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Vorwort

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Berichterstattung heißt hier Dokumentation des Transfers aktueller Ergebnisse aus mathematischer Forschungs- und Entwicklungsarbeit in industrielle Anwendungen und Softwareprodukte – und umgekehrt, denn Probleme der Praxis generieren neue interessante mathematische Fragestellungen.

Prof. Dr. Dieter Prätzel-Wolters
Institutsleiter

Kaiserslautern, im Juni 2001

Solving the Ordered One-Median Problem in the Plane

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Abstract

In this paper we propose a general approach solution method for the single facility ordered median problem in the plane. All types of weights (non-negative, non-positive, and mixed) are considered. The big triangle small triangle approach is used for the solution. Rigorous and heuristic algorithms are proposed and extensively tested on eight different problems with excellent results.

Keywords: planar location, global optimization, ordered median, big triangle small triangle method, bounds, numerical experiments

AMS: 90B85, 90C26

1 Introduction

Continuous location has achieved an important degree of maturity. Witnesses of it are the large number of papers and research books published within this field. In addition, this development has been also recognized by the mathematical community since the AMS code 90B85 is reserved for this area of research. Continuous location problems appear very often in economic models of distribution or logistics, in statistics when one tries to find an estimator from a data set or in pure optimization problems where one looks for the optimizer of a certain function. For a comprehensive overview the reader is referred to [11] or [5]. Despite the fact that many continuous location problems rely heavily on a common framework, specific solution approaches have been developed for each of the typical objective functions in location theory. To overcome this inflexibility and to work towards a unified approach to location theory the so called Ordered Median Problem (OMP) was developed (see [9] and references therein). Ordered Median Problems represent as special cases nearly all classical objective functions in locations theory, like Median, Cent-Dian, center and k-centra. More precisely, the 1-facility Ordered Median Problem in the plane can be formulated as follows: A vector of weights $\lambda_1, \lambda_2 \ldots \lambda_n$ is given. The problem is to find a location for a facility that minimizes the weighted sum of distances where the distance to the closest point to the facility is multiplied by the weight λ_1, the distance to the second closest, by λ_2, and so on, the distance to the farthest point is multiplied by λ_n. Many location problems can be formulated as an ordered one-median problem by selecting appropriate weights. For example, the vector for which all $\lambda_i = 1$ is the unweighted 1-median problem, the problem where $\lambda_n = 1$ and all others are equal to zero is the one center problem. Minimizing the range of distances is achieved by $\lambda_1 = -1$ and $\lambda_n = 1$ and all others are zero. Minimizing the median of distances is achieved by $\lambda_{(n+1)/2} = 1$ for odd n and $\lambda_{n/2} = \lambda_{n/2+1} = 0.5$ for even n and all others are equal to zero. The expropriation problem ([1]) seeks the expropriation of x% of the demand points and to maximize the distance from the facility to the closest non-expropriated point. This leads to $\lambda(x\%n+1) = -1$ and all other λs=0. However, the solution methods for continuous OMPs so far had been mainly discretization results obtaining finite dominating sets (see [12]). Moreover, a linear programming approach for a subclass of OMPs was developed (see [8]). In this paper we want to tackle the OMP with a general vector λ. Therefore, we have a global optimization problem which has to be addressed by specific global optimization methods. The Big Triangle Small Triangle (BTST) approach has shown to be very effective for solving difficult location problems ([6, 4]). The rest of the paper is organized as follows: After briefly summarizing the BTST approach we give in the following section some additional notation and basic results needed in the reminder of the paper. Different lower bounds for the 1-OMP are then developed in Section 3. Section 4 is devoted to extensive numerical experiments using the bounds from Section 3. The paper ends with some conclusions and an outlook to future research.

1.1 The BTST Approach

The framework of the BTST approach is summarized below. The complete details are given in [6]. A feasible region which consists of a finite number of convex polygons is given.

Phase 1: Each convex polygon is triangulated using the Delaunay triangulation [6]. The vertices of the triangles are the demand points and the vertices of the convex polygon. The union of the triangulations is the initial set of triangles.

Phase 2: Calculate a lower bound, LB, and an upper bound, UB, for each triangle. Let the smallest UB be $\overline{UB}$. Discard all triangles for which $LB \geq \overline{UB}(1-\epsilon)$.

Phase 3: Choose the triangle with the smallest UB and split it into four small triangles by connecting the centers of its sides. Calculate LB and UB for each triangle, and update $\overline{UB}$ if necessary. The large triangle and all triangles for which $LB \geq \overline{UB}(1-\epsilon)$ are discarded.

Stopping Criterion: The branch and bound is terminated when there are no triangles left. The solution $\overline{UB}$ is within a relative accuracy of ϵ from the optimum.

2 Analysis

2.1 Notation

Let:

n	be the number of demand points.
A	$=(a_1,\ldots,a_n)$ be a vector of real numbers.
$a_{(k)}$	be the k^{th} smallest value in the vector A.
λ	be the vector of weights.
$d_i(X)$	be the distance from location X to demand point i.
$d(X,Y)$	be the distance between two points X and Y. If not stated otherwise we are using by default the Euclidean metric.
$f_\lambda(X)$	$= \sum_{i=1}^{n} \lambda_i d_{(i)}(X)$ be the value of the objective function for a given λ at point X.
T	be a given triangle with vertices T_1, T_2, T_3.

2.2 Basic Results

In order to derive good bounds for the Ordered median function we first prove that the function $f_\lambda(X)$ satisfies the Lipschitz condition. That means that there exists a constant $m > 0$ independent of X such that for any two points X and Y $|f_\lambda(Y) - f_\lambda(X)| \leq m d(X,Y)$. Consider a sorted vector $A = (a_1 \leq a_2, \ldots, \leq a_n)$. For a given $\epsilon \geq 0$ a vector (not necessarily sorted) B satisfies $|a_i - b_i| \leq \epsilon$ for $i = 1, \ldots, n$. We prove the following Lemma.

Lemma 1: *For the sorted vector B, $|b_{(k)} - a_k| \leq \epsilon$ for $k = 1, \ldots, n$.*

Proof: For all $i \leq k$ $b_i \leq a_i + \epsilon \leq a_k + \epsilon$ because $a_i \leq a_k$. There are at least k b's satisfying $b_l \leq a_k + \epsilon$. Therefore, $b_{(k)} \leq a_k + \epsilon$. Similarly, for all $i \geq k$ $b_i \geq a_i - \epsilon \geq a_k - \epsilon$ because

$a_i \geq a_k$. There are at least $n-k$ b's satisfying $b_l \geq a_k - \epsilon$. Therefore, $b_{(k)} \geq a_k - \epsilon$. Consequently, $|b_{(k)} - a_k| \leq \epsilon$ for $k = 1, \ldots, n$. □

Theorem 1: *The Lipschitz condition: For a given vector λ, there exists a constant $m > 0$ such that for any two points X and Y, $|f_\lambda(Y) - f_\lambda(X)| \leq m\, d(X,Y)$.*

Proof: Let $D(X)$ be the sorted vector $(d_1(X) \leq d_2(X) \leq \ldots \leq d_n(X))$. By the triangle inequality $|d_i(Y) - d_i(X)| \leq d(X,Y)$. By applying Lemma 1 with $\epsilon = d(X,Y)$ we get that $|d_{(i)}(Y) - d_i(X)| \leq d(X,Y)$. $f_\lambda(X) = \sum\limits_{i=1}^{n} \lambda_i d_i(X)$.

$$|f_\lambda(Y) - f_\lambda(X)| = |\sum_{i=1}^{n} \lambda_i d_{(i)}(Y) - \sum_{i=1}^{n} \lambda_i d_i(X)| = |\sum_{i=1}^{n} \lambda_i (d_{(i)}(Y) - d_i(X))| \leq$$

$$\leq \sum_{i=1}^{n} |\lambda_i||(d_{(i)}(Y) - d_i(X))| \leq \sum_{i=1}^{n} |\lambda_i| d(X,Y) = \left(\sum_{i=1}^{n} |\lambda_i|\right) d(X,Y).$$

The theorem follows by using $m = \sum\limits_{i=1}^{n} |\lambda_i|$. □

Corollary 1: $|f_\lambda(Y) - f_\lambda(X)| \leq \left(\sum\limits_{i=1}^{n} |\lambda_i|\right) d(X,Y)$.

Note that the bound $m = \sum\limits_{i=1}^{n} |\lambda_i|$ can be tight. Consider a point X and a line passing through X. When all demand points are on the line, those with a positive λ on one side and those with a negative λ on the other side (and farther away then the points on the other side), then for Y close to X on the same line, the triangle inequality for the distances to demand points is an equality, the order of the distances does not change, and $|f_\lambda(Y) - f_\lambda(X)| = \left(\sum\limits_{i=1}^{n} |\lambda_i|\right) d(X,Y)$. Such an example can be extended to include demand points not on the line when their $\lambda = 0$ as long as the order of distances does not change by moving from X to Y. It is interesting that if there is only one non-zero λ, the bound can be tight almost all the time. Even two non-zero λs may lead quite commonly to a tight lower bound when X is on the line connecting the two points.

Another interesting property is about the special case when the vector λ has only two consecutive positive λs with the rest of the λs are equal to zero. Consider the vector $\lambda^{(1)}$ with $\lambda_{k-1} = \alpha$, $\lambda_k = 1 - \alpha$ for a given k and $0 \leq \alpha \leq 0.5$, and all other λs are equal to zero. We prove that the optimal solution to a problem $\min\{f_{\lambda^{(1)}}(X)\}$ is the same as minimizing the function based on a vector $\lambda^{(2)}$ where $\lambda_k = 1$ and all other λs are equal to zero.

