m)!}(-1)^m\int_{-1}^{1}\frac{(n+2)!}{(n+2+m)!}(1+t)^{n+2+m}m!(-1)^m\,\mathrm{d}t \\
&= \frac{n!}{2^m(n-m)!}\cdot\frac{(n+2)!}{(n+3+m)!}\,2^{n+3+m}.
\end{aligned}
$$

Hence, the side condition becomes

$$
\frac{2n+1}{4\pi}\cdot a^{n+1}\cdot V_{n,j} = a^{n+3}\sum_{m=0}^{n}\varrho_{m,n,j}\,\frac{n!(n+2)!}{(n-m)!(n+3+m)!}.
$$

Furthermore, using the Parseval identity one finds

$$
\begin{aligned}
\int_0^a r^2\left(\varrho_{n,j}(r)\right)^2\,\mathrm{d}r &= \int_{-1}^{1}\left(\frac{a(1+t)}{2}\right)^2\left(\varrho_{n,j}((1+t)a/2)\right)^2\,\frac{a}{2}\,\mathrm{d}t \\
&= \frac{a^3}{8}\sum_{m=0}^{\infty}\varrho_{m,n,j}^2\,\left\|P_m^{(0,2)}\right\|^2_{\mathrm{L}^2_{(0,2)}} \\
&= \frac{a^3}{8}\sum_{m=0}^{\infty}\varrho_{m,n,j}^2\,\frac{8}{2m+3},
\end{aligned}
$$

where the norm of the Jacobi polynomials can be found, for instance, in [Szegö, 1939, p. 68]. Since no conditions are given for $\varrho_{m,n,j}$ with $m > n$, these coefficients must vanish for a minimum. Thus, the discussed problem is reduced to a finite-dimensional optimisation problem with the Lagrangian function

$$
\begin{aligned}
L\left(\varrho_{n,j},\lambda\right) :=& \sum_{m=0}^{n}\varrho_{m,n,j}^2\,\frac{1}{2m+3} \\
&+\lambda\left(\sum_{m=0}^{n}\varrho_{m,n,j}\,\frac{n!(n+2)!}{(n-m)!(n+3+m)!}-\frac{2n+1}{4\pi a^2}\,V_{n,j}\right),
\end{aligned}
$$

where $\varrho_{n,j} := (\varrho_{0,n,j},\ldots,\varrho_{n,j,n})$. In addition to the side condition, the necessary conditions for an optimum are

$$
0 = \frac{\partial}{\partial\varrho_{m,n,j}}L\left(\varrho_{n,j},\lambda\right) = 2\varrho_{m,n,j}\,\frac{1}{2m+3}+\lambda\,\frac{n!(n+2)!}{(n-m)!(n+3+m)!}.
$$

Hence,

$$\varrho_{m,n,j} = -\frac{1}{2}\,(2m+3)\lambda\,\frac{n!(n+2)!}{(n-m)!(n+3+m)!}\,, \quad m = 0,\dots,n.$$

Inserting this into the side condition one obtains

$$\sum_{m=0}^{n}\left(-\frac{1}{2}\,(2m+3)\lambda\,\frac{n!(n+2)!}{(n-m)!(n+3+m)!}\,\frac{n!(n+2)!}{(n-m)!(n+3+m)!}\right) = \frac{2n+1}{4\pi a^2}\,V_{n,j},$$

which is equivalent to

$$\lambda = -\frac{2n+1}{2\pi a^2}\left(\sum_{m=0}^{n}(2m+3)\left(\frac{n!(n+2)!}{(n-m)!(n+3+m)!}\right)^2\right)^{-1} V_{n,j}.$$

Consequently,

$$\begin{aligned}\varrho_{m,n,j} \;&=\; \frac{2n+1}{4\pi a^2}\,(2m+3)\,\frac{n!(n+2)!}{(n-m)!(n+3+m)!}\\ &\quad\times\left(\sum_{l=0}^{n}(2l+3)\left(\frac{n!(n+2)!}{(n-l)!(n+3+l)!}\right)^2\right)^{-1} V_{n,j}.\end{aligned} \tag{9}$$

Finally, this unique candidate for a minimiser also satisfies a second-order sufficient condition since the Hessian of the Lagrangian function

$$\left(\nabla_{\varrho_{n,j}}\otimes\nabla_{\varrho_{n,j}}\right)L\left(\varrho_{n,j},\lambda\right) = \left(\frac{2}{2m+3}\,\delta_{ml}\right)_{m,l=0,\dots,n}$$

is positive definite (δ_{ml} is the Kronecker delta). Hence, the derived function is the unique minimal-$\mathrm{L}^2(\mathcal{B})$-norm solution of the inverse gravimetric problem. ■

Note that the factorials appearing in the formula in (9) can be further simplified. However, the present form of Equation (6) appears more convenient for numerical calculations.

Whereas there is no known constraint for a unique solution with a satisfactory physical motivation[1], the harmonicity constraint has a mathematical motivation. Due to the complementarity of the harmonic and the anharmonic part, the calculation of a harmonic density yields a unique part of the whole unknown function (like the unique determination of the x-component of an unknown vector out of data which are invariant with respect to the y- and the z-component of the unknown vector). The rest has to be calculated out of other data, which can, at least partially, be carried out by solving the inverse problem of the normal mode tomography.

[1] An inequality constraint $0 \le \varrho(x) \le \varrho_{\max}$, $x \in \mathcal{B}$, is an undisputable physical necessity. However, it still remains a challenge for future research to include this condition into the theory and numerics of the calculation of a continuous (i.e., not discrete), high-dimensional, global solution of the inverse gravimetric problem.

