Technische Universität Kaiserslautern
Fachbereich Mathematik

Aspects of Optimal Capital Structure and Default Risk

Sarp Kaya Acar

Vom Fachbereich Mathematik
der Technischen Universität Kaiserslautern
zur Verleihung des akademischen Grades
Doktor der Naturwissenschaften
(Doctor rerum naturalium, Dr.rer.nat)
genehmigte Dissertation

1. Gutachter: Prof. Dr. Ralf Korn

2. Gutachter: Dr. habil. Jörg Wenzel

Vollzug der Promotion: 11.09.2006

To my Dear Mom and Dad...

Contents

1 Leland's Optimal Capital Structure Model **15**

1.1 Introduction 15

1.2 The Model Setup 16

1.3 PDE-Approach to Price Contingent Claims 18

1.4 Pricing Firm Derivatives 20

1.4.1 Debt Value 20

1.4.2 Total Firm Value 22

1.4.3 The Equity Value 24

1.5 Endogenous Default 26

1.6 Two Step Optimisation Problem of Total Firm Value 29

1.6.1 The First Step 30

1.6.2 An Equivalent Step to the First Step 31

1.6.3 The Second Step 33

1.7 Optimal Leverage 33

1.8 Credit Spread 34

1.9 Comparative statistics for debt and equity value value 37

2 Extended Optimal Capital Structure Models **41**

2.1 Motivation 41

2.1.1 Choosing the underlying 42

2.1.2 Relaxing Perpetuity 42

2.1.3 A Realistic Tax Regime 44

2.2 Diffusion case 46

2.2.1 Preliminaries 47

2.2.2 Pricing Firm Derivatives . . . 51
2.2.3 Endogenous Default . . . 58
2.2.4 Optimal Leverage . . . 61
2.2.5 Credit Spreads . . . 64
2.2.6 Comparative Statistics . . . 69
2.3 Jump Diffusion Case . . . 72
2.3.1 Preliminaries . . . 74
2.3.2 Pricing Firm Derivatives . . . 83
2.3.3 Endogenous Default . . . 87
2.3.4 Optimal Leverage . . . 89
2.3.5 Default Probabilities . . . 90
2.3.6 Credit Spreads . . . 93
2.4 Conclusion . . . 100

3 An extension of the Libor Market Model with Default Risk 103
3.1 Introduction . . . 103
3.2 Default model . . . 104
3.3 Bond prices and basic rates . . . 105
3.4 Dynamics of Rates under the Spot Martingale Measure . . . 107
3.4.1 The two factor model . . . 110
3.4.2 The multi factor model . . . 111
3.5 Forward and Survival Measures . . . 113
3.5.1 The Spot Martingale Measure . . . 114
3.5.2 T_k-Forward Measure . . . 115
3.5.3 T_k-Survival Measure . . . 116
3.6 Dynamics of the Rates under the Survival Measures . . . 119
3.7 Basic Credit Derivatives . . . 123
3.7.1 Credit Default Swaps . . . 125
3.7.2 Credit Default Swaption . . . 128
3.7.3 Schönbucher's CDSwaption Formula . . . 129
3.7.4 An approximation for the valuation of CDSwaptions using multiple factors and inhomogeneous volatilities . . . 131

3.7.5 Numerical Results 136
3.8 Conclusion 140

A Modigliani-Miller Results 141
A.1 Introduction 141
A.2 Propositions 144

Preface

The credit risk is the possibility that a counterparty in a financial contract will not fulfill her obligations stated in the contract. In the literature, there are two different approaches to model the credit risk: the structural approach, and the intensity based or reduced form approach.

Structural models are concerned with modeling and pricing the credit risk that is assigned to a particular firm. Default events are due to the movements of the firm's value (assets) relative to some random or non-random default triggering threshold. Consequently, the major issue within this framework is the modeling of the evolution of the firm's value and firm's capital structure by making explicit assumptions. Therefore, the structural approach is also referred to as the firm value approach. The default event is modeled as the first hitting time of the underlying firm value process to an exogenously or endogenously specified level, which is called the bankruptcy level. Then, by considering the corporate liabilities of the firm as contingent claims on the firm's value, closed form expressions for their prices are derived. In most of the firm value models, it is assumed that in a frictionless market the firm's value is independent from the financial decisions of the firm. This is postulated by Modigliani and Miller [MM58] in a well known theorem, (MM-Proposition 1, see Appendix A), which tells us that when bankruptcy costs, tax advantages, transactions costs, agency costs and other frictions in the market are omitted, the firm's value is not affected from being an all-equity, all-debt or a leveraged[1] firm. On the contrary, if one of these frictions are considered in the market, the firm's decision on how it finances its businesses will become important[2]. In this case, additionally to the firm's value another quantity of the firm is considered, namely the total firm value[3]. The total firm value is equal to the firm value plus the net effect of the frictions of the market, depending on

[1]Finances its businesses both by issuing debt and equity.

[2]The effect of the frictions on the financial decisions of a firm are discussed in details in Appendix A.

[3]In such an environment, one should make a clear difference between the **firm value** and the **total firm value** notions. The latter one is the value of the firm, after the debt is issued. This quantity is also known as **levered firm value**. Through out this thesis, we interchangeably use both names, but they refer to the same quantity. The former one, also called as the **unlevered firm value**, is the value of an all-equity firm. From now on when we write firm value, we refer to the unlevered firm value.

the financial decision of the firm. Therefore, the total firm value is dependent on how the firm's business is financed, so that the optimal capital structure of the firm becomes a relevant issue. The structural approach can be divided into two, according to how the bankruptcy level is specified.

The exogenous bankruptcy level refers to the case when the bankruptcy event is due to some protective covenant. For instance, the bankruptcy is triggered when the asset value reaches the exogenously specified principal value of the debt or an exogenously specified threshold process (which can be deterministic or stochastic). Merton [Mer74] considered a case where the default event can only occur at maturity, if the firm's assets are lower than the face value (principal value) of the debt. But recognizing that a firm may default well before the maturity of the debt, Black and Cox [BC76] alternatively assumed that the firm goes bankrupt, when the value of its assets hits for the first time some lower, time dependent threshold. Further first passage time models with exogenous default are proposed by Longstaff and Schwartz [LS95], Colin-Duffrense and Goldstein [CDG01].

The notion of the endogenous bankruptcy covers the situations, when the bankruptcy is declared by the equity holders. Therefore, the bankruptcy level is specified by some additional conditions in the model. The endogenous bankruptcy approach is widely used in the framework of the optimal capital structure, which takes into account market frictions (e.g. costs of bankruptcy, tax advantages, agency costs, etc.) and examines the optimal proportion of the debt over the total firm value, i.e. the optimal leverage. Within the framework of the optimal capital structure approach, equity holders choose the bankruptcy level in such a way that their value is maximised. The endogenous specification of the default level enables the analysis of the optimal capital structure. Brennan and Schwartz [BS78] provide the first quantitative examination of the optimal capital structure of a firm, by utilizing numerical techniques to determine the optimal leverage when the firm value follows a diffusion process with constant volatility. The problem of the optimal capital structure and its endogenous default barrier has been considered in a series of papers by Leland [Lel94], Leland [Lel95], Leland and Toft [LT96]. Other important papers of this framework are Kane, Marcus, and McDonald [KMM84], Mella-Barral and Perraudin [MBP97], Fischer, Heinkel, and Zechner [FHZ89], Goldstein, Ju, and Leland [GNL98], Chen and Kou [CK05], Hilberink and Rogers [HR02].

Structural models of the credit risk relies mostly on diffusion processes to model the evolution of the firm value. Although it is analytically tractable to work with a diffusion process, such kind of models implies that short term credit spread is equal to zero[4] due to the zero instantaneous default probability of a healthy firm under a continuous process. Zhou [Zho01] consideres a jump-diffusion process with normally distributed jump-heights and observes empirically supported credit spread shapes.

In the second class of credit risk models, namely reduced form models neither the value of the firm nor its capital structure are modeled at all, and the default events are specified in terms of some exogenously given jump processes. Typically, the default time is defined as the first jump time of a Poisson process with a random or non-random intensity. Therefore, the default time is a totally inaccessible stopping time, which means that in this group of models the default is modeled as a suprising, unexpected event. Reduced models of the credit risk incorporates non-zero short term credit spreads, due to the inaccessibility of the default time. The first model of this type was developed by Jarrow and Turnbul [JT95], Duffie and Singleton [DS99]. They considered the case, where the default event is driven by a Poisson process with constant intensity. Lando [Lan94] modeled the default due to a Poisson process with stochastic intensity by considerding the Cox-processes. Other important papers of reduced form models are, Jarrow, Lando, and Turnbul [JLT97], Madan and Unal [MÜ98], Schönbucher [Sch00].

There are also links between the structural approach and the reduced form approach, if one incorporates different information sets into the structural models and for the later Duffie and Lando [DL01] and Jarrow and Protter [JP04].

The structure of the thesis is as follows. In Chapter 1, we shall cover Leland's [Lel94] model and see how the closed form solution of the firm's contingent claims can be derived by modeling the (unlevered) firm value as a geometric Brownian motion. Moreover the

[4]Vanishing short term credit spreads is not empirically supported. In other words, even for a short maturity bond a positive credit spread is observed in the market; see Sarig and Warga [SW89].

optimal leverage of a firm shall be analysed by using a two step optimisation procedure of the total firm value. The main contribution of the thesis is introduced in Chapter 2, where we shall overcome some inconsistencies of Leland [Lel94] by using the **E**arnings **B**efore **I**nterest and **T**axes (EBIT) process as the underlying instead of the (unlevered) firm value process. By following Leland and Toft [LT96], Goldstein, Ju, and Leland [GNL98], Hibrenik and Rogers [HR02], we shall suggest two modeling dynamics of the EBIT process, namely a diffusion process and a jump-diffusion process, whose jumps are double exponentially distributed. We shall observe that the incorporation of jumps in a structural model implies the empirically supported well-known fact that short term credit spreads do not converge to zero. The effect of the jump risk on the optimal leverage ratios shall be also analysed. Chapter 3 is devoted to Schönbucher's [Sch05b] model, where the default risk is incorporated in the Libor market model by using a reduced form model. We shall state Schönbucher's closed form solution for the price of an European credit default swap call option (CDSwaption) and derive an approximative CDSwaption price in a more general framework. We compare our approximative formula with Schönbucher's one and with the Monte Carlo simulations[5].

Acknowledgements

Firstly, I would like to thank Prof. Dr. Ralf Korn, who gave me the honor to be his Phd. student and write this thesis under his supervision. During this work, his new ideas and remarks always enlightened my way and avoid me getting lost in this huge research area. A very special thanks goes to Kalina Natcheva, who was always ready to discuss my work and helped me with her brilliant ideas, whenever I was stuck. The last chapter of the thesis, is a part of a joint work with Hypo Vereinsbank. Dr. Jörg Wenzel and Dr. Martin Krekel introduced the field to me and gave the opportunity to work on it. I would like to thank also Prof. Dr. Chris Rogers and Ass. Prof. Dr. Christian Riis Flor for the useful discussions during their visits in Technische Universität Kaiserslautern and Prof. Dr. Hayri Krezlioglu during my visit in Middle East Technical University, Ankara. I would like to thank also the participants of the workshops "Recent Developments in

[5] Thanks to Dr.Wenzel for providing the Monte Carlo code for Libor market model with risk framework.

Financial and Acturial Mathematics”, ETH Zürich and ”21st European Conference on Operational Research”, University of Iceland, Reykjavik for the remarkable comments on the subject. Especially, I thank to Matthias Scherer for sharing with me the results of his thesis and the useful discussion on aspects of the jump-diffusion modeling. Not on the last place, I thank also my colleague Evren Baydar, who read the work and made very useful comments on it.

I am very grateful to my family; Ülkü, Hilmi Bener, Ömer Mert and Kalina. Their love and support always kept me on track.

Chapter 1

Leland's Optimal Capital Structure Model

1.1 Introduction

In his seminal paper Leland [Lel94] considers costs and benefits of the debt and examines their effect on the capital structure of the firm. The debt becomes beneficial because there is a tax advantage to the debt. On the other hand, the debt has a cost such that if the firm bankrupts, it pays bankruptcy costs proportional to its value. In such an environment, the amount of the issued debt is important for a firm, because it has two effects on the total value of the firm. Issuing debt increases the total firm value because of the tax advantages to the debt, and decreases the total firm value because of the possible direct and indirect bankruptcy costs[1]. By introducing these two frictions in the market, MM-Proposition 1 (see Proposition A.2.1) becomes invalid. Therefore, the firm must choose the optimal level of its debt. This optimal level is measured by the so called leverage value. It is the ratio between the debt value and the total firm value. Hence the natural question is

> *What should be the optimal leverage (i.e. optimal ratio of debt and firm value) in order to maximise the firm value?*

[1]Direct bankruptcy costs can be seen as the amount paid to the institutions, tracking the bankruptcy procedure. Indirect bankruptcy costs are the credit spreads paid to the debt holders to bear the default probability.

1.2 The Model Setup

The model is set in a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$, where P is some subjective probability measure. It is assumed that the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions[2]. In this probability space, we consider a market that consists of two assets: the money market account with a price process M and the firm value with a price process V.

The dynamics of the price process M is given by the following Ordinary Differential Equation (ODE)

$$dM(t) = M(t)r\,dt,$$

where r is the riskless interest rate, which is assumed to be constant.

The dynamics of the price process V under the measure P is given by the following the Stochastic Differential Equation (SDE)

$$dV(t) = V(t)(\mu dt + \sigma d\widetilde{W}(t)), \tag{1.1}$$

where μ is the drift and σ is the diffusion coefficient of the firm value, which are also assumed to be constant. $\widetilde{W}(t)$ is a one dimensional P-Brownian motion.

Our market is assumed to be perfect in the sense that assets are completely divisible and trading takes place in continuous time, individuals are small traders i.e. they take the prices in the market, assets can be sold long and short, borrowing and lending rates are identical[3]. Furthermore, the market is assumed to be complete and arbitrage free. Therefore, there exists[4] an unique equivalent measure Q, i.e., the price processes of the traded assets discounted by the money market account M are martingales under the measure Q.

The dynamics of the firm value under the measure Q is given as follows

$$dV(t) = V(t)(rdt + \sigma dW^Q(t)). \tag{1.2}$$

[2]See Jacod and Shiryaev [JS00].

[3]This rate is constant, r.

[4]For the respective proofs of the connection between arbitrage-free markets and the existence of a martingale measure, we refer to Harrison and Kreps [HK79] and Harrison and Pilska [HP] and for a detailed treatment in continuous set-up, we refer to Björk [Bjö98].

From now on we work only under the measure Q, therefore in our notations we drop the superscript Q in the Brownian motion. Note that the equation (1.2) is satisfied if and only if the firm value is a traded asset. But this assumption may lead to arbitrage opportunities. We will come to this point later on. From now on we will work only under equivalent martingale measures, therefore we drop the superscript Q in our notations.

The value of corporate securities are dependent on the underlying firm value but they are time independent. This environment can be fulfilled by rolling over the debt. For instance, today one invests on a bond which has a 10 year maturity and tomorrow she invests on a bond with the same features as the previously invested one, thus the maturity of the bond remains 10 years. Although this assumption is far from being realistic, one can consider console bonds issued by governments or sinking fund provisions[5] with an almost zero retirement rate. The perpetuity assumption of securities will be relaxed in the next chapter, when we introduce the extended optimal capital structure models.

There are two agency groups in the model; equity holders and debt holders. Further, any conflicts within equity holders and debt holders are omitted.

Once debt is issued by a firm, its face value remains static through the time, which means debt calling or renegotiations are not included in the model. See Goldstein, Ju and Leland [GNL98] and Christensen *et al.* [CFLM01] for dynamic capital structure models. The debt holders receive perpetual, continuous coupon rate C and the firm takes back the τ_c portion of the coupon payments as tax advantages to the debt.

The debt is subject to default and the default event is trigged, when V hits an endogenously determined bankruptcy level. Let V_B denote the bankruptcy level and τ_B denote the default time. Then, the default time can be mathematically represented as follows

$$\tau_B = \inf \{ t : V(t) = V_B \}.$$

The endogenous nature of the bankruptcy level is incorporated in the model as follows. It is determined optimally by the equity holders, who have limited liability. In other words,

[5]Sinking fund provisions are quite common in corporate debt issues; see Smith and Wagner [SW79]. The structure of these securities are the retirement of the debt principal on a regular basis. The perpetual environment corresponds to a very slowly retired sinking fund provision.

equity holders declare bankruptcy, when the equity value drops to zero. They would not declare bankruptcy, as long as they could dilute their equity value and service the debt payments. Such kind of a strategy[6] is intuitive for equity holders, since at the default event the debt holders pay the α portion of the firm value as bankruptcy costs and take over the firm. As a consequence, at the default event equity holders lose all their rights on the firm.

In the following two sections, we will deal with the question on how to price CCs of a firm. We assume that the firm's CCs underly on the state variable V.

1.3 PDE-Approach to Price Contingent Claims

In the following theorem, we will state a general partial differential equation (PDE) in order to price contingent claims of a firm. The derivation of the PDE can be also found in Merton [Mer74].

Theorem 1.3.1. *For any Contingent Claim (CC), with a price process denoted by $F(V,t)$ and a continuous payout rate denoted by $h(V,t)$. If $F \in \mathbb{C}^2$ with respect to V and $F \in \mathbb{C}$ with respect to t, then $F(V,t)$ satisfies the following PDE;*

$$\frac{1}{2}\sigma^2 V^2 \frac{\partial^2 F(V,t)}{\partial V^2} + rV\frac{\partial F(V,t)}{\partial V} - rF(V,t) + \frac{\partial F(V,t)}{\partial t} + h(V,t) = 0. \tag{1.3}$$

Proof. If we apply the Itô formula[7];

$$\begin{aligned} dF(V,t) &= \frac{\partial F}{\partial t}\,dt + \frac{\partial F}{\partial V}\,dV_t + \frac{1}{2}\,\frac{\partial^2 F}{\partial V^2}\,d\langle V\rangle_t \\ &= F_t\,dt + F_V\,(rV - d(V,t))\,dt + F_V V\sigma dW_t^Q + \frac{1}{2}\,F_{VV}V^2\sigma^2 dt \\ &= \underbrace{(F_t + (rV - d(V,t))\,F_V + \frac{1}{2}\,\sigma^2 V^2 F_{VV})}_{=:\mu_F} dt + \sigma V F_V dW_t^Q. \end{aligned} \tag{1.4}$$

Since the market is arbitrage free

$$\mu_F = r\,F - h(V,t),$$

[6]Something is better than nothing.
[7]Subscripts denote the partial derivatives.

which means the drift in (1.4) must be equal to the risk free rate corrected for the payout rate $h(V,t)$. Hence we get the PDE (1.3). $\square$

Remark 1.3.1. *We can also derive the PDE* (1.3) *by replicating the CC (since the market is complete) in the market. Namely, we can form a riskless portfolio by buying ϕ_t^1 amount of the underlying, ϕ_t^2 amount of the money market account and selling the corporate security short. Let us define $Y(t)$ as follows*

$$Y(t) := \phi_t^1 V(t) + \phi_t^2 B(t) - F(V,t),$$

where B_t is the money market account (i.e. $dB_t = rB_t dt$), ϕ_t^1 and ϕ_t^2 are self financing admissible trading strategies. By applying again Itô formula tools we can get the PDE (1.3)*.*

We already mentioned that corporate securities in which we are interested, are time independent. Thus PDE (1.3) becomes an ordinary differential equation (ODE)

$$\frac{1}{2}\sigma^2 V^2 \frac{\partial^2 F(V)}{\partial V^2} + rV\frac{\partial F(V)}{\partial V} - rF(V) + h(V) = 0. \tag{1.5}$$

This ODE is the Euler's equation and its well known solution is given by the next Corollary.

Corollary 1.3.2. *The general solution of ODE* (1.5) *is given by*

$$F_{GS}(V) = A_1 V + A_2 V^{-x}, \tag{1.6}$$

where $x = 2r/\sigma^2$. A_1, A_2 are to be specified by the boundary conditions. A particular solution F_{PS} depends on the payout rate $h(V)$. Therefore, $F(V)$ satisfies

$$F(V) = F_{PS} + A_1 V + A_2 V^{-x}. \tag{1.7}$$

Proof. To find the general solution, we substitute V^x in (1.5)

$$\begin{aligned}
&\frac{1}{2}\sigma^2 V^2 x(x-1)V^{x-2} + rVxV^{x-1} - rV^x = 0\\
&\frac{1}{2}\sigma^2 x(x-1) + rx - r = 0\\
&\frac{1}{2}\sigma^2 x^2 + (r - \frac{1}{2}\sigma^2)x - r = 0.
\end{aligned}$$

The roots of the characteristic polynomial is $x_1 = 1$, $x_2 = -2r/\sigma^2$. Hence, the general solution takes the form

$$F_{GS}(V) = A_1 V + A_2 V^{-x},$$

where $x = 2r/\sigma^2$. □

Notice that any time independent contingent claim (with an equity financed payout) can be priced by using Corollary 1.3.2.

1.4 Pricing Firm Derivatives

In this section, we consider the contingent claims such as equity, corporate debt, tax benefits, bankruptcy costs and present value of 1 unit of money contingent on the future bankruptcy and price them using the general PDE-approach, explained in Section 1.3.

1.4.1 Debt Value

As we already mentioned, the firm issues debt in order to finance its businesses. It is assumed that once debt is issued its total principal value stays constant, i.e., debt calling and renegotiations are excluded. Moreover, the firm issues only console bond, i.e, the maturity of the bond is infinite[8]. In the literature, this type of debt is also called the perpetual debt. The holders of the perpetual debt receive perpetual coupon C. When there is bankruptcy, debt holders take over the firm but they have to pay a portion α of firm value V as bankruptcy costs. Hence, when $V = V_B$, they receive $(1-\alpha)V_B$. This threshold is not chosen by debt holders[9]. Therefore, the debt value, $D(V)$ is a time independent firm's derivative underlying on the firm value. We can price it by using the ODE (1.5), which has a solution given by Corollary 1.3.2.

When $V \to \infty$, there can not be bankruptcy, therefore the debt value attains a limit

[8]It is a very strict assumption. We will relax it in Chapter 2.
[9]We shall explore, how it is chosen in Section 1.5.

which can be written as

$$\lim_{V\to\infty} D(V) = \int_0^\infty C\,e^{-rt}\,dt \;=\; \frac{C}{r}.$$

Hence, we have the following boundary conditions

$$\begin{array}{lllll} V = V_B & \Longrightarrow & D(V) = (1-\alpha)V_B & \Longrightarrow & A_2 = \left[(1-\alpha)V_B - \frac{C}{r}\right] V_B^x, \\ V \to \infty & \Longrightarrow & D(V) \to \frac{C}{r} & \Longrightarrow & A_1 = 0,\;\; F_{PS} = \frac{C}{r}. \end{array}$$

If we substitute A_1, A_2 and F_{PS} into equation (1.7), we get

$$D(V, V_B, C) = \frac{C}{r}\left(1 - \left(\frac{V}{V_B}\right)^{-x}\right) + (1-\alpha)\,V_B\left(\frac{V}{V_B}\right)^{-x}. \tag{1.8}$$

The first term is the total coupon payment, the second term is the present value of the firm, contingent on the future bankruptcy, after bankruptcy costs and coupons until the bankruptcy time are paid. The following definition and theorem state the interpretation of $(V/V_B)^{-x}$ term in the above equation.

Definition 1.4.1. *Let $p_B(V)$ denote the present value of one unit of money contingent on the future bankruptcy. The holder of this claim receives one unit of money when the equity value drops to V_B. It can also be considered as the limiting hitting probability of the process V to V_B.*

Theorem 1.4.1. *The present value of one unit of money contingent on the future bankruptcy is equal to*

$$p_B(V) = \left(\frac{V}{V_B}\right)^{-x}.$$

Proof. This claim is time independent, thus it satisfies ODE (1.5) and the functional (1.7)

$$p_B(V) = \underbrace{p_{B_{PS}}(V)}_{=0} + A_1 V + A_2 V^{-x}. \tag{1.9}$$

Boundary conditions are

$$\begin{array}{lllll} V \to \infty & \Longrightarrow & p_B(V) \to 0 & \Longrightarrow & A_1 = 0, \\ V = V_B & \Longrightarrow & p_B(V_B) = 1 & \Longrightarrow & A_2 = V_B^x. \end{array}$$

When we substitute A_1 and A_2 into equation (1.9), we get

$$p_B(V) = \left(\frac{V}{V_B}\right)^{-x}.$$

□

1.4.2 Total Firm Value

As we mentioned before, the firm value is affected by the debt issuance in two ways; it increases because of the tax advantages and decreases because of the bankruptcy costs. According to the MM-Proposition 3 (see Appendix A), the total firm value, denoted by $\nu(V)$ is the asset value plus the net effect of the debt issuance; namely the difference between the tax benefits and bankruptcy costs

$$\nu(V) = V + TB(V) - BC(V). \tag{1.10}$$

Therefore in order to find the levered firm value, one has to price these two effects.

