Scheduling and Location (ScheLoc): Makespan Problem with Variable Release Dates

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While in classical scheduling theory the locations of machines are assumed to be fixed we will show how to tackle location and scheduling problems simultaneously. Obviously, this integrated approach enhances the modeling power of scheduling for various real-life problems. In this paper, we present in an exemplary way theory and a solution algorithm for a specific type of a scheduling and a rather general, planar location problem, respectively. More general results and a report on numerical tests will be presented in a subsequent paper.

Keywords: Machine Scheduling, Location Theory, Algorithmics, Gauge Distances

1 Introduction

Scheduling and location theory are equally important areas of operations research with a wealth of applications. For many of these applications it is obvious, that dealing with these problems in the usual *sequential* manner (i.e., taking the output of one of the problems as input of the other) weakens the model and should be replaced by an *integrated* approach (i.e., solving both problems simultaneously). The latter problem, which we call *ScheLoc* was first introduced by Hennes and Hamacher [4] where machines can be located anywhere on a network. A more detailed investigation on this type of ScheLoc was given by Hennes [3].

The focus of this study is to investigate and analyze planar ScheLoc (P-ScheLoc) problems, where machines can be located anywhere in a given planar region. In this short note, we restrict ourselves to the single machine case. Starting from a general formulation and the main concepts of this new class, a specific P-ScheLoc problem – makespan problem with variable release dates – is discussed in detail. We give a first formal description of this problem and derive two conditions to detect optimal solutions directly. Moreover, we present a problem reformulation using a modified version of the Earliest Release Date (ERD) rule. An important tool for solving this problem is the construction of release date bisectors and ordered regions. In our problem formulation the release dates are shown to be be representable by a special type of distance functions, so-called gauges. Using these results, we develop an efficient solution algorithm based on Linear Programming (LP) for polyhedral gauges, which also include as special cases the rectilinear and maximum distances. Finally, complexity results and some concluding remarks are presented.

The results are based on diploma theses of Elvikis [1] and Kalsch [5].

2 Basics

We are given a set $\mathcal{J} = \{1, \dots, n\}$ of *jobs* with nonnegative *processing times* p_i, $i \in \mathcal{J}$, which must be scheduled nonpreemptively on a single machine M. In addition, we assume that M can be placed anywhere in the plane $\mathbb{R}^2$ and that each job $i \in \mathcal{J}$ has a given *storage location* $a_i \in \mathbb{R}^2$. Hence the general *Single Machine Planar ScheLoc (1-P-ScheLoc) Problem* consists of choosing a *machine location* $X \in \mathbb{R}^2$, under the constraint that the set of jobs $\mathcal{J}$ is completely processed and that all processing conditions are satisfied. Our goal is to optimize some scheduling objective function which depends not only on the sequence of jobs, but also on the choice of X.

In 1-P-ScheLoc problems, each job $i \in \mathcal{J}$ is additionally characterized by the following parameters. The *storage arrival time* $\sigma_i \geq 0$ represents the time at which job i is available at its storage location a_i. If $\sigma_i = 0$, then i is already available at its storage at the beginning of the processing sequence. The *travel speed* $v_i > 0$ represents the rate of motion of job i , or equivalently the rate of change of position, expressed as distance per unit time. Hence after job i is available at its storage location a_i, we can start to move i from its storage to the machine M. The time at which i can start its processing is given by its arrival time at M. It is obvious that this time can be interpreted as the job release date. Now, let $dist_i$ be a general distance function on $\mathbb{R}^2$ corresponding to a_i and $\tau_i := \frac{1}{v_i} > 0$, then $r_i(X) := \sigma_i + \tau_i \; dist_i(a_i, X)$ is called the *variable release date* of job i for M dependent on its machine location $X \in \mathbb{R}^2$. Moreover, the sequence in which the jobs are to be processed on the machine is defined by a permutation π of $\{1, \ldots, n\}$, where $\pi(j) = i$ means that job i is the j^{th} job in the processing order. The set of all permutations of $\{1, \ldots, n\}$ is denoted by Π_n. Then for each sequence $\pi \in \Pi_n$ and each machine location $X \in \mathbb{R}^2$, we can easily calculate the completion times for all jobs $i \in \mathcal{J}$ using the following recursive formula