For the purpose of the following Lemma and Theorem we define for a given k and $0 \leq \alpha \leq 0.5$: $\lambda^{(1)}$: $\lambda_{k-1} = \alpha$, $\lambda_k = 1 - \alpha$, and $\lambda_i = 0 \ \forall i \neq k-1, k$. $\lambda^{(2)}$: $\lambda_k = 1$, and $\lambda_i = 0 \ \forall i \neq k$.

Lemma 2: $f_{\lambda^{(1)}}(X) \leq f_{\lambda^{(2)}}(X)$.

Proof: The Lemma follows from the inequality $\alpha d_{(k-1)}(X) + (1-\alpha)d_{(k)}(X) \leq \alpha d_{(k)}(X) + (1-\alpha)d_{(k)}(X) = d_{(k)}(X)$. □

Note that Lemma 2 is true for any $0 \leq \alpha \leq 1$.

Theorem 2: *The solution to the problem based on the vector $\lambda^{(1)}$ defined above and $0 \leq \alpha \leq 0.5$, is the same as the solution to the problem based on $\lambda^{(2)}$ defined above.*

Proof: Suppose that at an optimal solution X, $d_{(k)}(X) > d_{(k-1)}(X)$ (by definition $d_{(k)}(X) \geq d_{(k-1)}(X)$) and we first reach a contradiction for $0 \leq \alpha < 0.5$. For $\epsilon = \left[d_{(k)}(X) - d_{(k-1)}(X)\right]/2$ consider a point X' "in the direction" of demand point $k' = (k)$ such that $d(X, X') = \epsilon$ and $d_{k'}(X') = d_{k'}(X) - \epsilon$. By the triangle inequality $|d_i(X') - d_i(X)| \leq \epsilon \ \forall i$. The distance $d_{k'}(X')$ remains in position k' because for $i > k'$ $d_{(i)}(X) \geq d_{k'}(X)$ by definition and $d_{(i)}(X') \geq d_{(i)}(X) - \epsilon \geq d_{(k)}(X) - \epsilon = d_{k'}(X')$. Therefore, demand point k' cannot increase in the ranking. It cannot decrease in the ranking if $\epsilon = \left[d_{(k)}(X) - d_{(k-1)}(X)\right]/2$. Also, for $i \leq (k-1)$, $d_{(i)}(X') \leq d_{(i)}(X) + \epsilon \leq d_{(k-1)}(X) + \epsilon$ because $d_{(i)}(X) \leq d_{(k-1)}(X)$ by definition. Now,

$$
\begin{aligned}
f_{\lambda^{(1)}}(X') &= \alpha d_{(k-1)}(X') + (1-\alpha)d_{(k)}(X') \leq \alpha[d_{(k-1)}(X) + \epsilon] + (1-\alpha)[d_{(k)}(X) - \epsilon] \\
&= f_{\lambda^{(1)}}(X) + (2\alpha - 1)\epsilon \leq f_{\lambda^{(1)}}(X)
\end{aligned}
$$

because $\alpha \leq 0.5$. This inequality must be an equality by the optimality of X. It cannot be an equality for $\alpha < 0.5$ and we have reached a contradiction for $0 \leq \alpha < 0.5$. Therefore, there is an optimal solution X^* for which $d_{(k)}(X^*) = d_{(k-1)}(X^*)$. X^* is optimal for minimizing $f_{\lambda^{(1)}}(X)$. Also, $f_{\lambda^{(1)}}(X^*) = f_{\lambda^{(2)}}(X^*)$. By Lemma 2 X^* must be the optimal solution to minimizing $f_{\lambda^{(2)}}(X)$. What is left is to prove the theorem for $\alpha = 0.5$. We proved that for $\alpha = 0.5 - \delta$ for any small δ, $f_{\lambda^{(1)}}(X^*) = f_{\lambda^{(2)}}(X^*)$. Since $f_{\lambda^{(1)}}(X)$ is a continuous function of α, $\lim_{\delta \to 0}\{f_{\lambda^{(1)}}(X^*) - f_{\lambda^{(2)}}(X^*)\} = 0$. □

Theorem 2 is useful to show that the 2-centra problem is equivalent to the 1-center problem and that the problem of minimizing the median of distances for an even n, is equivalent to minimizing the $(n/2+1)^{\text{th}}$ distance. A similar Theorem exists for maximization of positive λs. The solution is the same as the solution of maximizing the *smaller* distance.

We propose to solve the ordered one-median problem for any set of λs by a general approach employing the Big Triangle Small Triangle (BTST) approach ([6]). In order to implement the BTST approach, we need a lower bound for the value of the objective function for a facility located in a triangle. Note that by the design of the algorithm, no demand point is in the interior of such a triangle. Consider a particular triangle T whose vertices are T_1, T_2, T_3. For simplicity of notation, all the formulas below refer to triangle T without explicitly denoting such values as a function of T.

3 Lower Bounds in a Triangle

In this section we propose several rigorous and heuristic lower bounds.

3.1 Rigorous Lower Bounds

3.1.1 The Lipschitz Lower Bound

Consider a triangle T with vertices T_1, T_2, T_3. Let r be the 1-center value of the objective function for the solution to the problem based on the demand points T_1, T_2, T_3. We show how to calculate r for Euclidean distances. Let the sides of the triangle be a, b, and c with c being the largest side. If $a^2+b^2 \le c^2$, then $r = c/2$ because the triangle is obtuse. Otherwise, $r = \frac{abc}{4s}$ where s is the area of the triangle which can be found by Heron's Theorem $s = \sqrt{p(p-a)(p-b)(p-c)}$ where $p = (a+b+c)/2$. For every point $X \in T$, there exist a vertex T_j, such that $d(X, T_j) \le r$.

Theorem 3: *For any $X \in T$,*

$$f_\lambda(X) \ge LB_{Lip} = \min_{j=1,2,3} \{f_\lambda(T_j)\} - \left(\sum_{i=1}^{n} |\lambda_i|\right) r$$

Proof: Let T_j be the vertex closest to X. By the definition of r $d(X, T_j) \le r$. By Corollary 1

$$|f_\lambda(T_j) - f_\lambda(X)| \le \left(\sum_{i=1}^{n} |\lambda_i|\right) d(X, T_j) \le \left(\sum_{i=1}^{n} |\lambda_i|\right) r$$

Therefore,

$$f_\lambda(X) \geq f_\lambda(T_j) - \left(\sum_{i=1}^{n} |\lambda_i|\right) r \geq \min_{j=1,2,3} (f_\lambda(T_j)) - \left(\sum_{i=1}^{n} |\lambda_i|\right) r$$

□

3.1.2 A Lower Bound Based on Individual Distances

The shortest possible distance between demand point i and any point in the triangle δ_i can be easily evaluated ([4]). The longest possible distance Δ_i is obviously measured to one of the three vertices. These are defined as the vectors δ and Δ, respectively. For a point $X \in T$, $\delta_i \leq d_i(X) \leq \Delta_i$.

Theorem 4: *For any* $1 \leq k \leq n$, $\delta_{(k)} \leq d_{(k)}(X)$.

Proof: For any $i \leq k$, $\delta_{(i)} \leq d_{(i)}(X) \leq d_{(k)}(X)$. Therefore, there exist at least k $\delta_{(j)} \leq d_{(k)}(X)$. Consequently, $\delta_{(k)} \leq d_{(k)}(X)$. □

A similar Theorem holds for upper bounds, i.e., replacing δ by Δ. The following lower bounds are easy to calculate.

Case I: All Weights are Non-Negative

$$LB_1 = \sum_{i=1}^{n} \lambda_i \delta_{(i)} \tag{1}$$

Case II: All Weights are Non-Positive

$$LB_2 = \sum_{i=1}^{n} \lambda_i \Delta_{(i)} \tag{2}$$

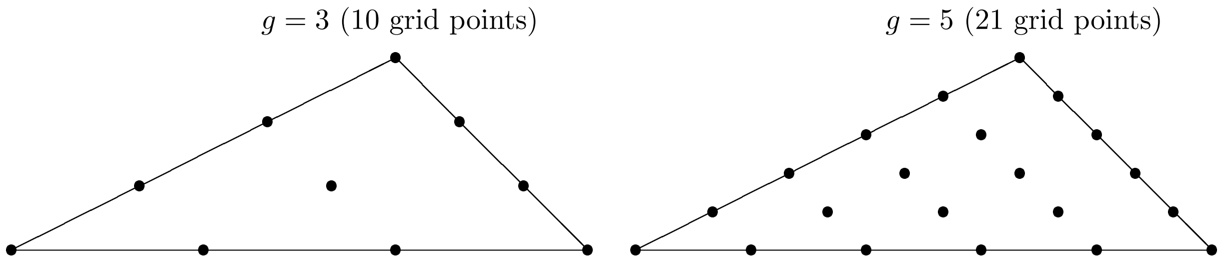
Case III: Mixed Weights

$$LB_3 = \sum_{i=1}^{n} \left[\max\{\lambda_i, 0\}\delta_{(i)} + \min\{\lambda_i, 0\}\Delta_{(i)}\right] \tag{3}$$

3.2 Heuristic Lower Bounds

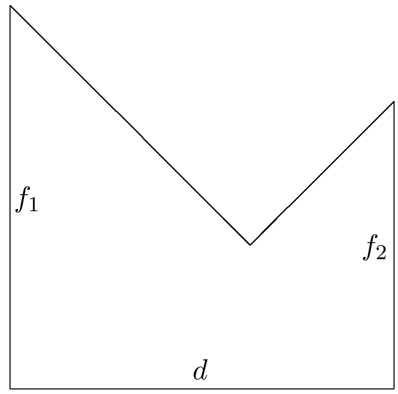
We experimented with many heuristic lower bounds and found the following one to be the most effective. One of the reasons that LB_1, LB_2, or LB_3 may not be that tight is that the shortest distance from demand points to the triangle is measured to different points on the boundary of

