A piecewise analytic solution for Jiang's model of elastoplasticity

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Abstract

In this article, we present an analytic solution for Jiang's constitutive model of elastoplasticity. It is considered in its stress controlled form for proportional loading under the assumptions that the one-to-one coupling of the yield radius and the memory surface radius is switched off, that the transient hardening is neglected and that the ratchetting exponents are constant.

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1 Introduction

The existence of analytic solutions for some kind of nonlinear equations is not just an academic question. In fact, it is very useful to have analytical solutions as benchmark tests, in order to perform accurate and reliable numerical simulations. In the field of elastoplastic material (or 'stress-strain') laws, including nonlinear hysteresis and memory effects, it is a very tricky task to find some, cf. [1, 2, 3, 11].
Until now, no analytic solutions of Jiang's constitutive model [5, 6, 7, 8, 9] are available, for proportional loading however, numerical benchmarks are exposed in [7]. We give an explicit example of an analytical solution for the special case of arbitrary proportional loading, where we assume the following simplifications. The resulting equations are given in (1), ..., (7). (We follow the notation in [10].)

- The ratchetting exponent in the nonlinear term of the evolution of the ith backstress is set to a constant value
$$\chi_i \equiv \text{const} \geq 0.$$
- For the coupling between the yield surface radius ρ and the memory surface radius R, given by $\rho = \rho_\infty(1 + a_\rho \exp(b_\rho R))$, we assume $a_\rho = 0$, which means
$$\rho \equiv \rho_\infty \equiv \text{const} > 0,$$
i. e. the evolution of the memory surface does not influence the evolution of the yield surface radius. Thus the hardening modulus $H = n : \mathrm{d}\alpha/\mathrm{d}\xi + \mathrm{d}\rho/\mathrm{d}\xi$ simplifies to $H = n : \mathrm{d}\alpha/\mathrm{d}\xi$, since $\rho > 0$ is now constant.

- Transient hardening/softening is switched off, which means that we assume $a_i^j = 0$ for all i and j in the relation for the coefficients $c_i = c_i^\infty(1 + \sum_j a_i^j \exp(-b_i^j \xi))$, i.e.

$$c_i \equiv c_i^\infty \equiv \text{const} > 0.$$

Now Jiang's stress controlled constitutive model, which describes the movements in the deviatoric stress space, takes on the following form.

(J1) The total (deviatoric) stress is additively decomposed into the *total (deviatoric) backstress* and the '*(deviatoric) stop*'

$$s(t) = \alpha + \beta_d. \tag{1}$$

(J2) The *total (deviatoric) backstress* is additively decomposed into I *partial (deviatoric) backstresses*

$$\alpha = \alpha_1 + \ldots + \alpha_I. \tag{2}$$

(J3) The backstress evolutions (or the '*hardening rules*') are given by the nonlinear relationships

$$\dot{\alpha}_i = c_i r_i \left(n - \left(\frac{\|\alpha_i\|}{r_i} \right)^{\chi_i + 1} \frac{\alpha_i}{\|\alpha_i\|} \right) \dot{\xi} \qquad (i = 1, \ldots, I). \tag{3}$$

(J4) The *(deviatoric) yield surface normal* is given by

$$n = \frac{\beta_d}{\rho}. \tag{4}$$

(J5) The *hardening modulus* is defined by

$$H = \begin{cases} n : \dfrac{\mathrm{d}\alpha}{\mathrm{d}\xi} & \text{if } \|\beta_d\| = \rho \text{ and } \dot{s} : n > 0 \\ \infty & \text{otherwise} \end{cases}. \tag{5}$$