3.2 Normal Mode Tomography

Eigenoscillations of the Earth can be divided into spheroidal (or poloidal) modes, denoted by ${}_kS_l$, and toroidal (or torsional) modes, denoted by ${}_kT_l$. (Note that in literature usually the notation ${}_nS_l$ and ${}_nT_l$ is used.) The index $k \in \mathbb{N}_0$ designates the overtone index. For fixed $l \in \mathbb{N}_0$ the mode with $k = 0$ has the lowest frequency and is called fundamental mode, the modes with $k > 0$ are called overtones.
Furthermore, each displacement field is related to a spherical harmonic of degree l, i.e. $Y_{l,m},\ m \in \{1, \ldots, 2l+1\}$. In case of a spherically symmetric, non-rotating Earth model, the frequency of a certain mode is independent of the order m of the corresponding spherical harmonic. This yields $2l + 1$ modes with the same (degenerate) frequency. These $2l + 1$ modes are called singlets and are combined into one multiplet. Since the real Earth is not spherically symmetric, each singlet has a slightly different frequency. This phenomenon is named "splitting". Each multiplet has its own unique splitting function, which gives a local average of the Earth's three-dimensional heterogeneity. More precisely, the value of the splitting function at a particular direction $\xi \in \Omega$ can be interpreted as the degenerate frequency perturbation that the multiplet would experience if the spherically averaged Earth structure was identical to the radially dependent structure located in the direction ξ.

Since the splitting function σ (corresponding to one fixed multiplet) only depends on the direction, it can be regarded as an element of $\mathrm{L}^2(\Omega)$. It, thus, can be expanded as follows

$$\sigma = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \sigma_{n,j} Y_{n,j} \ .$$

Its coefficients $\sigma_{n,j}$, $n \in \mathbb{N}_0$, $j \in \{1, \ldots, 2n + 1\}$, are related to material properties of the Earth by

$$\sigma_{n,j} = \int_0^a \tilde{\mathbf{k}}_n(r) \cdot \delta\tilde{\mathbf{m}}_{n,j}(r)\, \mathrm{d}r \ , \tag{10}$$

where

$$\delta\tilde{\mathbf{m}} := \left(\frac{\delta\alpha}{\alpha_m}, \frac{\delta\beta}{\beta_m}, \frac{\delta\varrho}{\varrho_m} \right)$$

represents the relative deviation of the shear velocity, the compressional velocity and the mass density from a given reference model (usually PREM [Dziewonski and Anderson, 1981, 1984]). Furthermore,

$$\tilde{\mathbf{k}}_n := \left(K_n^\alpha, K_n^\beta, K_n^\varrho \right) \ , \quad n \in \mathbb{N}_0,$$

are the corresponding sensitivity kernels (see e.g. [Dahlen and Tromp, 1998, Li et al., 1991, Woodhouse and Dahlen, 1978]). For the later purposes the terms are rearranged as follows

$$\begin{aligned} \sigma_{n,j} &= \int_0^a \mathbf{k}_n(r) \cdot \delta\mathbf{m}_{n,j}(r)\,\mathrm{d}r \ , \\ \delta\mathbf{m} &:= (\delta\alpha, \delta\beta, \delta\varrho) \ , \\ \mathbf{k}_n &:= \left(\frac{K_n^\alpha}{\alpha_m}, \frac{K_n^\beta}{\beta_m}, \frac{K_n^\varrho}{\varrho_m} \right) \ . \end{aligned}$$

A linear functional mapping $\delta\mathbf{m}$ to $\sigma(\xi_i) \in \mathbb{R}$ can now be defined by

$$\begin{aligned} \mathcal{F}^i \delta\mathbf{m} := \sigma(\xi_i) &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \sigma_{n,j} Y_{n,j}(\xi_i) \\ &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \int_0^a \mathbf{k}_n(r) \cdot \delta\mathbf{m}_{n,j}(r)\,\mathrm{d}r\, Y_{n,j}(\xi_i) \ . \end{aligned} \tag{11}$$

Note that it is often assumed (see e.g. [Ishii and Tromp, 2001, Li et al., 1991]) that there exist scaling factors $c_\alpha, c_\beta > 0$ such that

$$\frac{\delta\varrho}{\varrho_m} = c_\alpha \frac{\delta\alpha}{\alpha_m} = c_\beta \frac{\delta\beta}{\beta_m} \ .$$

Using this property one can also define a functional mapping $\delta\varrho$ to $\sigma(\xi_i) \in \mathbb{R}$:

$$\mathcal{F}^i \delta\varrho := \sigma(\xi_i) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \int_0^a \frac{K_n(r)}{\varrho_m(r)}\, \delta\varrho_{n,j}(r)\,\mathrm{d}r\, Y_{n,j}(\xi_i) \ ,$$

where

$$K_n := \frac{1}{c_\alpha} K_n^\alpha + \frac{1}{c_\beta} K_n^\beta + K_n^\varrho \ .$$

There are several methods for estimating splitting function coefficients from the mode spectra, e.g. iterative spectral fitting [Resovsky and Ritzwoller, 1998], receiver stripping [Masters et al., 2000] and peak shift observation [Widmer-Schnidrig, 2002]. An overview and links for downloading some coefficients can be found on the “Reference Earth Model Website” (http://mahi.ucsd.edu/Gabi/rem.dir/surface/rem.surf.html).