Figure 1: Grid Points in a Triangle



the triangle (mostly to one of the three vertices). Suppose for explanation purposes that one third of demand points are closest to each vertex. The shortest distance to a "center" of the triangle is larger by about the radius of the circumscribing circle for all demand points. If the minimum in the triangle occurs at one of the vertices, then one third of the demand points are bounded accurately, but two thirds of the demand points are bounded below by about a side of the triangle. In other words, the lower bound deviates from the minimum possible value in the triangle by a value proportional to the size of the triangle. This is also true for LB_{Lip} where the deviation is close to a constant multiple of the radius of the circumscribing triangle. When the triangle is divided into four smaller triangles, the discrepancy is about halved. This explains why LB_1, LB_2, or LB_3 may not be so tight. We therefore attempted to construct a lower bound based on the actual value of the objective function at various points in the triangle. The idea is to follow the Lipschitz lower bound, but estimate the value of m rather than apply the upper bound for it. This makes our approach a heuristic lower bound. We followed the idea in [3] where a grid of points is constructed in the triangle and the value of the objective function is calculated at each point in the grid. The problem analyzed in [3] is very contrived and does not satisfy the Lipschitz condition. Therefore, the approach suggested here could not be applied in [3]. Consider a grid with a parameter g. g is the number of segments dividing each side of the triangle (see Figure 1). The number of grid points is $h = (g + 1)(g + 2)/2$. In our experiments we used $g = 3$ and $g = 5$ leading to 10 and 21 grid points, respectively. Let the grid points be $X_1, \ldots, X_h$. The value of the objective function at each grid point $f_\lambda(X_i)$ is calculated. The value of m in the triangle to be used in Theorem 1 is

Figure 2: The Lower Bound on a Segment



estimated as

$$m = \max_{1 \le i < j \le h} \left\{ \frac{|f_\lambda(X_i) - f_\lambda(X_j)|}{d(X_i, X_j)} \right\} \tag{4}$$

Now suppose that two points at distance d from one another have the values of the objective function f_1 and f_2 at the ends of the segment (see Figure 2). Given the Lipschitz constant m, what is the minimum possible value of the objective function in the segment? At distance x from the left end of the segment, the bound is $\max\{f_1 - mx\ ,\ f_2 - m(d-x)\}$ which obtains its minimum at a point where $f_1 - mx = f_2 - m(d-x)$. Simple algebraic manipulations lead to a lower bound of $(f_1 + f_2 - md)/2$. The suggested heuristic lower bound LB_H is calculated as follows:

1. Calculate m by (4).

2. Find $f_{\min} = \min_{1 \le i \le h} \{f_\lambda(X_i)\}$.

3. The minimum $f_{\min}$ is obtained at grid point k.

4. $LB_H = \min_{1 \le i \ne k \le h} \{[f_{\min} + f_\lambda(X_i) - md(X_i, X_k)]/2\}$.

Note, that since we estimate the Lipschitz constant m in the triangle by (4) the real Lipschitz constant might be larger. Therefore, the heuristic lower bound LB_H might exclude the real optimal solution. The quality of this bound will be checked in the next section. As an upper bound in the triangle we use $f_{\min}$.

Since we do only have a heuristic lower bound, we need to prove that this lower bound is never larger than the chosen upper bound. This is done in the next theorem.

Theorem 5: $LB_H \le f_{\min}$.

Proof: By (4) $f_\lambda(X_i) - f_{\min} \le md(X_i, X_k)$. Therefore, $f_{\min} + f_\lambda(X_i) - md(X_i, X_k) \le 2f_{\min}$ and thus $LB_H \le f_{\min}$. $\square$

4 Computational Experiments

Programs in Fortran[1] using double precision arithmetic were coded, compiled by Intel 9.0 FORTRAN Compiler, and ran on a 2.8GHz Pentium IV desk top computer with 256MB RAM. The solutions were recorded to twelve digits after the decimal point. The program can store up to 500,000 triangles at any stage of the algorithm. If the limit on the number of triangles is reached, the program terminates with the best solution found so far and an error message. In all the experiments we never used such results. The accuracy ϵ was selected such that such an issue is not encountered. We tested problems with up to 10,000 demand points randomly generated in a unit square. Ten values of n between 10 and 10,000 were used and ten problems generated for each value of n for a total of 100 test problems for each experiment. For each problem we established the "best known" solution obtained by all experiments. We first tested problems with random λs in (0,1) to establish the prefered algorithms. We then tested the problem of minimizing the median of distances both by applying Theorem 2 and not applying it to find out if applying Theorem 2 is effective for the solution of this problem. Then we tested the truncated mean problem by removing the top 20% and the bottom 20% of the distances and minimizing the sum of the "middle" 60% of the distances. Minimizing the median distance can be viewed as a truncated mean with only one or two distances left in the objective function. Both of these objectives are suitable for cases where outliers need to be ignored. Then we experimented with the k-centra objective which is minimizing the average of the k largest distances. This model is an extension of the 1-center model where only the largest distance is minimized. We used $k = \max\{0.1n, 5\}$ points. All the previous examples have no negative λs. We tested the algorithms on four additional problems that include negative λs. The expropriation problem ([1]) translates into maximizing the $(0.2n+1)^{\text{th}}$ distance

[1]We thank Atsuo Suzuki for his Fortran program that finds the triangulation based on [13] subroutines first developed in [10].

Table 1: Results for Rigorous Lower Bounds (random λs in (0,1))

	Iterations			Max # Triangles			Time (sec.)			Fraction Above Best Known*		
n	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
LB_{Lip}, $\epsilon = 10^{-4}$												
10	2,278	56,365	12,096	815	29,588	5,694	0.02	4.78	0.58	5.3×10^{-10}	7.2×10^{-6}	2.2×10^{-6}
20	5,181	28,672	12,766	1,921	16,500	5,899	0.11	1.52	0.42	1.1×10^{-8}	2.5×10^{-6}	7.4×10^{-7}
50	6,113	19,930	12,877	2,415	8,694	5,440	0.53	1.69	1.16	7.5×10^{-8}	1.2×10^{-6}	4.4×10^{-7}
100	8,692	59,714	23,782	3,259	31,899	11,434	2.05	17.2	5.91	6.5×10^{-8}	5.0×10^{-7}	2.5×10^{-7}
200	23,132	40,576	31,709	11,316	19,632	16,026	11.95	23.92	18.04	3.9×10^{-8}	4.3×10^{-7}	1.7×10^{-7}
500	24,832	62,365	43,454	11,871	33,895	23,625	35.94	91.05	64.23	1.0×10^{-8}	5.4×10^{-7}	1.5×10^{-7}
1000	38,553	77,130	59,021	18,579	44,586	32,178	144.22	281.83	211.83	2.4×10^{-8}	2.1×10^{-7}	1.2×10^{-7}
2000	45,649	109,803	70,659	25,225	64,624	40,712	413.08	975.34	633.90	1.9×10^{-8}	2.7×10^{-7}	1.1×10^{-7}
5000	56,175	91,525	76,563	28,599	55,890	43,627	1836.20	3177.53	2561.14	1.3×10^{-9}	1.7×10^{-7}	8.8×10^{-8}
10000	62,779	87,219	75,174	34,695	52,979	41,995	5252.06	7552.39	6265.88	1.4×10^{-8}	1.6×10^{-7}	6.4×10^{-8}
LB_1, $\epsilon = 10^{-4}$												
10	1,219	20,355	5,516	447	10,848	2,540	0.00	0.61	0.10	1.8×10^{-9}	8.2×10^{-6}	2.5×10^{-6}
20	2,436	13,424	6,373	773	7,499	2,972	0.03	0.36	0.13	2.0×10^{-8}	2.1×10^{-6}	6.3×10^{-7}
50	2,995	11,037	6,962	1,158	4,782	2,862	0.19	0.75	0.46	1.3×10^{-7}	1.6×10^{-6}	7.1×10^{-7}
100	4,804	24,330	11,772	1,928	12,647	5,510	0.75	4.38	1.99	4.7×10^{-8}	8.9×10^{-7}	3.8×10^{-7}
200	12,735	21,887	16,992	5,970	11,911	8,644	5.12	8.33	6.83	1.3×10^{-8}	3.8×10^{-7}	1.2×10^{-7}
500	13,132	32,975	23,902	5,981	19,637	13,020	13.47	35.25	25.12	3.0×10^{-8}	2.1×10^{-7}	9.3×10^{-8}
1000	24,113	41,250	31,924	11,814	24,347	17,798	65.39	108.08	81.13	2.5×10^{-8}	2.2×10^{-7}	9.2×10^{-8}
2000	26,251	58,199	37,235	14,757	35,458	21,238	171.47	357.88	234.40	1.2×10^{-9}	2.8×10^{-7}	1.2×10^{-7}
5000	33,067	47,028	41,014	19,058	28,277	23,811	763.44	1145.17	982.82	5.6×10^{-8}	4.0×10^{-7}	1.5×10^{-7}
10000	37,881	45,828	41,378	21,337	25,049	23,461	2338.03	2724.34	2506.65	1.9×10^{-8}	2.9×10^{-7}	1.1×10^{-7}
LB_1, $\epsilon = 10^{-5}$												
10	1,721	173,852	28,135	457	102,890	14,688	0.02	48.22	5.18	0	4.9×10^{-7}	1.6×10^{-7}
20	3,747	37,007	22,358	933	17,437	10,011	0.06	2.19	1.12	7.6×10^{-9}	1.7×10^{-7}	6.3×10^{-8}
50	6,047	48,953	22,104	1,738	25,405	9,793	0.41	5.80	2.12	1.4×10^{-8}	1.1×10^{-7}	5.3×10^{-8}
100	11,675	72,567	35,068	4,098	28,278	13,818	1.86	17.14	7.13	6.2×10^{-9}	5.5×10^{-8}	2.1×10^{-8}
200	34,982	58,832	44,911	11,182	25,155	17,403	14.62	29.50	19.95	1.3×10^{-9}	2.8×10^{-8}	8.6×10^{-9}
500	33,590	128,480	75,392	11,006	62,683	33,764	35.16	155.38	85.56	0	1.6×10^{-8}	5.1×10^{-9}
1000	86,464	192,144	127,894	39,231	101,831	61,681	240.34	541.86	340.20	1.6×10^{-9}	1.0×10^{-8}	4.5×10^{-9}
2000	97,542	362,232	191,745	46,146	216,909	106,153	633.66	2374.72	1237.97	7.4×10^{-10}	5.9×10^{-9}	2.5×10^{-9}
5000	204,059	332,806	270,602	111,436	194,069	151,983	4508.22	8089.12	6306.33	1.0×10^{-10}	6.2×10^{-9}	2.4×10^{-9}
10000	228,254	399,232	324,169	126,179	233,908	186,628	12962.77	21825.97	17971.63	4.4×10^{-10}	2.8×10^{-9}	1.3×10^{-9}