$$C_{\pi(1)}(X) = r_{\pi(1)}(X) + p_{\pi(1)}, \tag{1}$$

$$C_{\pi(j)}(X) = \max\{C_{\pi(j-1)}(X)\ ,\ r_{\pi(j)}(X)\} + p_{\pi(j)} \quad \forall j \in \{2, \ldots, n\}, \tag{2}$$

where $p_{\pi(j)}$ defines the processing time of job $\pi(j)$. Finally, the maximum completion time (or makespan) in $X \in \mathbb{R}^2$ is given by

$$C_{max}(X) = \max\{C_1(X), \ldots, C_n(X)\} = C_{\pi(n)}(X). \tag{3}$$

To illustrate the modeling potential of this approach, we concentrate on a specific 1-P-ScheLoc problem, the makespan problem with variable release dates.

3 The Problem

In general, the single machine makespan problem with variable release dates (*1-MPVRD*) can be formulated using (1)-(3):

$$\begin{aligned}
\min \quad & C_{\pi(n)}(X) && \\
\text{s.t.} \quad & C_{\pi(j)}(X) \geq C_{\pi(j-1)}(X) + p_{\pi(j)} \quad \forall j \in \{2, \ldots, n\} && (4)\\
& C_{\pi(j)}(X) \geq r_{\pi(j)}(X) + p_{\pi(j)} \quad \forall j \in \{1, \ldots, n\} && (5)\\
& \pi \in \Pi_n && (6)\\
& X \in \mathbb{R}^2 && (7)
\end{aligned}$$

where completion time formula (1)-(2) is explicitly represented by constraints (4)-(5). It easy to see that if $p_i = 0$ for all $i \in \mathcal{J}$, then 1-MPVRD reduces to a classical 1-center facility location problem. If we fix X a priori, then we only have to solve a classical makespan problem with fixed release dates. Furthermore, for a given sequence $\pi \in \Pi_n$, we only have to solve a 1-facility location problem to obtain an optimal machine location. For convex distance functions and a fixed sequence $\pi \in \Pi_n$, it is obvious that the objective function $C_{\pi(n)}(X)$ is convex on $\mathbb{R}^2$. Note that, (5) can be replaced by $C_i(X) \geq r_i(X) + p_i$ for all $i \in \mathcal{J}$.

The following two sufficient criteria describe situations, where one of the job locations is an optimal ScheLoc location for the machine. They are proved using the trivial lower bound $LB := \min_{i \in \mathcal{J}}\{\sigma_i\} + \sum_{i \in \mathcal{J}} p_i$.

Proposition 1. *If there exists a job $i \in \mathcal{J}$ with $i \in argmin\{\sigma_s : s = 1, \dots, n\}$ and $\sigma_i + p_i \geq r_s(a_i) = \sigma_s + \tau_s\ dist_s(a_s, a_i)$ for all $s \in \{1, \dots, n\}$ then a_i is an optimal machine location and $\pi^* = (i, \pi^*(2), \pi^*(3), \dots, \pi^*(n)) \in \Pi_n$ with $\pi^*(s) \neq i$, $s \in \{2, \dots, n\}$, defines an optimal job sequence.*

Proposition 2. *Let $\pi^* \in \Pi_n$ be an optimal sequence in $X = a_i$ with $i \in argmin\{\sigma_s : s = 1, \dots, n\}$. If $\sigma_i + \sum_{j=1,\dots,l} p_{\pi^*(j)} \geq r_{\pi^*(l+1)}(a_i)$ for all $l \in \{1, \dots, n-1\}$, then a_i is an optimal machine location.*

In the following, we assume that neither of the preceding conditions hold such that we have to develop an efficient algorithm to solve ScheLoc.