1. **Bankruptcy costs, $BC(V)$:** Bankruptcy costs can be considered as a derivative of the firm. The holder of this claim tracks the bankruptcy procedure, when the default happens. They get the α amount of the firm value as bankruptcy costs. In other words, they hold a claim which pays off, when the bankruptcy occurs. This claim is time independent and underlies on the firm value. Therefore, it satisfies ODE (1.5), whose solution is given by the functional (1.7)

$$BC(V) = BC_{PS}(V) + A_1 V + A_2 V^{-x}. \tag{1.11}$$

Boundary conditions are

$$\begin{array}{lllll} V \to \infty & \Longrightarrow & BC(V) \to 0 & \Longrightarrow & BC_{PS} = 0,\ \ A_1 = 0, \\ V = V_B & \Longrightarrow & BC(V) = \alpha V_B & \Longrightarrow & A_2 = \alpha V_B^{x+1}. \end{array}$$

When we substitute A_1 and A_2 into (1.11), we get

$$BC(V, V_B) = \alpha\, V_B \left(\frac{V}{V_B}\right)^{-x}. \tag{1.12}$$

Bankruptcy costs are the present value of α portion of the firm value at the default.

2. **Tax benefits, $TB(V)$:** Tax benefits can also be considered as a derivative of the firm. When the firm issues debts, it automatically holds this CC, paying a specific constant amount of the coupon, $\tau_c C$, back to the firm as long as the firm is solvent. This claim is again time independent and satisfies ODE (1.5) and the functional (1.7)
$$TB(V) = TB_{PS}(V) \, + \, A_1 V \, + \, A_2 V^{-x}. \tag{1.13}$$
Boundary conditions are
$$\begin{aligned} V \to \infty \quad &\Longrightarrow \quad A_1 = 0, \; TB_{PS}(V) = \tfrac{\tau_c C}{r}, \\ V = V_B \quad &\Longrightarrow \qquad\qquad A_2 = \tfrac{-\tau_c C}{r} V_B^x. \end{aligned}$$
When we substitute A_1 and A_2 into (1.13), we get
$$TB(V, V_B, C) = \frac{\tau_c C}{r} \left(1 - \left(\frac{V}{V_B} \right)^{-x} \right). \tag{1.14}$$
The first term in equation (1.14) is the total coupon payment and the second term is the present value of the loss in case of the default.

   Leland models the tax benefits as a cash inflow of a firm. But in reality one of the biggest cash outflow of a firm are the taxes paid to the government. It would be more reasonable to model the tax advantages as a reduction on taxes paid to the government. In the model, we observe that an increase in the corporate tax rate τ_c yields also an increase in the total firm value, which can be seen as a draw back of the model. In the next section by following Goldstein, *et al.* [GNL98], we will model the tax advantage as a reduction in the total tax payment to the government and overcome this inconsistency.

The levered firm value can be obtained by substituting the equations (1.12) and (1.14) into the equation (1.10)
$$\nu(V, V_B, C) \, = \, V \, + \, \frac{\tau_c \, C}{r} \left(1 - \left(\frac{V}{V_B} \right)^{-x} \right) - \alpha \, V_B \left(\frac{V}{V_B} \right)^{-x}. \tag{1.15}$$
Note that the free parameters to choose for maximizing $\nu(V)$ are C and V_B. $v(V)$ would be maximized, holding C fixed, if we set V_B as low as possible. But V_B is chosen by

the equity holders, taking into account the limited liability assumption. This assumption forbids equity holders to set V_B arbitrarily small, because for very small values of V_B, equity value can be zero or even negative. In Section 1.5, we will see how small V_B can be chosen. Maximising the total firm value is a two step optimisation problem. Firstly, the optimal default level is chosen for an arbitrary coupon level, then the total firm value is maximised by manipulating the coupon level. The details of this procedure will be covered in section 1.6.

Remark 1.4.1. *Let us recall that firm value $V(t)$ is assumed to be a traded asset. To trade the firm value in the market is not only impossible but also (if it were possible) leads to arbitrage opportunities. If the levered firm value and the firm value coexist in the market and the firm value V is smaller than the levered one, i.e. $V(t) < \nu(V(t))$, then one can buy the assets of the firm for $V(t)$, lever the firm and sell it for $\nu(V(t))$ and vice versa if $\nu(V(t)) < V(t)$. Therefore, tax benefits must compensate bankruptcy costs in order to avoid arbitrage opportunities, which means there is no need to have debt in this setup.*

In the view of Remark 1.4.1, we shall assume that the firm value is a non-traded in Chapter 2. Moreover, we shall use another firm's quantity as the underlying asset, namely EBIT which is closely related to the firm value, hence also non-traded.

1.4.3 The Equity Value

Our next aim is to analyze the equity value[10], $EQ(V, V_B, C)$. It is the total firm value minus the debt value

$$\begin{aligned} EQ(V, V_B, C) &= \nu(V, V_B, C) - D(V, V_B, C) \\ &= \left(V - (1-\tau_c)\frac{C}{r}\right) + \left((1-\tau_c)\frac{C}{r} - V_B\right)\left(\frac{V}{V_B}\right)^{-x}. \end{aligned} \tag{1.16}$$

We will show that the representation (1.16) holds. The equity value can be seen as a CC, whose owner pays C amount of coupon and receives $\tau_C C$ amount of tax advantage.

[10]According to our purposes, we use the notations $EQ(V, V_B, C)$, $EQ(V, V_B)$, $EQ(V)$ interchangeably.

Therefore, the pricing PDE for the equity value, $EQ(V)$ can be written as follows

$$\frac{1}{2}\sigma^2 V^2 \frac{\partial^2 EQ(V)}{\partial V^2} + rV\frac{\partial EQ(V)}{\partial V} - rEQ(V) - (1-\tau_c)C = 0. \tag{1.17}$$

We already know that the solution of ODE (1.5) is given by the functional (1.7)

$$EQ(V) = EQ_{PS}(V) \,+\, A_1\, V \,+\, A_2\, V^{-x}, \tag{1.18}$$

where the coefficients can be found by employing the following boundary conditions

$$V \to \infty \implies EQ(V) = V - (1-\tau_c)\tfrac{C}{r} \implies A_1 = 1,\ EQ_{PS}(V) = -(1-\tau_c)\tfrac{C}{r},$$

$$V = V_B \implies EQ(V_B) = 0 \implies A_2 = \left[(1-\alpha)\, V_B \,-\, \tfrac{C}{r}\right] V_B^x.$$

If we substitute the above values into the equation (1.18), then we get the equation (1.16)

$$EQ(V, V_B) = \left(V - (1-\tau_c)\frac{C}{r}\right) + \left((1-\tau_c)\frac{C}{r} - V_B\right)\left(\frac{V}{V_B}\right)^{-x}.$$

The first term is the value of the unlevered firm less the net payments due to the debt issued. This term attains negative values for $V < (1-\tau_c)\frac{C}{r}$ but the debt could be still serviced. However, limited liability of equity forces equity holders to declare bankruptcy, when $EQ(V) < 0$ [11]. That is why the second term comes into play. It is like an option for equity holders to declare bankruptcy (or not). When $V = V_B$, $EQ(V_B) = 0$ which implies (from the limited liability restriction for equity holders) there is bankruptcy. So the option embedded in equity is the option to declare bankruptcy and avoid paying the debt payments C, whenever it is too costly. Since it is an option to be exercised by the firm, it must have a positive value, so that $V_B < (1-\tau_c)\frac{C}{r}$.

Proposition 1.4.2. *For $V_B < (1-\tau_c)\frac{C}{r}$ equity is a strictly convex function of V.*

Proof. Let us

$$\frac{\partial^2 EQ(V)}{\partial V^2} = \left((1-\tau_c)\frac{C}{r} - V_B\right) x(x+1)\left(\frac{V_B}{V}\right)^{x+2}\frac{1}{V_B^2}.$$

Clearly, for $V_B < (1-\tau_c)\frac{C}{r}$, we get $\frac{\partial^2 E(V)}{\partial V^2} > 0$, which proves the convexity and also justifies the option nature of equity. □

[11] In continuous time, the coupon Cdt paid over the infinitesimal, dt, is itself infinitesimal. Therefore the value of equity is simply needs to be positive. In discrete time, value of equity at each period must exceed the coupon Cdt to be paid that period.

1.5 Endogenous Default

For any choice of V_B, the equation (1.16) is equal to 0 at $V = V_B$. But the bankruptcy level V_B is still unknown. In the literature, this problem is known as a free boundary problem[12]. We need a third boundary condition for ODE (1.17) to specify the free boundary. This problem was first solved by Paul Samuelson [Sam60], by introducing the so called smooth pasting condition

$$\frac{\partial EQ(V, V_B)}{\partial V}\Big|_{V=V_B^*} = 0, \tag{1.19}$$

which claims that at the optimal default level V_B^* not only the equity value is zero, but also the partial derivative of equity with respect to V at V_B^* is equal to zero. By applying smooth pasting condition into the equity value, given by equation (1.16), we obtain the optimal bankruptcy level as follows

$$V_B^* = (1 - \tau_c)\,\frac{C}{r}\,\frac{x}{x+1}, \tag{1.20}$$

where $x = 2\,r/\sigma^2$.

Remark 1.5.1.

1. *Notice that V_B^* given in equation* (1.20) *satisfies the postulate of the Proposition 1.4.2.*

2. *V_B^* is independent of α. It increases as C increases. It decreases as τ_c, r, σ increases.*

Next, we will state a theorem on the optimality of V_B^* and prove it in two different ways. For the first part of the proof, we need to adapt the ideas of Dixit and Pindyck [DP93] and the second part is proved with similar arguments as in Chen and Kou [CK05].

Theorem 1.5.1. *"Smooth Pasting Condition"*
The bankruptcy level, V_B^*, given by equation (1.20) *is the optimal one for the equity holders.*

[12] Notice that there is an analogy to the pricing of American options, where one has to determine the continuation region separated from the stopping region by the free boundary, see Wilmott [Wil06]. More information about the free boundary problem and a general treatment of it can be found in Crank [Cra87].

Proof. 1^{st} *way:* By approximating the underlying firm value $V(t)$, by a discrete random walk we will show that stopping at V_B^* is optimal.

Let us recall the discrete random walk as an approximation to the firm value. From Cox, Ross, Rubinstein [CRR79] setup;

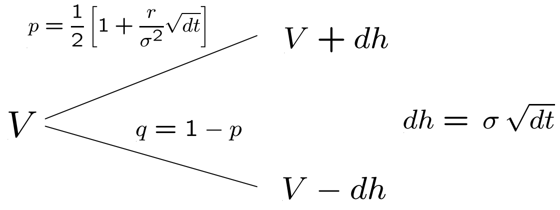


Figure 1.1: Discrete approximation of the firm value in the CRR [CRR79] setup.

To prove the theorem, we use a contradiction argument on the violation of the smooth pasting condition under the assumption of the optimality of stopping at $V = V_B^*$. Then, there are two cases;

1. Case $EQ_V(V_B^*) > 0$: Let us suppose at time t default happens, i.e., $V = V_B^*$. Then our strategy is as follows. We do not stop the process right away, but wait for the next infinitesimal time interval dt. Let us denote the value of V at time $t + dt$ as V_B^{*+}. If $V_B^{*+} = V_B^* + dh$, where $dh = \sigma\, dt$, we continue. If $V_B^{*+} = V_B^* - dh$, we stop.

   By following this strategy and using the discrete approximation to the firm value, explained in Figure 1.5, we obtain

$$\begin{aligned}
\mathbb{E}_Q\left(EQ\left(V_B^{*+}\right)\middle|\mathcal{F}_t\right) &= \left(\frac{1}{2}\left(1 + \frac{r}{\sigma^2}\sqrt{dt}\right)\, EQ(V_B^* + dh) \,+\, q.0\right) \\
&= \left(\frac{1}{2}\left(1 + \frac{r}{\sigma^2}\sqrt{dt}\right)\left(EQ(V_B^*) + EQ_V(V_B^*)\sqrt{dt} + o(\sqrt{dt})\right)\right) \\
&= \underbrace{\frac{1}{2}\, EQ(V_B^*)}_{=0} + \frac{1}{2}\underbrace{EQ_V(V_B^*)}_{>0}\sqrt{dt} + \frac{r}{\sigma^2}\underbrace{EQ(V_B^*)}_{=0}\sqrt{dt} + o(\sqrt{dt}) > 0.
\end{aligned}$$

   Under our assumption $EQ_V\left(V_B^*\right) > 0$, this alternative strategy offers a better gain than supposedly optimal strategy of stopping at $V = V_B^*$, which is a contradiction to the assumption that V_B^* is the optimal stopping default boundary, thus $EQ_V(V_B^*)$ can not be bigger than 0.

2. Case $EQ_V(V_B^*) < 0$: If $EQ_V(V_B^*) < 0$, then for $V > V_B^*$ stopping would be optimal, which is a contradiction.

Therefore, the optimal bankruptcy level is the one which satisfies the smooth pasting condition (1.19).

2^{nd} *way:* We will show it in several steps.

1. Step: The optimal default barrier, denoted by V_B^o, must satisfy $V_B^o \geq V_B^*$.

   From the limited liability assumption of equity holders, we have $EQ(V, V_B^o) > 0$ $\forall\, V > V_B^o$. Define $l = V/V_B$. Then we obtain
   $$EQ(V, V_B^o) = \frac{V_B^o}{l} - (1-\tau_c)\frac{C}{r} + \left((1-\tau_c)\frac{C}{r} - V_B^o\right) l^{-x} \geq 0.$$
   Hence $\forall\, V > V_B^o$, the following inequality is satisfied,
   $$V_B^o \left(\frac{1}{l} - l^{-x}\right) \geq \frac{(1-\tau_c)\, C}{r} - \frac{(1-\tau_c)\, C}{r} l^{-x}.$$
   In particular,
   $$V_B^o \geq \lim_{l \downarrow 1} \frac{\frac{(1-\tau_c)\, C}{r} - \frac{(1-\tau_c)\, C}{r} l^{-x}}{\left(\frac{1}{l} - l^{-x}\right)}.$$
   From l'Hospital's rule,
   $$V_B^o \geq (1-\tau_c)\, \frac{C}{r}\, \frac{x}{x+1}$$

2. Step: $\partial EQ(V, V_B)/\partial V_B < 0, \quad \forall\, V \geq V_B$.

   $$\frac{\partial EQ(V, V_B)}{\partial V_B} = \left(\frac{V_B}{V}\right)^x \left(\frac{(1-\tau_c)Cx}{V_B r} - (1+x)\right).$$
   Note that the second term on the right hand side is smaller than or equal to zero, hence we get
   $$\frac{\partial EQ(V, V_B)}{\partial V_B} < 0, \quad \forall\, V \geq V_B.$$
   Note that step 2 implies that for $V_B^* < V_B^1 < V_B^2 < V$, we have $EQ(V, V_B^1) > EQ(V, V_B^2)$.

3. Step: $\partial EQ(V,V_B)/\partial V \geq 0, \quad \forall\, V \geq V_B$.

$$\frac{\partial EQ(V,V_B)}{\partial V} = 1 + \left(\frac{V_B}{V}\right)^{x+1}\left(\frac{(1-\tau_c)Cx}{V_B r} - x\right).$$

Note that second term on the right hand side is positive since $(1-\tau_c)C/r \geq V_B$, hence we get

$$\frac{\partial EQ(V,V_B)}{\partial V} \geq 0, \quad \forall\, V \geq V_B.$$

By using the above three steps one can show that V_B^* is the optimal solution as follows. First, V_B^* satisfies $EQ(V,V_B^*) \geq 0$, $\forall V \geq V_B^*$, since $EQ(V_B^*,V_B^*) = 0$ and EQ is non-decreasing in V from third step. Second any $V_B \in (V_B^*, V]$ can not deliver better value for equity holders from second step and any V_B smaller than V_B^* is excluded by first step. □

Now, we are able to find the optimal values of the equity, debt and total firm for any coupon level C, we substitute the equation (1.27) into the equations (1.16), (1.15), (1.8).

$$EQ(V,V_B^*,C) = V - (1-\tau_c)\frac{C}{r}\left[1 - \left(\frac{C}{V}\right)^x A_1\right], \tag{1.21}$$

$$D(V,V_B^*,C) = \frac{C}{r}\left[1 - \left(\frac{C}{V}\right)^x A_2\right], \tag{1.22}$$

$$\nu(V,V_B^*,C) = V + \frac{\tau_c C}{r}\left[1 - \left(\frac{C}{V}\right)^x A_3\right], \tag{1.23}$$

where

$$\begin{aligned} A_1 &= \left((1-\tau_c)\frac{x}{r(x+1)}\right)^x \frac{1}{1+x}, \\ A_2 &= A_1\Big(1 + x - (1-\alpha)(1-\tau_c)x\Big), \\ A_3 &= A_1\left(1 + x + \frac{\alpha(1-\tau_c)x}{\tau_c}\right). \end{aligned}$$

1.6 Two Step Optimisation Problem of Total Firm Value

As we already mentioned, the two free parameters to maximise the total firm value are V_B and C. Therefore, the general maximisation problem is

$$\max_{V_B,C}\ \nu(V,V_B,C). \tag{1.24}$$

Leland [Lel94], and Leland *et al.* [LT96] did not solve the above maximisation problem, but they used a two step optimisation procedure to find the maximum total firm value, which can be described as follows. Firstly, the optimal default level is chosen by the equity holders for any coupon level C and in the next step the total firm value is maximised for C. However, this approach might deliver a lower maximum total firm value then maximising it simultaneously with respect to V_B and C, i.e. the maximisation problem given by (1.24). Leland [Lel98] used the difference between these two approaches as an explanation for the agency costs. The next theorem states some properties of $\nu(V, V_B, C)$.

Theorem 1.6.1.

a) For fixed C, $\nu(V, V_B, C)$ is a decreasing function of V_B.

b) For the optimal default level, given by equation (1.20)*, $\nu(V, V_B^*, C)$ is a concave function with respect to C.*

Proof. It is trivial to prove for part a) that $\partial\nu(V, V_B)/\partial V_B < 0$ and for part b) that $\partial^2\nu(V, V_B)/\partial C^2 < 0$. □

In the following subsections, we shall describe the steps of the optimisation procedure. The crucial result is, the optimal default level, chosen by equity holders and given by equation (1.20) also maximises the total firm value for any coupon level C.

1.6.1 The First Step

The first step is to maximise the total firm value with respect to the default boundary V_B for a given coupon level C. From Theorem 1.6.1 a), the total firm value is a decreasing function of V_B, for fixed C. Therefore, the total firm value would be maximised by setting V_B as low as possible. The optimal default level is determined by the equity holders taking into account their limited liability, therefore the lowest possible value of V_B is given by equation (1.20). Hence, the first step maximisation problem can be stated as follows

$$\begin{aligned} &\max_{V_B} \ \nu(V, V_B, C) \\ &s.t. \qquad EQ(V, V_B) \geqslant 0, \qquad \forall\, V \geqslant V_B. \end{aligned} \tag{1.25}$$

The solution to maximisation problem (1.25), is given by the equation (1.23).

1.6.2 An Equivalent Step to the First Step

Instead of problem (1.25), if we consider the maximisation problem of the equity value

$$\max_{V_B} \; EQ(V, V_B). \tag{1.26}$$

The solution to the problem (1.26) can be found by checking the first and second order conditions.

First order condition:

$$\frac{\partial EQ(V,V_B)}{\partial V_B} = -\left(\frac{V}{V_B}\right)^{-x} + \left((1-\tau_c)\frac{C}{r} - V_B\right) x \left(\frac{V_B}{V}\right)^{x-1} \frac{1}{V} = 0$$

$$\Longrightarrow \quad V_B^* = (1-\tau_c)\frac{C}{r}\frac{x}{x+1}. \tag{1.27}$$

Second order condition: One can show that

$$\frac{\partial^2 EQ(V,V_B)}{\partial V_B^2} < 0,$$

which implies that $EQ(V, V_B)$ has a local maximum[13] at V_B^*, given by the equation (1.27).

Therefore, $V_B^* = (1-\tau_c)\frac{C}{r}\frac{x}{x+1}$ is an optimal choice for equity holders to maximise their equity value, but we have already seen that this optimal default level also maximises the total firm value by taking into account the so called limited liability constraint. The equivalence of these two problems is not a coincidence.

Equivalence of the two Problems

Equations (1.20) and (1.27) are equal, which implies the problem (1.25) and (1.26) delivers the same total firm value. What does this mean economically? We refer to Leland [Lel94],

[13] It is also the global maximum, since $\partial EQ(V,V_B)/\partial V_B < 0$, $\forall\, V_B < V_B^*$ and $\partial EQ(V,V_B)/\partial V_B > 0$, $\forall\, V_B > V_B^*$

footnote 20; "The equivalence of two conditions brings us to an important point, which is the notion of an incentive compatible contract. In an incentive compatible contract, the firm declares bankruptcy, when there is really no other way out. Before the debt issuance, equity holders wish to maximise the firm value subject to the limited liability of the equity. They achieve this by choosing the optimal default level V_B^*, satisfying the smooth pasting condition. After the debt is issued, equity holders will have no incentive to declare bankruptcy at a different V, since V_B also satisfies the ex-post optimal condition (first order condition) for maximising the equity value". Next theorem gives us the mathematical connection between the smooth pasting condition and the first order condition.

Theorem 1.6.2. *Let $f(x,y)$ be a differentiable function, concave in its second argument, for $0 \le x \le y$. Let y^* be the y, which maximises $f(x,y)$ in y. Assume that there exists a differentiable function $g(y)$ such that for $x = y$, one has $f(y,y) = g(y)$. Then the smooth pasting condition*

$$\frac{\partial f(x,y)}{\partial x}\Big|_{x=y,y=y^*} = \frac{dg(y)}{dy}\Big|_{y=y^*}$$

is satisfied.