* Best known found in this table and Table 2

which consists of one negative λ and the rest of them equal to zero. The anti-k-centrum problem maximizes the average distance to the k closest points. We used $k = \max\{0.1n, 5\}$ points. For these problems we have to use the lower bound LB_2 rather than LB_1. Two additional problems: minimizing the inter-quartile range and minimizing the range have one positive and one negative λ. These problems require the use of LB_3 as a lower bound.

4.1 Experiments with Random λs

We first tested the various lower bounds for problems with λ_i randomly generated in (0, 1). We tested six algorithms, three of them applying a rigorous lower bound, and three heuristics. The

Table 2: Results for Heuristic Approaches (random λs in (0,1))

	Iterations			Max # Triangles			Time (sec.)			Fraction Above Best Known*		
n	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
Objective function evaluated at one million points												
10							0.39	0.45	0.43	2.5×10^{-6}	1.6×10^{-4}	5.8×10^{-5}
20							1.17	1.39	1.21	2.8×10^{-6}	1.2×10^{-4}	3.3×10^{-5}
50							5.47	6.44	6.06	3.8×10^{-6}	3.9×10^{-5}	1.9×10^{-5}
100							16.28	17.98	17.14	1.1×10^{-6}	3.9×10^{-5}	1.3×10^{-5}
200							42.48	44.97	43.95	3.5×10^{-6}	2.4×10^{-5}	1.2×10^{-5}
500							113.56	117.52	115.84	8.0×10^{-7}	1.7×10^{-5}	7.0×10^{-6}
1000							282.73	293.72	288.16	1.9×10^{-6}	1.4×10^{-5}	8.5×10^{-6}
2000							715.03	754.33	730.69	2.0×10^{-6}	1.3×10^{-5}	7.4×10^{-6}
5000							2547.39	2738.05	2635.22	2.2×10^{-6}	6.2×10^{-6}	4.0×10^{-6}
10000							6257.66	6742.52	6495.20	2.2×10^{-6}	3.7×10^{-6}	2.9×10^{-6}
LB_H, $g=3$, $\epsilon=10^{-8}$												
10	105	14,000	2,728	28	3,708	736	0.00	0.42	0.08	0	3.8×10^{-9}	6.9×10^{-10}
20	154	6,703	1,956	29	1,831	525	0.02	0.33	0.11	0	2.0×10^{-9}	5.6×10^{-10}
50	161	661	310	26	152	67	0.05	0.19	0.09	0	3.0×10^{-9}	9.2×10^{-10}
100	153	746	310	26	125	54	0.14	0.55	0.25	0	1.5×10^{-9}	4.5×10^{-10}
200	193	422	263	36	91	51	0.48	0.84	0.63	0	1.0×10^{-9}	2.6×10^{-10}
500	150	434	235	32	89	50	1.83	3.09	2.22	0	3.3×10^{-9}	4.7×10^{-10}
1000	163	333	244	27	101	55	7.72	9.58	8.49	0	9.8×10^{-10}	2.4×10^{-10}
2000	123	541	264	26	125	59	31.73	45.23	36.60	0	1.6×10^{-9}	3.9×10^{-10}
5000	136	274	190	37	68	49	271.50	295.77	281.82	0	1.1×10^{-9}	3.6×10^{-10}
10000	107	485	222	30	125	58	1278.39	1442.52	1350.01	1.4×10^{-11}	8.6×10^{-10}	4.8×10^{-10}
LB_H, $g=5$, $\epsilon=10^{-8}$												
10	117	17,023	3,186	29	4,483	878	0.00	1.03	0.19	0	3.9×10^{-10}	4.9×10^{-11}
20	164	7,521	2,264	31	2,081	611	0.02	0.78	0.26	0	0	0
50	190	735	358	28	163	73	0.09	0.44	0.21	0	2.0×10^{-11}	2.0×10^{-12}
100	184	914	377	32	164	66	0.36	1.36	0.61	0	1.1×10^{-10}	1.4×10^{-11}
200	235	489	321	42	90	59	1.17	1.97	1.51	0	7.3×10^{-10}	1.0×10^{-10}
500	179	594	300	38	102	63	4.09	8.00	5.28	0	3.1×10^{-10}	7.0×10^{-11}
1000	217	446	321	49	118	73	17.62	22.39	19.67	0	2.5×10^{-10}	3.6×10^{-11}
2000	158	756	351	35	160	79	68.58	108.28	82.21	0	9.2×10^{-11}	9.2×10^{-12}
5000	187	373	263	47	122	76	581.16	644.20	608.07	0	8.3×10^{-12}	8.3×10^{-13}
10000	155	662	313	45	173	86	2711.98	3130.09	2883.98	0	0	0

* Best known found in this table and Table 1

three rigorous algorithms are based on: the Lipschitz lower bound LB_{Lip} using a relative accuracy of $\epsilon = 10^{-4}$, and LB_1 with $\epsilon = 10^{-4}$ and $\epsilon = 10^{-5}$. The results are summarized in Table 1. Three heuristic approaches are reported in Table 2. A grid of 1000 by 1000 points in the unit square was generated and the best value of the objective function among the values on the 1,000,000 grid points was reported as a heuristic approach. We also tested the LB_H bound using $g = 3$ and $g = 5$ grids and a relative accuracy of $\epsilon = 10^{-8}$. By examining Tables 1 and 2 we conclude that the heuristic lower bound LB_H using $g = 5$ performed best even though it is not guaranteeing optimality. It found the best known solution within a relative accuracy of 7.3×10^{-10} for all 100 problems. It actually found the best known solution for 83 of the 100 problems (the LB_H with

Table 3: Results for Minimizing the Median Without Applying Theorem 2

	Iterations			Max # Triangles			Time (sec.)			Fraction Above Best Known*		
n	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
LB_{Lip}, $\epsilon = 10^{-4}$												
10	504	588,399	60,712	192	413,990	42,255	0.00	619.50	61.97	0	2.6×10^{-5}	7.5×10^{-6}
20	771	172,013	18,727	229	120,278	12,701	0.00	53.95	5.42	0	9.7×10^{-6}	3.5×10^{-6}
50	510	13,659	2,750	165	8,126	1,385	0.05	1.17	0.23	1.1×10^{-6}	1.3×10^{-5}	6.4×10^{-6}
100	415	1,070	608	131	373	221	0.09	0.20	0.13	5.9×10^{-6}	2.0×10^{-5}	1.2×10^{-5}
200	502	5,879	1,531	210	3,731	724	0.31	3.45	0.89	0	1.4×10^{-5}	6.6×10^{-6}
500	919	1,749	1,216	443	1,042	601	1.61	2.91	2.03	8.2×10^{-7}	1.5×10^{-5}	7.6×10^{-6}
1000	1,113	2,931	1,904	657	1,769	1,096	5.48	12.48	8.33	3.0×10^{-6}	2.0×10^{-5}	1.1×10^{-5}
2000	1,702	5,664	3,213	940	3,947	2,019	23.58	57.97	36.73	3.9×10^{-7}	2.4×10^{-5}	1.4×10^{-5}
5000	3,951	9,507	6,562	2,601	5,903	4,101	202.98	376.08	285.95	3.0×10^{-6}	3.0×10^{-5}	1.3×10^{-5}
10000	7,960	18,162	13,005	5,065	11,316	8,186	1010.07	1818.36	1436.83	8.7×10^{-8}	2.8×10^{-5}	1.6×10^{-5}
LB_1, $\epsilon = 10^{-4}$												
10	185	570,123	58,229	69	403,220	40,920	0.00	584.58	58.47	0	2.3×10^{-5}	8.7×10^{-6}
20	263	125,995	13,302	115	86,930	9,013	0.00	29.14	2.93	0	1.8×10^{-5}	5.3×10^{-6}
50	190	2,522	712	63	1,360	332	0.02	0.16	0.05	5.0×10^{-7}	2.1×10^{-5}	6.6×10^{-6}
100	126	298	189	52	104	73	0.03	0.06	0.04	7.3×10^{-6}	1.8×10^{-5}	1.2×10^{-5}
200	171	1,946	556	68	1,256	270	0.09	0.86	0.26	0	1.7×10^{-5}	6.7×10^{-6}
500	298	622	430	173	395	238	0.53	0.89	0.67	3.0×10^{-6}	3.4×10^{-5}	1.2×10^{-5}
1000	414	1,104	693	259	712	415	2.23	4.14	2.94	1.7×10^{-6}	2.1×10^{-5}	1.1×10^{-5}
2000	586	2,282	1,251	373	1,628	855	9.73	20.14	13.83	8.5×10^{-7}	2.1×10^{-5}	6.9×10^{-6}
5000	1,720	3,921	2,744	1,165	2,621	1,915	92.17	139.58	115.37	5.5×10^{-6}	2.1×10^{-5}	1.3×10^{-5}
10000	3,415	8,014	5,706	2,270	6,769	3,913	448.22	686.28	576.82	1.2×10^{-6}	1.7×10^{-5}	9.0×10^{-6}
LB_H, $\epsilon = 10^{-8}$												
10	155	192,913	19,881	31	51,110	5,199	0.00	40.22	4.06	0	6.6×10^{-10}	6.6×10^{-11}
20	261	118,477	12,394	36	34,580	3,528	0.03	23.83	2.45	0	0	0
50	245	1,788	674	33	215	88	0.16	0.92	0.38	0	1.0×10^{-9}	1.5×10^{-10}
100	179	534	282	33	79	53	0.31	0.77	0.47	0	1.0×10^{-9}	1.0×10^{-10}
200	181	126,314	13,025	39	38,092	3,883	0.98	524.38	54.22	0	1.1×10^{-9}	1.4×10^{-10}
500	256	904	507	81	250	130	4.80	11.66	7.28	0	3.0×10^{-9}	8.7×10^{-10}
1000	368	1,005	640	92	328	188	20.80	37.69	27.69	0	5.3×10^{-10}	1.0×10^{-10}
2000	500	1,449	843	159	727	293	90.73	148.16	112.07	0	9.7×10^{-10}	1.1×10^{-10}
5000	422	2,158	1,052	153	797	392	650.80	1006.80	783.23	0	3.0×10^{-9}	7.6×10^{-10}
10000	711	3,695	1,637	299	864	572	2982.88	4664.06	3612.19	0	2.5×10^{-9}	4.1×10^{-10}