However, the splitting functions represent global averages. Due to the heterogeneity of the data structure a more advanced and, possibly, more precise way would be the use of the anomalies $\delta\omega$ of the spectrum. The frequency shift $\delta\omega$ of a fixed multiplet on the great circle with pole $\eta \in \Omega$ is given by

$$\delta\omega(\eta) := \omega(\eta) - \overline{\omega} \ ,$$

where ω is the peak frequency and $\overline{\omega}$ is the mean frequency of the multiplet. This frequency shift can be interpreted as a great circle average of the splitting function (see [Jordan, 1978, Widmer-Schnidrig, 2002]):

$$\delta\omega(\eta) = \frac{1}{2\pi} \oint_\eta \sigma(\xi)\,\mathrm{d}\Delta(\xi) \ , \tag{12}$$

where $\oint_\eta$ denotes the integration around the great circle with pole η. This domain of the integral refers to the (approximate) surface paths of a propagating seismic wave.
From [Backus, 1964] it is known that

$$\frac{1}{2\pi}\oint_\eta Y_{n,j}(\xi)\,\mathrm{d}\Delta(\xi) = P_n(0)Y_{n,j}(\eta)\ ,$$

such that

$$\begin{aligned}
\delta\omega(\eta) &= \frac{1}{2\pi}\oint_\eta \sigma(\xi)\,\mathrm{d}\Delta(\xi) \\
&= \frac{1}{2\pi}\oint_\eta \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}\sigma_{n,j}Y_{n,j}(\xi)\,\mathrm{d}\Delta(\xi) \\
&= \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} P_n(0)\sigma_{n,j}Y_{n,j}(\eta)\ .
\end{aligned}$$

Note that the Legendre polynomials fulfill $P_n(0) = 0$ if and only if n is odd. This has the consequence that it is not possible to obtain Earth structures of odd harmonic degree from the spectra of isolated multiplets.
A linear functional mapping $\delta\mathbf{m}$ to $\delta\omega(\xi_i) \in \mathbb{R}$ can be defined by

$$\mathcal{F}^i\delta\mathbf{m} := \delta\omega(\xi_i) = \sum_{\substack{n=0\\ n\ \mathrm{even}}}^{\infty}\sum_{j=1}^{2n+1} P_n(0)\int_0^a \mathbf{k}_n(r)\cdot\delta\mathbf{m}_{n,j}(r)\,\mathrm{d}r\,Y_{n,j}(\xi_i)\ . \tag{13}$$

4 A Spline Method for the Combined Inversion

In the following, a vectorial spline method is constructed for the case that the given data of the interpolation problem are represented in terms of continuous linear functionals depending on three unknown functions with the domain $\mathcal{B}$. For this purpose, a known method for the corresponding interpolation of a single function on $\mathcal{B}$ introduced in [Amirbekyan, 2006, Amirbekyan and Michel, 2008] is used.

4.1 Sobolev Spaces

The spline method is based on the following reproducing kernel Hilbert spaces on $\mathcal{B}$, see [Amirbekyan, 2006, Amirbekyan and Michel, 2008] for further details. Let $\{A_{m,n}\} := \{A_{m,n}\}_{m,n\in\mathbb{N}_0}$ be a sequence of real numbers. Consider the set $\mathcal{E} := \mathcal{E}(\{A_{m,n}\};\mathcal{B})$ of all functions $F \in \mathrm{L}^2(\mathcal{B})$ satisfying $\langle F, G_{m,n,j}\rangle_{\mathrm{L}^2(\mathcal{B})} = 0$ for all $m,n \in \mathbb{N}_0$ with $A_{m,n} = 0$ and

$$\sum_{\substack{m,n=0\\ A_{m,n}\neq 0}}^{\infty}\sum_{j=1}^{2n+1} A_{m,n}^{-2}\,\langle F, G_{m,n,j}\rangle_{\mathrm{L}^2(\mathcal{B})}^2 < +\infty\ .$$

An inner product $(\cdot,\cdot)_{\mathcal{H}(\{A_{m,n}\};\mathcal{B})}$ on $\mathcal{E}$ can be defined by

$$(F_1,F_2)_{\mathcal{H}(\{A_{m,n}\};\mathcal{B})} := \sum_{\substack{m,n=0\\ A_{m,n}\neq 0}}^{\infty} \sum_{j=1}^{2n+1} A_{m,n}^{-2} \left\langle F_1, G_{m,n,j}\right\rangle_{\mathrm{L}^2(\mathcal{B})} \left\langle F_2, G_{m,n,j}\right\rangle_{\mathrm{L}^2(\mathcal{B})} .$$

The associated norm is given by

$$\|F\|_{\mathcal{H}(\{A_{m,n}\};\mathcal{B})} := \left((F,F)_{\mathcal{H}(\{A_{m,n}\};\mathcal{B})}\right)^{1/2} .$$

The Sobolev space $\mathcal{H}(\{A_{m,n}\};\mathcal{B})$ is defined as the completion of $\mathcal{E}(\{A_{m,n}\};\mathcal{B})$ with respect to $(\cdot,\cdot)_{\mathcal{H}(\{A_{m,n}\};\mathcal{B})}$, i.e.,

$$\mathcal{H}(\{A_{m,n}\};\mathcal{B}) := \overline{\mathcal{E}(\{A_{m,n}\};\mathcal{B})}^{\|\cdot\|_{\mathcal{H}(\{A_{m,n}\};\mathcal{B})}} .$$

$\mathcal{H}(\{A_{m,n}\};\mathcal{B})$ equipped with the inner product $(\cdot,\cdot)_{\mathcal{H}(\{A_{m,n}\};\mathcal{B})}$ is a Hilbert space.
A sequence $\{A_{m,n}\}\subset\mathbb{R}$ is said to be summable with respect to the basis of type I if