Recall that 1-MPVRD reduces for a given machine location $X \in \mathbb{R}^2$, to a classical makespan problem with fixed release dates $r_i(X) = r_i$, $i \in \mathcal{J}$. In this case, we can use the well-known ERD rule to obtain an optimal job sequence. Thus, for every machine location $X \in \mathbb{R}^2$ we can easily obtain an optimal job sequence $\pi \in \Pi_n$ using the ***ScheLoc ERD rule***: For machine location $X \in \mathbb{R}^2$, sort the jobs $i \in \mathcal{J}$ in increasing order of their release dates $r_i(X)$, i.e., $r_{\pi(1)}(X) \leq \ldots \leq r_{\pi(n)}(X)$. Thus, 1-MPVRD can be reformulated using the provided ScheLoc ERD rule:

$$\begin{aligned} \min \quad & C_{\pi(n)}(X) \\ \text{s.t.} \quad & (4) - (7) \\ & r_{\pi(1)}(X) \leq \ldots \leq r_{\pi(n)}(X) \qquad (8) \end{aligned}$$

Here it should be noted that the objective function is in general non-convex on $\mathbb{R}^2$ (see Example 1).

Example 1. Consider two jobs with storage locations $a_1 = (0, 0)$ and $a_2 = (10, 5)$ with rectilinear distance l_1. Moreover, let $p_1 = 1$ and $p_2 = 15$, $\sigma_1 = \sigma_2 = 0$ and $\nu_1 = \nu_2 = 1$:

$$C_{max}(a_1) = \max\{r_2(a_1), r_1(a_1) + p_1\} + p_2 = \max\{15, 0 + 1\} + 15 = 30$$

$$C_{max}(a_2) = \max\{r_1(a_2), r_2(a_2) + p_2\} + p_1 = \max\{15, 0 + 15\} + 1 = 16$$

$$C_{max}(0.5 \cdot (a_1 + a_2)) = \max\{7.5, 7.5 + 1\} + 15 = \max\{7.5, 7.5 + 15\} + 1 = 23.5$$

If we assume that our distance functions are convex, then it is easy to see that the objective function is also convex in each of the regions in which the sequence of inequalites (8) does not change.

4 Geometrical Properties: Bisectors and Ordered Regions

Let $i, j \in \mathcal{J}$ with $i \neq j$. Then the set $B^{i,j} := \{X \in \mathbb{R}^2 \mid r_i(X) = r_j(X)\}$ is called the *release date bisector* with respect to job i located in a_i and job j located in a_j.

The bisectors divide the plane into various *(release date -) ordered regions* $O_\pi := \{X \in \mathbb{R}^2 \mid r_{\pi(1)}(X) \leq \ldots \leq r_{\pi(n)}(X)\}$ defined by permutations $\pi \in \Pi_n$ (see Figure 1). In each O_π, the order of the release dates does not change. Note that, ordered regions are in general neither convex nor connected.

For each $X \in O_\pi$ an optimal job sequence of problem 1-MPVRD is obtained by π. Thus solution of a location problem for all $n!$ permutations of possible ordered regions solves the ScheLoc problem 1-MPVRD. As we will show subsequently, the efficiency of this approach follows, since for a large class of distance functions, only a ***polynomial number of these ordered regions*** needs to be considered, since for many sequences π we have that $O_\pi = \emptyset$, which means that these π can not be optimal sequences.

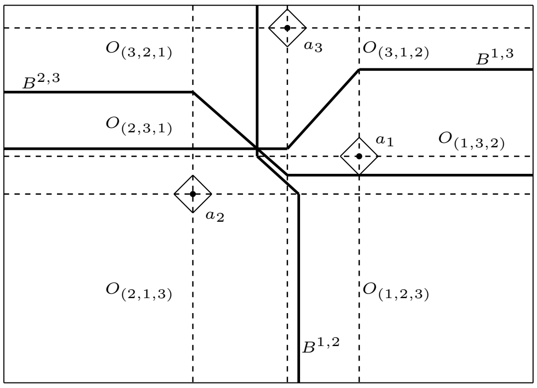

Figure 1: Fundamental directions, Bisectors and Ordered regions generated by a_1, a_2, a_3 associated with $dist_i = l_1$, $i = 1, 2, 3$