* Best known found in this table and Table 4

$g = 3$ found it for the other 17 problems). The lower bound LB_1 performed better than LB_{Lip}. However, using $\epsilon = 10^{-5}$ for LB_1 required many iterations and many triangles, resulting in longer run times. For further experiments we tested LB_{Lip}, LB_1 with $\epsilon = 10^{-4}$, and LB_H with $g = 5$ and $\epsilon = 10^{-8}$. In some cases, we used smaller values of ϵ if there were no convergence issues. However, in all experiments with LB_H $\epsilon = 10^{-8}$ was used.

4.2 Experimenting with Minimizing the Median

The results for minimizing the median are depicted in Tables 3 and 4. For the heuristic lower bound LB_H we obtained about the same quality results in about the same running time. However, the

Table 4: Results for Minimizing the Median Applying Theorem 2

	Iterations			Max # Triangles			Time (sec.)			Fraction Above Best Known*		
n	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
LB_{Lip}, $\epsilon = 10^{-8}$												
10	393	186,205	19,579	85	53,239	5,499	0.00	31.56	3.16	0	4.1×10^{-9}	1.8×10^{-9}
20	490	130,868	13,949	84	34,746	3,615	0.00	15.50	1.57	1.1×10^{-11}	4.1×10^{-9}	1.8×10^{-9}
50	511	2,243	1,065	113	325	219	0.05	0.17	0.09	2.2×10^{-11}	4.5×10^{-9}	1.7×10^{-9}
100	454	896	617	121	311	206	0.11	0.17	0.13	0	5.2×10^{-9}	2.1×10^{-9}
200	516	149,033	15,807	185	48,763	5,151	0.33	108.59	11.39	1.3×10^{-11}	4.1×10^{-9}	2.4×10^{-9}
500	974	1,944	1,344	427	1,122	600	1.64	3.19	2.20	0	4.7×10^{-9}	1.7×10^{-9}
1000	1,347	2,944	2,026	659	1,695	1,084	6.31	12.55	8.77	2.7×10^{-10}	3.3×10^{-9}	2.3×10^{-9}
2000	1,919	6,095	3,477	955	4,010	2,035	25.48	61.58	39.04	0	2.8×10^{-9}	1.5×10^{-9}
5000	4,065	9,937	6,809	2,626	6,255	4,254	207.02	388.88	294.31	0	5.5×10^{-9}	2.3×10^{-9}
10000	8,307	19,855	13,828	5,215	11,696	8,621	1012.09	1913.97	1467.54	0	3.1×10^{-9}	1.3×10^{-9}
LB_1, $\epsilon = 10^{-8}$												
10	81	75,055	7,819	15	18,877	1,939	0.00	6.67	0.67	9.8×10^{-12}	3.1×10^{-9}	1.5×10^{-9}
20	164	43,033	4,553	26	9,373	979	0.00	2.48	0.26	1.4×10^{-11}	5.7×10^{-9}	2.2×10^{-9}
50	109	584	246	33	79	56	0.02	0.06	0.03	0	6.2×10^{-9}	2.6×10^{-9}
100	135	213	168	38	112	64	0.03	0.06	0.05	1.2×10^{-9}	4.1×10^{-9}	2.4×10^{-9}
200	140	34,878	3,781	66	8,957	991	0.14	24.20	2.65	0	4.0×10^{-9}	1.6×10^{-9}
500	291	796	457	162	387	227	0.81	1.56	1.09	1.2×10^{-9}	6.7×10^{-9}	3.1×10^{-9}
1000	482	1,122	697	255	698	410	3.67	6.45	4.62	0	4.6×10^{-9}	1.8×10^{-9}
2000	644	2,277	1,298	350	1,606	856	16.06	33.31	22.91	1.3×10^{-9}	6.8×10^{-9}	3.3×10^{-9}
5000	1,738	4,010	2,795	1,186	2,598	1,949	144.77	219.14	180.35	0	5.6×10^{-9}	1.8×10^{-9}
10000	3,527	8,516	6,018	2,367	7,059	4,103	701.95	1112.75	920.40	6.6×10^{-10}	4.2×10^{-9}	2.1×10^{-9}
LB_H, $\epsilon = 10^{-8}$												
10	121	29,130	3,171	19	8,576	901	0.00	2.09	0.23	1.6×10^{-12}	2.1×10^{-9}	9.8×10^{-10}
20	130	39,590	4,243	24	10,151	1,056	0.02	4.95	0.53	1.8×10^{-12}	3.1×10^{-9}	1.4×10^{-9}
50	144	1,288	442	23	174	74	0.09	0.66	0.25	0	1.9×10^{-9}	7.4×10^{-10}
100	136	401	220	33	96	60	0.25	0.59	0.38	0	1.8×10^{-9}	6.9×10^{-10}
200	143	53,888	5,637	52	15,792	1,657	0.86	220.25	23.26	0	3.5×10^{-9}	1.1×10^{-9}
500	273	810	397	94	303	147	4.97	10.66	6.24	0	4.5×10^{-9}	1.5×10^{-9}
1000	350	879	589	102	396	226	20.22	33.58	26.34	4.9×10^{-10}	4.4×10^{-9}	1.7×10^{-9}
2000	469	1,536	816	183	837	341	91.03	153.48	110.48	8.8×10^{-11}	3.7×10^{-9}	1.2×10^{-9}
5000	445	2,366	1,111	189	972	464	654.73	1051.05	795.95	0	3.1×10^{-9}	1.1×10^{-9}
10000	760	3,067	1,635	344	952	633	3012.70	4337.06	3616.02	0	3.9×10^{-9}	1.5×10^{-9}

* Best known found in this table and Table 3

performance of the rigorous algorithms were much improved when applying Theorem 2. Run times were not changed by much but the accuracy was improved from $\epsilon = 10^{-4}$ to $\epsilon = 10^{-8}$ yielding results much closer to the best known result. Using LB_1 performed better than using LB_{Lip}. The best results were obtained by LB_H, but at a longer running time.

4.3 Experiments with the Truncated Mean Problems

The results for the truncated mean problems are depicted in Table 5. The heuristic approach using LB_H performed the best with respect to both the quality of the solution and running time. The lower bound LB_1 provided comparable to better results than LB_{Lip}, in a much shorter run time.