(J6) The *accumulating plastic strain rate* is given by

$$\dot{\xi} = \frac{\dot{s}(t) : n}{H}. \tag{6}$$

(J7) The *(deviatoric) plastic strain rate* is proportional to the yield surface normal

$$\dot{\varepsilon}^{pl} = \dot{\xi}\, n. \tag{7}$$

Here $s = \operatorname{dev} \circ \sigma : [0, T] \to \mathbb{R}^{3\times 3}_{sd}$ is the given (deviatoric) stress input and the (deviatoric) plastic strain $\varepsilon^{pl} : [0, T] \to \mathbb{R}^{3\times 3}_{sd}$ is the output. $\mathbb{R}^{3\times 3}_{s} = \{\tau \in \mathbb{R}^{3\times 3} : \tau = \tau^T\}$ is the space of symmetric 3×3-tensors, $\mathbb{R}^{3\times 3}_{sd} = \{\operatorname{dev} \tau : \tau \in \mathbb{R}^{3\times 3}_{s}\}$ is the deviatoric projection of the former, $\mathbb{R}^{3\times 3}_{sd} = \operatorname{dev} \mathbb{R}^{3\times 3}_{s}$, where $\operatorname{dev}(\tau) = \tau - \operatorname{tr}(\tau)I/3$ for $\tau \in \mathbb{R}^{3\times 3}_{s}$. The constant $\rho > 0$ denotes the radius of the Mises cylinder in the deviatoric stress space. The model (1), ..., (7) is written in differential equation form

$$\big(\dot{\varepsilon}^{pl}, \dot{\xi}, \dot{\alpha}, \dot{\alpha}_1, \ldots, \dot{\alpha}_i\big) = \mathcal{J}_s\big(t, \alpha_1, \ldots, \alpha_I\big)$$

with discontinuous right-hand side. The differential equations for $\dot\varepsilon^{pl}$, $\dot\xi$ and $\dot\alpha$ do *not* depend on ε^{pl}, ξ and α. Thus they are pure integrations. It follows from the constitutive equations (1), ..., (7), that

$$\dot\varepsilon^{pl} = 0, \qquad \dot\xi = 0, \qquad \dot\alpha_i = 0,$$

iff the Mises yield condition $\|\beta_d\| = \rho$, $\dot s(t) : n > 0$ (i. e. iff $H = \infty$) is not satisfied. Iff it is satisfied (i. e. iff $0 < H < \infty$), it follows that

$$\dot\xi > 0, \qquad \dot\varepsilon^{pl} \neq 0.$$

Here we apply the practical convention $x/\infty = 0$ for each real number x. In the proportional (or 'uniaxial') case, the hardening modulus H is proportional to the slope $\mathrm{d}s/\mathrm{d}\varepsilon^{pl}$ of the stress-strain phase curve $s(t) \leftrightarrow \varepsilon^{pl}(t)$. It is important to note that the ith partial backstress evolution (3) can be equivalently rewritten in the form

$$\frac{\mathrm{d}\alpha_i}{\mathrm{d}t} = c_i\left(r_i n - \left(\frac{\|\alpha_i\|}{r_i}\right)^{\chi_i}\alpha_i\right)\dot\xi, \tag{8}$$

where the zeros in the denominator are lifted.

Remark 1.1 (a) Let us denote by $\cdot_h = \mathrm{hyd}(\cdot)$ the hydrostatic component, by $\cdot_d = \mathrm{dev}(\cdot)$ the deviatoric component of a tensor and by $s = \sigma_d$ the deviatoric part of the stress input. Jiang material is assumed plastically incompressible, i. e. if the prescribed initial values $\alpha(0)$ and $\varepsilon^{pl}(0)$ are deviatoric, it follows from $\dot\varepsilon^{pl} \in \mathbb{R}(s-\alpha)$ in (7) and $\dot\alpha \in \mathbb{R}(s-\alpha) + \mathbb{R}\alpha$ in (2), (3), that $\varepsilon^{pl}_h \equiv 0$, $\alpha_h \equiv 0$, i. e. ε^{pl} and α remain purely deviatoric in $[0,T]$. There holds $\sigma = \sigma_h + \sigma_d$, $\sigma = \alpha + \beta$. Since $\alpha_h = 0$, $\varepsilon^{pl}_h = 0$, it follows that $\sigma_h = \beta_h$, $s = \alpha + \beta_d$. As the action of Hooke's tensor $C = (c_{ijkl})$, $c_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ with Lamé's constants $\lambda, \mu > 0$, decouples into hydrostatic and deviatoric components, i. e.