The considered class of distance functions is the class of polyhedral gauges with respect to a_i, $i \in \mathcal{J} = \{1, \ldots, n\}$, defined by $\gamma_{\mathcal{B}_i}(X) := \inf\{\lambda > 0 \mid X \in \lambda \mathcal{B}_i\}$ (see e.g. Minkowski [6], Nickel and Puerto [7]). Here $\mathcal{B}_i$ is the *unit ball* of the gauge given by a polytope in $\mathbb{R}^2$, i.e., a convex, compact polyhedral set, containing the origin in its interior. A polyhedral gauge is a convex distance function and even a norm (called *block norm*), if it is additionally symmetric. Examples for block norms are the rectilinear distance l_1 and the maximum distance l_∞, both having polyhedral unit balls ($\mathcal{B}_{l_1}$ and $\mathcal{B}_{l_\infty}$) with four extreme points.

Denote the set of *extreme points* of the polytope $\mathcal{B}_i \subseteq \mathbb{R}^2$ by $Ext(\mathcal{B}_i) = \{e^i_g \mid g = 1, \ldots, G_i\}$. Moreover, we define $\mathcal{G}_i := \{1, \ldots, G_i\}$, $i \in \{1, \ldots, n\}$, and $G := \max\{G_i \mid i = 1, \ldots, n\}$. The half-lines ξ^i_g, $g \in \mathcal{G}_i$, $i \in \{1, \ldots, n\}$, starting at the origin 0 and passing through an extreme point $e^i_g \in Ext(\mathcal{B}_i)$ are called *fundamental directions*. Moreover, we define Γ^i_g as the *fundamental cone* generated by two consecutive fundamental directions ξ^i_g and ξ^i_{g+1}, where $\xi^i_{G_i+1} := \xi^i_1$. Clearly, $\bigcup_{g \in \mathcal{G}_\rangle} \Gamma^i_g = \mathbb{R}^2$ for every $i \in \{1, \ldots, n\}$ (see Figure 2).

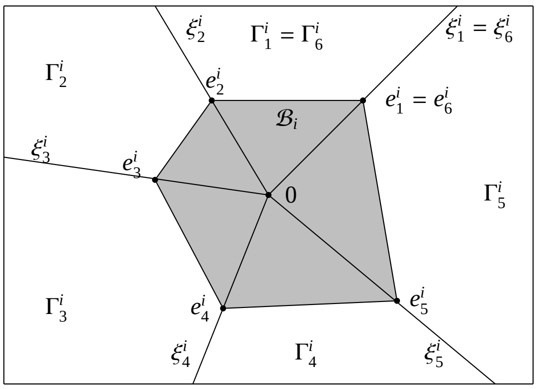

Figure 2: Fundamental directions and cones generated by the extreme points of the convex polyhedron $\mathcal{B}_i$

The *polar set* $\mathcal{B}^\circ_i$ of $\mathcal{B}_i$ is defined by $\mathcal{B}^\circ_i := \{X \in \mathbb{R}^2 \mid < X, p > \leq 1, \ \forall\, p \in \mathcal{B}_i\}$. (Here and in the following, we use the denotation $< X, p >$ for the inner product $x_1 p_1 + x_2 p_2$ in $\mathbb{R}^2$.) Its set of extreme points is denoted by $Ext(\mathcal{B}^\circ_i) = \{e^{i^\circ}_g \mid g = 1, \ldots, G_i\}$. For example, the polar set corresponding to $\mathcal{B}_{l_1}$ is $\mathcal{B}_{l_\infty}$, and vice versa.

Lemma 1. (Ward and Wendell [10]) *For all $i \in \{1, \ldots, n\}$ and $X \in \mathbb{R}^2$ the polyhedral gauge $\gamma_{\mathcal{B}_i}(X)$ can be computed by $\gamma_{\mathcal{B}_i}(X) = max\{< e^{i^\circ}_g, X > \mid e^{i^\circ}_g \in Ext(\mathcal{B}^\circ_i)\}$.*

Lemma 2. (Thisse et al. [9]) *For $i \in \{1, \ldots, n\}$ let $\mathcal{B}_i \subseteq \mathbb{R}^2$ be a polytope and $\gamma_{\mathcal{B}_i}$ its corresponding polyhedral gauge. Then $\gamma_{\mathcal{B}_i}$ is a linear function on every fundamental cone Γ^i_g, $g \in \mathcal{G}_i$.*

The ***region-wise linearity of polyhedral gauges*** is one of the reasons why the ScheLoc algorithm of this paper is efficient. The other is the fact that only polynomially many regions (sequences) need to be considered in our 1-MPVRD ScheLoc problem.