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{m,n}^2 (2n+1)(4m+2n+3)\binom{m+n+\frac{1}{2}}{m}^2 < +\infty .$$

$\{A_{m,n}\}\subset\mathbb{R}$ is said to be summable with respect to the basis of type II if

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{m,n}^2\, m^5 n < +\infty .$$

The sequences $\{A_{m,n}\}$ used below are always assumed to be summable with respect to the chosen basis. The summability automatically guarantees that every element of the Hilbert space $\mathcal{H}(\{A_{m,n}\};\mathcal{B})$ corresponds to a continuous bounded function such that $\mathcal{H}(\{A_{m,n}\};\mathcal{B})\subset \mathrm{C}(\mathcal{B})$ in case of type I and $\mathcal{H}(\{A_{m,n}\};\mathcal{B})\subset \mathrm{C}(\mathcal{B}\setminus\{0\})$ in case of type II. Note that also in case II, all functions in the Sobolev space are bounded in $\mathcal{B}$.
$\mathcal{H} := \mathcal{H}(\{A_{m,n}\};\mathcal{B})$ has a unique reproducing kernel $K_{\mathcal{H}}:\mathcal{B}\times\mathcal{B}\to\mathbb{R}$ satisfying

(i) $K_{\mathcal{H}}(x,\cdot)$, $K_{\mathcal{H}}(\cdot,x)\in\mathcal{H}$ for all $x\in\mathcal{B}$,

(ii) $(F,K_{\mathcal{H}}(x,\cdot))_{\mathcal{H}} = (F,K_{\mathcal{H}}(\cdot,x))_{\mathcal{H}} = F(x)$ for all $F\in\mathcal{H}$ and $x\in\mathcal{B}$.

This kernel is given by

$$K_{\mathcal{H}}(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} A_{m,n}^2 G_{m,n,j}(x) G_{m,n,j}(y) , \quad x,y\in\mathcal{B} .$$

Note that the kernel can also be used on the whole ball $\mathcal{B}$ if the basis of type II is chosen.

4.2 Vectorial Splines

Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be three (not necessarily different[2]) Sobolev spaces as introduced in the previous subsection. Consider the space $\mathfrak{H} := \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$. An inner product on $\mathfrak{H}$ can be defined by

$$(f, g)_{\mathfrak{H}} := (F_1, G_1)_{\mathcal{H}_1} + (F_2, G_2)_{\mathcal{H}_2} + (F_3, G_3)_{\mathcal{H}_3} \ ,$$

where $f := (F_1, F_2, F_3)$ and $g := (G_1, G_2, G_3)$ are elements of $\mathfrak{H}$. Note that $(\mathfrak{H}, (\cdot,\cdot)_{\mathfrak{H}})$ is complete.

One can define a tensorial kernel $\mathfrak{K}$, which inherits some reproducing kernel properties from the scalar kernels $K_{\mathcal{H}_i}$, $i = 1, 2, 3$.

Definition 4 *Let $K_{\mathcal{H}_i} : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ be the unique reproducing kernel of $\mathcal{H}_i$, $i = 1, 2, 3$. The tensorial kernel $\mathfrak{K} : \mathcal{B} \times \mathcal{B} \to \mathbb{R}^{3\times 3}$ is defined by*

$$\mathfrak{K}(x, y) := \begin{pmatrix} K_{\mathcal{H}_1}(x, y) & 0 & 0 \\ 0 & K_{\mathcal{H}_2}(x, y) & 0 \\ 0 & 0 & K_{\mathcal{H}_3}(x, y) \end{pmatrix} \ , \ x, y \in \mathcal{B} \, .$$

Note that if one parameter $x \in \mathcal{B}$ is chosen arbitrary but fixed each row of $\mathfrak{K}(\cdot, x)$ is an element of $\mathfrak{H}$ since $K_{\mathcal{H}_i}(\cdot, x)$ is an element of $\mathcal{H}_i$, $i = 1, 2, 3$. Therefore, a functional $\mathcal{F} \in \mathcal{L}(\mathfrak{H}, \mathbb{R})$ can be applied rowwise:

$$\mathcal{F}\mathfrak{K}(\cdot, x) := \begin{pmatrix} \mathcal{F}(K_{\mathcal{H}_1}(\cdot, x)\,,\, 0\,,\, 0) \\ \mathcal{F}(0\,,\, K_{\mathcal{H}_2}(\cdot, x)\,,\, 0) \\ \mathcal{F}(0\,,\, 0\,,\, K_{\mathcal{H}_3}(\cdot, x)) \end{pmatrix} \ .$$

This property can be used for the definition of splines on $\mathfrak{H}$. This definition includes the use of bounded linear functionals. For the discussed application, this will be a mixture of some functionals of type (5) and some of type (11) or (13).

Definition 5 *Let $N \in \mathbb{N}$ and let $\mathcal{F} := \{\mathcal{F}^1, \ldots, \mathcal{F}^N\} \subset \mathcal{L}(\mathfrak{H}, \mathbb{R})$ be a linearly independent system of bounded linear functionals from $\mathfrak{H}$ into $\mathbb{R}$. A function $s \in \mathfrak{H}$ of the form*

$$s(x) = \sum_{k=1}^{N} a_k \mathcal{F}^k \mathfrak{K}(\cdot, x) \ , \quad x \in \mathcal{B} \ ,$$

$a = (a_1, \ldots, a_N)^{\mathrm{T}} \in \mathbb{R}^N$, is called (vectorial) spline in $\mathfrak{H}$ relative to $\mathcal{F}$. Such splines are collected in the space $\mathrm{spline}(\mathfrak{H}; \mathcal{F})$.