Proof. The prerequisites of the theorem are

$$\frac{\partial^2 f(x,y)}{\partial y^2} \le 0, \quad 0 \le x \le y \quad \text{and} \quad y^* = \arg\max_y f(x,y),$$

which imply that the first order condition with respect to y is satisfied at $y = y^*$

$$\frac{\partial f(x,y)}{\partial y}\Big|_{y=y^*} = 0.$$

In particular, for $x = y$

$$\frac{\partial f(x,y)}{\partial y}\Big|_{x=y,y=y^*} = 0$$

is also satisfied. Let us consider the total derivative of $f(x,y)$ with respect to y along the boundary $x = y$,

$$\frac{df(y,y)}{dy} = \frac{dg(y)}{dy} = \frac{\partial f(x,y)}{\partial x}\frac{dx}{dy}\Big|_{x=y} + \frac{\partial f(x,y)}{\partial y}\Big|_{x=y}.$$

For $y = y^*$,

$$\frac{dg(y)}{dy}\Big|_{y=y^*} = \frac{\partial f(x,y)}{\partial x}\Big|_{x=y,y=y^*} + \underbrace{\frac{\partial f(x,y)}{\partial y}\Big|_{x=y,y=y^*}}_{=0}$$

$$\Longrightarrow \quad \frac{\partial f(x,y)}{\partial x}\Big|_{x=y,y=y^*} = \frac{dg(y)}{dy}\Big|_{y=y^*}.$$

□

Remark 1.6.1. *In our case, the equity value $EQ(V, V_B)$ corresponds to the function $f(x, y)$. Note that $EQ(V_B, V_B) = 0$, corresponding to the function $g(y)$. Hence, the smooth pasting condition is given as follows*

$$\frac{\partial f(x,y)}{\partial x}\Big|_{x=y,y=y^*} = \frac{dg(y)}{dy}\Big|_{y=y^*} \quad \Longrightarrow \quad \frac{\partial EQ(V,V_B)}{\partial V}\Big|_{V=V_B} = 0.$$

1.6.3 The Second Step

From Theorem 1.6.1 b), the total firm value is concave with respect to C. Then, the second step optimisation problem can be stated as

$$C^* = \arg\max_C \ \nu(V, V_B^*, C) \tag{1.28}$$

$$\Rightarrow C^* = V\left(\frac{1}{(1+x)A_3}\right)^{1/x}.$$

Substituting C^*, into equation (1.23), we obtain the optimal total firm value

$$\nu(V, V_B^*, C^*) \quad = \quad V\left(1 + \left(\frac{\tau_c}{r}\right)\left(\frac{1}{(1+x)A_3}\right)^{1/x}\left(\frac{x}{(1+x)}\right)\right). \tag{1.29}$$

Note that for the optimal coupon rate C^*, the optimal bankruptcy level becomes

$$V_B^* = (1-\tau_c)\,\frac{C^*}{r}\,\frac{x}{x+1}.$$

1.7 Optimal Leverage

By substituting C^* in equation (1.22), we find the optimal debt value

$$D(V, V_B^*, C^*) = V\left(\frac{1}{(1+x)A_3}\right)^{1/x}\left(1 - A_2\frac{1}{((1+x)A_3)}\right)\,\frac{1}{r}. \tag{1.30}$$

Hence, the optimal leverage is given by

$$L(V, V_B^*, C^*) = \frac{D(V, V_B^*, C^*)}{\nu(V, V_B^*, C^*)}.$$

Next, we plot the total firm value with respect to the leverage for different corporate tax rates τ_c and observe an inconsistency of the model. For bigger levels of corporate tax τ_c, the total firm value is bigger, which is very unrealistic. The problem with the model is that tax benefits are modeled as an inflow of funds, rather than a reduction of outflow of funds. In reality, firms pay taxes to the government and tax benefits are the reductions from what they pay. In the next chapter, we will see how one can overcome this problem by using another underlying rather than the unlevered firm value.

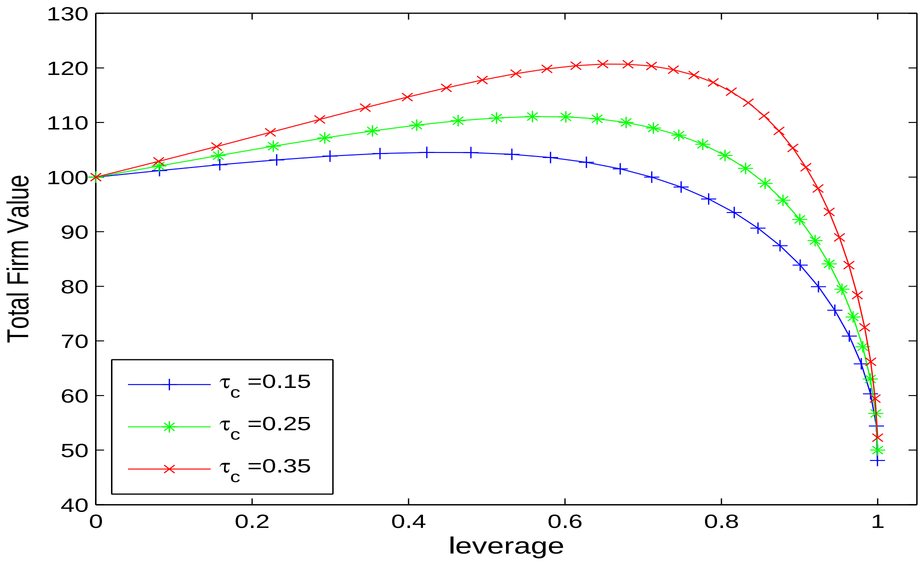


Figure 1.2: Total firm value versus leverage for different corporate tax rates. The other parameters are $r = 0.06$, $\sigma = 0.3$, $\alpha = 0.5$, $\tau = 0.15$, $V_0 = 100$.

1.8 Credit Spread

Let R denote the the interest rate that risky debt pays. Then, we obtain

$$R = \frac{C}{D(V)} = \frac{C}{\frac{C}{r}\left(1 - \left(\frac{C}{V}\right)^x A_2\right)} = \frac{r}{\left(1 - \left(\frac{C}{V}\right)^x A_2\right)}. \tag{1.31}$$

$\left(1-\left(\frac{C}{V}\right)^x A_2\right)$ has the interpretation of a risk adjustment factor that the firm must pay to compensate debt holders for the risk taken. The yield spread is given by

$$R - r = \frac{\left(\frac{C}{V}\right)^x A_2}{1-\left(\frac{C}{V}\right)^x A_2}.$$

In figure 1.3, the credit spread is plotted with respect to the leverage for different volatilities. We observe that when the riskiness of the firm increases the credit spread, paid to the debt holders to bear the risk of bankruptcy, increases.

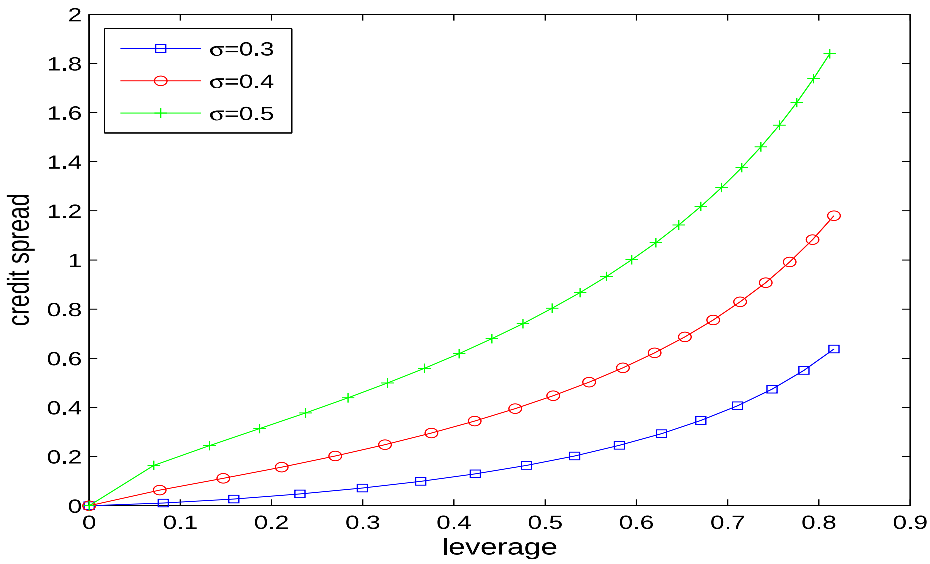


Figure 1.3: Credit spread versus leverage for different volatilities. The other parameters are $r = 0.06$, $\alpha = 0.5$, $\tau_c = 0.15$, $V_0 = 100$.

In Figure 1.4 and Figure 1.5, the credit spread is plotted with respect to the leverage for different bankruptcy costs and different corporate tax levels respectively. We observe that increasing bankruptcy costs implies increasing credit spreads and increasing corporate tax implies decreasing credit spreads[14].

In this model, we can not explore the behavior of credit spreads for different, especially short maturities, due to the fact that we use a rolling procedure that simulates a constant time to maturity. However, it is important to see the behavior of the credit spread for short

[14] The second unrealistic observation is because of the modeling of tax advantages, as we have already mentioned in several places, for example see Figure 1.2.

maturities in order to observe the classical structural firm value inconsistency, namely for short maturities the credit spread is almost zero. In Chapter 2, we will be able to observe this inconsistency, moreover we shall suggest a model to overcome it.

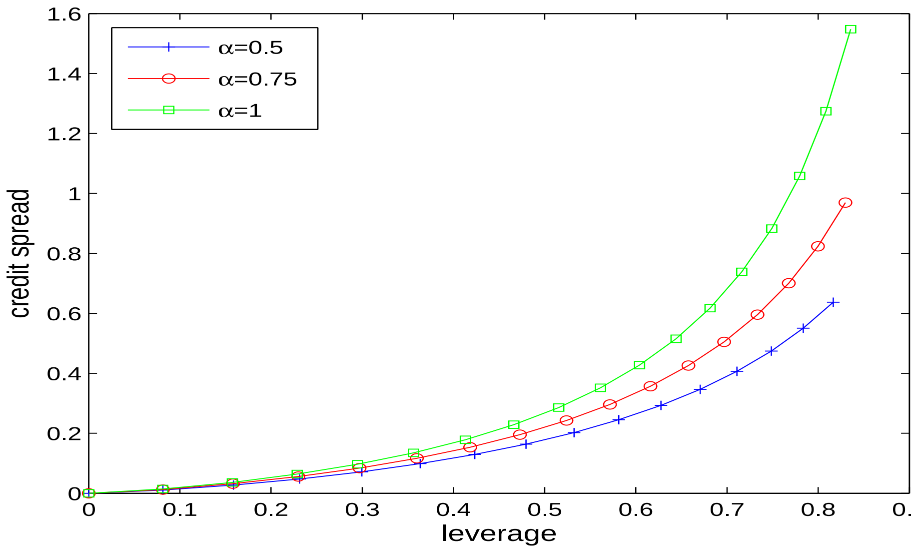


Figure 1.4: Credit spread versus leverage for different bankruptcy costs. The other parameters are $r = 0.06$, $\sigma = 0.3$, $\tau_c = 0.15$, $V_0 = 100$.

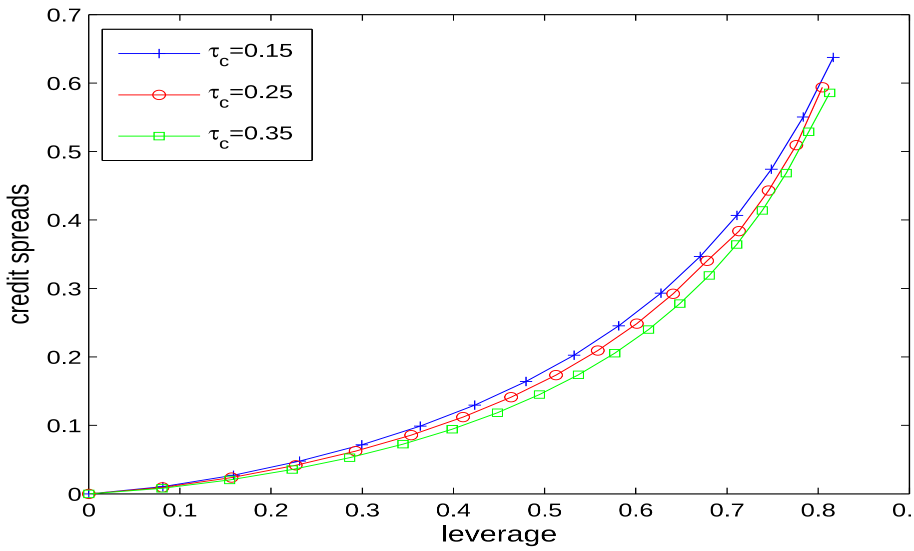


Figure 1.5: Credit spread versus leverage for different volatilities. The other parameters are $r = 0.06$, $\sigma = 0.3$, $\alpha = 0.5$, $V_0 = 100$.

1.9 Comparative statistics for debt and equity value value

Our base case parameters are $r = 0.06$, $\sigma = 0.3$, $\alpha = 0.5$, $\tau = 0.15$, $V_0 = 100$.

Debt value and Debt Capacity

The debt value, given by equation (1.8), is a concave function with respect to V. One can easily verify it by showing that the following second order condition holds.

$$\frac{\partial^2 D(V,C)}{\partial V^2} = \underbrace{\left[(1-\alpha)(1-\tau_c)\frac{C}{r}\frac{x}{1+x} - \frac{C}{r}\right]}_{<0} x(1+x)C^x V^{-x-2}\left((1-\tau_c)\frac{x}{r(1+x)}\right)^x < 0.$$

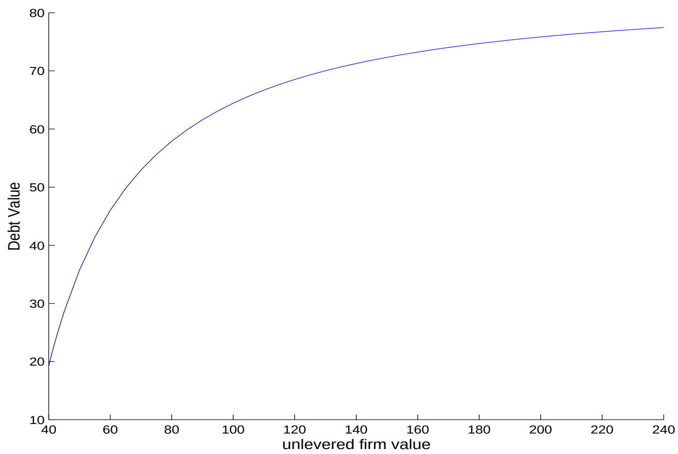


Figure 1.6: Debt value versus unlevered firm value.

The concavity of the debt value with respect to V confirms the risk aversity of debt holders. In Figure 1.7, we observe that for higher firm risk the total debt value decreases. Note that the debt is also a concave function with respect to C, since

$$\frac{\partial^2 D(V,C)}{\partial C^2} = \underbrace{\left[(1-\alpha)(1-\tau_c)\frac{1}{r}\frac{x}{1+x} - \frac{1}{r}\right]}_{<0}\left((1-\tau_c)\frac{x}{r(1+x)}\right)^x\left(\frac{C}{V}\right)^{x-1}\frac{x(x+1)}{V} < 0$$

holds. If V is close to V_B, the value of the debt will be very sensitive to bankruptcy costs. Lowering V_B will raise the value of the debt, since bankruptcy costs will be less imminent. From equation (1.20) lower coupon value, higher interest rate, higher asset volatility will all serve to lower V_B. For values of V close to V_B, this positive effect on $D(V)$ will dominate. It implies that $D(V)$ is eventually decreasing as the value of the coupon increases. This means that the debt has a capacity for the positive effect of the tax advantage. In other words, a firm can not benefit from the tax advantage for an arbitrary big amount of debt value. Let us denote the maximum debt value by $D_{\max}$. To find it we differentiate equation (1.22) with respect to C

$$\frac{\partial D(V)}{\partial C} = \frac{1}{r}\left[1 - \left(\frac{C}{V}\right)^{x} A_2\right] - \frac{C}{r}\frac{x}{V}\left(\frac{C}{V}\right)^{x-1} A_2 = 0,$$

which implies

$$C_{\max} = V\left[\frac{1}{(1+x)A_2}\right].$$

When we substitute C_{max} into the equation (1.22), we get the maximum value of the debt, i.e. debt capacity, as follows

$$D_{\max} = V\left[x\left(\frac{1}{A_2}\right)^{1/x}\left(\frac{1}{1+x}\right)^{-(1+1/x)}\right]\frac{1}{r}.$$

In Figures 1.7, 1.8 and 1.9, the debt value is plotted with respect to the coupon value for different volatilities, bankruptcy rates and tax rates respectively. We observe that debt has a maximum capacity, which decreases as the volatility and the bankruptcy rate increases and tax rate decreases.

Equity value

The equity value, given by equation (1.16), is convex with respect to V and C. We have already shown in Subsection 1.4.3 the convexity with respect to the firm value V, which justifies the option nature of the equity value. In order to prove the convexity with respect to the coupon value C, one can check the second partial derivative of equity value with respect to C.

$$\frac{\partial^2 EQ}{\partial C^2} = \frac{(1-\tau_c)}{r} x(x+1)\frac{A_1}{V} > 0.$$

Figure 1.10 plots how the equity value is related to the firm value and the coupon payment.

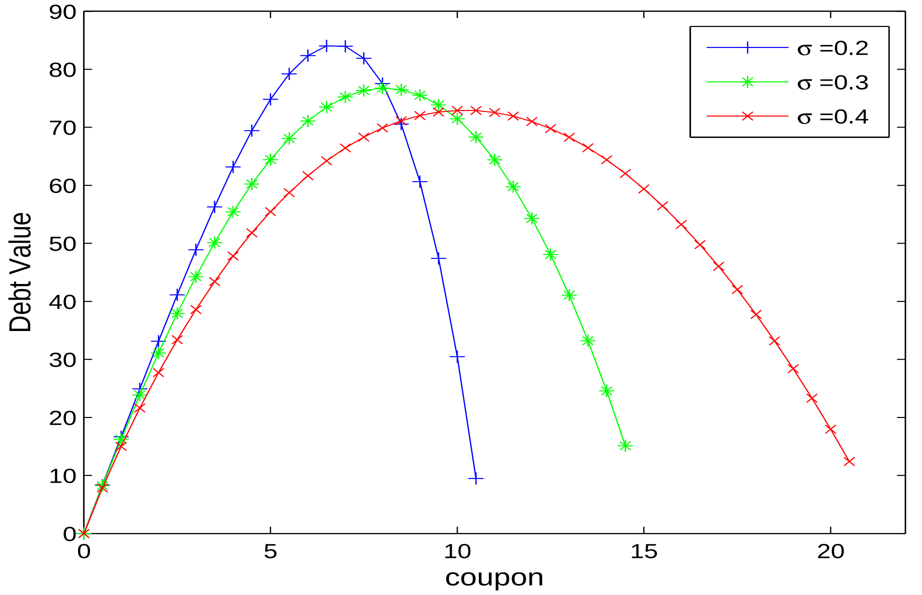


Figure 1.7: Debt value versus coupon for different volatilities.

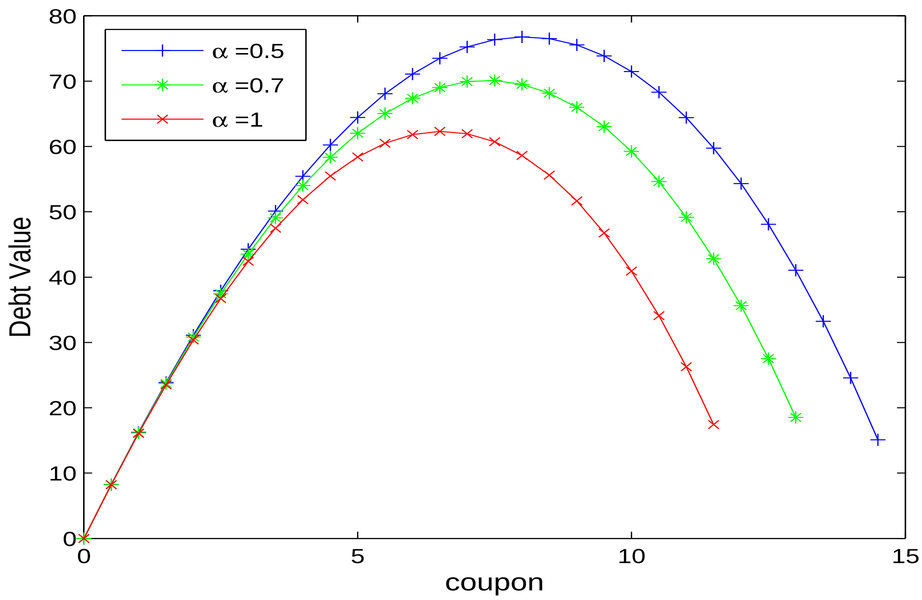


Figure 1.8: Debt value versus coupon for different bankruptcy rates.

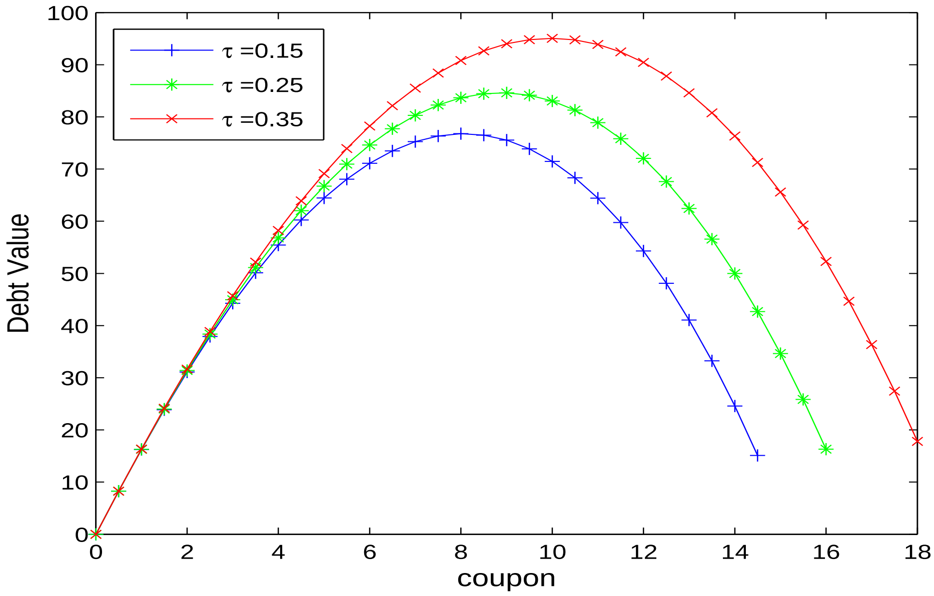
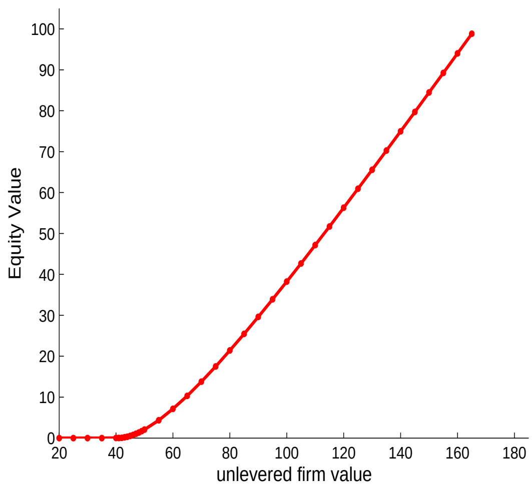
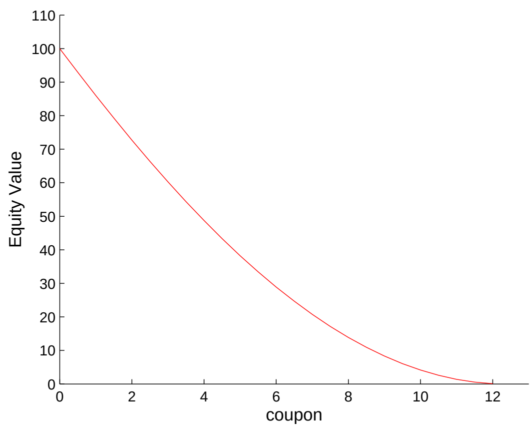


Figure 1.9: Debt value versus coupon for different tax rates.



(a) Equity value versus unlevered firm value.



(b) Equity value versus coupon.

Figure 1.10: The parameters are the same as in base case.

Chapter 2

Extended Optimal Capital Structure Models

2.1 Motivation

As we have seen from the previous chapter, to analyse the optimal capital structure of a firm Leland [Lel94] and also many other researchers used the value of the firm as the underlying process and assumed that it is a tradable asset, but this approach implies some inconsistencies. Firstly, the unlevered firm value and levered firm value can not exist simultaneously. If we assume that they coexist, arbitrage opportunities might appear in the market.

Moreover, in these models the tax advantage of a firm is not modeled realistically. More precisely, Leland and Toft [LT96] introduced the tax advantage as a cash inflow of a firm, but in reality tax advantage is a reduction on the cash outflow of a firm to the government.

In Leland's model debt is assumed to be perpetual. Although it is hard to create the perpetual environment, this assumption has another drawback; such as the behavior of the credit spreads for short term debt can not be analysed.

In the following subsections, we shall see how one can overcome the drawbacks of Leland's model by specifying the appropriate dynamics of the underlying process, relaxing the perpetuity assumption, introducing a realistic tax regime.