Table 5: Results for Truncated Mean Problems

	Iterations			Max # Triangles			Time (sec.)			Fraction Above Best Known		
n	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
LB_{Lip}, $\epsilon = 10^{-4}$												
10	1,650	259,984	31,335	563	157,004	17,837	0.02	109.11	10.98	$2.3{\times}10^{-10}$	$5.5{\times}10^{-6}$	$1.3{\times}10^{-6}$
20	5,723	119,622	51,646	2,630	70,677	28,752	0.12	24.45	7.34	0	$6.9{\times}10^{-7}$	$1.8{\times}10^{-7}$
50	16,906	137,102	52,828	7,865	82,960	29,070	1.27	41.88	10.41	$1.2{\times}10^{-10}$	$1.0{\times}10^{-6}$	$2.2{\times}10^{-7}$
100	19,040	117,316	45,735	9,213	64,330	24,223	4.02	45.34	14.39	$1.3{\times}10^{-10}$	$6.4{\times}10^{-7}$	$2.0{\times}10^{-7}$
200	37,555	134,113	70,768	18,808	73,490	39,049	20.86	97.30	46.73	$1.3{\times}10^{-11}$	$3.0{\times}10^{-7}$	$1.1{\times}10^{-7}$
500	52,190	155,973	88,842	27,227	83,147	49,291	75.44	254.75	137.59	$3.7{\times}10^{-9}$	$1.2{\times}10^{-7}$	$5.0{\times}10^{-8}$
1000	67,407	222,163	125,882	34,755	119,124	68,627	242.67	886.28	467.69	$2.7{\times}10^{-9}$	$5.6{\times}10^{-8}$	$1.8{\times}10^{-8}$
2000	114,225	365,666	181,835	63,187	223,470	104,912	1032.42	3427.48	1664.17	$1.8{\times}10^{-10}$	$2.2{\times}10^{-8}$	$7.8{\times}10^{-9}$
5000	157,347	307,170	205,697	86,551	185,331	115,880	5305.47	10118.98	6714.02	$4.2{\times}10^{-11}$	$1.2{\times}10^{-8}$	$5.5{\times}10^{-9}$
10000	187,568	272,920	218,415	105,853	157,138	125,090	14923.53	21452.39	17788.13	$1.8{\times}10^{-10}$	$9.1{\times}10^{-9}$	$2.8{\times}10^{-9}$
LB_1, $\epsilon = 10^{-4}$												
10	820	150,008	17,870	291	85,987	9,754	0.00	37.09	3.74	0	$3.7{\times}10^{-6}$	$8.1{\times}10^{-7}$
20	2,690	65,829	29,034	1,140	40,943	17,067	0.05	8.03	2.55	0	$7.2{\times}10^{-7}$	$1.2{\times}10^{-7}$
50	9,336	78,007	28,434	4,308	44,615	15,046	0.50	14.69	3.44	$1.8{\times}10^{-10}$	$1.2{\times}10^{-6}$	$2.5{\times}10^{-7}$
100	11,427	50,696	23,131	5,610	29,320	12,505	1.75	11.95	4.61	$6.3{\times}10^{-10}$	$6.6{\times}10^{-7}$	$2.7{\times}10^{-7}$
200	22,317	69,050	38,181	10,838	33,010	20,311	8.83	33.09	17.15	$8.1{\times}10^{-11}$	$2.5{\times}10^{-7}$	$8.2{\times}10^{-8}$
500	31,463	74,047	48,187	17,320	38,440	26,620	33.39	82.03	51.96	$5.4{\times}10^{-9}$	$1.5{\times}10^{-7}$	$5.0{\times}10^{-8}$
1000	39,113	126,165	69,729	20,332	77,512	39,597	98.53	347.12	180.61	$3.2{\times}10^{-9}$	$4.0{\times}10^{-8}$	$1.6{\times}10^{-8}$
2000	61,660	160,282	97,755	32,231	100,161	55,269	384.81	1033.53	617.45	$2.9{\times}10^{-9}$	$3.3{\times}10^{-8}$	$9.6{\times}10^{-9}$
5000	84,774	194,591	116,290	48,708	124,304	67,777	1904.98	4462.00	2650.98	0	$1.2{\times}10^{-8}$	$6.4{\times}10^{-9}$
10000	101,329	159,366	124,718	56,513	95,367	71,852	5463.72	8651.84	6981.73	$7.5{\times}10^{-11}$	$1.1{\times}10^{-8}$	$2.5{\times}10^{-9}$
LB_H, $\epsilon = 10^{-8}$												
10	121	19,379	6,655	21	5,513	1,861	0.00	1.19	0.39	0	0	0
20	226	7,318	2,695	39	2,147	730	0.03	0.86	0.31	0	$2.3{\times}10^{-10}$	$2.3{\times}10^{-11}$
50	193	5,267	2,210	26	1,550	614	0.09	2.42	1.18	0	0	0
100	140	1,948	661	24	499	169	0.28	2.83	1.02	0	0	0
200	160	1,165	409	35	306	101	1.00	4.64	1.87	0	0	0
500	141	920	337	27	220	77	3.81	11.17	5.65	0	0	0
1000	140	411	267	26	105	55	15.19	22.75	18.47	0	0	0
2000	124	339	215	29	107	47	69.45	80.47	74.34	0	0	0
5000	129	224	159	30	55	40	567.61	607.84	587.25	0	$2.3{\times}10^{-9}$	$2.3{\times}10^{-10}$
10000	89	179	135	26	47	36	2679.22	2905.08	2796.72	0	0	0

4.4 Experiments with k-centra problems

The results for the k-centra problems are depicted in Table 6. The heuristic approach based on LB_H clearly provided the best results. However, it required the longest running time. The two rigorous algorithms provided comparable results but LB_1 was faster.

4.5 Experiments with Non-Positive Weights

For this type of problems with no positive weights we used the expropriation problem and the anti-k-centrum problem. The results are depicted in Tables 7 and 8. These problems are relatively easy for all algorithms. The best approach is using the rigorous lower bound LB_2 with an accuracy of $\epsilon = 10^{-10}$. The heuristic approach provided good results but required much longer running time.

Table 6: Results for the k-centra Problems

	Iterations			Max # Triangles			Time (sec.)			Fraction Above Best Known		
n	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
LB_{Lip}, $\epsilon = 10^{-4}$												
10	1,184	85,410	14,193	419	54,744	8,172	0.00	12.30	1.35	1.9×10^{-10}	6.0×10^{-6}	1.7×10^{-6}
20	813	25,460	4,798	300	14,681	2,455	0.02	1.22	0.17	1.4×10^{-7}	7.5×10^{-6}	2.6×10^{-6}
50	727	5,092	2,157	252	2,809	987	0.05	0.36	0.17	4.9×10^{-7}	5.8×10^{-6}	2.6×10^{-6}
100	2,601	7,437	4,758	1,188	3,828	2,362	0.61	1.64	1.00	1.4×10^{-7}	2.6×10^{-6}	8.3×10^{-7}
200	3,607	14,160	7,314	1,743	9,003	3,947	1.95	8.16	3.93	3.9×10^{-8}	9.6×10^{-7}	5.5×10^{-7}
500	2,799	16,223	7,147	1,210	8,879	3,804	4.48	22.94	10.25	1.0×10^{-8}	7.1×10^{-7}	2.9×10^{-7}
1000	5,728	22,075	10,718	2,943	13,967	6,086	21.53	79.89	39.45	8.5×10^{-9}	4.0×10^{-7}	1.4×10^{-7}
2000	8,162	20,307	11,589	4,459	11,583	6,625	81.27	190.55	110.69	3.8×10^{-11}	1.4×10^{-7}	4.7×10^{-8}
5000	8,581	14,915	10,433	4,876	8,858	5,947	353.72	579.12	416.21	7.4×10^{-9}	1.0×10^{-7}	4.5×10^{-8}
10000	9,127	20,802	11,173	4,729	12,528	6,285	1077.08	2099.14	1265.55	8.6×10^{-10}	3.8×10^{-8}	1.4×10^{-8}
LB_1, $\epsilon = 10^{-4}$												
10	585	65,646	9,906	219	40,997	5,664	0.00	7.36	0.79	5.4×10^{-10}	5.7×10^{-6}	1.6×10^{-6}
20	377	17,691	2,851	135	9,110	1,395	0.02	0.59	0.08	7.7×10^{-8}	5.1×10^{-6}	2.2×10^{-6}
50	308	3,094	1,041	108	1,550	468	0.02	0.19	0.07	5.7×10^{-7}	8.3×10^{-6}	3.8×10^{-6}
100	1,138	3,596	2,485	504	1,916	1,260	0.20	0.58	0.39	8.2×10^{-8}	2.8×10^{-6}	1.3×10^{-6}
200	1,586	7,268	3,511	831	4,386	1,932	0.62	3.08	1.37	2.8×10^{-8}	1.6×10^{-6}	5.8×10^{-7}
500	1,612	8,422	3,784	769	4,659	2,065	2.00	8.73	3.99	8.4×10^{-9}	5.8×10^{-7}	2.9×10^{-7}
1000	3,068	12,214	5,583	1,798	7,813	3,328	8.89	31.28	15.04	2.7×10^{-8}	3.3×10^{-7}	1.7×10^{-7}
2000	4,224	9,465	6,034	2,213	5,774	3,446	31.52	66.17	43.04	0	2.0×10^{-7}	4.9×10^{-8}
5000	4,761	7,609	5,769	2,766	4,104	3,277	164.55	228.89	188.13	5.0×10^{-9}	6.3×10^{-8}	2.9×10^{-8}
10000	4,821	9,925	6,032	2,714	6,006	3,490	520.81	832.97	598.44	2.2×10^{-9}	6.7×10^{-8}	2.4×10^{-8}
LB_H, $\epsilon = 10^{-8}$												
10	122	31,235	8,395	18	8,749	2,317	0.00	2.06	0.53	0	0	0
20	95	28,541	5,926	19	8,452	1,679	0.02	4.06	0.77	0	0	0
50	120	2,323	544	20	421	92	0.08	1.17	0.28	0	0	0
100	166	6,149	1,475	34	1,805	395	0.28	9.03	2.21	0	0	0
200	191	4,359	1,063	31	1,439	295	1.03	16.12	4.10	0	0	0
500	110	2,102	419	24	632	101	3.38	21.53	6.33	0	0	0
1000	135	1,348	345	28	380	83	15.14	43.50	20.25	0	0	0
2000	123	1,041	453	23	262	114	66.77	127.59	88.76	0	1.4×10^{-10}	1.4×10^{-11}
5000	117	227	161	28	44	33	563.34	613.64	587.10	0	0	0
10000	111	337	154	25	62	35	2705.39	2944.73	2804.64	0	0	0

Run times for the expropriation problem are shorter than those reported in [1]. The $n = 10,000$ problem was solved there in 6740 seconds to an accuracy of $\epsilon = 10^{-5}$ while it was solved (see Table 7) in about 260 second using LB_2 to an accuracy of $\epsilon = 10^{-10}$. The computer used for the computations in our paper is about five times faster than the computer used in [1]. However, our program is still five times faster solving the problem to a better accuracy.