A spline interpolation problem can be formulated in the following way: A system of linearly independent functionals $\mathcal{F} := \{\mathcal{F}^1, \ldots, \mathcal{F}^N\} \subset \mathcal{L}(\mathfrak{H}, \mathbb{R})$, $N \in \mathbb{N}$, and a vector $b = (b_1, \ldots, b_N)^{\mathrm{T}} \in \mathbb{R}^N$ are given. Determine $s \in \mathrm{spline}(\mathfrak{H}; \mathcal{F})$ such that

$$\mathcal{F}^i s = b_i \quad \text{for all } i = 1, \ldots, N$$

[2]The spaces may differ with respect to the chosen symbol $\{A_{m,n}\}$ and the used basis $\{G^{\mathrm{I/II}}_{m,n,j}\}$.

or, equivalently, determine $a \in \mathbb{R}^N$ such that

$$\sum_{j=1}^{N} a_j \mathcal{F}_y^i \mathcal{F}_x^j \mathfrak{K}(x,y) = b_i \quad \text{for all } i = 1, \ldots, N\,, \tag{14}$$

where $\mathcal{F}_x^j$ means that $\mathcal{F}^j$ is applied to the function $x \mapsto \mathfrak{K}(x,y)$ for each fixed y. This yields a function of y, to which $\mathcal{F}^i$ is eventually applied.
Obviously, (14) represents a system of linear equations with the matrix

$$\left(\mathcal{F}_y^i \mathcal{F}_x^j \mathfrak{K}(x,y)\right)_{i,j=1,\ldots,N}\,,$$

which is positive definite according to the following considerations, which are an analogue of known features of the scalar approach.

Lemma 6 *Let $\mathcal{F} \in \mathcal{L}(\mathfrak{H},\mathbb{R})$ be arbitrary. Then $y \mapsto \mathcal{F}_x \mathfrak{K}(x,y)$ is in $\mathfrak{H}$ and*

$$\mathcal{F}f = \left(f, \mathcal{F}_x \mathfrak{K}(x,\cdot)\right)_{\mathfrak{H}}$$

for all $f \in \mathfrak{H}$.

Proof. Let $\mathcal{F} \in \mathcal{L}(\mathfrak{H},\mathbb{R})$. According to Riesz' representation theorem (see e.g. [Yosida, 1980, p. 90]), there exists a function $h := (H_1, H_2, H_3) \in \mathfrak{H}$ with

$$\mathcal{F}f = (f,h)_{\mathfrak{H}} \quad \text{for all } f \in \mathfrak{H}\,. \tag{15}$$

Let $y \in \mathcal{B}$ be arbitrary but fixed. Equation (15) and the reproducing kernel property of $K_{\mathcal{H}_i}$, $i = 1, 2, 3$, lead to

$$\begin{aligned} \mathcal{F}_x\left(K_{\mathcal{H}_1}(x,y), 0, 0\right) &= \left(\left(K_{\mathcal{H}_1}(\cdot,y), 0, 0\right), h\right)_{\mathfrak{H}} \\ &= \left(K_{\mathcal{H}_1}(\cdot,y), H_1\right)_{\mathcal{H}_1} + (0, H_2)_{\mathcal{H}_2} + (0, H_3)_{\mathcal{H}_3} \\ &= H_1(y)\,. \end{aligned}$$

Analogously,

$$\begin{aligned} \mathcal{F}_x\left(0, K_{\mathcal{H}_2}(x,y), 0\right) &= H_2(y)\,, \\ \mathcal{F}_x\left(0, 0, K_{\mathcal{H}_3}(x,y)\right) &= H_3(y)\,. \end{aligned}$$

In summary, one gets

$$\mathcal{F}_x \mathfrak{K}(x,y) = \begin{pmatrix} \mathcal{F}_x\left(K_{\mathcal{H}_1}(x,y), 0, 0\right) \\ \mathcal{F}_x\left(0, K_{\mathcal{H}_2}(x,y), 0\right) \\ \mathcal{F}_x\left(0, 0, K_{\mathcal{H}_3}(x,y)\right) \end{pmatrix} = \begin{pmatrix} H_1(y) \\ H_2(y) \\ H_3(y) \end{pmatrix} = h(y)\,.$$

■

This result shows that this vectorial approach is a straight forward generalisation of the scalar case in [Amirbekyan, 2006, Amirbekyan and Michel, 2008]. The matrix of the system

of linear equations turns out to be a Gramian matrix and is, thus, positive definite, if it is regular.
Moreover, the proofs of the following minimum properties are completely analogous to the scalar case. They justify e.g. the name “spline” that is used for the interpolating functions. The usual Euclidean cubic splines have the advantage that they have minimal linearised curvature among all interpolating functions. More abstractly they minimise a certain norm in the set of all interpolating functions. The splines that are given here minimise the $\mathfrak{H}$-norm.

Theorem 7 (First Minimum Property) *Let $b \in \mathbb{R}^N$, $N \in \mathbb{N}$ be given and let $\mathcal{F} := \{\mathcal{F}^1, \ldots, \mathcal{F}^N\} \subset \mathcal{L}(\mathfrak{H}, \mathbb{R})$ be linearly independent. Furthermore, let $s^* \in \operatorname{spline}(\mathfrak{H}; \mathcal{F})$ be the unique interpolating spline with $\mathcal{F}^i s^* = b_i$ for all $i \in \{1, \ldots, N\}$. Then*

$$\|s^*\|_{\mathfrak{H}} = \min \left\{ \|f\|_{\mathfrak{H}} \;\middle|\; f \in \mathfrak{H} \text{ with } \mathcal{F}^i f = b_i \text{ for all } i = 1, \ldots, N \right\} ,$$

where s^* is the unique minimising function.