2.1.1 Choosing the underlying

In this chapter, we follow Goldstein, *et al.* [GNL98] and use the EBIT value[1] as the underlying asset, which can be seen as all the cash inflows of a firm. In addition to Goldstein, *et al.* [GNL98], we also assume that EBIT is a non-tradable asset. This approach has some positive implications. First of all, non-tradable EBIT assumption implies that we do not have any no-arbitrage restrictions that gives us its drift under the equivalent martingale measure, hence the arbitrage opportunities are excluded in the market. All the claims of a firm (equity, debt, government taxes, etc.) are subject to EBIT value and they are treated consistently. As a consequence, we shall see that the tax benefit to debt is modeled as a reduction on the total tax payments to the government. In contrast to the firm value, EBIT value does not cease to exist after a capital structure change occurs, which enables one to investigate dynamic capital structure of a firm[2]. EBIT is assumed to be invariant to changes in capital structure. It means that the distribution of EBIT-flow among the claimants does not affect the firm. It is the separation of financing and investment policy. These features make the EBIT framework ideal for investigating the optimal capital structure.

In fact, firm value process and the EBIT value are closely related. We will see in Subsection 2.2.2 that they differ only by a constant, which is in the literature known as the price earnings ratio.

2.1.2 Relaxing Perpetuity

In our model, the firm is partly financed by issuing debt. Debt is initially issued at time $t = 0$ with principal P and coupon C. The debt amount is hold constant by continuous retirement and reissue of it in the following way. At each moment in time, the firm has debt with constant principal P. The firm continuously rolls over a fraction m of debt. That is, it continuously retires outstanding debt principal at the rate $m\,P$. On the other

[1] As we have already mentioned EBIT stays for Earnings Before Interest and Taxes.

[2] Dynamic optimal capital strutrure frame work is not cover in this thesis but we keep the way open by deriving our results by using the EBIT value. For dynamic optimal capital models, see Christensen *et al.* [CFLM01], Goldstein *et al.* [GNL98].

hand, at each instant s, the firm issues new debt with principal p and coupon c. Since $m\,P$ is the amount of the debt principal, retired at each instant, we have

$$p\,dt = m\,P\,dt, \tag{2.1}$$

$$c\,dt = m\,C\,dt. \tag{2.2}$$

In other words, the debt retirement is replaced by newly issued debt of equal principal and coupon, so that the total principal and coupon remain constant. Note that the price of newly issued debt depends on the current underlying value. The real world equivalent of this debt structure is sinking fund provisions, on which a fraction of the principal of the debt value is retired on a regular basis. Smith and Wagner [SW79] stated that Sinking funds are quite common in corporate debt issues. With such a structure, the finite maturity debt can be examined with a time homogenous repayment of principal and coupon.

Let $p(s,t)$ and $c(s,t)$ denote respectively the principal outstanding and the coupon outstanding at time t of debt issued at time $s \leq t$, $(p(s,s) = p,\ c(s,s) = c)$. As t passes, the principal of the debt issued at any time $s \leq t$ is retired at a fractional rate m. Hence we have,

$$\frac{\partial\, p(s,t)}{\partial\, t} = -m\,p(s,t),$$

which implies,

$$\begin{aligned} p(s,t) &= p\,e^{-m(t-s)}, \\ c(s,t) &= c\,e^{-m(t-s)}. \end{aligned}$$

In order to confirm equations (2.1) and (2.2), consider the following equations

$$\begin{aligned} P &= \int_{-\infty}^{t} p(s,t)ds = \int_{-\infty}^{t} p\,e^{-m(t-s)}ds = \frac{p}{m}, \\ C &= \int_{-\infty}^{t} c(s,t)ds = \int_{-\infty}^{t} c\,e^{-m(t-s)}ds = \frac{c}{m}. \end{aligned}$$

At any time t a fraction of e^{-mt} of the initially issued debt will remain outstanding, with principal $e^{-mt}P$ and coupon $e^{-mt}C$.

Let the current time be s, one can easily see that the fraction of currently outstanding debt principal which is redeemed at time t in the future is

$$\frac{p(s,t)}{P} = m\, e^{-m\,(t-s)},$$

which implies the average maturity M of debt is

$$M = \int_0^\infty t\,(m\, e^{-m\,t})\, dt = \frac{1}{m}.$$

The motivation of the average maturity is as follows; the slower we roll over our debt, i.e. the smaller m is, the longer it takes to pay our debt back. As a special case, we can consider $m \to 0$, which implies $M \to \infty$, which is the perpetual debt case in Leland [Lel94].

Remark 2.1.1. *Another interpretation of the above explained average maturity setup is as follows. At each time instant, the constant total principal P of a firm is rolled over continuously with a rate m. In order to keep the total principal constant, a bond with the principal $m\,P$ is issued. The maturity of the newly issued bond is chosen from a exponentially distributed random variable with a mean $1/m$. Since the total debt value consists of these bonds, the average maturity of the total debt value is $1/m$.*

2.1.3 A Realistic Tax Regime

The debt becomes beneficial because there is a tax advantage to the debt. That is, the firm must pay corporate tax at the rate τ_c, of the earnings before taxes $(\delta_t - C)$. The remaining value is immediately paid out to the equity holders as dividends. They must pay the dividend tax rate τ_d. Debt holders, who receive constant coupon payments C, must pay the tax rate τ_i. Altogether $\tau_e := 1 - (1-\tau_c)(1-\tau_d)$ of the earnings before taxes (EBT) are taxed from the equity holder's point of view. With this tax regime, the tax benefit of the debt is modeled as a reduction in the tax rate paid to the government, rather than a cash flow into the firm (as in Leland's model). Let us first consider the case, where the firm is only financed by issuing equity. In Table 2.1, we see the total cash outflow that is paid to the government by the claim holders. It is important to note that the effective tax rate, on which the firm is taxed, is denoted by τ_e in the case of no debt.

When the firm issues debt, the cash out flow becomes as in the following Table 2.2. The term $(\tau_e - \tau_i)C$ is the tax advantage of the firm, when they issue debt. Note that τ_e must be greater than τ_i, in order to have tax advantage to debt.

	State	Equity holders
δ	$\tau_c\delta$	
$(1-\tau_c)\delta$	$\tau_d(1-\tau_c)\delta$	$(1-\tau_d)(1-\tau_c)\delta$
Net income	$\underbrace{(\tau_c+\tau_d-\tau_d\tau_c)}_{=\tau_e}\delta$	$(1-\tau_e)\delta$

Table 2.1: The amount of the tax rate paid to the government, when no debt is issued.

	Debt holders	State	Equity holders
C	$(1-\tau_i)C$	$\tau_i C$	
$(\delta - C)$		$\tau_c(\delta - C)$	
$(1-\tau_c)(\delta - C)$		$\tau_d(1-\tau_c)(\delta - C)$	$(1-\tau_d)(1-\tau_c)(\delta - C)$
Net income	$(1-\tau_i)C$	$\tau_e\,\delta - (\tau_e - \tau_i)C$	$(1-\tau_e)(\delta - C)$

Table 2.2: The amount of tax rate paid to the government, when debt is issued.

We are left with the question how to model the underlying process. In the following two sections, we shall consider two cases. The first case is the extension of the Leland's model in the sense that the underlying process changes but it is still modeled as a geometric Brownian (GBM) motion. As already mentioned, the main advantages of this extension are; excluding the possible arbitrage opportunities, modeling the tax advantage realistically and being able to analyse the term structure of credit spreads. In the second case, we go a step further and model the underlying as a jump-diffusion process, whose jump heights are double exponentially distributed.

2.2 Diffusion case

Let us consider the probability space, already introduced in Chapter 1, $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$, where P is some subjective probability measure. As we have already mentioned the EBIT value is assumed to be a non-traded asset, we consider a market consists of the money market account and a traded derivative of the EBIT, for example the stock (equity) value of the firm. We assume that this market is arbitrage-free and complete, hence an equivalent martingale measure Q exists. Then, by following Ericsson and Reneby [ER02], the Q-dynamics of the EBIT process can be written as follows

$$d\delta(t) = \delta(t)\big(\mu\, dt + \sigma\, dW(t)\big), \tag{2.3}$$

where μ and σ are constants and $W(t)$ is a one dimensional Q-Brownian motion. Notice that the drift term of the EBIT-process under the measure Q is not equal to r, since it is assumed to be a non-traded asset.

When process (2.3) hits an endogenously given down barrier δ_B, the firm goes bankrupt. Therefore we shall compute the distribution of the first passage time of the process (2.3) to a given barrier δ_B from above.

Next, let us define a process $X(t)$ such that

$$X(t) := \gamma t + \sigma W(t), \tag{2.4}$$

where $\gamma = \mu - 0.5\sigma^2$. Then, from Itô's formula[3], we have

$$\delta(t) = \delta\, e^{X(t)}.$$

Let us denote by τ_B the first hitting time of the process (2.3) to a given boundary δ_B. Then, we have

$$\begin{aligned} \tau_B &= \inf\{t : \delta(t) = \delta_B\} && (2.5)\\ &= \inf\big\{t : X(t) = z\big\}, && (2.6) \end{aligned}$$

[3] We refer to Karatzas and Shreve [KS00].

where

$$z = \ln\left(\frac{\delta_B}{\delta}\right).$$

Note that $X(t)$ is a Brownian motion with drift γt and starting point 0. We need some preliminary results on $X(t)$ in order to calculate firm derivatives and analyse credit spreads.

2.2.1 Preliminaries

The following theorem states the well known result of a density of the first passage time of a Brownian motion to a given boundary.

Theorem 2.2.1 (First Hitting Time of a Brownian Motion). *Let $W(t)$ be a one dimensional Brownian motion equipped with the usual filtration. Let $X(t)$ be given by equation* (2.4)

$$X(t) = \gamma t + \sigma W(t).$$

Then, the probability density function of the hitting time of $X(t)$ to the barrier z is given by

$$f(t) = \frac{|z|}{\sigma\sqrt{2\pi t^3}}\, e^{-\frac{1}{2}\frac{(z-\gamma t)^2}{\sigma^2 t}}. \tag{2.7}$$

Proof. see Karatzas and Shreve [KS00] □

Next corollaries are direct consequences of the above theorem. The first one states the probability distribution function of the running minimum of a Brownian motion with drift. Running minimum of $X(t)$ is defined as follows

$$m(t) := \inf_{0\le s\le t}\big\{\gamma s + \sigma W(s)\big\}.$$

Corollary 2.2.2. *The probability for $m(t)$ to remain above the threshold z is given by*

$$\Phi_z^X(t) := Q\big(m(t) > z\big) = \Phi\left(\frac{-z+\gamma t}{\sigma\sqrt{t}}\right) - e^{2\gamma z\sigma^{-2}}\Phi\left(\frac{z+\gamma t}{\sigma\sqrt{t}}\right). \tag{2.8}$$

Notice that $\Phi_z^X(t)$ can be also defined as the survival probability of the firm.

The next one states the Laplace transformation of the distribution function of τ.

Corollary 2.2.3 (Laplace Transformation of the First Passage Times). *For any $\alpha > 0$, the Laplace transformation of the first passage times of process* (2.3) *to a given boundary δ_B is given by*

$$\mathbb{E}_Q\left(e^{-\alpha\tau_B}\right) = \left(\frac{\delta}{\delta_B}\right)^{-x}, \tag{2.9}$$

where α is a real valued constant and x is given as follows

$$x = \frac{\gamma + \sqrt{\gamma^2 + 2\alpha\sigma^2}}{\sigma^2}.$$

Proof. Recall (2.6),

$$\tau_B = \inf\{t : X(t) = z\}.$$

From Theorem 2.2.1, the density of the first hitting time of $X(t)$ to z is

$$f(t) = \frac{z}{\sigma\sqrt{2\pi t^3}}\, e^{-\frac{1}{2}\frac{(z-\gamma t)^2}{\sigma^2 t}}.$$

Therefore, the Laplace transform of τ can be obtained

$$\begin{aligned}
\mathbb{E}_Q\left(e^{-\alpha\tau_B}\right) &= \int_0^\infty e^{-\alpha t} f(t)dt \\
&= \left(\frac{\delta_B}{\delta}\right)^{\frac{\gamma}{\sigma^2}-\lambda} \int_0^\infty \frac{z}{\sigma\sqrt{2\pi t^3}}\, e^{-\frac{1}{2}\frac{(z-\lambda\sigma^2 t)^2}{\sigma^2 t}} dt \\
&= \left(\frac{\delta_B}{\delta}\right)^{\frac{\gamma}{\sigma^2}-\lambda} \int_0^\infty g(t)dt, \qquad (2.10)
\end{aligned}$$

where $g(t)$ is the density function of the first passage time of a process with drift $\lambda\sigma^2 t$, where λ is given as follows

$$\lambda = \frac{\sqrt{\gamma^2 + 2\alpha\sigma^2}}{\sigma^2}.$$

In Theorem 1.4.1, we have already shown that the integral in equation (2.10) is equal to

$$\int_0^\infty g(t)dt = e^{\frac{-2\lambda\sigma^2 z}{\sigma^2}} = \left(\frac{\delta}{\delta_B}\right)^{-2\lambda}. \tag{2.11}$$

Substituting equation (2.11) into equation (2.10), we obtain the required Laplace transform

$$\mathbb{E}_Q\left(e^{-\alpha\tau_B}\right) = \left(\frac{\delta}{\delta_B}\right)^{-x},$$

where x is

$$x = \frac{\gamma + \sqrt{\gamma^2 + 2\alpha\sigma^2}}{\sigma^2}.$$

□

The next theorem states the partial derivative approach to price the firm derivatives under the average maturity setup.

Theorem 2.2.4. *For any firm's CC, underlying on equation* (2.3)*, we denote the price at time t of the initially issued CC by $F^0(\delta, t)$ with a continuous payout rate $h^0(V, t)$. Then*

1. *$F^0(\delta, t)$ satisfies the following partial differential equation*

$$\frac{1}{2}\sigma^2\delta^2 F^0_{\delta\delta}(\delta, t) + \mu\,\delta\, F^0_\delta(\delta, t) - r\,F^0(\delta, t) + F^0_t(\delta, t) + h^0(\delta, t) = 0. \tag{2.12}$$

2. *Moreover, by considering the average maturity setup, introduced in section 2.1.2, one can show that the above PDE takes the following ODE form,*

$$\frac{1}{2}\sigma^2\delta^2 F_{\delta\delta}(\delta) + \mu\,\delta\, F_\delta(\delta) - (r+m)F(\delta) + h(\delta) = 0, \tag{2.13}$$

where $F(\delta)$ is defined as the total outstanding contingent claim value at any future time t, i.e., $F(\delta) := e^{mt}F^0(\delta, t)$ and $h(\delta) := e^{mt}h^0(\delta, t)$

Proof. The first part of the theorem can be proved similarly as in Theorem 1.3.1.

For the second part, notice that $F(\delta)$ is defined as the total outstanding value of the contingent claim at any future time t. Therefore, from the average maturity environment defined in section 2.1.2, we have

$$F^0(\delta, t) = e^{-mt}F(\delta).$$

By substituting $e^{-mt}F(\delta)$ in PDE (2.13), we obtain the result. □

The following Corollary states the general solution of the ODE (2.13). It is a finite maturity setup analogous of Corollary 1.3.2, introduced in Chapter 1.

Corollary 2.2.5. *The general solution of ODE* (2.13) *is given by*

$$F_{GS}(\delta) = A_1\,\delta^{-x_1} \;+\; A_2\,\delta^{-x_2}, \tag{2.14}$$

where

$$x_1 \;=\; \frac{\gamma + \sqrt{\gamma^2 + 2(r+m)\sigma^2}}{\sigma^2}, \tag{2.15}$$

$$x_2 \;=\; \frac{\gamma - \sqrt{\gamma^2 + 2(r+m)\sigma^2}}{\sigma^2}. \tag{2.16}$$

Note that A_1 and A_2 depend on the specific contract and the particular solution F_{PS} depends on the payout rate $h(\delta)$. Therefore, the solution of ODE (2.13) *is given by*

$$F(\delta) = F_{PS} + A_1\,\delta^{-x_1} \;+\; A_2\,\delta^{-x_2}. \tag{2.17}$$

Proof. Similar to the proof of Corollary 1.3.2. Note that $x_1 > 0$ and $x_2 < 0$. □

The next corollary is a direct consequence of Theorem 1.3.1 and Corollary 1.3.2, whereas in this case the underlying is a non-traded asset.

Corollary 2.2.6. *If the firm's CC is time independent i.e $F(\delta, t) = F(\delta)$, then ODE* (2.13) *has the following form*

$$\frac{1}{2}\sigma^2\delta^2 F_{\delta\delta}(\delta) + \mu\delta F_{\delta}(\delta) - rF(\delta) + h(\delta) = 0, \tag{2.18}$$

which has a solution

$$F_{GS}(\delta) = A_1\,\delta^{-y_1} \;+\; A_2\,\delta^{-y_2}, \tag{2.19}$$

where

$$y_1 \;=\; \frac{\gamma + \sqrt{\gamma^2 + 2r\sigma^2}}{\sigma^2}, \tag{2.20}$$

$$y_2 \;=\; \frac{\gamma - \sqrt{\gamma^2 + 2r\sigma^2}}{\sigma^2}. \tag{2.21}$$

Note that A_1 and A_2 depend on the specific contract and the particular solution F_{PS} depends on the payout rate $h(\delta)$. Therefore, the solution of ODE (2.13) *is given by*

$$F(\delta) = F_{PS} + A_1\,\delta^{-y_1} \;+\; A_2\,\delta^{-y_2}. \tag{2.22}$$

Proof. Similar to Theorem 1.3.1. □

2.2.2 Pricing Firm Derivatives

In this section, we will derive prices of the firm derivatives by using both the PDE-approach and the martingale approach. In the view of Theorem 2.2.4 and Corollary 2.2.5, the PDE-approach is an easy tool to price firm's CC in diffusion case. We shall also introduce a martingale approach to derive the firm's CC prices. This approach will be useful in Section 2.3, where the underlying is modeled as a jump-diffusion process. Modelling by jump-diffusion processes leads to integro-differential equations for the prices of CCs. The martingale approach spares us to solve this integro-differential equations.

Firm value versus EBIT value

We have already mentioned that there is a relation between the firm value and the EBIT value. Let us now explore this relation. In order to do so, we consider the net present value of the future earnings of the EBIT value

$$\begin{aligned} V\big(\delta(t)\big) &= \mathbb{E}_Q\left(\int_t^\infty e^{-\hat{r}(s-t)}\,\delta(s)\,ds\,\Big|\,\mathcal{F}_t\right) \\ &= \mathbb{E}_Q\left(\int_t^\infty e^{-\hat{r}(s-t)}\,\delta(t)\;e^{(\mu-\frac{1}{2}\sigma^2)(s-t)+\sigma\,(W(s)-W(t))}\,ds\,\Big|\,\mathcal{F}_t\right) \\ &= \delta(t)\int_t^\infty e^{(-\hat{r}+\mu)(s-t)}\,ds \\ &= \frac{\delta(t)}{\hat{r}-\mu}, \end{aligned}$$

where the future earnings are discounted by the net interest rate $\hat{r} := (1-\tau_i)\,r$ and $\hat{r} > \mu$, i.e., the interest rate which is corrected by the tax losses. Hence, one can define the artificial firm value as

$$V_U\big(\delta(t)\big) = K\,\delta(t), \tag{2.23}$$

where

$$K := \frac{(1-\tau_e)}{\hat{r}-\mu} \tag{2.24}$$

and τ_e is the effective tax rate, paid by the equity holders to the state. Therefore, the EBIT value differs from the firm value just by a constant. We denote it by K. In the literature the constant K is known as the price-earnings ratio.

Note that Leland used the process V_U, modeled as a GBM, as the state variable and assumed that it is a traded asset (the drift rate of the corresponding SDE is r), which implies the levered firm value is also a traded asset. But, coexistence of levered and unlevered firm values as traded assets implies an arbitrage in the model. If unlevered firm value is smaller than the levered one, one can buy the unlevered firm value, lever it and sell again, which implies an arbitrage opportunity. Therefore, we assume that neither the firm value V nor the EBIT value δ are traded assets. Thus, we do not have any no-arbitrage restrictions, giving us their drifts under the equivalent martingale measure.

Debt Value

In this part, we will find the total debt value by using Theorem 2.2.4 and Corollary 2.2.5. Let us denote the time t value of the initially issued debt as $D^0(\delta, t)$. Since the holders of the initially issued debt receive a total payment (coupon plus return of principal) of $e^{-mt}\,(C + m\,P)$, from Theorem 2.2.4, $D^0(\delta, t)$ satisfies the following PDE

$$\frac{1}{2}\sigma^2\delta^2 D^0_{\delta\delta}(\delta, t) + \mu\,\delta D^0_{\delta}(\delta, t) - r\,D^0(\delta, t) + D^0_t(\delta, t) + e^{-mt}(1-\tau_i)\,(C + m\,P) = 0.$$

Let us define $D(\delta)$ as the total outstanding debt value at any future time t, then

$$D(\delta) = e^{mt} D^0(\delta, t).$$

$D(\delta)$ is time independent and satisfies ODE (2.13) with a payout rate $(1-\tau_i)\,(C + m\,P)$

$$\frac{1}{2}\sigma^2\delta^2 D_{\delta\delta}(\delta) + \mu\,\delta\,D_{\delta}(\delta) - (r+m)D(\delta) + (1-\tau_i)\,(C + m\,P) = 0.$$

Total debt value at any future time t can be found by employing Corollary 2.2.5

$$D(\delta) = D_{PS} + A_1\,\delta^{-x_1} \;+\; A_2\,\delta^{-x_2}.$$

Note that when $\delta \to \infty$, the default is unlikely to happen. Therefore debt holders are paid the whole amount of the coupons and the principal

$$\lim_{\delta\to\infty} D(\delta) = \int_0^{\infty} (1-\tau_i)\,(C + m\,P)\; e^{-(\hat{r}+m)t}\,dt \;=\; \frac{(1-\tau_i)\,(C + m\,P)}{m + \hat{r}}.$$

Hence, we have the following boundary conditions;

$$\delta \to \infty \implies D(\delta) \to \tfrac{(1-\tau_i)(C+m\,P)}{m+\hat{r}} \implies \quad A_2 = 0,\;\; D_{PS} = \tfrac{(1-\tau_i)(C+m\,P)}{m+\hat{r}}$$

$$\delta = \delta_B \implies D(\delta) = (1-\alpha)K\delta_B \implies A_1 = \left[(1-\alpha)K\delta_B - \tfrac{(1-\tau_i)(C+m\,P)}{m+\hat{r}}\right]\delta_B^{-x_1}.$$

When we substitute the coefficients A_1 and A_2, the particular solution D_{PS} into functional (2.17), we obtain the total debt value as follows,

$$D(\delta, \delta_B, P, C) = \frac{(1-\tau_i)\,(C \,+\; m\,P)}{m+\hat{r}}\left(1 - \left(\frac{\delta}{\delta_B}\right)^{-x_1}\right) + (1-\alpha)\,K\,\delta_B\,\left(\frac{\delta}{\delta_B}\right)^{-x_1},$$

where x_1 is given by the equation (2.15). Note that total debt value given above depends on the underlying EBIT value, the default level, the total principal level and the coupon level.