$$(C\eta)_d = C\eta_d, \quad (C\eta)_h = C\eta_h, \quad (C^{-1}\tau)_d = C^{-1}\tau_d, \quad (C^{-1}\tau)_h = C^{-1}\tau_h \qquad (\eta,\, \tau \in \mathbb{R}^{3\times3}_s),$$

we obtain for the strains $\varepsilon = \varepsilon^{el} + \varepsilon^{pl}$, $\varepsilon^{el} = C^{-1}\sigma$, consequently $\varepsilon^{el}_h = C^{-1}\beta_h = C^{-1}\sigma_h$, $\varepsilon^{el}_d = C^{-1}(\alpha + \beta_d) = C^{-1}s$. Therefore, it is sufficient to consider just the *deviatoric* stress controlled ($s \to \varepsilon^{pl}$) instead of the *total* stress controlled ($\sigma \to \varepsilon$) model.
(b) Jiang's model is a multisurface model. The total backstress α is additively decomposed into I backstresses, see (2). Thus we make the convention $\sum_i \cdot := \sum_{i=1}^I \cdot$. Clearly it would not cause any difficulties, if the finite sum $\sum_i \cdot$ was replaced by a continuous integral $\int_{i\in I} \cdot \mathrm{d}\mu(i)$ over an appropriate measure space, similarly as in [2] for Chaboche's model. That way, we would obtain a *continuous* version of Jiang's model with *smooth* uniaxial equivalent stress-strain curves. □

2 Derivation of piecewise analytic solution

We assume that for a fixed deviatoric unit normal $N \in \partial B_1(0) = \{\tau \in \mathbb{R}^{3\times3}_{sd}, \|\tau\| = 1\}$, the stress deviator $s = \mathrm{dev}\circ\sigma : [0,T] \to \mathbb{R}^{3\times3}_{sd}$ is given in the form

$$s(t) = S(t)N, \qquad \dot s(t) = \dot S(t)N$$

with a piecewise linear scalar function $S : [0,T] \to \mathbb{R}$. As Jiang's model is rate-independent, this assumption yields no loss of generality, as a reparametrisation with an appropriate strongly increasing absolutely continuous function $[0,T] \ni t \to \tau \in [0,T]$ shows. Simply use the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\tau}, \qquad \frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^{-1}.$$

For the same reason, using induction, it suffices to consider just *one* subinterval, reparametrised onto $[0,1]$, where S is affine linear. Thus we may w.l.o.g. assume that we have

$$S(t) = S(0) + t\Delta S, \qquad \dot{S}(t) \equiv \Delta S, \qquad \Delta S = S(1) - S(0).$$

If $\Delta S = 0$, clearly the non-moving constant initial conditions yield a solution of Jiang's equations in a trivial fashion. Thus we may w.l.o.g. assume $\Delta S \neq 0$. By replacing N by $-N$ if necessary, we may as well w.l.o.g. assume that

$$\dot{s}(t) : N = \dot{S}(t)\|N\|^2 > 0.$$

If $\tau \in [0,\infty)$ denotes the point in time, where the stress path intersects the current yield surface, we clearly have $H \equiv \infty$, $\|n\| < 1$, $\varepsilon^{pl} \equiv \varepsilon^{pl}(0)$, $\alpha_i \equiv \alpha_i(0)$, $\alpha \equiv \alpha(0)$, $\xi \equiv \xi(0)$ on $[0,\tau]$. If $\tau < 1$, active plastic yielding starts, and on the interval $(\tau, 1]$, a solution is constructed as follows.
Reparametrisation of the ith backstress evolution with respect to the accumulated plastic strain ξ in the case of active plastic flow $\dot{\xi} > 0$ yields

$$\frac{\mathrm{d}\alpha_i}{\mathrm{d}\xi} = \frac{\mathrm{d}\alpha_i}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\xi} = c_i\left(r_i n - \left(\frac{\|\alpha_i\|}{r_i}\right)^{\chi_i}\alpha_i\right), \qquad \frac{\mathrm{d}t}{\mathrm{d}\xi} = (\dot{\xi})^{-1}.$$