Furthermore, there is one and only one spline which is closest to a given function $f \in \mathfrak{H}$. This so-called best approximation is the spline which is given by the interpolation conditions.

Theorem 8 (Second Minimum Property) *Let $\mathcal{F} := \{\mathcal{F}^1, \ldots, \mathcal{F}^N\} \subset \mathcal{L}(\mathfrak{H}, \mathbb{R})$ be linearly independent, $N \in \mathbb{N}$, and let $f \in \mathfrak{H}$ be an arbitrary function. Moreover, let $s^* \in \operatorname{spline}(\mathfrak{H}; \mathcal{F})$ be the unique spline to satisfy $\mathcal{F}^i s^* = \mathcal{F}^i f$ for all $i \in \{1, \ldots, N\}$. Then*

$$\|s^* - f\|_{\mathfrak{H}} = \min \left\{ \|s - f\|_{\mathfrak{H}} \;\middle|\; s \in \operatorname{spline}(\mathfrak{H}; \mathcal{F}) \right\} ,$$

where s^* is the unique minimising spline.

5 Details of the Application to the Combined Inverse Problem

For the discussed application, the unknown vectorial function consists of the anomalies of the body wave velocities and the mass density, i.e.

$$s = (\delta\alpha, \delta\beta, \delta\varrho),$$

where these anomalies are considered with respect to a given radially symmetric Earth model $(\alpha_{\mathrm{m}}, \beta_{\mathrm{m}}, \varrho_{\mathrm{m}})$ such as PREM. The given data will be a mixture of gravity anomalies and normal mode anomalies. In the first case, the corresponding version of Equation (5) for the consideration of anomalies is, due to the linearity of the problem,

$$\mathcal{F}^k_{\mathrm{Gr}}(\delta\alpha, \delta\beta, \delta\varrho) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left(\frac{a}{|y_k|} \right)^n \frac{4\pi}{2n+1} \sqrt{\frac{a^3}{2n+3}} \left\langle \delta\varrho, G^{\mathrm{I}}_{0,n,j} \right\rangle_{\mathrm{L}^2(\mathcal{B})} \frac{1}{|y_k|} \, Y_{n,j} \left(\frac{y_k}{|y_k|} \right) .$$

If the whole potential V (i.e. not the potential associated to the anomaly $\delta\varrho$) is given, then the required data can be easily calculated via

$$\mathcal{F}^k_{\mathrm{Gr}}(\delta\alpha, \delta\beta, \delta\varrho) = V(y_k) - 4\pi \int_0^a \varrho_{\mathrm{m}}(r) \, r^2 \, \mathrm{d}r \, \frac{1}{|y_k|} ,$$

see [Michel, 1999]. The treatment of a representation in the basis of type II is not unique as Equation (9) shows. This is caused by the potential theoretic fact that a harmonic function (e.g. on a ball) is given by its values at the surface. Hence, for each index pair (n,j) the knowledge of $\delta\varrho_{m,n,j}$ for one $\hat{m} \in \{0,...,n\}$ only suffices to obtain $V_{n,j}$ uniquely. As a consequence, this also allows the calculation of the other expansion coefficients of the density for $m \in \{0,...,n\} \setminus \{\hat{m}\}$ out of $\delta\varrho_{\hat{m},n,j}$. It remains to investigate which choice performs best from the numerical point of view. With the notation $\mu(n) \in \{0,...,n\}$ for the chosen degree of the involved Jacobi polynomial, the corresponding functional can be written as

$$\begin{aligned}\mathcal{F}^k_{\mathrm{Gr}}(\delta\alpha,\delta\beta,\delta\varrho) &= \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}\left(\sum_{l=0}^{n}(2l+3)\left(\frac{n!(n+2)!}{(n-l)!(n+3+l)!}\right)^2\right)\sqrt{\frac{a^3}{2\mu(n)+3}}\\ &\quad\times\frac{4\pi}{2n+1}\,\frac{(n-\mu(n))!(n+3+\mu(n))!}{n!(n+2)!}\,\left\langle\delta\varrho, G^{\mathrm{II}}_{\mu(n),n,j}\right\rangle_{\mathrm{L}^2(\mathcal{B})}\left(\frac{a}{|y_k|}\right)^n\frac{1}{|y_k|}\,Y_{n,j}\left(\frac{y_k}{|y_k|}\right).\end{aligned}$$

Note that the previous notation of Section 3 is now changed in the sense that the functionals $\mathcal{F}^k$ now act on the vector $(\delta\alpha,\delta\beta,\delta\varrho)$. Moreover, for the normal mode tomography, Equation (11) becomes

$$\begin{aligned}\mathcal{F}^i_{\mathrm{Sp}}(\delta\alpha,\delta\beta,\delta\varrho) &= \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}\int_0^a\left(\frac{K^\alpha_n(r)}{\alpha_{\mathrm{m}}(r)},\frac{K^\beta_n(r)}{\beta_{\mathrm{m}}(r)},\frac{K^\varrho_n(r)}{\varrho_{\mathrm{m}}(r)}\right)\cdot(\delta\alpha_{n,j}(r),\delta\beta_{n,j}(r),\delta\varrho_{n,j}(r))\,\mathrm{d}r\\ &\quad\times Y_{n,j}\left(\xi_i\right).\end{aligned}$$