Next, we introduce the martingale approach to derive the above formula. Let $d(0;c,p)$ denote the value of the currently issued debt , paying continuous coupons c, and principal p. Then the value of the currently issued debt is equal to

$$\begin{aligned} d(0,c,p) = &\int_0^\infty e^{-\hat{r}t}\,e^{-mt}(1-\tau_i)\,(c+mp)\,\big(1 - F(t;\delta,\delta_B)\big)\,dt \\ &+ \int_0^\infty e^{-\hat{r}t}\,\frac{e^{-mt}p}{P}\,(1-\alpha)\,K\,\delta_B\,f(t;\delta,\delta_B)\,dt, \end{aligned} \tag{2.25}$$

where $f(t;\delta,\delta_B)$ denotes the density of the first passage time t to δ_B from δ, given in Theorem 2.2.1 and $F(t;\delta,\delta_B)$ be the cumulative distribution function of it, given in Theorem 2.2.2. The first term represents the discounted expected value of the continuously (exponentially) declining coupon plus principal repayment, which will be paid with probability $(1-F(.,\delta,\delta_B))$. The second term is the expected present value of the fraction of the value of the firm after bankruptcy costs are paid, if bankruptcy occurs at time t. Recalling that $p/P = m$, integrating by parts and simplifying gives

$$d(0,c,p) = \frac{c+m\,p}{\hat{r}+m}\left(1 - \int_0^\infty e^{-(\hat{r}+m)t}f(t;\delta,\delta_B)dt\right) + m(1-\alpha)K\delta_B\left(\int_0^\infty e^{-(\hat{r}+m)t}f(t;\delta,\delta_B)dt\right). \tag{2.26}$$

In Corollary 2.2.3, it is shown that

$$\int_0^\infty e^{-\alpha t}f(t;\delta,\delta_B)dt = \left(\frac{\delta}{\delta_B}\right)^{-x},$$

where α is a real valued constant and x is given as follows

$$x = \frac{\gamma + \sqrt{\gamma^2 + 2\alpha\sigma^2}}{\sigma^2}.$$

The value of the outstanding debt of the generation t, $t \leq 0$ is $e^{mt} d(0, c, p)$. Integrating over $-\infty \leq t \leq 0$ gives the total value of the outstanding debt , since all outstanding units of the debt sell for the same price (they carry the same coupon and principal), and the retirement of remaining units follows the same exponentially declining schedule

$$D(\delta, \delta_B, P, C) = \int_{-\infty}^{0} e^{m\,t}\, d(0, c, p)\, dt = \frac{d(0, c, p)}{m}.$$

Recalling that $P = p/m$ and $C = c/m$, gives

$$D(\delta, \delta_B, P, C) \quad = \quad \frac{(1-\tau_i)\,(C + \ m\,P)}{m + \hat{r}} \left(1 - \left(\frac{\delta}{\delta_B}\right)^{-x_1}\right) + (1-\alpha)\,K\,\delta_B \left(\frac{\delta}{\delta_B}\right)^{-x_1}, \tag{2.27}$$

where x_1 is given by equation (2.15). The first term in equation (2.27) can be interpreted as the present value of the coupon and principal payments of the debt, contingent on no bankruptcy, after the interest tax is paid to the government. The second term is the present value of the firm, contingent on the default, overtaken by the debt holders after bankruptcy costs are paid. Note that α portion of the firm value at the bankruptcy is paid to the government as bankruptcy costs.

By substituting δ_B, C, P into the equation (2.27), we get a closed form solution for total debt value.

In Remark 2.2.1, we state the relation of the debt value, given by equation (2.27) and the debt value, given by equation (1.8).

In order to analyse the credit spreads and to be able to calibrate[4] the parameters μ and σ of the underlying EBIT process by using the firm's existing bond prices in the market, we consider bonds issued by the firm.

[4]In this thesis, we shall not calibrate the parameters of the EBIT process, but make a short remark how to do it. See Remark 2.2.7.

Lemma 2.2.7. *Let $\overline{B}_\Theta(0,t;\rho,F)$ be the value of a defaultable bond issued by the firm at time 0, with maturity t, paying constant continuous coupon rate ρ and principal F. Then, we have*

$$\begin{aligned}\overline{B}_\Theta(0,t;\rho,F) \;&=\; \mathbb{E}_Q\left(\int_0^{t\wedge\tau_B}(1-\tau_i)\,\rho\,F\,e^{-\hat{r}s}\,ds\right) + \mathbb{E}_Q\left((1-\tau_i)\,F\,e^{-\hat{r}t}\,1_{\{t<\tau_B\}}\right)\\ &+\mathbb{E}_Q\left(F\,R\,e^{-\hat{r}t}\,1_{\{t\geq\tau_B\}}\right),\end{aligned} \tag{2.28}$$

where R is the recovery rate and given by

$$\frac{(1-\alpha)K\delta_B}{P}.$$

It is paid at the maturity, if default occurs.

The first term is the net coupon payments as long as default does not happen. The second term is the principal payment, in case default does not occur until maturity. The last term is the payment to the debt holder subject to the bankruptcy. Note that ρ is not defined as the realized coupon payments but the coupon rate. The realized coupon payments are equal to ρF. The subscript Θ in the notation of the defaultable bond indicates the dependence of the defaultable bonds on the parameter set $\Theta=(\mu,\sigma)$ of the underlying process, $\delta(t)$. Let $\overline{B}_\Theta(0,t;\rho,1)$ be the value of a bond issued by the firm at time 0, with maturity t, paying constant continuous coupon rate ρ and principal 1. Then by the scaling property of the bond prices with respect to the face value, we obtain the following relation

$$\overline{B}_\Theta(0,t;\rho,1)=\frac{\overline{B}_\Theta(0,t;\rho,F)}{F}.$$

The total value, at time 0, of all debt outstanding can be found by calculating the following integral,

$$\begin{aligned}D(\delta,\delta_B,P,C)=&\int_0^\infty p(0,t)\overline{B}_\Theta(0,t;\rho,1)dt\\ =&\int_0^\infty p\,e^{-mt}\,\mathbb{E}_Q\left(\int_0^{t\wedge\tau_B}(1-\tau_i)\,\rho\;e^{-\hat{r}s}\,ds\right)\,dt\\ &+\int_0^\infty p\,e^{-mt}\,\mathbb{E}_Q\left((1-\tau_i)\,e^{-\hat{r}t}\,1_{\{t<\tau_B\}}\right)\,dt+\int_0^\infty p\,e^{-mt}\,\mathbb{E}_Q\left(e^{-\hat{r}t}\,R\,1_{\{t\geq\tau_B\}}\right)\,dt\\ =&p\,(1-\tau_i)\,\rho\,\mathbb{E}_Q\left(\int_0^{\tau_B}e^{-\hat{r}s}\left(\int_s^\infty e^{-mt}\,dt\right)\,ds\right)+(1-\tau_i)\,p\,\mathbb{E}_Q\left(\int_0^{\tau_B}e^{-(\hat{r}+m)t}\,dt\right)\\ &+p\,R\,\mathbb{E}_Q\left(\int_{\tau_B}^\infty e^{-(m+\hat{r})t}\,dt\right).\end{aligned}$$

Substituting R on the right hand side of the last equation and computing the integrals yields,

$$D(\delta, \delta_B, P, C) = \frac{(1-\tau_i)P(\rho+m)}{m+\hat{r}} \mathbb{E}_Q \left(1 - e^{-(m+\hat{r})\tau_B}\right) + (1-\alpha)\, K\, \delta_B\, \mathbb{E}_Q \left(e^{-(m+\hat{r})\tau_B}\right).$$

When we substitute the above expectations by using Corollary 2.2.3, we obtain

$$D(\delta, \delta_B, P, C) = \frac{(1-\tau_i)P(\rho+m)}{m+\hat{r}} \left(1 - \left(\frac{\delta}{\delta_B}\right)^{-x_1}\right) + (1-\alpha)\, K\, \delta_B \left(\frac{\delta}{\delta_B}\right)^{-x_1}, \quad (2.29)$$

where x_1 is given by equation (2.15).

Remark 2.2.1.

1. *The total amount of the coupon payment is equal to* $C = \rho\, P$. *Therefore, equations* (2.29) *and* (2.27) *agree.*

2. *If EBIT were a traded asset, then for* $m = 0$, *the debt value, given by equation* (2.29) *would agree with the debt vale, given by equation* (1.8), *since* $V = K\,\delta$ *and* $x_1 = 2r/\sigma^2$ *(in the case of tradable EBIT and* $m = 0$*).*

Total Firm Value

The total firm value can be written as the sum of the unlevered firm value and tax benefits minus bankruptcy costs,

$$\nu(\delta) = K\delta + TB(\delta) - BC(\delta). \quad (2.30)$$

As in the previous chapter, tax benefits and bankruptcy costs can be seen as firm derivatives. These value functions include the benefits and costs in all future periods. They are time independent, because their cash flows and boundary conditions are not functions of time. Therefore, they can be valued by using ODE (2.18) and its functional solution (2.22),

$$\begin{aligned} BC(\delta) &= BC_{PS} + A_1^{BC}\,\delta^{-y_1} + A_2^{BC}\,\delta^{-y_2}, \\ TB(\delta) &= TB_{PS} + A_1^{TB}\,\delta^{-y_1} + A_2^{TB}\,\delta^{-y_2}. \end{aligned}$$

Next, we explore the corresponding boundary conditions to specify the above coefficients and the particular solution of the ODE.

- **Bankruptcy Costs**:

  The corresponding pricing ODE is given by

$$\frac{1}{2}\,\sigma^2\,\delta^2\,BC_{\delta\delta}(\delta)+\mu\,\delta\,BC_{\delta}(\delta)-r\,BC(\delta)=0,$$

  with the following boundary conditions

$$\delta\to\infty \implies BC(\delta)\to 0 \implies A_2^{BC}=0,\; BC_{PS}=0,$$

$$\delta=\delta_B \implies BC(\delta)=\alpha K\delta_B \implies A_1^{BC}=\alpha K\delta_B^{y_1+1}.$$

  Therefore, we obtain

$$BC(\delta,\delta_B)=\alpha\,K\,\delta_B\,\left(\frac{\delta}{\delta_B}\right)^{-y_1}.$$

- **Tax Benefits:**

  The corresponding pricing ODE is given by

$$\frac{1}{2}\,\sigma^2\,\delta^2\,TB_{\delta\delta}(\delta)+\mu\,\delta\,TB_{\delta}(\delta)-r\,TB(\delta)=0,$$

  with the following boundary conditions

$$\delta\to\infty \implies A_2^{BC}=0,\; TB_{PS}(\delta)=\tfrac{(\tau_e-\tau_i)C}{r},$$

$$\delta=\delta_B \implies A_1^{BC}=\tfrac{-(\tau_e-\tau_i)C}{r}\delta_B^{x}.$$

  Therefore, we obtain

$$TB(\delta,\delta_B,C)=(\tau_e-\tau_i)\,\frac{C}{\hat{r}}\,\left(1-\left(\frac{\delta}{\delta_B}\right)^{-y_1}\right).$$

When we substitute the value of the tax benefits and the bankruptcy costs into equation (2.30), we obtain

$$\nu(\delta,\delta_B,C)=K\,\delta+\,(\tau_e-\tau_i)\,\frac{C}{\hat{r}}\,\left(1-\left(\frac{\delta}{\delta_B}\right)^{-y_1}\right)-\alpha\,K\,\delta_B\,\left(\frac{\delta}{\delta_B}\right)^{-y_1}, \qquad (2.31)$$

where y_1 is given by the equation (2.20).

One can also use the martingale approach to derive the total firm value,

$$\begin{aligned}\nu(\delta,\delta_B,C) &= V_U(\delta)\,+\,\mathbb{E}_Q\left(\int_0^{\tau_B}(\tau_e-\tau_i)\,C\,e^{-\hat{r}s}\,ds\right)-\alpha\,K\,\delta_B\,\mathbb{E}_Q\left(e^{-\hat{r}\tau_B}\right)\\ &= K\,\delta+\,(\tau_e-\tau_i)\,\frac{C}{\hat{r}}\,\mathbb{E}_Q\left(1-e^{-\hat{r}\tau_B}\right)-\alpha\,K\,\delta_B\,\mathbb{E}_Q\left(e^{-\hat{r}\tau_B}\right). \qquad (2.32)\end{aligned}$$

The first term is the unlevered firm value, the second term is the value of tax benefits and the last term corresponds to the value of bankruptcy costs. When we substitute the above expectations by using Corollary 2.2.3, we again obtain equation (2.31).

Remark 2.2.2. *If EBIT were a traded asset, then the total firm value, given by equation* (2.31) *would agree with the total firm vale, given by equation* (1.15)*, since* $V = K\,\delta$ *and* $y_1 = 2r/\sigma^2$ *(in the case of tradable EBIT).*

Equity Value

The equity value is the difference between the total firm value and the debt value. Therefore, we have

$$\begin{aligned} EQ(\delta,\delta_B,P,C) &= \nu(\delta,\delta_B,C) - D(\delta,\delta_B,P,C) \\ &= K\,\delta + (\tau_e-\tau_i)\,\frac{C}{\hat r}\left(1-\left(\frac{\delta}{\delta_B}\right)^{-y_1}\right) - \alpha\,K\,\delta_B\left(\frac{\delta}{\delta_B}\right)^{-y_1} \\ &\quad -\frac{(1-\tau_i)\,(C\,+\,m\,P)}{m+\hat r}\left(1-\left(\frac{\delta}{\delta_B}\right)^{-x_1}\right) - (1-\alpha)\,K\,\delta_B\left(\frac{\delta}{\delta_B}\right)^{-x_1}. \end{aligned} \tag{2.33}$$

Notice that for any value of δ_B, we have $EQ(\delta_B,\delta_B)=0$, which means that the limited liability restriction of equity holders is respected but the boundary level has not been specified yet. As in the previous chapter, we have a free boundary problem. Therefore, we need an additional condition. In the following section, we shall explore, which additional condition is to be chosen.

Remark 2.2.3. *If EBIT were a traded asset, then for* $m=0$*, the equity value, given by equation* (2.33) *would agree with the equity vale, given by equation* (1.16)*, since* $V = K\,\delta$ *and* $x_1 = y_1 = 2r/\sigma^2$ *(in the case of tradable EBIT and m=0).*

2.2.3 Endogenous Default

As in the previous chapter, the default barrier is chosen endogenously by the equity holders. Their aim is to maximise their equity value and they achieve it by employing the smooth pasting condition,

$$\frac{\partial\, EQ(\delta, \delta_B)}{\partial\, \delta}\bigg|_{\delta=\delta_B} = 0.$$

Let δ_B^* denote the optimal default level. Then, by applying the smooth pasting condition into the equation (2.33), we obtain

$$\delta_B^* = \frac{\frac{(1-\tau_i)(C+m\,P)x_1}{m+\hat{r}} - \frac{(\tau_e-\tau_i)Cy_1}{\hat{r}}}{K(1+\alpha y_1 + (1-\alpha)x_1)}. \tag{2.34}$$

Remark 2.2.4. *δ_B^* is an increasing function of C, τ_c, τ_d, α and decreasing function of τ_i, m.*

Theorem 2.2.8. *"Smooth Pasting Condition"*
The default level, δ_B^*, given by equation (2.34) *is the optimal one for the equity holders.*

Proof. The proof is very similar to the case in previous chapter, but it is less trivial to show the steps.

1. Step: Let us denote the optimal default barrier δ_B^o, then $\delta_B^o \geq \delta_B^*$ is satisfied.

   From the limited liability assumption of equity holders, we have $EQ(\delta, \delta_B^o) > 0$ $\forall\, \delta > \delta_B^o$. Define $l = \delta/\delta_B$. Then, we obtain

$$\begin{aligned} EQ(\delta, \delta_B^o) &= \frac{V_B^o}{l} + (\tau_e - \tau_i)\frac{C}{\hat{r}}\left(1 - l^{-y_1}\right) - \alpha\, \delta_B^o\, l^{-y_1} \\ &\quad - \frac{(1-\tau_i)\,(C\, +\, m\, P)}{m+\hat{r}}\left(1 - l^{-x_1}\right) - (1-\alpha)\, K\, \delta_B^o\, l^{-x_1} \geq 0. \end{aligned}$$

   Hence $\forall\, \delta > \delta_B^o$, the following inequality is satisfied,

$$\delta_B^o K\left(\frac{1}{l} - \alpha\, l^{-y_1} - (1-\alpha)l^{-x_1}\right) \geq \frac{(1-\tau_i)\,(C\, +\, m\, P)}{m+\hat{r}}\left(1 - l^{-x_1}\right) - (\tau_e - \tau_i)\frac{C}{\hat{r}}\left(1 - l^{-y_1}\right),$$

   since $V_B^o = \delta_B^o K$. In particular,

$$\delta_B^o \geq \lim_{l \downarrow 1} \frac{\frac{(1-\tau_i)\,(C+m\,P)}{m+\hat{r}}\left(1 - l^{-x_1}\right) - (\tau_e - \tau_i)\frac{C}{\hat{r}}\left(1 - l^{-y_1}\right)}{K\left(\frac{1}{l} - \alpha\, l^{-y_1} - (1-\alpha)l^{-x_1}\right)}.$$

   From l'Hospital's rule,

$$\delta_B^o \geq \frac{\frac{(1-\tau_i)\,(C+m\,P)}{m+\hat{r}}\, x_1 - (\tau_e - \tau_i)\frac{C}{\hat{r}}\, y_1}{K\,(1 + \alpha\, y_1 + (1-\alpha)x_1)}.$$

2. Step: $\partial EQ(\delta,\delta_B)/\partial\delta_B < 0, \quad \forall\,\delta \geq \delta_B$.

   In order to prove this step we use the following obvious lemma

   Lemma 2.2.9. *Let $f(z) = z^{\alpha}(A+B) - z^{\beta}(C+D)$, where $A, B, C, D \in \mathbb{R}$. If $0 \leq \alpha \leq \beta$, $A+B \geq C+D$, then $f(z) \geq 0$, $\forall\, 0 \leq z \leq 1$.*

$$\begin{aligned}\frac{\partial EQ(\delta,\delta_B)}{\partial\delta_B} \quad &= \quad -\Bigg[\frac{(\tau_e-\tau_i)C}{\hat{r}\,\delta_B} y_1 \left(\frac{\delta_B}{\delta}\right)^{y_1} + \alpha K(1+y_1)\left(\frac{\delta_B}{\delta}\right)^{y_1} \\ &\quad - \frac{(1-\tau_i)\,(C\,+\,m\,P)}{(m+\hat{r})\delta_B} x_1 \left(\frac{\delta_B}{\delta}\right)^{x_1} + (1-\alpha)K(1-x_1)\left(\frac{\delta_B}{\delta}\right)^{x_1}\Bigg].\end{aligned}$$

   By using lemma 2.2.9 and step 1, we obtain that

$$\frac{\partial EQ(\delta,\delta_B)}{\partial\delta_B} < 0, \quad \forall\,\delta \geq \delta_B.$$

   Note that step 2 implies that for $\delta_B^* < \delta_B^1 < \delta_B^2 < \delta$, we have $EQ(\delta,\delta_B^1) > EQ(\delta,\delta_B^2)$.

3. Step: $\partial EQ(\delta,\delta_B)/\partial\delta \geq 0, \quad \forall\delta \geq \delta_B$.

$$\begin{aligned}\frac{\partial EQ(\delta,\delta_B)}{\partial\delta} \quad &= \quad K + \frac{(\tau_e-\tau_i)C}{\hat{r}\,\delta_B} y_1 \left(\frac{\delta_B}{\delta}\right)^{y_1+1} + \alpha K y_1 \left(\frac{\delta_B}{\delta}\right)^{y_1+1} \\ &\quad - \frac{(1-\tau_i)\,(C\,+\,m\,P)}{(m+\hat{r})\delta_B} x_1 \left(\frac{\delta_B}{\delta}\right)^{x_1+1} + (1-\alpha)K x_1 \left(\frac{\delta_B}{\delta}\right)^{x_1+1}.\end{aligned}$$

   Again by using lemma 2.2.9 and step 1, we obtain the result

$$\frac{\partial EQ(\delta,\delta_B)}{\partial\delta_B} \geq 0, \quad \forall\,\delta \geq \delta_B.$$

By using the above three steps one can show that δ_B^* is the optimal solution as follows. First, δ_B^* satisfies $EQ(\delta,\delta_B^*) \geq 0$, $\forall\delta \geq \delta_B^*$, since $EQ(\delta_B^*,\delta_B^*) = 0$ and EQ is non-decreasing in δ from third step. Second any $\delta_B \in (\delta_B^*, \delta]$ can not deliver better value for equity holders from second step and any δ_B smaller than δ_B^* is excluded by first step. □

When we substitute equation (2.34) into equations (2.27), (2.32), (2.33), we obtain closed form solutions of the debt value, the total firm value and the equity value for given coupon C and principal values P.

2.2.4 Optimal Leverage

In this subsection, the optimal leverage of the firm shall be analysed by taking into account the two step optimisation problem introduced in the previous chapter (see Chapter 1, Section 1.6). Note that in our model the free parameters to maximise the total firm value are P and δ_B. The coupon payment C is either specified exogenously or it is set such that the debt is sold at par.

Definition 2.2.1. *The debt is sold at par, if at the issuance time the total value of the debt is equal to the total principal value, i.e,*

$$D(\delta, \delta_B, P, C) = P. \tag{2.35}$$

If δ is the asset value when the debt is first issued, this constraint requires that C is the smallest solution to equation (2.35). As before, the first step to find the optimal leverage is the choice of the optimal default level, δ_B^* by the equity holders. This optimal default barrier is given by equation (2.34), which also maximises the total firm value, subject to the limited liability constraint of equity holders. Therefore, we can write

$$\nu(\delta, \delta_B^*, C) = \max_{\delta_B} \nu(\delta, \delta_B, C).$$

Clearly, the optimal δ_B^* depends on P, i.e., $\delta_B^* = \delta_B^*(P)$. In the second stage, the firm maximises its value by manipulating P. The following proposition states the relation between ν and P.

Theorem 2.2.10. *After plugging the optimal δ_B^* into $\nu(\delta, \delta_B, C)$, we have that $\nu(\delta, \delta_B^*(P), C)$ is a concave function with respect to P.*

Proof. It can be easily verified that

$$\frac{\partial^2 \nu(\delta, \delta_B^*(P), C)}{\partial P^2} < 0.$$

$\square$

Therefore, we have

$$P^* = \arg\max_{P} \nu(\delta, \delta_B^*, C).$$

Let us denote the optimal total firm value and the optimal debt value respectively by $\nu(\delta, \delta_B^*, C)$ and $D(\delta, \delta_B^*, P^*, C)$. Then the optimal leverage is given by

$$L(\delta, \delta_B^*, P^*, C) = \frac{D(\delta, \delta_B^*, P^*, C)}{\nu(\delta, \delta_B^*, C)}.$$

In this section, we shall use the following basic parameters in our numerical examples

$\delta_0 = 20,\ \mu = 0.02,\ \sigma = 0.3$
$r = 6\%,\ \alpha = 50\%,\ \tau_c = 25\%,\ \tau_i, \tau_c = 15\%$

Table 2.3: Basic parameters, used in the numerical examples in diffusion case.

Figure 2.1 plots the relationship between the total firm value ν and the leverage ratios, for debt maturities from 5 to 100 years. We observe that for short debt maturities, increasing

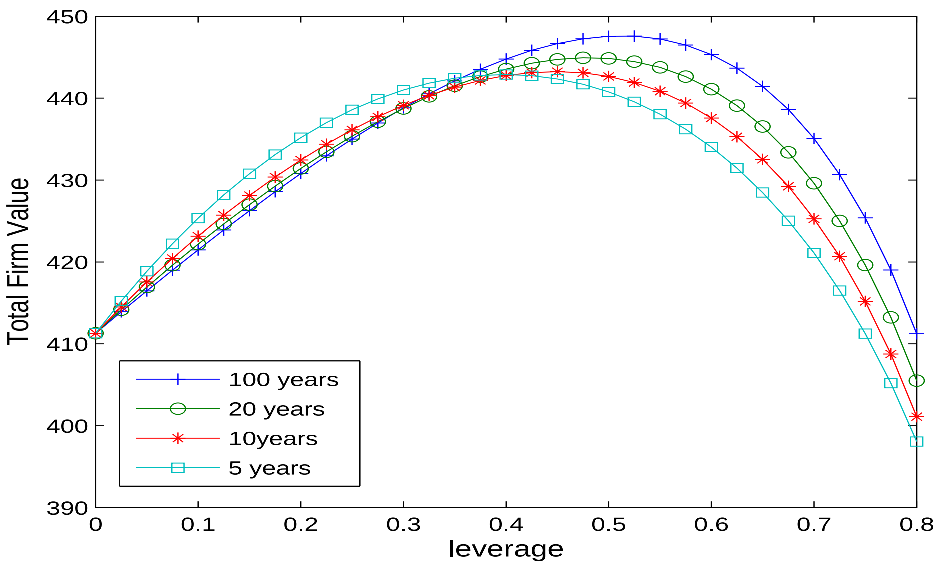


Figure 2.1: Firm value versus leverage.

leverage ratios yields bigger total firm values, because the increasing coupon values do not provoke bankruptcy quickly enough for bankruptcy costs to dominate and to offset the tax gains. However, for large leverage ratios (around up 40%), the bankruptcy costs starts dominating the tax advantage and for longer average maturities, the total firm value becomes bigger. We observe the maximum firm value for the 100 year (console) average maturity, which implies that the bigger the average maturity is, the higher leverage ratios can a company bear and the more tax advantage it can enjoy.