For the evolution of the total backstress, we consequently get

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\xi} = \sum_i c_i\left(r_i n - \left(\frac{\|\alpha_i\|}{r_i}\right)^{\chi_i}\alpha_i\right).$$

Following the approach in [3, 7], we try the ansatz

$$\alpha_i(\xi) = f_i(\xi)N, \qquad \beta_d(\xi) = \rho N, \tag{9}$$

and claim that this yields a solution of Jiang's constitutive equations in the active plastic regime, if the scalar functions f_i are chosen appropriately. Inserting (9) into the constitutive equations (1), ..., (7) gives

$$s = (f+\rho)N, \qquad \dot{s} = \frac{\mathrm{d}f}{\mathrm{d}\xi}\dot{\xi}N, \qquad \frac{\mathrm{d}s}{\mathrm{d}\xi} = \frac{\mathrm{d}f}{\mathrm{d}\xi}N, \qquad \dot{s} : n = \frac{\mathrm{d}f}{\mathrm{d}\xi}\dot{\xi},$$

$$\dot{\beta}_d \equiv 0, \qquad \frac{\mathrm{d}\beta_d}{\mathrm{d}\xi} \equiv 0, \qquad n \equiv N,$$

$$\dot{\alpha}_i = \dot{f}_i N, \qquad \frac{\mathrm{d}\alpha_i}{\mathrm{d}\xi} = \frac{\mathrm{d}f_i}{\mathrm{d}\xi}N, \qquad \alpha_i : n = \alpha_i : N = f_i, \qquad \|\alpha_i\| = |f_i|,$$

$$f = \sum_i f_i, \qquad \alpha = fN, \qquad \alpha : n = \alpha : N = f, \qquad \|\alpha\| = |f|.$$

The hardening modulus H becomes

$$H = n : \frac{\mathrm{d}\alpha}{\mathrm{d}\xi} = \sum_i c_i \Big(r_i \|n\|^2 - \Big(\frac{\|\alpha_i\|}{r_i}\Big)^{\chi_i} \alpha_i : n \Big) = \sum_i c_i \Big(r_i - \Big(\frac{|f_i|}{r_i}\Big)^{\chi_i} f_i \Big).$$

We obtain

$$H\dot{\xi} = \dot{s} : n = \frac{\mathrm{d}f}{\mathrm{d}\xi}\,\dot{\xi},$$

which is equivalent to

$$H = \frac{\mathrm{d}f}{\mathrm{d}\xi}, \qquad \dot{\xi} = \frac{\dot{s} : n}{H}, \qquad \text{if } \dot{\xi} > 0.$$

Thus, the ansatz (9) yields a solution of Jiang's equations, if and only if the functions f_i satisfy

$$\frac{\mathrm{d}f_i}{\mathrm{d}\xi} N = c_i \Big(r_i n - \Big(\frac{|f_i|}{r_i}\Big)^{\chi_i} f_i N \Big).$$

Multiplication with $N = n$ and $r_i^{\chi_i}$ gives

$$r_i^{\chi_i} \frac{\mathrm{d}f_i}{\mathrm{d}\xi} = c_i \Big(r_i^{\chi_i + 1} - \mathrm{sign}(f_i) f_i^{\chi_i + 1} \Big),$$

since $\|N\|^2 = 1$. With the substitutions

$$\varphi_i = \frac{1}{r_i}\, f_i, \qquad \mathrm{d}\varphi_i = \frac{1}{r_i}\,\mathrm{d}f_i, \qquad \zeta_i = c_i\,\xi, \qquad \mathrm{d}\zeta_i = c_i\,\mathrm{d}\xi, \qquad w_i = \chi_i + 1 \tag{10}$$

we see, that (9) yields a solution of Jiang's equations, if and only if each function $\varphi_i(\zeta_i)$ satisfies the ordinary scalar differential equation

$$\frac{\mathrm{d}\varphi_i}{\mathrm{d}\zeta_i} = \Big(1 - \mathrm{sign}(\varphi_i)\varphi_i^{w_i} \Big). \tag{11}$$

Since the constants χ_i are non-negative, we have $w_i \geq 1$. Thus, the following elementary lemma for $\varphi = \varphi_i$ and $\zeta = \zeta_i$ may be applied. Thus (9) with (10) yield a solution of Jiang's equations during active plastic flow.