and Equation (13) can be written as

$$\begin{aligned}\mathcal{F}^i_{\mathrm{Fr}}(\delta\alpha,\delta\beta,\delta\varrho) &= \sum_{\substack{n=0\\ n\text{ even}}}^{\infty}\sum_{j=1}^{2n+1}\int_0^a\left(\frac{K^\alpha_n(r)}{\alpha_{\mathrm{m}}(r)},\frac{K^\beta_n(r)}{\beta_{\mathrm{m}}(r)},\frac{K^\varrho_n(r)}{\varrho_{\mathrm{m}}(r)}\right)\cdot(\delta\alpha_{n,j}(r),\delta\beta_{n,j}(r),\delta\varrho_{n,j}(r))\,\mathrm{d}r\\ &\quad\times P_n(0)\,Y_{n,j}\left(\xi_i\right),\end{aligned}$$

where

$$P_n(0) = \frac{1}{2^n}\,(-1)^{n/2}\,\frac{n!}{[(n/2)!]^2},\quad n\in\mathbb{N}_0,\ n\text{ even},$$

see [Freeden et al., 1998, p. 41]. Note that for both choices of orthonormal bases for $\mathrm{L}^2(\mathcal{B})$ the unknown scalar functions $\delta\alpha$, $\delta\beta$, $\delta\varrho$ admit an expansion of the form

$$\delta F(r\xi) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}\underbrace{\sum_{m=0}^{\infty}c_{m,n}P_{m,n}(r)\left\langle\delta F, G_{m,n,j}\right\rangle_{\mathrm{L}^2(\mathcal{B})}}_{=\delta F_{n,j}(r)}Y_{n,j}(\xi)$$

in the sense of $\mathrm{L}^2(\mathcal{B})$, where $G_{m,n,j}(r\xi) = c_{m,n}P_{m,n}(r)Y_{n,j}(\xi)$ almost everywhere.
For the calculation of the splines, the spline basis functions and the matrix of the system of linear equations have to be calculated. For this purpose, expressions of the form

$$\begin{aligned}\mathcal{F}^k\mathfrak{K}(\cdot,x) &= \begin{pmatrix} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} A_{m,n,1}^2 \mathcal{F}^k\left(G_{m,n,j}^{\mathrm{t}(1)},0,0\right) G_{m,n,j}^{\mathrm{t}(1)}(x) \\ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} A_{m,n,2}^2 \mathcal{F}^k\left(0,G_{m,n,j}^{\mathrm{t}(2)},0\right) G_{m,n,j}^{\mathrm{t}(2)}(x) \\ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} A_{m,n,3}^2 \mathcal{F}^k\left(0,0,G_{m,n,j}^{\mathrm{t}(3)}\right) G_{m,n,j}^{\mathrm{t}(3)}(x) \end{pmatrix} \\ &= \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} A_{m,n,i}^2 \mathcal{F}^k\left(G_{m,n,j}^{\mathrm{t}(i)}\cdot(\delta_{ip})_{p=1,2,3}\right) G_{m,n,j}^{\mathrm{t}(i)}(x)\right)_{i=1,2,3} \end{aligned} \tag{16}$$

and

$$\mathcal{F}_x^l\mathcal{F}_y^k\mathfrak{K}(y,x) = \mathcal{F}_x^l\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} A_{m,n,i}^2 \mathcal{F}^k\left(G_{m,n,j}^{\mathrm{t}(i)}\cdot(\delta_{ip})_{p=1,2,3}\right) G_{m,n,j}^{\mathrm{t}(i)}(x)\right)_{i=1,2,3}.$$

have to be calculated (note the symmetry of the kernel $\mathfrak{K}$, i.e. $\mathfrak{K}(x,y) = \mathfrak{K}(y,x)$). Here, $\mathrm{t}(i) \in \{\mathrm{I},\mathrm{II}\}$ refers to the possibility to choose different basis systems for the construction of the Sobolev spaces $\mathcal{H}_i$, $i = 1,2,3$. Note that the freedom to choose different sequences $\{A_{m,n,i}\}_{m,n\in\mathbb{N}_0}$, $i = 1,2,3$ has also been taken into account here.
Due to the previous considerations the identity in (16) is explicitly given for all relevant kinds of functionals. In each case, one gets an expression of the form

$$\mathcal{F}^k\left(G_{m,n,j}^{\mathrm{t}(i)}\cdot(\delta_{ip})_{p=1,2,3}\right) = C_{i,m,n,k}Y_{n,j}(\xi_k) \tag{17}$$