4.6 Experiments with Mixed Weights

For this type of problems we tested the objectives of minimizing the inter-quartile range and minimizing the range. The results are depicted in Tables 9 and 10. The rigorous approaches could be solved using an accuracy of $\epsilon = 10^{-6}$. The two rigorous approaches provided comparable

Table 7: Results for Expropriation Problems

n	Iterations Min.	Iterations Max.	Iterations Ave.	Max # Triangles Min.	Max # Triangles Max.	Max # Triangles Ave.	Time (sec.) Min.	Time (sec.) Max.	Time (sec.) Ave.	Fraction Below Best Known Min.	Fraction Below Best Known Max.	Fraction Below Best Known Ave.
LB_{Lip}, $\epsilon = 10^{-6}$												
10	104	829	331	12	167	70	0.00	0.02	0.00	0	2.6×10^{-7}	7.7×10^{-8}
20	128	653	291	28	124	63	0.00	0.02	0.01	0	2.1×10^{-7}	7.5×10^{-8}
50	133	1,302	427	29	319	100	0.00	0.12	0.04	0	3.6×10^{-7}	7.0×10^{-8}
100	114	1,770	556	25	551	145	0.03	0.41	0.13	0	1.7×10^{-7}	2.5×10^{-8}
200	154	1,417	336	35	447	90	0.12	0.70	0.22	0	2.0×10^{-7}	2.6×10^{-8}
500	120	5,706	851	26	2,008	260	0.50	8.61	1.56	0	2.5×10^{-7}	3.3×10^{-8}
1000	123	1,932	502	31	665	139	2.09	8.75	3.44	0	7.9×10^{-8}	1.7×10^{-8}
2000	96	599	334	23	178	76	9.41	13.81	11.63	0	2.1×10^{-7}	2.1×10^{-8}
5000	134	8,599	1,228	34	3,238	438	82.38	356.34	118.20	0	1.4×10^{-7}	2.6×10^{-8}
10000	92	5,197	853	24	2,433	353	388.58	815.78	454.92	0	1.0×10^{-7}	2.2×10^{-8}
LB_2, $\epsilon = 10^{-10}$												
10	37	259	82	12	44	23	0.00	0.02	0.00	0	8.2×10^{-11}	2.4×10^{-11}
20	35	274	82	10	39	24	0.00	0.02	0.00	0	8.3×10^{-11}	2.1×10^{-11}
50	36	413	87	12	53	29	0.00	0.02	0.01	0	8.2×10^{-11}	2.3×10^{-11}
100	34	74	44	12	50	20	0.00	0.02	0.02	0	9.0×10^{-11}	4.3×10^{-11}
200	35	165	53	11	29	19	0.05	0.09	0.06	0	7.7×10^{-11}	4.6×10^{-11}
500	32	81	42	7	22	15	0.25	0.31	0.28	0	7.5×10^{-11}	3.8×10^{-11}
1000	30	46	36	8	21	13	1.19	1.27	1.24	0	9.8×10^{-11}	5.5×10^{-11}
2000	31	49	35	7	22	14	5.89	6.31	6.06	0	8.6×10^{-11}	6.4×10^{-11}
5000	32	911	127	9	105	27	52.44	71.38	55.85	0	9.6×10^{-11}	4.3×10^{-11}
10000	28	52	37	9	17	14	255.31	274.14	263.40	0	9.8×10^{-11}	5.9×10^{-11}
LB_H, $\epsilon = 10^{-8}$												
10	44	512	175	5	76	27	0.00	0.03	0.01	0	5.1×10^{-9}	1.2×10^{-9}
20	66	199	134	16	32	24	0.00	0.03	0.02	0	1.9×10^{-9}	6.9×10^{-10}
50	80	600	229	13	98	35	0.05	0.33	0.14	0	1.4×10^{-9}	3.7×10^{-10}
100	84	1,611	407	14	229	64	0.19	2.47	0.68	0	5.7×10^{-10}	1.1×10^{-10}
200	84	1,002	211	13	173	36	0.64	3.69	1.10	0	4.6×10^{-9}	5.2×10^{-10}
500	84	6,114	769	15	1,191	144	3.19	64.41	10.12	0	5.8×10^{-10}	7.8×10^{-11}
1000	53	1,546	381	8	337	74	12.95	51.42	21.10	0	5.7×10^{-10}	1.0×10^{-10}
2000	74	673	223	11	134	36	65.56	99.84	74.65	0	2.7×10^{-10}	2.7×10^{-11}
5000	80	1,172	342	7	168	62	560.72	799.52	626.62	0	2.7×10^{-8}	2.9×10^{-9}
10000	42	3,309	534	7	1,207	164	2693.02	4661.02	3009.55	0	1.7×10^{-9}	1.8×10^{-10}

results but the algorithm based on LB_3 required shorter running times. The heuristic approach provided the best results (it found the best known solution for 99 of 100 problems for minimizing the interquartile range and for all 100 problems of minimizing the range). However, one result of minimizing the inter-quartile range, yielded an inferior solution which is less accurate than ϵ. This is the only case out of 900 runs that the heuristic approach clearly did not find a solution within the specified accuracy from optimum.

It should be noted that the special lower bound constructed in [2] was much faster than our approach. It solved the $n = 10,000$ problem in only 2.2 seconds to an accuracy of $\epsilon = 10^{-10}$!

Table 8: Results for Anti-k-centrum Problems

	Iterations			Max # Triangles			Time (sec.)			Fraction Below Best Known		
n	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
LB_{Lip}, $\epsilon = 10^{-6}$												
10	139	2,185	678	30	430	133	0.00	0.02	0.00	0	1.2×10^{-7}	3.3×10^{-8}
20	406	3,225	1,256	95	552	244	0.00	0.06	0.02	0	1.0×10^{-7}	4.4×10^{-8}
50	613	4,109	1,649	105	918	337	0.05	0.25	0.12	0	1.2×10^{-7}	4.7×10^{-8}
100	190	16,447	3,246	93	4,376	780	0.05	3.97	0.76	0	4.7×10^{-8}	1.5×10^{-8}
200	168	10,502	3,626	34	2,476	864	0.14	6.14	1.98	0	3.4×10^{-8}	5.2×10^{-9}
500	131	3,422	652	51	634	169	0.53	5.48	1.29	0	0	0
1000	191	9,141	1,314	46	2,710	353	2.39	31.42	6.06	0	0	0
2000	224	6,649	969	55	1,766	239	10.44	65.83	17.10	0	0	0
5000	171	1,078	416	40	193	81	84.14	113.81	91.83	0	0	0
10000	138	370	262	39	95	62	388.81	428.34	407.66	0	0	0
LB_2, $\epsilon = 10^{-10}$												
10	72	2,097	541	23	227	73	0.00	0.02	0.00	0	4.7×10^{-11}	1.9×10^{-11}
20	210	3,704	1,069	67	392	132	0.00	0.05	0.02	0	2.4×10^{-11}	2.8×10^{-12}
50	343	4,133	1,322	61	583	183	0.02	0.19	0.07	0	1.5×10^{-11}	1.5×10^{-12}
100	78	11,793	2,497	68	1,585	344	0.02	1.94	0.40	0	6.9×10^{-11}	1.4×10^{-11}
200	76	9,415	3,128	42	1,215	430	0.06	3.73	1.15	0	7.3×10^{-11}	3.2×10^{-11}
500	103	564	191	44	161	93	0.33	0.80	0.42	4.8×10^{-11}	9.1×10^{-11}	7.0×10^{-11}
1000	76	833	227	27	255	63	1.36	2.98	1.68	3.0×10^{-11}	8.3×10^{-11}	6.5×10^{-11}
2000	88	742	192	24	247	54	6.44	10.17	6.99	5.6×10^{-11}	9.1×10^{-11}	7.0×10^{-11}
5000	91	238	135	11	27	19	53.97	58.97	56.10	2.7×10^{-11}	8.7×10^{-11}	5.8×10^{-11}
10000	61	158	120	13	26	18	260.50	279.86	268.92	5.0×10^{-11}	8.3×10^{-11}	6.4×10^{-11}
LB_H, $\epsilon = 10^{-8}$												
10	46	334	142	5	49	20	0.00	0.02	0.01	0	7.6×10^{-10}	1.8×10^{-10}
20	86	575	247	16	83	36	0.02	0.05	0.03	0	1.4×10^{-9}	2.3×10^{-10}
50	97	546	310	14	71	45	0.06	0.27	0.18	0	3.4×10^{-10}	1.5×10^{-10}
100	57	1,482	529	14	251	83	0.14	2.45	0.88	0	4.5×10^{-10}	1.4×10^{-10}
200	59	3,976	839	10	567	124	0.58	14.17	3.39	0	1.1×10^{-10}	2.2×10^{-11}
500	73	1,367	235	11	177	39	3.06	16.73	4.74	0	0	0
1000	87	800	213	10	132	31	14.08	29.73	16.91	0	0	0
2000	69	1,959	301	11	273	42	64.75	177.84	78.79	0	0	0
5000	72	231	132	8	29	17	552.91	599.41	577.59	0	0	0
10000	48	154	109	10	21	14	2678.09	2877.61	2771.32	0	0	0

4.7 Summary of Results

Among the rigorous algorithms, the lower bound based on individual distances (LB_1, LB_2, or LB_3) performed better than LB_{Lip}. In most cases the algorithms provided comparable results but LB_{Lip} required longer running times. The heuristic algorithm provided the best value of the objective function for most problems but generally required longer running times. It exhibited the most consistent performance and seems to be the "most practical" approach for general problems. While the rigorous bounds need to be modified for norms different from Euclidean (such as calculating the shortest distance to a triangle for LB_1 and LB_3, calculating the value of r for LB_{Lip}), no special treatment for calculating LB_H is required for using other planar norms in the definition of the problem.