Remark 2.2.5. *Leland [Lel95] introduced a limit on the tax deductibility of coupons to overcome the inconsistency of observing bigger total firm value for shorter average maturities, see Figure 2.1. He assumed that the tax benefits are lost, if the coupon payments exceed the* 20% *of the initial firm value. His aim is to limit the tax deductibility of coupons, when coupons exceed profits. Without such a level, debt capacity using shorter term debt may be arbitrarily large, since* $D(\delta, \delta_B, C, P)$ *is a monotone increasing function of* C*, when debt is sold at par.*

Although, it is a straight forward modeling issue, we will not consider the asset level at which the firm loses the tax deductibility of coupons. Alternatively, we use an exogenously given fixed coupon rate in our numerical examples .

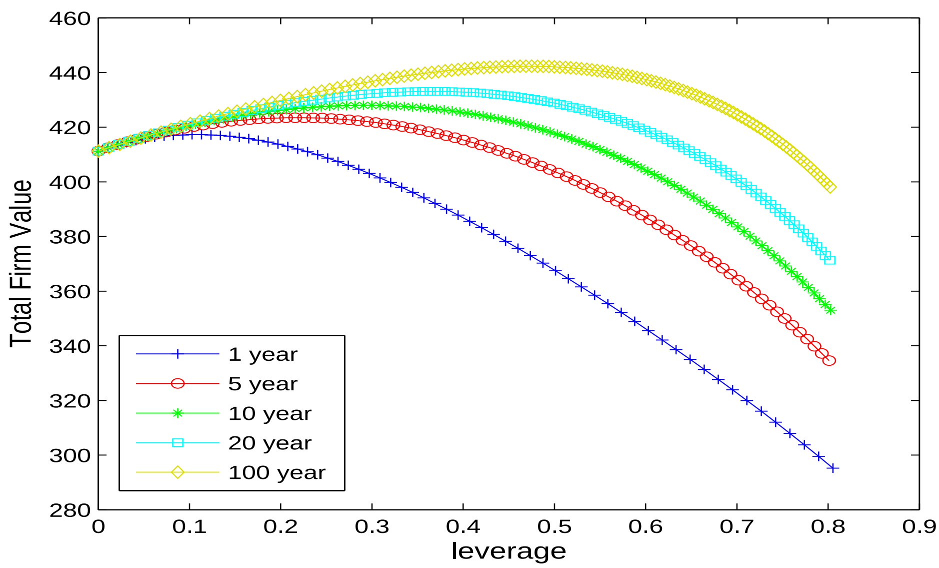


Figure 2.2: Total firm value versus leverage, for different average maturities and a fixed coupon rate 6.091%.

In figure 2.2, we observe that the tax advantage does not any more reach unrealistic big amounts, and hence the firm value is not bigger for smaller debt maturities. The coupon rate is chosen to be 6.091%, which is the par coupon rate for riskfree bonds with semi-annual coupon payments when the continuously compounded interest rate is 6%.

In figure 2.3, we observe the relation between the total firm value and the corporate tax. In contrast to the previous chapter, we do not observe the unrealistic increase in the total firm value as the corporate tax increases, since the tax advantage to debt is modeled as a reduction on the total tax payment to the government, i.e, increasing corporate tax implies bigger amount of tax payment to the government and hence the total firm value decreases.

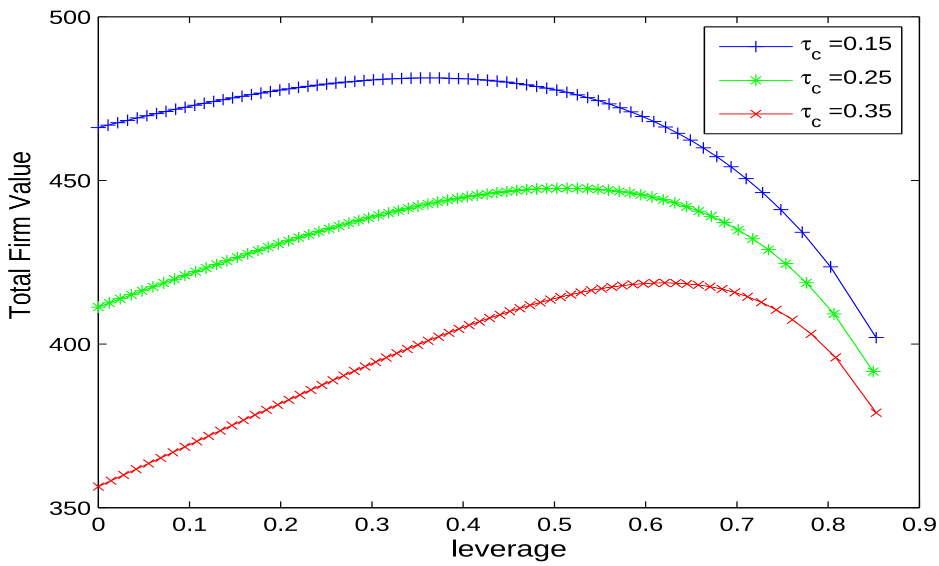


Figure 2.3: Total firm value versus leverage for different corporate tax rates, average debt maturity is 100 years (console debt) and coupon is set by solving equation (2.35).

2.2.5 Credit Spreads

The aim of this subsection is twofold. First, we shall explore the behavior of the credit spreads by using the firm bonds. As a result, we state and prove a theorem, which claims that credit spread tends to zero as maturity tends to zero. Second, we shall analyse the credit spread of the total debt value by plotting it with respect to the leverage levels.

Recall the defaultable bond price is given by

$$\begin{aligned}\overline{B}_{\Theta}(0,t;\rho,1) =&\mathbb{E}_Q\left(\int_0^{t\wedge\tau_B}(1-\tau_i)\,\rho\,1\,e^{-\hat{r}s}\,ds\right)+\mathbb{E}_Q\big((1-\tau_i)\,1\,e^{-\hat{r}t}\,1_{\{t<\tau_B\}}\big)\\ &+\mathbb{E}_Q\left(e^{-\hat{r}t}\,R\,1_{\{t\geq\tau_B\}}\right)\\ =&(1-\tau_i)\left(1-\frac{\rho}{\hat{r}}\right)\,e^{-\hat{r}t}Q\,(t<\tau_B)+(1-\tau_i)\frac{\rho}{\hat{r}} \qquad (2.36)\end{aligned}$$

$$big(1-\mathbb{E}_Q\left(e^{-\hat{r}\tau_B}1_{\{t\geq\tau_B\}}\right)\big)$$

$$+\,e^{-\hat{r}t}R\,Q\,(t\geq\tau_B)\,. \qquad (2.37)$$

The default (survival) probabilities are known form the Theorem 2.2.1. The only unknown quantity left is the expectation in the above equation. If $\delta(t)$ were a traded asset, we could have calculated the above expectation analytically by using a change of measure technique, introduced by Ericsson and Reneby [ER06]. Since this is not the case, we will approximate the expectation by employing a numerical integration method. Let us consider the expectation term in equation (2.37).

$$\mathbb{E}_Q\left(e^{-\hat{r}\tau_B}1_{\{t\geq\tau_B\}}\right)=\int_0^t e^{-\hat{r}u}f(u)\,du=\int_0^t g(u)\,du \qquad (2.38)$$

where $g(t):=e^{-\hat{r}t}f(t)$ and $f(t)$ is the first hitting time of the process $X(t)$ to the given boundary z. See Theorem 2.2.1. The following proposition states the approximation of the defaultable bond price, given by equation 2.37

Proposition 2.2.11. *By using the extended Trapezoid rule*[5] *to approximate the integral in equation* (2.38)*, we have*

$$\begin{aligned}\overline{B}_{\Theta}(0,t;\rho,1) =&(1-\tau_i)\left(1-\frac{\rho}{\hat{r}}\right)\,e^{-\hat{r}t}Q\,(t<\tau_B)+e^{-\hat{r}t}\,R\,Q\,(t\geq\tau_B)\\ &+(1-\tau_i)\frac{\rho}{\hat{r}}\left(1-\frac{t}{2N}\big(g(t_0)+g(t_{N-1})\big)+\frac{t}{N}\sum_{i=2}^{N-2}g(t_i)\right)+o\left(\frac{t^3|g''(M)|}{N^2}\right),\end{aligned}$$

where N is the discretisation number of the interval $[0,t]$, $0=t_0<t_1<\ldots<t_{N-1}=t$ are the equidistant discretisation points and M maximises $|g''(x)|$ in $[0,t]$.

In all structural models, the conditional probability that default occurs before $t+h$, conditioned on no default until time t, converges to zero as h does. The convergence rate is $o(h)$ in the pure diffusion models. Next corollary states this fact.

[5] We refer to Press *et al.* [PTVF02]

Corollary 2.2.12.

$$\lim_{h\searrow 0}\frac{1}{h}\,Q\big(\tau_B\le t+h|\mathcal{F}_t\big)=\lim_{h\searrow 0}\frac{1}{h}\left(\Phi\left(\frac{z-\gamma h}{\sigma\sqrt{h}}\right)+e^{2\gamma z\sigma^{-2}}\Phi\left(\frac{z+\gamma h}{\sigma\sqrt{h}}\right)\right)=0. \tag{2.39}$$

Proof. It is a direct consequence of Theorem 2.2.2. Apply the l'Hospital's rule and observe for a solvent company with $z=\ln(\delta_B/\delta)$ □

Remark 2.2.6. *In Section 2.3, we will model the underlying as a jump-diffusion process. As a consequence, we shall see that limit* (2.39) *will be non-zero, i.e, the convergence rate will be $O(h)$.*

Note that above corollary allows us to write, for an infinitesimal time interval $(0,ds]$,

$$Q\big(\tau_B\in(0,ds]\big)=0.$$

In pure diffusion models, this fact forces credit spreads to tend to zero as maturity decreases to zero. Let us define a risk-free bond, $B(0,t;\rho,1)$ with maturity t and unit principal, paying constant continuous coupons ρ. It can be replicated by the riskless zero coupon bonds in the market and its price is given as follows

$$B(0,t;\rho,1)=(1-\tau_i)(e^{-\hat{r}t}+\int_0^t\rho\,e^{-\hat{r}s}ds). \tag{2.40}$$

The corporate spread, we examine is the difference between the yield to maturity on the corporate bond and risk-free interest rate. Therefore, corporate yield spread, denoted by $s(t)$ is equal to

$$s(t):=-\frac{\ln(\overline{B}_\Theta(0,t;\rho,1))-\ln(B(0,t;\rho,1))}{t}. \tag{2.41}$$

Corollary 2.2.13. *The credit spread defined above tends to zero as maturity tends to zero, i.e.*

$$s(0)=\lim_{t\downarrow 0}s(t)=0.$$

Proof. For $t\in(0,ds]$, by using Corollary 2.2.12, we get

$$\begin{aligned}\overline{B}_\Theta(0,t;\rho,1)&=(1-\tau_i)\rho\frac{1}{\hat{r}}\,\big(1-e^{-\hat{r}t}\big)+(1-\tau_i)\,e^{-\hat{r}t}Q\big(\tau_B\notin(0,t]\big)+R\,e^{-\hat{r}t}Q\big(\tau_B\in(0,t]\big)\\&=(1-\tau_i)\rho\frac{1}{\hat{r}}\,\big(1-e^{-\hat{r}t}\big)+(1-\tau_i)\,e^{-\hat{r}t}.\end{aligned} \tag{2.42}$$

By substituting equations (2.40) and (2.42) into equation (2.41) as $t \downarrow 0$ and applying l'Hospital's rule, we obtain the result. □

In figure 2.4, the credit spread of a bond with continuous coupon payments ρ equal to 8.162%[6] is plotted. It is assume that the firm, issuing the bond, has 5 years average debt maturity. The riskless interest rate is set to 8% and the other parameters are the same as in Table 2.3. The defaultable bond prices are approximated by the Monte Carlo simulation. The risk-free bonds are calculated form equation (2.40). We observe that the assertion in Proposition 2.2.13 holds, namely credit spreads is almost zero for short maturities (at least for several months).

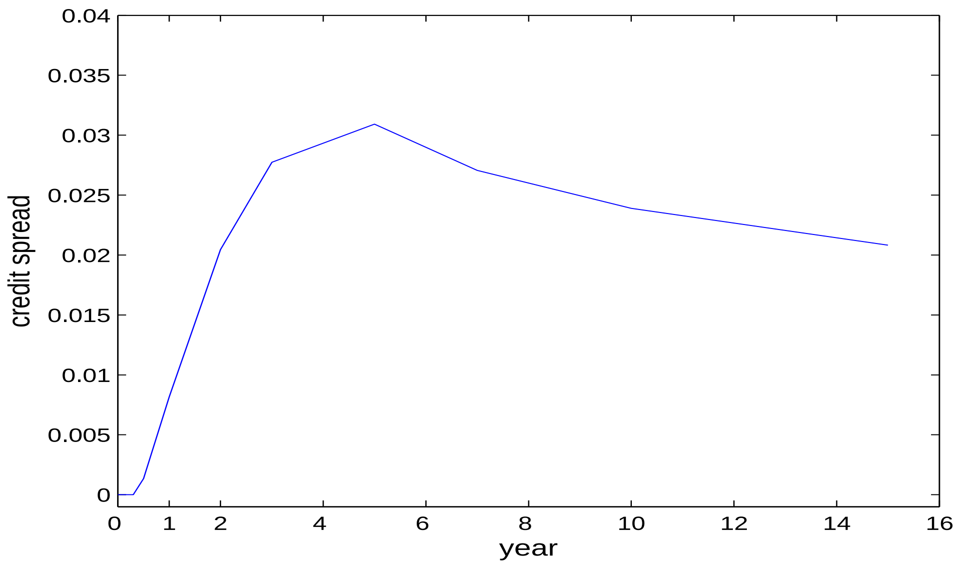


Figure 2.4: Credit spread curve of a firm bond. The firm has a leverage level of 70%, its average debt maturity is 5 years, $r = 8\%$, $\rho = 8.162\%$. The other parameters are the same as in Table 2.3.

By following Leland [Lel95], the credit spread of the total debt is defined as $C/P - r$. Figure 2.5 plots the credit spread of the total debt with respect to the leverage by using the parameters in Table 2.2. The coupon payment is set such that C is the solution of

[6]It is the par coupon rate for risk-free bonds with semi-annual coupon payments when the continuously compounded interest rate is 8%.

equation (2.35). We obverse that the credit spread of the total debt increases as leverage increases, since an increase in the leverage implies a higher default probability. Therefore, the debt holders demand higher amount of credit spreads to bear the likely default. We also observe that as the average maturity of the debt decreases, the credit spread increases, since for short maturities it is less probable for the firm to turn over the debt. Therefore, also in this case debt holders demand high credit spreads. Notice that even for small leverage levels the credit spread is non-zero.

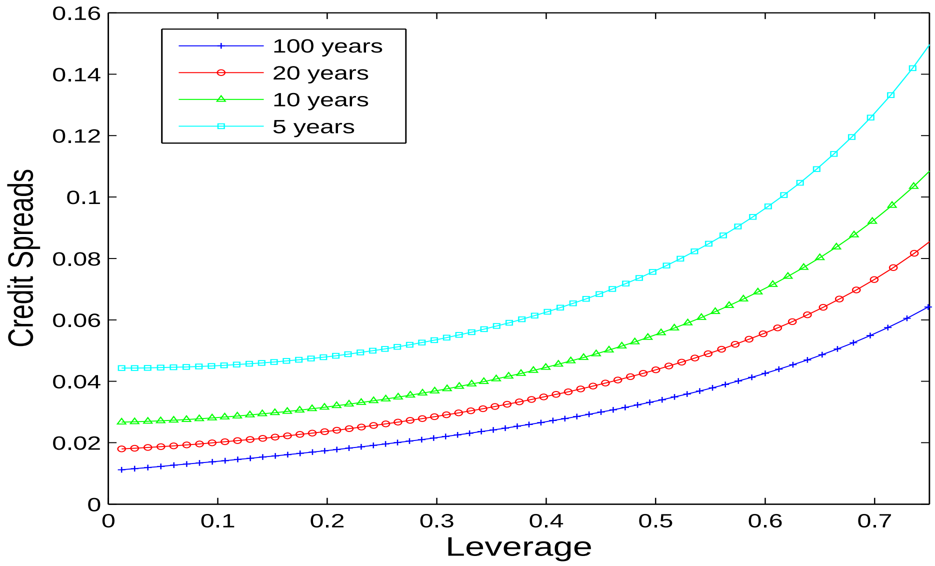


Figure 2.5: Credit spreads versus leverage for different maturities in diffusion case. The coupon is set by solving equation (2.35). The other parameters are the same as in Table 2.3.

Remark 2.2.7. *By using Proposition 2.2.11, we can numerically calculate the model bond prices and calibrate them to the market ones to obtain the drift term μ and the volatility σ of the EBIT process. Let us denote the market price of firm's issued bonds by $\overline{B}^M_{i,\Theta}(0,t;\rho,1)$, $i=1,\ldots,N$. The model prices for each bond are given by Proposition 2.2.11, depending on the parameter set Θ. Then these model prices are calibrated to the market as follows,*

$$\Theta = \arg\min_{\Theta} \sum_{i=1}^{N} \left(\overline{B}^M_{i,\Theta}(0,t;\rho,1) - \overline{B}_{i,\Theta}(0,t;\rho,1) \right)^2 .$$

2.2.6 Comparative Statistics

Debt Value and Debt Capacity

In figure 2.6, the debt value D is plotted as a function of leverage (D/ν) for different maturities, given that P and debt value D coincide at current value $\delta = 20$. The parameters are the same as in Table 2.3. We observe that the debt capacity is the maximal value of the debt value curve. Note that the debt capacity is smaller for shorter maturities. Maximal debt value occurs at higher coupon levels (denoted C_{max}) for shorter term debt, but at approximately the same leverage (80%-85%) for different maturities of debt.

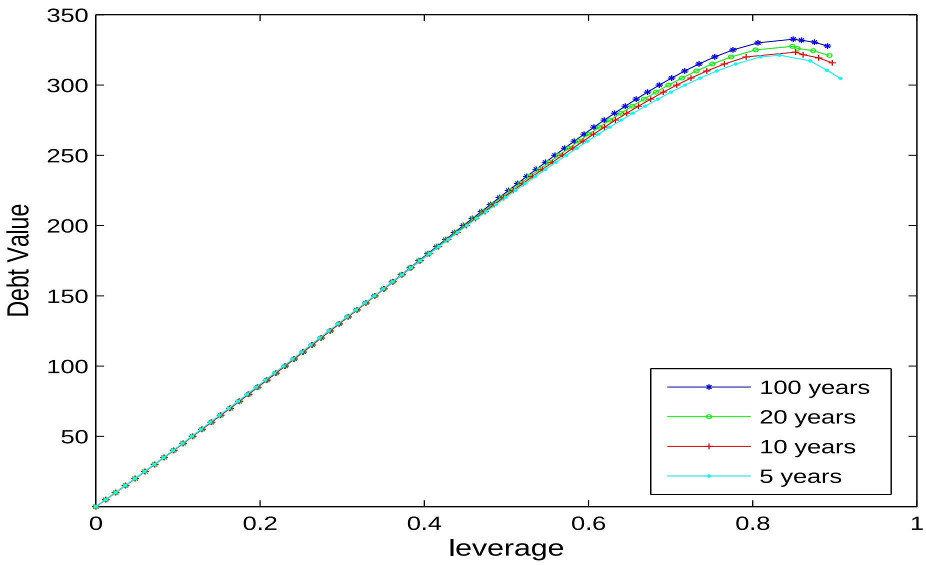


Figure 2.6: Debt value versus leverage for different maturities. The coupon is set by solving equation (2.35). The other parameters are the same as in Table 2.3.

As volatility σ and/or bankruptcy costs α increase, the debt value decreases, since the default becomes more possible and bankruptcy costs will be more imminent. In Figure 2.7 and Figure 2.8, we observe these facts. In Figure 2.9 and Figure 2.10, we plot the debt value with respect to the leverage for different corporate tax and interest tax levels respectively. It can be observed that as τ_c increases, debt value decreases, since δ_B^* is an increasing function with respect to τ_c. On the other hand, increasing τ_i implies increasing debt value. By recalling equation (2.27), one expects that the debt value decreases as τ_i

increases. However, we already mentioned in Remark 2.2.4 that the optimal default level decreases as τ_i increases and this effect dominates.

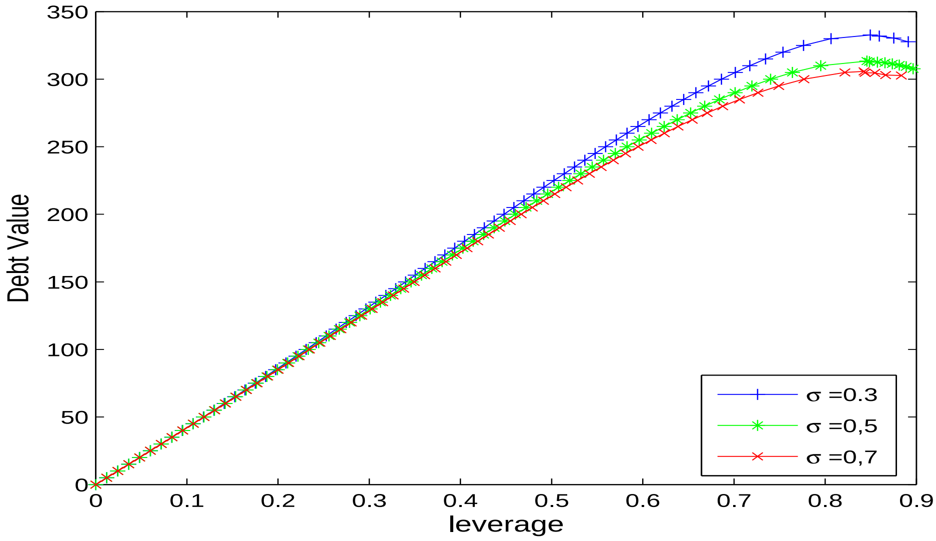


Figure 2.7: Debt value versus leverage for different volatilities. The average maturity is 100 years and coupon is set by solving equation (2.35). The other parameters are the same as in Table 2.3.

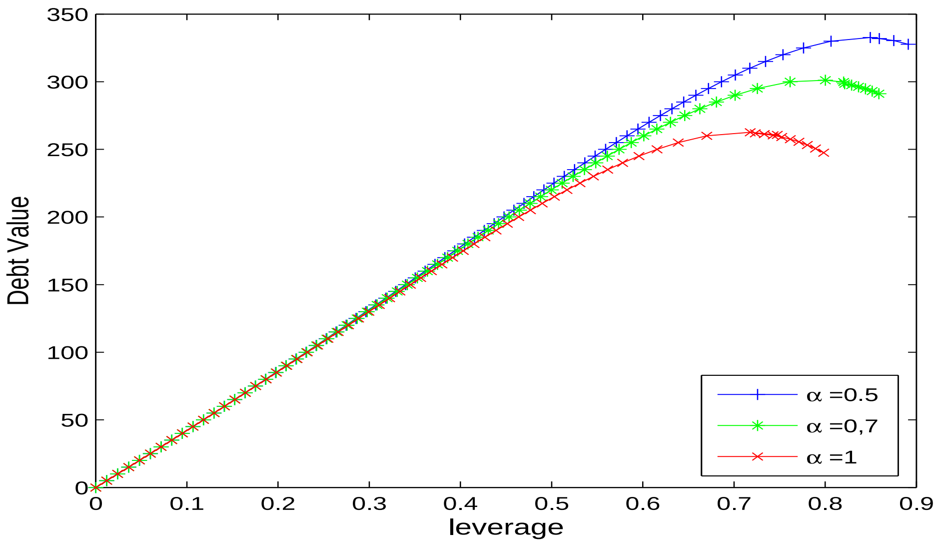


Figure 2.8: Debt value versus leverage for different bankruptcy rates. The average maturity is 100 years and coupon is set by solving equation (2.35). The other parameters are the same as in Table 2.3.