for a given point $\xi_k \in \Omega$. For instance,

$$\begin{aligned}\mathcal{F}_{\mathrm{Gr}}^k\left(G_{m,n,j}^{\mathrm{I}}\cdot(\delta_{ip})_{p=1,2,3}\right) &= \delta_{i3}\delta_{m0}\left(\frac{a}{|y_k|}\right)^n \frac{4\pi}{2n+1}\sqrt{\frac{a^3}{2n+3}}\,\frac{1}{|y_k|}\,Y_{n,j}\left(\frac{y_k}{|y_k|}\right), \\ \mathcal{F}_{\mathrm{Gr}}^k\left(G_{m,n,j}^{\mathrm{II}}\cdot(\delta_{ip})_{p=1,2,3}\right) &= \delta_{i3}\delta_{m\mu(n)}\left(\sum_{l=0}^{n}(2l+3)\left(\frac{n!(n+2)!}{(n-l)!(n+3+l)!}\right)^2\right) \\ &\times\sqrt{\frac{a^3}{2\mu(n)+3}}\,\frac{4\pi}{(2n+1)}\,\frac{(n-\mu(n))!(n+3+\mu(n))!}{n!(n+2)!}\left(\frac{a}{|y_k|}\right)^n\frac{1}{|y_k|}\,Y_{n,j}\left(\frac{y_k}{|y_k|}\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_{\mathrm{Sp}}^k\left(G_{m,n,j}^{\mathrm{I}},0,0\right) &= \sqrt{\frac{4m+2n+3}{a^{2n+3}}}\int_0^a \frac{K_n^{\alpha}(r)}{\alpha_{\mathrm{m}}(r)}\,P_m^{(0,n+1/2)}\left(2\frac{r^2}{a^2}-1\right)r^n\,\mathrm{d}r\,Y_{n,j}(\xi_k), \\ \mathcal{F}_{\mathrm{Sp}}^k\left(G_{m,n,j}^{\mathrm{II}},0,0\right) &= \sqrt{\frac{2m+3}{a^3}}\int_0^a \frac{K_n^{\alpha}(r)}{\alpha_{\mathrm{m}}(r)}\,P_m^{(0,2)}\left(2\frac{r}{a}-1\right)\mathrm{d}r\,Y_{n,j}(\xi_k), \\ \mathcal{F}_{\mathrm{Fr}}^k\left(G_{m,n,j}^{\mathrm{II}},0,0\right) &= P_n(0)\sqrt{\frac{2m+3}{a^3}}\int_0^a \frac{K_n^{\alpha}(r)}{\alpha_{\mathrm{m}}(r)}\,P_m^{(0,2)}\left(2\frac{r}{a}-1\right)\mathrm{d}r\,Y_{n,j}(\xi_k),\end{aligned}$$

etc. The radial integrals which occur for the normal mode tomography have to be calculated numerically. The general structure in (17) in combination with the fact that both types of basis functions of $\mathrm{L}^2(\mathcal{B})$ involve a spherical harmonic make the addition theorem for spherical harmonics applicable. As a consequence, the series of a spline basis function in (16) has the form

$$\mathcal{F}^k\mathfrak{K}(\cdot,x) = \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{m,n,i}^2 C_{i,m,n,k} c_{m,n} P_{m,n}(|x|)\,\frac{2n+1}{4\pi}\,P_n\left(\xi_k\cdot\frac{x}{|x|}\right)\right)_{i=1,2,3},$$

which can be handled numerically (as a truncated series) by calculating the involved Jacobi polynomials via their recurrence formula (see [Szegö, 1939, p. 71]) and computing the series in Legendre polynomials using the Clenshaw algorithm (see [Clenshaw, 1955]). Note that these spline basis functions are, if they are restricted to a sphere, zonal functions, i.e. the restricted function only depends on the distance between the inserted point and ξ_k. This corresponds to the isotropy of both involved inverse problems.
Finally, the derivation of an explicit formula for the matrix entries is straight-forward:

$$\begin{aligned}\mathcal{F}_x^l\mathcal{F}_y^k\mathfrak{K}(y,x) &= \sum_{i=1}^{3}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} A_{m,n,i}^2 C_{i,m,n,k} C_{i,m,n,l} Y_{n,j}(\xi_k) Y_{n,j}(\xi_l)\\ &= \sum_{i=1}^{3}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{m,n,i}^2 C_{i,m,n,k} C_{i,m,n,l}\,\frac{2n+1}{4\pi}\,P_n(\xi_k\cdot\xi_l).\end{aligned}$$

Thus, the calculation of the matrix entries is more easy than the computation of the spline basis, since the matrix entries do not involve the Jacobi polynomials. Note that different kinds of functionals (from inverse gravimetry and normal mode tomography) can be mixed here, as long as the associated coefficients $C_{i,m,n,p}$ and points ξ_p are chosen correspondingly.

6 Conclusions and Outlook

Theoretical aspects of the inverse gravimetric problem and the normal mode tomography were summarised. In particular, the representation of the available (external) information in terms of functionals acting on the unknown material quantities $(\delta\alpha, \delta\beta, \delta\varrho)$ was investigated in detail. In addition, a vectorial spline method was developed. Since this approach can be regarded as a further development of spline methods which have already successfully been applied to the inverse gravimetric problem and the seismic traveltime tomography (separately), the proposed method promises to be able to tackle the problem of a combined inversion of gravity data and spectral anomalies of the normal modes. Note that due to the finite dimensions of the spline spaces the method also represents a regularisation (see also [Amirbekyan, 2006] for further theoretical investigations of such regularisations).
In a forthcoming publication numerical aspects will be investigated. The very different levels of resolution of the available data (the gravity field is available at a very high resolution

due to satellite missions whereas the knowledge of normal mode anomalies is comparatively poor) have to be taken into account. It should be avoided that the gravity data override the seismic data in the numerical calculations. Since the gravitational anomalies are essentially influenced by density anomalies at the upper-most layer only, normal mode data will not be able to improve the obtained model there. However, normal mode anomalies provide a perspective to improve the Earth model at higher depths, in particular at the mantle.
Finally, the inclusion of seismic traveltime tomography is an interesting challenge. Since the traveltimes depend on the unknown velocities in a non-linear way, the presented method cannot be applied directly to this problem. However, in [Amirbekyan, 2006, Amirbekyan and Michel, 2008] the applicability of such a spline method to the linear problem of calculating the slowness was demonstrated. At present, this requires, however, a separate resolution of the problems and the combination of the obtained velocity functions afterwards. A further development of a corresponding localised method for non-linear problems is a challenge for future research.

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