Table 9: Results for Minimizing Inter-Quartile Range

	Iterations			Max # Triangles			Time (sec.)			Fraction Above Best Known		
n	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
LB_{Lip}, $\epsilon = 10^{-6}$												
10	503	6,034	1,425	128	1,134	305	0.00	0.06	0.01	3.1×10^{-8}	3.4×10^{-7}	1.5×10^{-7}
20	623	21,724	4,337	157	4,698	913	0.02	0.58	0.10	2.1×10^{-8}	2.8×10^{-7}	1.3×10^{-7}
50	718	3,388	1,544	197	662	326	0.06	0.30	0.12	1.9×10^{-9}	2.0×10^{-7}	9.7×10^{-8}
100	797	3,579	1,934	293	802	496	0.19	0.73	0.41	5.1×10^{-8}	2.1×10^{-7}	1.2×10^{-7}
200	912	11,801	2,689	349	4,817	1,006	0.55	6.61	1.51	3.0×10^{-8}	3.1×10^{-7}	1.7×10^{-7}
500	1,066	3,117	2,262	509	987	754	1.81	4.73	3.47	3.2×10^{-8}	2.6×10^{-7}	1.2×10^{-7}
1000	1,338	38,426	9,805	482	24,074	5,452	6.48	139.03	36.18	0	2.1×10^{-7}	8.6×10^{-8}
2000	2,014	7,117	4,122	975	3,001	1,795	26.19	71.16	44.66	1.9×10^{-8}	2.5×10^{-7}	9.7×10^{-8}
5000	3,034	7,265	5,403	1,371	5,285	2,996	182.30	306.05	251.69	5.8×10^{-8}	3.9×10^{-7}	2.1×10^{-7}
10000	5,578	13,801	7,890	2,994	6,712	4,334	827.30	1431.97	1000.22	6.1×10^{-8}	2.6×10^{-7}	1.6×10^{-7}
LB_3, $\epsilon = 10^{-6}$												
10	168	4,205	738	49	789	145	0.00	0.05	0.01	4.5×10^{-8}	2.9×10^{-7}	1.6×10^{-7}
20	198	10,778	1,985	54	2,384	438	0.00	0.25	0.05	1.5×10^{-8}	2.7×10^{-7}	1.2×10^{-7}
50	239	2,034	639	66	427	147	0.02	0.20	0.06	0	2.0×10^{-7}	9.8×10^{-8}
100	282	2,068	707	97	483	195	0.08	0.47	0.17	6.1×10^{-8}	2.7×10^{-7}	1.9×10^{-7}
200	331	7,628	1,272	132	3,335	525	0.23	4.45	0.78	1.2×10^{-8}	3.8×10^{-7}	1.2×10^{-7}
500	324	1,826	1,028	204	480	344	0.84	3.08	1.86	1.7×10^{-8}	3.6×10^{-7}	1.3×10^{-7}
1000	502	33,554	5,320	210	20,065	3,006	3.70	124.91	21.24	1.3×10^{-10}	3.7×10^{-7}	1.5×10^{-7}
2000	782	3,338	1,678	358	1,344	763	16.05	38.03	24.01	4.8×10^{-9}	3.9×10^{-7}	1.6×10^{-7}
5000	1,189	3,069	2,300	668	1,909	1,271	123.31	179.16	156.42	4.9×10^{-8}	2.7×10^{-7}	1.4×10^{-7}
10000	2,296	5,933	3,470	1,312	3,266	2,059	592.52	864.00	677.27	3.9×10^{-8}	5.2×10^{-7}	1.9×10^{-7}
LB_H, $\epsilon = 10^{-8}$												
10	120	520	263	21	64	42	0.00	0.03	0.02	0	0	0
20	122	2,476	669	31	247	84	0.02	0.25	0.08	0	0	0
50	243	490	358	39	119	64	0.14	0.30	0.21	0	9.7×10^{-8}	9.7×10^{-9}
100	219	669	464	65	201	104	0.44	1.03	0.75	0	0	0
200	257	1,466	523	77	211	128	1.31	5.98	2.31	0	0	0
500	268	855	560	75	211	138	4.94	11.28	7.79	0	0	0
1000	306	22,415	3,200	95	7,741	993	19.84	546.50	89.10	0	0	0
2000	371	1,391	817	122	455	243	82.55	147.48	110.79	0	0	0
5000	398	1,505	849	135	613	324	654.20	866.00	741.43	0	0	0
10000	434	2,161	956	178	692	341	2901.73	3872.80	3237.33	0	0	0

5 Conclusions and Future Research

We constructed a general algorithm to solve the ordered one median problem in the plane. Two rigorous algorithms and one heuristic algorithm were designed and tested. Extensive computational experiments with eight different problems demonstrated the effectiveness of the procedure. The selected problems consisted of positive λs, negative λs, and mixed λs to demonstrate the versatility of the approach.

Our procedures are superior to other approaches unless a special structure of a particular problem can be exploited. For example, the Weiszfeld algorithm [14] for the solution of the Weber problem or the Elzinga-Hearn algorithm [7] for the solution of the 1-center problem are more

Table 10: Results for Minimizing The Range

n	Iterations Min.	Iterations Max.	Iterations Ave.	Max # Triangles Min.	Max # Triangles Max.	Max # Triangles Ave.	Time (sec.) Min.	Time (sec.) Max.	Time (sec.) Ave.	Fraction Above Best Known Min.	Fraction Above Best Known Max.	Fraction Above Best Known Ave.
						LB_{Lip}, $\epsilon = 10^{-6}$						
10	311	8,787	2,205	72	1,617	452	0.00	0.09	0.02	2.1×10^{-8}	3.0×10^{-7}	1.2×10^{-7}
20	400	6,649	1,981	87	1,713	478	0.00	0.16	0.04	1.8×10^{-8}	2.1×10^{-7}	1.2×10^{-7}
50	312	1,563	849	78	346	203	0.02	0.11	0.07	4.7×10^{-9}	3.4×10^{-7}	1.9×10^{-7}
100	481	1,136	739	109	361	182	0.11	0.22	0.16	6.8×10^{-8}	2.8×10^{-7}	1.5×10^{-7}
200	286	1,175	762	90	311	185	0.20	0.67	0.46	2.5×10^{-8}	3.8×10^{-7}	1.3×10^{-7}
500	274	1,641	895	72	533	250	0.70	2.73	1.60	2.0×10^{-8}	3.4×10^{-7}	1.6×10^{-7}
1000	292	15,256	2,331	81	6,204	831	2.64	53.69	9.76	1.6×10^{-8}	3.1×10^{-7}	1.1×10^{-7}
2000	235	1,082	673	69	524	196	10.52	18.44	14.59	9.0×10^{-8}	2.6×10^{-7}	1.4×10^{-7}
5000	280	1,252	615	86	501	201	87.22	115.09	98.30	5.6×10^{-8}	1.6×10^{-7}	1.1×10^{-7}
10000	248	1,199	557	68	536	177	400.45	489.11	431.68	1.5×10^{-8}	3.7×10^{-7}	2.0×10^{-7}
						LB_3, $\epsilon = 10^{-6}$						
10	92	4,829	868	21	895	188	0.00	0.06	0.01	3.5×10^{-8}	3.2×10^{-7}	1.7×10^{-7}
20	129	5,217	992	29	995	219	0.00	0.14	0.02	1.3×10^{-8}	2.7×10^{-7}	1.1×10^{-7}
50	100	608	304	26	132	79	0.00	0.06	0.03	1.3×10^{-8}	3.1×10^{-7}	1.5×10^{-7}
100	134	519	313	45	139	79	0.05	0.11	0.08	2.2×10^{-8}	2.3×10^{-7}	1.2×10^{-7}
200	93	602	327	26	170	84	0.09	0.38	0.24	3.8×10^{-8}	5.0×10^{-7}	2.0×10^{-7}
500	85	997	439	23	374	137	0.47	1.81	1.01	7.0×10^{-8}	3.5×10^{-7}	1.3×10^{-7}
1000	94	8,410	1,171	30	4,241	516	2.12	31.30	5.96	3.2×10^{-8}	4.9×10^{-7}	2.0×10^{-7}
2000	76	446	257	23	241	82	9.48	13.12	11.32	1.6×10^{-8}	2.5×10^{-7}	1.4×10^{-7}
5000	84	718	258	31	299	95	82.55	100.73	89.57	1.2×10^{-7}	3.8×10^{-7}	2.1×10^{-7}
10000	89	490	228	27	295	88	398.08	444.28	417.48	4.6×10^{-8}	2.7×10^{-7}	1.6×10^{-7}
						LB_H, $\epsilon = 10^{-8}$						
10	73	881	292	14	98	40	0.00	0.05	0.02	0	0	0
20	101	782	324	17	136	48	0.02	0.08	0.04	0	0	0
50	93	447	209	15	59	35	0.06	0.25	0.12	0	0	0
100	132	227	170	20	59	32	0.25	0.36	0.31	0	0	0
200	93	412	195	16	56	30	0.66	2.02	1.08	0	0	0
500	81	415	232	15	74	39	3.12	6.61	4.65	0	0	0
1000	119	3,057	486	18	426	71	14.56	84.69	23.63	0	0	0
2000	79	281	179	16	56	32	64.16	79.12	71.81	0	0	0
5000	86	273	166	19	70	34	561.97	616.00	587.81	0	0	0
10000	75	271	148	16	57	30	2697.44	2910.27	2801.00	0	0	0

efficient than our general approach. They both exploit the special structure of these particular problems.

All our experiments were conducted for Euclidean distances. As future research we suggest to test these algorithms on problems (even the same eight problems) based on other distance measures. Also, a possibly better Lipschitz lower bound may be obtained if the constant m in the triangle can be improved.

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