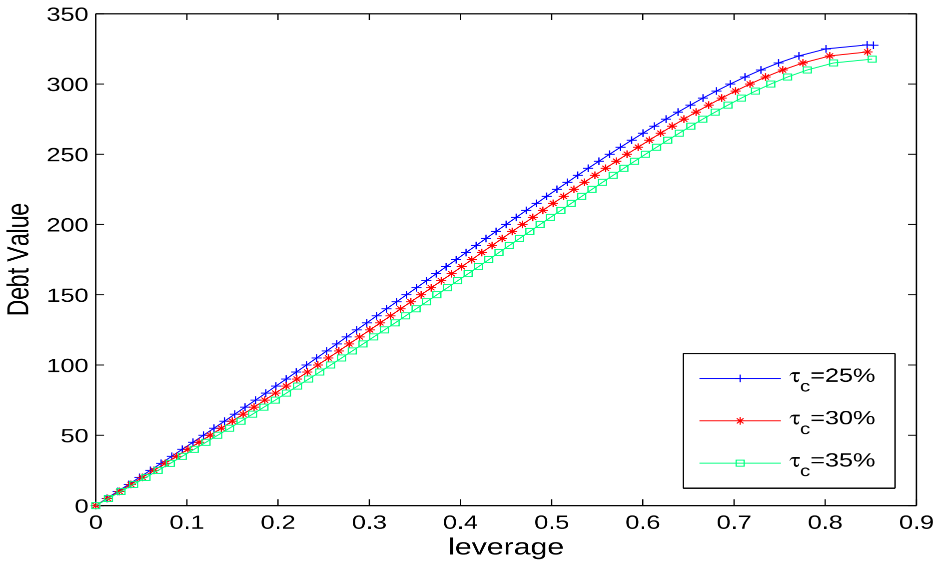


Figure 2.9: Debt value versus leverage for different corporate tax levels. The average maturity is 100 years and coupon is set by solving equation (2.35). The other parameters are the same as in Table 2.3.

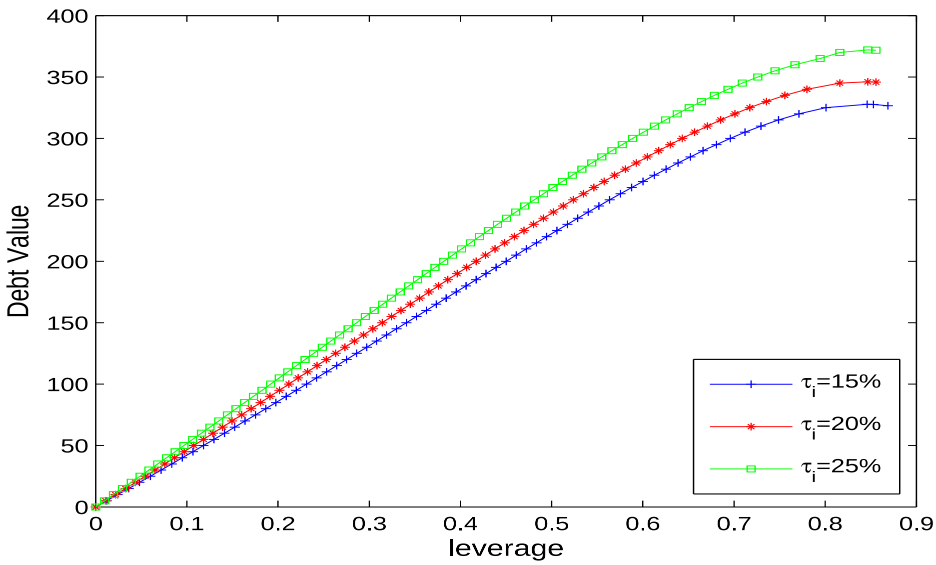


Figure 2.10: Debt value versus leverage for different interest tax levels. The average maturity is 100 years and coupon is set by solving equation (2.35). The other parameters are the same as in Table 2.3.

2.3 Jump Diffusion Case

Leland *et al.* [LT96] modeled the firm value process with a GBM, by taking into account the maturity of the debt. The disadvantage of their approach is that it introduces almost negligible credit spread for short maturities. But it is a well known fact that even for short maturities market induces non-zero credit spread. To include this fact in our model, we introduce a jump-diffusion process to model our underlying asset, namely EBIT value.

The EBIT value is modeled directly under an equivalent martingale measure Q as a jump-diffusion process. It is assumed to be a non-tradable asset. The Q-dynamics of the EBIT process is given by the following SDE

$$d\delta(t) = \delta(t-)\Big[\,(\mu - \lambda\Xi)\;dt + \sigma\,dW(t) + \big(J(t)-1\big)dN(t)\Big], \tag{2.43}$$

where $W(t)$ is the Brownian motion under Q, $N(t)$ is the compound Poisson process, whose marks, $Y(t) := \log J(t)$, are double exponentially distributed with the following probability density function

$$f_Y(y) = p_1\;\xi_1\;e^{-\xi_1 y}\;1_{\{y\geq 0\}} + p_2\;\xi_2\;e^{\xi_2 y}\;1_{\{y<0\}}, \tag{2.44}$$

where p_1 and p_2 are the probability of up and down jumps, they satisfy $p_1 + p_2 = 1$. The up-jump intensity and down-jump intensity are denoted by ξ_1 and ξ_2, repectively. They satisfy $\xi_1 > 1$ and $\xi_2 > 0$. The intensity of the poisson process $N(t)$ is assumed to be constant λ. The drift μ and diffusion σ coefficients are constant. Ξ is the expected value of $(J(t)-1)$. $W(t)$, $N(t)$, $Y(t)$ are assumed to be mutually independent. The expectation of the jump height can be easily computed

$$\Xi := \mathbb{E}_Q\big(J(t)-1\big) = \frac{p_1\,\xi_1}{\xi_1 - 1} + \frac{p_2\,\xi_2}{\xi_2 + 1} - 1.$$

Let us consider the same market, introduced in Section 2.2, namely there are two assets: the money market account and a tradeable derivative of the EBIT value (for instance the equity or bond.) Adding jumps in our underlying asset implies that the market is incomplete. In other words, we can not replicate the CCs by using the existing tradable assets in the market, since there is an infinite number of randomness in the market but just a finite number of assets to hedge this randomness. Therefore, there are infinitely many equivalent martingale measures[7] in the market and the market "decides" which equivalent martingale measure is to use via the given market prices.

Remark 2.3.1. *Note that an alternative representation of the last term in equation* (2.43) *is*

$$d\left(\sum_{i=1}^{N(t)}(J_i - 1)\right).$$

In equation (2.43), $J(t)$ *denotes the piecewise constant, left-continuous time interpolation of the sequence* J_i.

The default boundary and the default time are respectively denoted by δ_B and τ_B, where δ_B will be determined endogenously by the equity holders, taking into account the limited liability and τ_B is defined as follows,

$$\tau_B := \inf\{t \geq 0 : \delta(t) \leq \delta_B\}. \tag{2.45}$$

When the process (2.43) crosses the down-barrier δ_B, the firm goes bankrupt. Therefore, we shall compute the distribution of the first passage time of the process, given in equation (2.43). If we had a geometric Brownian motion, we could easily find it, either by using the reflection principle of Brownian motion or by calculating the Laplace transforms[8] of it. But the jump term in our model may incur undershoot, which means when a jump-diffusion process crosses the boundary level, sometimes it hits exactly on the boundary and sometimes it crosses the boundary. The undershoot causes some problems for computing the distribution of the first passage times analytically[9]. Firstly, one needs the exact distribution of the undershoot. This is possible to find, if the jump heights are distributed with exponential type distributions because of the memoryless property of such kind of distributions. Secondly, the dependence structure between the undershoot and the first passage time must be known. Under exponentially distributed jump heights, these two random variables are conditionally independent, conditioned on the undershoot.

[7] For the connection between an incomplete market and infinitely many equivalent measures, we refer to Björk [Bjö98].

[8] We refer to Karatzas and Shreve [KS00].

[9] See Kou and Wang [KW03]

2.3.1 Preliminaries

In this subsection, we shall formulate the above facts in a proposition and derive the Laplace transform of the first passage time of the process given by equation 2.43 to a given boundary (δ_B). But first, let us introduce some notations. Recall equation (2.43),

$$d\delta(t) = \delta(t-)\Big[(\mu - \lambda\Xi)\, dt + \sigma dW(t) + \big(J(t) - 1\big) dN(t)\Big].$$

If we apply the generalized Itô formula or by the or by the Doleans-Dade exponential formula[10] for the jump-diffusion processes to the function $F(t, \delta(t)) := \log \delta(t) \ \in \mathbb{C}^2$, then we obtain

$$\delta(t) = \delta \ e^{(\mu - 0.5\sigma^2 - \lambda\Xi)t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i}. \tag{2.46}$$

Let us define the stochastic process $X(t)$ as the natural logarithm of equation (2.46), then we have

$$X(t) := \gamma t + \sigma\, W(t) + \sum_{i=1}^{N(t)} Y_i, \tag{2.47}$$

where $\gamma = \mu - 0.5\,\sigma^2 - \lambda\Xi$.

Proposition 2.3.1. *The following properties of $X(t)$ can be found in Cont and Tankov [CT04].*

- *The infinitesimal generator of $X(t)$, for all $u(x) \in \mathbb{C}^2$, is*

$$\Lambda u(x) = \frac{1}{2}\,\sigma^2\, u''(x) \ + \ \gamma\, u'(x) \ + \ \lambda \int_{-\infty}^{\infty} \big(u(x+y) - u(x)\big)\, f_Y(y)\, dy. \tag{2.48}$$

- *Expectation and variance of $X(t)$ are*

$$\mathbb{E}_Q\left(X(t)\right) = \left(\gamma + \lambda\left(\frac{p_1}{\xi_1} - \frac{p_2}{\xi_2}\right)\right) t, \qquad var\big(X(t)\big) = \left(\sigma^2 + \lambda\left(\frac{p_1}{\xi_1^2} + \frac{p_2}{\xi_2^2}\right)\right) t.$$

- *The moment generating function of $X(t)$ is*

$$\mathbb{E}_Q(e^{\theta X(t)}) = e^{G(\theta)t},$$

where

$$G(x) = \gamma\, x \ + \ \frac{1}{2}\, x^2\, \sigma^2 \ + \ \lambda\left(\frac{p_1\,\xi_1}{\xi_1 - x} + \frac{p_2\,\xi_2}{\xi_2 + x} - 1\right). \tag{2.49}$$

[10] For the generalised Itô formula and the Doleans-Dade exponential formula for the jump-diffusion processes, we refer to Brémaud [Bre81].

The moment generating function of $X(t)$ can be found by either applying the Levy-Khintichin formula or directly computing the above expecation. The function $G(x)$ plays an important role in the derivation of the Laplace transform of the first passage times. Therefore, we shall states some facts about the function $G(x)$, which can be found in Kou and Wang [KW03].

Lemma 2.3.2. *For any $\alpha > 0$, $G(x) = \alpha$ has exactly four roots; $\beta_{1,\alpha}$,$\beta_{2,\alpha}$,$-\beta_{3,\alpha}$, $-\beta_{4,\alpha}$ where $0 < \beta_{1,\alpha} < \xi_1 < \beta_{2,\alpha} < \infty$ and $0 < \beta_{3,\alpha} < \xi_2 < \beta_{4,\alpha} < \infty$.*

Moreover, let the overall drift of the jump-diffusion process be

$$\bar{u} = \gamma + \lambda \left(\frac{p_1}{\xi_1} - \frac{p_2}{\xi_2} \right). \tag{2.50}$$

Then as $\alpha \to 0$,

$$\beta_{3,\alpha} \to \begin{cases} \beta_3^*, & if\ \bar{u} > 0; \\ 0, & if\ \bar{u} \leq 0, \end{cases}, \quad and \quad \beta_{4,\alpha} \to \beta_4^*,$$

where $-\beta_3^*$ and $-\beta_4^*$ are defined as the unique roots of $G(x) = \alpha$, as $\alpha \to 0$, i.e.,

$$G(-\beta_3^*) = 0, \qquad G(-\beta_4^*) = 0, \qquad 0 < \beta_3^* < \xi_2 < \beta_4^* < \infty.$$

Proof. Figure 2.11 plots the graph of $G(x)$. One can easily verify the first claim of the lemma with standard arguments from analysis. The second claim of the lemma, i.e., the limiting results when $\alpha \to 0$ follow easily once we note that $G(0) = \lambda(p_1 + p_2 - 1) = 0$ and $G'(0) = \bar{u}$. $\square$

The next lemma gives the analytic solution for the above roots. It can be found in Kou, Petrella and Wang [KPW05].

Lemma 2.3.3. *The equation $G(x) = \alpha$ is indeed a quartic polynomial*

$$ax^4 + bx^3 + cx^2 + dx + e = 0,$$

where

$$a = \sigma^2, \;\; b = 2\gamma - \sigma^2(\xi_1 - \xi_2), \;\; c = -\sigma^2\,\xi_1\,\xi_2 - 2\gamma\,(\xi_1 - \xi_2) - 2\lambda - 2\alpha$$
$$d = -2\gamma\,\xi_1\,\xi_2 - 2\,\lambda\,p_1\,(\xi_1 + \xi_2) + 2\lambda\,\xi_1 + 2\,\alpha\,(\xi_1 - \xi_2), \;\; e = 2\alpha\,\xi_1\,\xi_2.$$

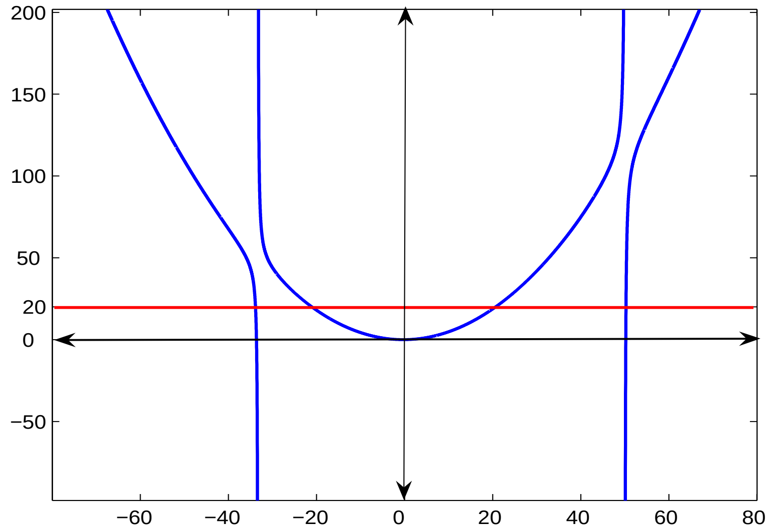


Figure 2.11: Graph of $G(x) = 20$ for $\xi_1 = 50$ and $\xi_2 = 33.3$.

The roots of the above polynomial are given by

$$\beta_{1,\alpha} = -\frac{b}{4a} + \frac{n_1 - n_3}{2}, \ \beta_{2,\alpha} = -\frac{b}{4a} + \frac{n_1 + n_3}{2}, \ \beta_{3,\alpha} = \frac{b}{4a} + \frac{n_1 - n_2}{2}, \ \beta_{4,\alpha} = \frac{b}{4a} + \frac{n_1 + n_2}{2},$$

where

$$\begin{aligned}
&n_1 = \sqrt{B_3 + C_0 + C_1}, \ n_2 = \sqrt{B_4 - C_0 - C_1 - \frac{B_5}{4n_1}}, \ n_3 = \sqrt{B_4 - C_0 - C_1 + \frac{B_5}{4n_1}},\\
&B_0 = c^2 - 3b\,d + 12a\,e, \ B_1 = 2c^3 - 9b\,c\,d + 27a\,d^2 + 27b^2\,e - 72a\,c\,e,\\
&B_2 = \sqrt{B_1^2 - 4B_0^3}, \ B_3 = \frac{b^2}{4a^2} - \frac{2c}{3a}, \ B_4 = \frac{b^2}{2a^2} - \frac{4c}{3a}, \ B_5 = \frac{4b\,c}{a^2} - \frac{8d}{a} - \left(\frac{b}{a}\right)^3,\\
&\widetilde{B} = \sqrt[3]{B_1 + B_2}, \ C_0 = \frac{\sqrt[3]{2}B_0}{3a\,\widetilde{B}}, \ C_1 = \frac{\widetilde{B}}{3\sqrt[3]{2}a}.
\end{aligned}$$

Proof. The technique to solve the quartic equation was first developed by Ferrari. We refer to Borwein and Erdèlyi [BE95]. □

The default time (2.45) can be rewritten as

$$\tau_B = \inf\Big\{t \geq 0 : X(t) \leq b\Big\}, \tag{2.51}$$

where

$$b := \log\left(\frac{\delta_B}{\delta}\right). \tag{2.52}$$

The following proposition state that the jump diffusion process (2.47) and the default time (2.51) are conditionally independent and the process (2.47) is conditionally memoryless, conditioned on the undershoot.

Proposition 2.3.4. *For any $x < 0$*

$$\begin{aligned} Q\big(\tau_B \le t,\, X(\tau_B) - b \le x\big) &= e^{\xi_2 x} Q\big(\tau_B \le t,\, X(\tau_B) - b < 0\big) &(2.53)\\ Q\big(X(\tau_B) - b \le x \,|\, X(\tau_B) - b < 0\big) &= e^{\xi_2 x}. &(2.54)\end{aligned}$$

Furthermore, conditional on $X_{\tau_B} - b < 0$, the stopping time τ_B and the undershoot are independent; more precisely, for any $x < 0$,

$$\begin{aligned} &Q\big(\tau_B \le t,\, X(\tau_B) - b \le x \,|\, X(\tau_B) - b < 0\big)\\ = \;&Q\big(\tau_B \le t, \,|\, X(\tau_B) - b < 0\big)\, Q\big(X(\tau_B) - b \le x \,|\, X(\tau_B) - b < 0\big). \qquad (2.55)\end{aligned}$$

Proof. We refer to Kou and Wang [KW03] and follow the steps to prove it for our case. Firstly, we will prove equation (2.53) and equation (2.54). Let us denote $T_1, T_2, \ldots$ the arrival times of the Poisson process $N(t)$. Let A_i be the event that the process $X(t)$ has not crossed the barrier b until time T_i,

$$A_i := \big\{\min_{0 \le s < T_i} X_s > b\big\}.$$

We consider the probability distribution of the undershoot, when the process $X(t)$ crosses the barrier b. Note that the default barrier b is an absorbing barrier. In other words; when the process $X(t)$ crosses the barrier, it ceases to exist. Therefore, the following equalities hold

$$\begin{aligned} Q(\tau_B \le t,\, X_{\tau_B} - b \le x) &= Q\left(\bigsqcup_{i=1}^{\infty}\Big\{T_i \le t,\, X_{T_i} - b \le x,\, A_i\Big\}\right)\\ &= \sum_{i=1}^{\infty} Q\Big(T_i \le t,\, X_{T_i} - b \le x,\, A_i\Big) =: \sum_{n=1}^{\infty} Q_i,\end{aligned}$$

where[11]

$$
\begin{aligned}
Q_i &= Q\Big(T_i \le t,\, X_{T_i} - b \le x,\, A_i\Big) \\
&= Q\ \Big(Q\Big(T_i \le t,\, X_{T_i} - b \le x,\, A_i\Big)\,\big|\, \mathcal{F}_{T_i^-},\, T_i\Big) \\
&= \mathbb{E}_Q\ \Big(Q\Big(X_{T_i} \le b + x \,|\, \mathcal{F}_{T_i^-},\, T_i\Big)\, 1_{\{A_i,\, T_i \le t\}}\Big) \\
&= \mathbb{E}_Q\ \left(Q\Big(Y_i \le b + x - \gamma t - \sigma W(t) - \sum_{j=1}^{i-1} Y_j \,\big|\, \mathcal{F}_{T_i^-},\, T_i\Big)\ 1_{\{A_i,\, T_i \le t\}}\right) \\
&= \mathbb{E}_Q\ \left(\int_{-\infty}^{b+x-\gamma t-\sigma W(t)-\sum_{j=1}^{i-1} Y_j} p_2\, \xi_2\, e^{\xi_2 s} ds\ 1_{\{A_i,\, T_i \le t\}}\right) \\
&= e^{\xi_2 x}\, \mathbb{E}_Q\ \Big(p_2\, e^{\xi_2 (b-\gamma t-\sigma W(t)-\sum_{j=1}^{i-1} Y_j)}\ 1_{\{A_i,\, T_i \le t\}}\Big) \\
&= e^{\xi_2 x}\, \mathbb{E}_Q\ \Big(Q\Big(X_{T_i} < b \,|\, \mathcal{F}_{T_i^-, T_i}\Big)\ 1_{\{A_i,\, T_i \le t\}}\Big) \\
&= e^{\xi_2 x}\, Q\ (T_i \le t,\, X_{T_i} < b,\, A_i)\,.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
Q(\tau_B \le t,\, X_{\tau_B} - b \le x) &= \sum_{i=1}^{\infty} Q\Big(T_i \le t,\, X_{T_i} - b \le x,\, A_i\Big) \\
&= e^{\xi_2 x} \sum_{i=1}^{\infty} Q\Big(T_i \le t,\, X_{T_i} - b \le 0,\, A_i\Big) \\
&= e^{\xi_2 x} Q(\tau_B \le t,\, X_{\tau_B} - b < 0),
\end{aligned}
$$

which proves equation (2.53). The equality (2.54) follows by letting $t \to \infty$ and observing that, on the set $\{X_{\tau_B} < b\}$ the hitting time τ_B is finite by definition. The equality (2.55) holds since

$$
\begin{aligned}
Q(\tau_B \le t,\, X_{\tau_B} - b \le x \,|\, X_{\tau_B} - b < 0) &= \frac{Q(\tau_B \le t,\, X_{\tau_B} - b \le x)}{Q(X_{\tau_B} - b < 0)} \\
&= e^{\xi_2 x}\, \frac{Q(\tau_B \le t,\, X_{\tau_B} - b < 0)}{Q(X_{\tau_B} - b < 0)} \\
&= Q(\tau_B \le t,\, |\, X_{\tau_B} - b < 0) Q(X_{\tau_B} - b \le x \,|\, X_{\tau_B} - b < 0).
\end{aligned}
$$

□

[11] $\bigsqcup$ stands for the disjoint union.

The following theorem is an adaptation of the Kou and Wang [KW03]'s result in our case. It states the Laplace transformation of the distribution of the first passage times of the process (2.47) to an (endogenously) given boundary b.

Theorem 2.3.5 (Laplace Transformation of the First Passage Times). *For any $\alpha > 0$, the Laplace transformation of the first passage times of the process, defined by equation* (2.47) *to the boundary b, defined in equation* (2.52), *is given by*

$$\mathbb{E}_Q(e^{-\alpha\tau_B}) = \underbrace{\frac{(\beta_{3,\alpha} - \xi_2)}{\xi_2} \frac{\beta_{4,\alpha}}{(\beta_{3,\alpha} - \beta_{4,\alpha})}}_{=:A(\alpha)} e^{\beta_{3,\alpha} b} + \underbrace{\frac{(\xi_2 - \beta_{4,\alpha})}{\xi_2} \frac{\beta_{3,\alpha}}{(\beta_{3,\alpha} - \beta_{4,\alpha})}}_{=:B(\alpha)} e^{\beta_{4,\alpha} b}, \quad (2.56)$$

where $-\beta_{3,\alpha}$, $-\beta_{4,\alpha}$ are the roots of $G(x) = \alpha$, where $G(x)$ is given by equation (2.49).

Proof. Let us recall equation (2.48) and consider any function $g(\cdot)$ in $\mathbb{C}^2$. Then, from the Ito's formula, it follows that

$$\mathbb{E}_Q\left(e^{-\alpha(t\wedge\tau_B)} g(X_{t\wedge\tau_B})\right) = g(0) + E_Q\Bigg(\int_0^{t\wedge\tau_B} e^{-\alpha s}\Big[-\alpha g(X_s) + \gamma g'(X_s) + \frac{1}{2}\sigma^2 g''(X_s) + \lambda \int_{-\infty}^{\infty} \Big(g(X_s + y) - g(X_s)\Big) f_Y(y)\, dy\Big] ds\Bigg).$$

On the set $\{\tau_B < \infty\}$, as $t \to \infty$, we have

$$\mathbb{E}_Q\left(e^{-\alpha\,\tau_B} g(X_{\tau_B})\right) = g(0) + E_Q\Bigg(\int_0^{\tau_B} e^{-\alpha s}\Big[-\alpha g(X_s) + \gamma g'(X_s) + \frac{1}{2}\sigma^2 g''(X_s) + \lambda \int_{-\infty}^{\infty} \Big(g(X_s + y) - g(X_s)\Big) f_Y(y)\, dy\Big] ds\Bigg).$$

To find $\mathbb{E}_Q(e^{-\alpha\tau_B})$ on the set $\{\tau_B < \infty\}$, we choose the function $g(x)$ so that the below conditions are satisfied

i) $g(x) = 1$, for $x \le b$,

ii) $-\alpha\, g(x) + \gamma\, g'(x) + \frac{1}{2}\sigma^2\, g''(x)\, \lambda \int_{-\infty}^{\infty} \Big(g(x+y) - g(x)\Big) f_Y(y)\, dy = 0$, for $x > b$.