Lemma 2.1 (Nondegenerate case) *Let $1 \leq w < \infty$ denote any real number. We consider the scalar differential equation*

$$\frac{\partial\varphi}{\partial\zeta} = F(\varphi) := \begin{cases} 1 + (-\varphi)^w & \text{for } \varphi < 0 \\ 1 & \text{for } \varphi = 0 \\ 1 - \varphi^w & \text{for } \varphi > 0 \end{cases}. \tag{12}$$

It has the unique solution $\zeta \mapsto \varphi(\zeta)$, which is implicitly given as the inverse function of the mapping

$$\varphi \mapsto \zeta(\varphi) = C + \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{\varphi^{nw+1}}{nw+1} & \text{for } \varphi > 0 \\ 0 & \text{for } \varphi = 0 \\ \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(-\varphi)^{nw+1}}{nw+1} & \text{for } \varphi < 0 \end{cases}, \tag{13}$$

the latter being defined for $-1 \le \varphi < 1$. Both functions $\zeta \to \varphi$ and $\varphi \to \zeta$ are strongly increasing with the boundary behaviour

$$\zeta(\varphi = -1) = \zeta_{\min}(w), \qquad \zeta(\varphi) \overset{\varphi \to 1}{\longrightarrow} +\infty,$$

where

$$\zeta_{\min}(w) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{nw+1}. \tag{14}$$

For any given values $-1 \le \varphi_0 < 1$ and $\zeta_{\min}(w) \le \zeta_0 < \infty$, there exists one and only one real constant C, such that the initial condition $\varphi(\zeta_0) = \varphi_0$ – or equivalently $\zeta(\varphi_0) = \zeta_0$ – holds. Especially for $\varphi = 0$, we obtain $C = \zeta(0) = \zeta_0$.

Proof: The function $F : \mathbb{R} \to \mathbb{R}$ is differentiable an the whole real axis, such that its derivative

$$\frac{\mathrm{d}F}{\mathrm{d}\varphi} = \begin{cases} -w(-\varphi)^{\chi} & \text{for } \varphi < 0 \\ 0 & \text{for } \varphi = 0 \\ -w\varphi^{\chi} & \text{for } \varphi > 0 \end{cases}$$

is still continuous on $\mathbb{R}$. Thus F is of class $C^1(\mathbb{R}, \mathbb{R})$ and existence and uniqueness of the solution of (12) is guaranteed, as F is locally Lipschitz continuous by the mean value theorem. We separate the variables. Recall that the function $z \mapsto 1/(1-z)$ is analytic on the complex unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and that there holds

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (z \in \mathbb{D}) \tag{15}$$

by Neumann (or geometric series) expansion. For $0 \le \varphi < 1$, we arrive with Neumann expansion (15) at

$$\zeta(\varphi) \in \int \mathrm{d}\zeta = \int \frac{\mathrm{d}\varphi}{1-\varphi^w} = \int \sum_{n=0}^{\infty} \varphi^{nw} \mathrm{d}\varphi = \sum_{n=0}^{\infty} \frac{\varphi^{nw+1}}{nw+1} + C.$$

For $-1 < \varphi < 0$, we find with the aid of the substitution $\psi = -\varphi$, $\mathrm{d}\psi = -\mathrm{d}\varphi$, with Neumann expansion (15)

$$\zeta(\varphi) \in \int \frac{\mathrm{d}\varphi}{1+(-\varphi)^w} = \int \frac{-\mathrm{d}\psi}{1-(-\psi^w)} = \int \sum_{n=0}^{\infty} (-1)^{n+1} \psi^{nw} \mathrm{d}\psi = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(-\varphi)^{nw+1}}{nw+1} + C.$$

The value of ζ for $\varphi = -1$ exists because of the Leibniz criterion for infinite series with alternating summands. The fact that $\varphi \mapsto \varphi(\zeta)$ and $\zeta \mapsto \zeta(\varphi)$ are strongly increasing follows from $F = \mathrm{d}\varphi/\mathrm{d}\zeta > 0$ for $-1 \le \varphi < 1$. ■

Lemma 2.1 gives an *explicit analytical* representation of the numerical solution considered in [7] in the case $0 \le \chi_i < \infty$. As well the degenerate case $\chi_i = \infty$ can be considered, where the formal convention

$$x^{\infty} = \begin{cases} 1 & \text{for } x = 1 \\ 0 & \text{for } -1 < x < 1 \\ \text{undefined} & \text{for } x = -1 \end{cases}$$

for any real number $-1 \le x \le 1$ is made in (8) and (11).