Theorem 1. *The number of nonempty ordered regions $|\ \Pi^{ord}_n := \{\pi \in \Pi_n \mid O_\pi \neq \emptyset\}\ |$ is polynomially bounded by $O(n^4 G^2)$.*

Rodríguez-Chía et al. [8] proved this result for polyhedral gauges without weights. Since the main argument in their proof is the linearity of polyhedral gauges on every fundamental cone established in Lemma 2, it can easily be extended to release dates, which are generated by polyhedral gauges $\gamma_{\mathcal{B}_i}$ using additional multiplicative (τ_i) and additive (σ_i) weights.

The preceding geometrical insights combined with linear programming yield an efficient solution algorithm for ScheLoc. This is shown in the next section.

5 Polynomial ScheLoc Algorithm

For all sequences $\pi \in \Pi^{ord}_n$ consider the following parametric linear program $LP(\pi)$:

$$\min \quad C_{\pi(n)}(X)$$

$$\text{s.t.} \quad C_{\pi(j)}(X) \geq C_{\pi(j-1)}(X) + p_{\pi(j)} \quad \forall j \in \{2, ..., n\} \tag{9}$$

$$C_i(X) \geq \sigma_i + \tau_i \ < e^{i^\circ}_g, X - a_i > + p_i \quad \forall e^{i^\circ}_g \in Ext(\mathcal{B}^\circ_i) \ \ \forall i \in \mathcal{J} \tag{10}$$

$$X \in \mathbb{R}^2 \tag{11}$$

From Lemma 1 we get $r_i(X) = \sigma_i + \tau_i\, \gamma_{\mathcal{B}_i}(X - a_i) = \sigma_i + \tau_i\ \max\{< e^{i^\circ}_g, X > \mid\ e^{i^\circ}_g \in Ext(\mathcal{B}^\circ_i)\}$. For each $\pi \in \Pi^{ord}_n$ let X^*_π be an optimal solution to $LP(\pi)$. If $X^*_\pi \in O_\pi$, then we know that π is a local optimal sequence in X^*_π. If $X^*_\pi \notin O_\pi$, then it is obvious that we can easily find another sequence $\overline{\pi} \in \Pi^{ord}_n$, by evaluating and sorting the release dates in X^*_π in increasing order, such that $C_{\overline{\pi}(n)}(X^*_\pi) \leq C_{\pi(n)}(X^*_\pi)$, which means that π is dominated by $\overline{\pi}$. Thus, for each $\pi \in \Pi^{ord}_n$ we only have to find an optimal machine location by solving the parametric linear program $LP(\pi)$ and output the globally best solution.

The complexity of this algorithm is characterized by the determination of Π^{ord}_n and the complexity of solving the corresponding linear programs $LP(\pi)$. Both can be done in polynomial time.

6 Conclusion

The class of ScheLoc problems is a new approach to scheduling with variable machine locations. In this paper we have introduced the 1-P-ScheLoc makespan problem where the release dates are depending on the distance between the (given) locations of the jobs and the (unknown) location of the machine. If this distance is given by polyhedral gauges, we showed that ScheLoc can be reduced to the solution of K linear programs, where K is a polynomial in the number of jobs and extreme points of the unit balls describing the gauge. Special cases include ScheLoc problems with respect to rectilinear or maximum distances.

In [2] we show that the ScheLoc problem introduced in this paper can be considered as a special case of a broader class. In addition to the plane tessellation and LP algorithm we also propose an alternative algorithm which is based on the computation of a finite dominating set (FDS), i.e., a finite set of candidate solutions. Numerical tests will compare the different approaches.

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