Lemma 2.2 (Degenerate case) *Let $w = \infty$. We consider the scalar differential equation, given by*

$$\frac{\partial \varphi}{\partial \zeta} = \begin{cases} 1 & \text{for } -1 < \varphi < 1 \\ 0 & \text{for } \varphi = 1 \end{cases} . \tag{16}$$

For any $\zeta_0 \in \mathbb{R}$ and any given initial value $\varphi(\zeta_0) = \varphi_0$, satisfying $-1 < \varphi_0 < 1$, it has the solution

$$\zeta \mapsto \varphi(\zeta) = \begin{cases} \zeta - \zeta_0 + \varphi_0 & \text{for } \zeta < 1 + \zeta_0 - \varphi_0 \\ 1 & \text{for } \zeta \geq 1 + \zeta_0 - \varphi_0 \end{cases} . \tag{17}$$

Solution is to be understood here in the sense of distributions.

Proof: Clear. ■

Let a sequence $(w_n) \subset [0, \infty)$ such that $w_n \xrightarrow{n} \infty$ and $(\zeta_0, \varphi_0) \in [\zeta_{\inf}, \infty) \times (-1, 1)$ be given, where

$$\zeta_{\inf} = \inf_{n \in \mathbb{N}} \zeta_{\min}(w_n), \qquad \zeta_{\min}(w) \text{ defined by (14).}$$

It can be shown, that for the solutions $\varphi_n : [\zeta_0, \infty) \to (-1, 1)$ corresponding to (12) with the initial value (ζ_0, φ_0) and the exponent w_n, defined by (13), it holds that

$$\|\varphi_n - \varphi_\infty\|_\infty = \sup_{\zeta \in [\zeta_0, \infty)} |\varphi_n(\zeta) - \varphi_\infty(\zeta)| \xrightarrow{n} 0,$$

where $\varphi_\infty : [\zeta_0, \infty) \to (-1, 1]$ is the solution (17) of differential equation (16), corresponding to the initial value (ζ_0, φ_0). This means, that the functions φ_n converge uniformly, thus pointwise, towards φ_∞ on the interval $[\zeta_0, \infty)$ for $n \to \infty$. This observation has been made numerically in [7].

Remark 2.3 (Benchmark) *If the solution, which was constructed in lemma 2.1, is used as an analytical benchmark for a numerical implementation of Jiang's model, it is important to know, where to truncate the series (13). The error, which is caused by truncation after the Nth summand, $N \in \mathbb{N}_0$, can be estimated by*

$$\left| \zeta(\varphi) - \sum_{n=0}^{N} \frac{\varphi^{nw+1}}{nw+1} \right| = \sum_{n=N+1}^{\infty} \frac{\varphi^{nw+1}}{nw+1} \leq \varphi^{(N+1)w} \sum_{n=0}^{\infty} \varphi^{nw} = \frac{\varphi^{(N+1)w}}{1 - \varphi^w}$$

for $0 < \varphi < 1$. In the same way, we find the useful estimate

$$\left| \zeta(\varphi) - \sum_{n=0}^{N} \frac{(-1)^{n+1}(-\varphi)^{nw+1}}{nw+1} \right| \leq \frac{(-\varphi)^{(N+1)w}}{1 - (-\varphi)^w}$$

for $-1 < \varphi < 0$. Simply use (15) and the triangle inequality for derivation. The right-hand sides in both estimates are vanishing for $N \to \infty$. The procedure for reversing the expansions in (13), in order to receive the inverse mapping $\zeta \mapsto \varphi(\zeta)$, is standard and can be found e. g. in [4, Section III.2]. □

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