A natural candidate for $g(x)$ is

$$g(x) = A(\alpha)\, e^{\Theta_1 x} + B(\alpha)\, e^{\Theta_2 x}, \quad (2.57)$$

where Θ_1 and Θ_2 are the two roots of $G(x) = \alpha$ and $A(\alpha)$, $B(\alpha)$ are two coefficients which are to be specified by using the above conditions $i)$ and $ii)$.
From $i)$, we get

$$g(b) = A(\alpha)\, e^{\Theta_1 b} + B(\alpha)\, e^{\Theta_2 b}. \tag{2.58}$$

From $ii)$, we find the corresponding roots as $-\beta_{3,\alpha}$ and $-\beta_{4,\alpha}$. Moreover, we get the following equation

$$-A(\alpha)\, \frac{\xi_2}{\xi_2 + \Theta_1}\, e^{\Theta_1 b} - B(\alpha)\, \frac{\xi_2}{\xi_2 + \Theta_2}\, e^{\Theta_2 b} + 1 \,=\, 0. \tag{2.59}$$

When we solve equations (2.58) and (2.59) together, we find

$$A(\alpha) = \frac{(\beta_{3,\alpha} - \xi_2)}{\xi_2}\, \frac{\beta_{4,\alpha}}{(\beta_{3,\alpha} - \beta_{4,\alpha})}, \qquad B(\alpha) = \frac{(\xi_2 - \beta_{4,\alpha})}{\xi_2}\, \frac{\beta_{3,\alpha}}{(\beta_{3,\alpha} - \beta_{4,\alpha})},$$

which implies

$$g(x) = \begin{cases} 1 & , \; x \le b \\ \frac{(\beta_{3,\alpha} - \xi_2)}{\xi_2}\, \frac{\beta_{4,\alpha}}{(\beta_{3,\alpha} - \beta_{4,\alpha})}\, e^{\beta_{3,\alpha}(b-x)} + \frac{(\xi_2 - \beta_{4,\alpha})}{\xi_2}\, \frac{\beta_{3,\alpha}}{(\beta_{3,\alpha} - \beta_{4,\alpha})}\, e^{\beta_{4,\alpha}(b-x)} & , \; x > b. \end{cases}$$

However, note that $g(x) \notin C^2$ at $x = b$, hence we can not apply the Itô's formula directly to the process $\{e^{-\alpha t}\, g(X_t);\, t \ge 0\}$. But one can define a sequence of functions $\{g_n(x);\, n = 1, 2, \ldots\}$ such that

$$g_n(x) = \begin{cases} 1 & , \; x < b - \frac{1}{n} \\ g(x) & , \; x \ge b, \end{cases}$$

which has the following properties;

$i)$ g_n is smooth everywhere,

$ii)$ $|-\alpha\, g_n(x) + \Lambda\, g_n(x)| \le \lambda\, p_2\, \xi_2 / n \,\rightarrow\, 0$, for $n \rightarrow \infty$, for all $x > b$,

$iii)$ $g_n(x) \rightarrow g(x)$, as $n \rightarrow \infty$, for all x,

$iv)$ $0 < g_n(x) < 2$, for all x.

Applying Itô formula for jump processes to $\{e^{-\alpha(t\wedge\tau_B)}\, g_n(X_{t\wedge\tau_B}\}$, we get

$$\begin{aligned} e^{-\alpha(t\wedge\tau_B)}\, g_n(X_{t\wedge\tau_B}) \;&=\; \int_0^{t\wedge\tau_B} e^{-\alpha s}\Big(-\alpha\, g_n(X_s) \,+\, \frac{1}{2}\,\sigma^2\, g_n''(X_s) \,+\, \gamma\, g_n'(X_s)\Big)\, ds \\ &\quad + \int_0^{t\wedge\tau_B} e^{-\alpha s}\, \sigma\, g_n'(X_s)\, dW_s - \int_0^{tt\wedge\tau_B} e^{-\alpha s}\big(g_n(X_s + Y) - g_n(X_s)\big) dN(s) \end{aligned}$$

We define the process

$$M_t^{(n)} \;:=\; e^{-\alpha(t\wedge\tau_B)}\, g_n(X_{t\wedge\tau_B}) - \int_0^{t\wedge\tau_B} e^{-\alpha s}\Big(-\alpha g_n(X_s) \,+ \Lambda g_n(X_s)\Big)\, ds,$$

which is a local martingale. Moreover, it is bounded

$$M_t^{(n)} \,\leq\, 2 \,+\, \frac{\lambda\, p_2\, \xi_2}{n}\, t.$$

Therefore $M_t^{(n)}$ is a martingale starting from $M_0^n = g_n(0) = g(0)$.

By dominated convergence theorem, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}_Q(M_t^{(n)}) \;&=\; \mathbb{E}_Q\left(\lim_{n\to\infty} e^{-\alpha(t\wedge\tau_B)}\, g_n(X_{t\wedge\tau_B})\right)\\
&\quad -\mathbb{E}_Q\Big(\int_0^{t\wedge\tau_B} \Big[\underbrace{\lim_{n\to\infty} e^{-\alpha s}\,(-\alpha\, g_n(X_s) + \Lambda g_n(X_s))}_{=0}\Big]\, ds\Big)\\
&=\; \mathbb{E}_Q\left(e^{-\alpha(t\wedge\tau_B)}\, g(X_{t\wedge\tau_B})\right).
\end{aligned}$$

On the other hand, from the martingale property of $M_t^{(n)}$,

$$\lim_{n\to\infty} \mathbb{E}_Q(M_t^{(n)}) = \lim_{n\to\infty} g_n(0) = g(0).$$

Hence, we have on the set $\{\tau_B < \infty\}$

$$\mathbb{E}_Q\left(e^{-\alpha(t\wedge\tau_B)}\, g(X_{t\wedge\tau_B})\right) = g(0).$$

$$\mathbb{E}_Q\left(e^{-\alpha(t\wedge\tau_B)}\, g(X_{t\wedge\tau_B})\right) = \mathbb{E}_Q\left(e^{-\alpha(t\wedge\tau_B)}\, g(X_{t\wedge\tau_B})\, 1_{\{\tau_B<\infty\}}\right) + \underbrace{\mathbb{E}_Q\left(e^{-\alpha(t\wedge\tau_B)}\, g(X_{t\wedge\tau_B})\, 1_{\{\tau_B=\infty\}}\right)}_{\longrightarrow 0, \text{ as } t\to\infty}$$

$$\begin{aligned}
&\Rightarrow \mathbb{E}_Q\Big(e^{-\alpha\tau_B}\, \underbrace{g(X_{\tau_B})}_{=1}\, 1_{\{\tau_B<\infty\}}\Big) = g(0), \text{ as } t\to\infty\\
&\Rightarrow \mathbb{E}_Q\left(e^{-\alpha\tau_B}\right) \;=\; g(0).
\end{aligned}$$

$\square$

The following Corollary gives the condition, under which the probability of default in finite time is one, i.e., default time is finite almost surely.

Corollary 2.3.6. *We have* $Q(\tau_B < \infty) = 1$ *if and only if* $\bar{u} \leq 0$, *where* $\bar{u}$ *is defined in equation* (2.50).

Proof. By Lemma 2.3.2, if $\bar{u} \leq 0$, then $\beta_{3,\alpha} \to 0$ and $\beta_{4,\alpha} \to \beta_4^*$ as $\alpha \to 0$. Thus

$$Q(\tau_B < \infty) = \lim_{\alpha \to 0} \mathbb{E}_Q\left(e^{-\alpha \tau_B}\right) = 1.$$

□

Remark 2.3.2. *The interpretation of Corollary 2.3.6 is intuitive. Let us recall that* $\bar{u}$ *is the drift of the process* $X(t)$. *The negative drift of the process* $X(t)$, *guaranties the default in finite time.*

Corollary 2.3.7. *For any* $\alpha > 0$ *and* $\Theta \in \mathbb{R}$, *we have*

$$\mathbb{E}_Q\left(e^{-\alpha\,\tau_B + \Theta\, X_{\tau_B}}\, 1_{\{\tau_B < \infty\}}\right) = e^{\Theta\, b}\left[\underbrace{\frac{(\beta_{3,\alpha} - \xi_2)(\beta_{4,\alpha} + \Theta)}{(\beta_{3,\alpha} - \beta_{4,\alpha})(\xi_2 + \Theta)}}_{=:C(\alpha,\Theta)} e^{\beta_{3,\alpha} b} + \underbrace{\frac{(\xi_2 - \beta_{4,\alpha})(\beta_{3,\alpha} + \Theta)}{(\beta_{3,\alpha} - \beta_{4,\alpha})(\xi_2 + \Theta)}}_{=:D(\alpha,\Theta)} e^{\beta_{4,\alpha} b}\right]. \tag{2.60}$$

Proof.

$$\begin{aligned}
&\mathbb{E}_Q\Big(e^{-\alpha\,\tau_B + \Theta\, X_{\tau_B}}\, 1_{\{\tau_B < \infty\}}\Big) \\
&= \mathbb{E}_Q\left(e^{-\alpha\,\tau_B + \Theta\, X_{\tau_B}}\, 1_{\{X_{\tau_B} = b,\ \tau_B < \infty\}}\right) + e^{\Theta b}\,\mathbb{E}_Q\left(e^{-\alpha\,\tau_B + \Theta\,(X_{\tau_B} - b)}\, 1_{\{X_{\tau_B} < b,\ \tau_B < \infty\}}\right) \\
&= e^{\Theta b}\,\mathbb{E}_Q\left(e^{-\alpha\,\tau_B}\, 1_{\{X_{\tau_B} = b\}}\right) + e^{\Theta b}\,\frac{\xi_2}{\xi_2 + \Theta}\,\mathbb{E}_Q\left(e^{-\alpha\,\tau_B}\, 1_{\{X_{\tau_B} < b\}}\right).
\end{aligned}$$

In the last equality, the conditional independence and memoryless of double exponential distribution is used. The rest of the proof follows from Theorem 2.3.5 and Kou and Wang [KW03].

□

2.3.2 Pricing Firm Derivatives

In this subsection, we shall derive formulas for the prices of firm's CCs. In the last chapter, we have seen that the Laplace transformation of the first passage times is sufficient to derive the equity value, the debt value and the total firm value. On the other hand, one needs the probability distribution of the first passage times to obtain bond prices. In Theorem 2.3.5, we prove that due to the memoryless property of the double exponential distribution, it is possible to calculate the Laplace transform of $Q(\tau_B \leq t)$. One can numerically invert $Q(\tau_B \leq t)$ from this transformation by noticing that

$$\int_0^\infty e^{-\alpha t} Q(\tau_B \leq t) dt = \frac{1}{\alpha} \int_0^\infty e^{-\alpha t} dQ(\tau_B \leq t) = \frac{1}{\alpha} \mathbb{E}_Q \left(e^{-\alpha \tau_B} \right)$$

is satisfied and equation (2.56) represents an analytical expression of the expectation in the above equation.

In this subsection, we shall derive closed form solutions of the debt value, total firm value and equity value by using the martingale approach, i.e. using Theorem 2.3.5 and Corollary 2.3.7. We shall employ Gaver-Stehfest numerical inversion algorithm to get approximated default probabilities. Moreover, we shall employ a Monte Carlo simulation to estimate the default probabilities, relying on the Brownian bridge concept[12], so that we can compare the numerical inversion algorithm results with the Monte Carlo results. We shall also investigate the optimal leverage and corporate credit spreads of a firm and compare them in the two cases of diffusion and jump-diffusion modeling. At the end of the subsection, we shall make some remarks about the calibration of the parameters of the underlying EBIT process.

Total firm value and EBIT value

As in the diffusion case, the total firm value and the EBIT value are closely related. Let us consider the net present value of the future earnings of the EBIT value

[12] Such kind of Monte Carlo simulation is first introduced by Metwally and Atiya [MA02].

$$
\begin{aligned}
V\big(\delta(t)\big) &= \mathbb{E}_Q\left(\int_t^\infty e^{-\hat{r}(s-t)}\,\delta(s)\,ds\,\bigg|\,\mathcal{F}_t\right)\\
&= \mathbb{E}_Q\left(\int_t^\infty e^{-\hat{r}(s-t)}\,\delta(t)\;e^{(\mu-\frac{1}{2}\sigma^2-\lambda\Xi)(s-t)+\sigma\,W_{(s-t)}+\sum_{i=1}^{N(s)-N(t)}\,Y_i}\,ds\,\bigg|\,\mathcal{F}_t\right)\\
&= \delta_t\int_t^\infty e^{(-\hat{r}+\mu-\lambda\Xi)(s-t)+\lambda\;\frac{p_1\,\xi_1}{\xi_1-1}+\frac{p_2\,\xi_2}{\xi_2+1}-1\;\,(s-t)}\,ds\\
&= \frac{\delta(t)}{\hat{r}-\mu+\lambda\Xi-\lambda\left(\frac{p_1\,\xi_1}{\xi_1-1}+\frac{p_2\,\xi_2}{\xi_2+1}-1\right)}\\
&= \frac{\delta(t)}{\hat{r}-\mu},
\end{aligned}
$$

where $\hat{r}=(1-\tau_i)\,r$ and $\hat{r}>\mu$. Similarly to the diffusion case, the artificial unlevered firm value becomes

$$V_U\big(\delta(t)\big) = K\,\delta(t), \tag{2.61}$$

where

$$K := \frac{(1-\tau_e)}{\hat{r}-\mu}. \tag{2.62}$$

Notice that the price earnings ration in jump-diffusion case is the same as in the duffision case, given by equation (2.24).

Debt Value

As in Section 2.2.2, let $d(0;c,p)$ denote the value of the currently issued debt, paying continuous coupons c, and principal p. Then, the value of the currently issued debt is equal to

$$
\begin{aligned}
d(0,c,p) = &\int_0^\infty e^{-\hat{r}t}e^{-mt}(1-\tau_i)(c+mp)\big(1-F(t;\delta,\delta_B)\big)dt\\
&+m\,(1-\alpha)\,K\,\mathbb{E}_Q\left(\delta_{\tau_B}e^{-(\hat{r}-m)\tau_B}\right).
\end{aligned}
$$

The first term represents the discounted expected value of the continuously (exponentially) declining coupon plus principal repayment, which will be paid with probability $(1-F(.,\delta,\delta_B))$. Second term is the expected present value of the fraction of the firm value after bankruptcy costs are paid. Note further that at the default event the EBIT value might not be equal to δ_B, since its value can be smaller than δ_B, i.e. an undershoot happens, therefore the EBIT value at the default time is a random variable. The total value, at time 0, of all debt outstanding is

$$D(\delta, \delta_B, P, C) = \int_0^\infty e^{-mt} d(0; c, p)\, dt.$$

Calculating the above integral delivers,

$$D(\delta, \delta_B, P, C) = \frac{(1-\tau_i)\,(C\,+\,m\,P)}{m+\hat{r}}\,\mathbb{E}_Q\left(1 - e^{-(m+\hat{r})\tau_B}\right) + (1-\alpha)\,K\,\delta\,\mathbb{E}_Q\left(e^{X_{\tau_B}-(m+\hat{r})\tau_B}\right). \tag{2.63}$$

Next, we consider bonds issued by the firm in order to analyse the credit spreads and to be able to calibrate[13] the underlying parameter set $\Theta = (\mu, \sigma, \lambda, p_1, p_2, \xi_1, \xi_2)$ by using the existing bond prices in the market.

The following lemma states the value of the bond issued by the firm, when the underlying is modeled as a jump-diffusion process with given dynamics in equation (2.43).

Lemma 2.3.8. *Let $B_\Theta(0, t; \rho, F)$ be the value of a defaultable bond issued by the firm at time 0, with maturity t, paying constant continuous coupons ρ and principal F. Then we have*

$$\begin{aligned} \overline{B}_\Theta(0, t; \rho, F) &= \mathbb{E}_Q\left(\int_0^{t\wedge\tau_B} (1-\tau_i)\,\rho\,F\,e^{-\hat{r}s}\,ds\right) + \mathbb{E}_Q\left((1-\tau_i)F\,e^{-\hat{r}t}\,1_{\{t<\tau_B\}}\right) \\ &\quad + \mathbb{E}_Q\left(F\,R\,e^{-\hat{r}t} 1_{\{t\geq\tau_B\}}\right), \end{aligned} \tag{2.64}$$

where R is the recovery rate and given by

$$\frac{(1-\alpha)K\delta_{\tau_B}}{P}.$$

It is paid at the maturity, if default occurs.

The interpretation of the lemma is the same as in Lemma 2.2.7. However, the last term is again different than the diffusion analogous, since the EBIT value at default might be different than the default barrier, i.e., there can be an undershoot. Hence in this

[13] As in the diffusion, we will not calibrate the parameters but make a short remark how to do it. See Remark 2.3.3.

case recovery rate is a random variable. The dependence of the bond prices on the underlying's parameter set is indicated by Θ, i.e., on $\Theta = (\mu, \sigma, \lambda, p_1, p_2, \xi_1, \xi_2)$. From the scaling property, we again obtain

$$\overline{B}_\Theta(0, t; \rho, 1) = \frac{\overline{B}_\Theta(0, t; \rho, F)}{F}. \tag{2.65}$$

Therefore, the debt value can be rewritten as follows

$$D(\delta, \delta_B, P, C) = \int_0^\infty p(0, t)\overline{B}_\Theta(0, t; \rho, 1)dt.$$

Calculating the above integral gives

$$D(\delta, \delta_B, P, C) = \frac{(1-\tau_i)\, P\,(\rho\, +\, m)}{m + \hat{r}}\, \mathbb{E}_Q\left(1 - e^{-(m+\hat{r})\tau_B}\right) + (1-\alpha)\, K\, \delta\, \mathbb{E}_Q\left(e^{X_{\tau_B} - (m+\hat{r})\tau_B}\right). \tag{2.66}$$

The above expectations are known by Theorem 2.3.5 and Corollary 2.3.7. Note again that the total coupon payment is equal to $C = \rho\, P$.

Total Firm Value

The total firm value can be written as the sum of the unlevered firm value and tax benefits minus bankruptcy costs

$$\begin{aligned} \nu(\delta, \delta_B, C) &= V_U(\delta) + \mathbb{E}_Q\left(\int_0^{\tau_B} (\tau_e - \tau_i)\, C\, e^{-\hat{r}s}\, ds\right) - \alpha\, K\, \delta\, \mathbb{E}_Q\left(e^{X_{\tau_B} - \hat{r}\tau_B}\right) \\ &= K\,\delta + (\tau_e - \tau_i)\,\frac{C}{\hat{r}}\,\mathbb{E}_Q\left(1 - e^{-\hat{r}\tau_B}\right) - \alpha\, K\, \delta\, \mathbb{E}_Q\left(e^{X_{\tau_B} - \hat{r}\tau_B}\right). \end{aligned} \tag{2.67}$$

The above expectations are known by Theorem 2.3.5 and Corollary 2.3.7.

Equity Value

The equity value can be written as the difference between the firm value and the equity value of the firm

$$\begin{aligned} EQ(\delta, \delta_B, P, C) &= \nu(\delta, \delta_B, C) - D(\delta, \delta_B, P, C) \\ &= K\,\delta + (\tau_e - \tau_i)\,\frac{C}{\hat{r}}\,\mathbb{E}_Q\left(1 - e^{-\hat{r}\tau_B}\right) - \alpha\, K\, \delta\, \mathbb{E}_Q\left(e^{X_{\tau_B} - \hat{r}\tau_B}\right) \\ &\quad - \frac{(1-\tau_i)\,(C\, +\, m\, P)}{m + \hat{r}}\,\mathbb{E}_Q\left(1 - e^{-(m+\hat{r})\tau_B}\right) - (1-\alpha)\, K\, \delta\, \mathbb{E}_Q\left(e^{X_{\tau_B} - (m+\hat{r})\tau_B}\right). \end{aligned} \tag{2.68}$$

The above expectations are known by Theorem 2.3.5 and Corollary 2.3.7.

2.3.3 Endogenous Default

As in the diffusion case, the optimal default level is chosen endogenously by the equity holders to maximize their share value. As already shown, the chosen level also maximizes the total firm value. This level can be found by the smooth pasting principal. The smooth pasting condition in the jump-diffusion case is justified by the recent paper by Chen and Kou [CK05].

One can rewrite the equity value given by equation (2.68), by using Theorem 2.3.5 and Corollary 2.3.7, and introducing the following notations.

Notation:

$$\begin{aligned} f_1(y) &:= \mathrm{E}_Q(e^{-y\tau_B}) \\ &= A(y)\left(\frac{\delta_B}{\delta}\right)^{\beta_{3,y}} + B(y)\left(\frac{\delta_B}{\delta}\right)^{\beta_{4,y}}. \end{aligned} \tag{2.69}$$

$$\begin{aligned} f_2(y,\Theta) &:= \mathrm{E}_Q\left(e^{\Theta\, X(\tau_B) - y\,\tau_B}\; 1_{\{\tau_B<\infty\}}\right) \\ &= \left(\frac{\delta_B}{\delta}\right)^{\Theta}\left[C(y,\Theta)\left(\frac{\delta_B}{\delta}\right)^{\beta_{3,y}} + D(y,\Theta)\left(\frac{\delta_B}{\delta}\right)^{\beta_{4,y}}\right], \end{aligned} \tag{2.70}$$

where $-\beta_{3,y},\ -\beta_{4,y}$ are the roots of $G(x) = y$, and

$A(y) = \frac{(\beta_{3,y}-\xi_2)}{\xi_2}\,\frac{\beta_{4,y}}{(\beta_{3,y}-\beta_{4,y})}$, $\qquad B(y) = \frac{(\xi_2-\beta_{4,y})}{\xi_2}\,\frac{\beta_{3,y}}{(\beta_{3,y}-\beta_{4,y})}$,

$C(y,\Theta) = \frac{(\beta_{3,y}-\xi_2)\,(\beta_{4,y}+\Theta)}{(\beta_{3,y}-\beta_{4,y})\,(\xi_2+\Theta)}$, $\qquad D(y,\Theta) = \frac{(\xi_2-\beta_{4,y})\,(\beta_{3,y}+\Theta)}{(\beta_{3,y}-\beta_{4,y})\,(\xi_2+\Theta)}$.

By using notations (2.69) and (2.70), the equity value (2.68) can be rewritten as follows;

$$\begin{aligned} EQ\big(\delta,\delta_B,P,C\big) &= K\,\delta + (\tau_e-\tau_i)\,\frac{C}{\hat r}\,\big(1-f_1(\hat r)\big) - \alpha\,K\,\delta\,f_2(\hat r,1) \\ &\quad - \frac{(1-\tau_i)\,(\,C\,+\,m\,P)}{m+\hat r}\,\big(1-f_1(\hat r+m)\big) - (1-\alpha)\,K\,\delta\,f_2(\hat r+m,1). \end{aligned}$$

To find the optimal default level, let us employ the smooth pasting condition,

$$\frac{\partial\, EQ(\delta,\delta_B)}{\partial\,\delta}\bigg|_{\delta=\delta_B} = 0.$$

Defining δ_B^* as δ_B which satisfies the above equality, we obtain

$$\delta_B^* = \frac{\left[\frac{(1-\tau_i)(C+m\,P)}{m+\hat{r}}\,\zeta(\hat{r}+m) - (\tau_e - \tau_i)\,\frac{C}{\hat{r}}\zeta(\hat{r})\right]}{K\big(1+\alpha\,\eta(\hat{r}) \,+\, (1-\alpha)\,\eta(\hat{r}+m)\big)}, \tag{2.71}$$

where

$$\begin{aligned}\zeta(\alpha) &:= \beta_{3,\alpha}\,A(\alpha) \,+\, \beta_{4,\alpha}\,B(\alpha),\\ \eta(\alpha) &:= \beta_{3,\alpha}\,C(\alpha,1) \,+\, \beta_{4,\alpha}D(\alpha,1).\end{aligned}$$

Theorem 2.3.9. *The bankruptcy level δ_B^*, given in equation* (2.71) *is the optimal one for the equity holders.*

Proof. The proof is similar to the proof of Theorem 2.2.8. For the interested reader, we refer to Chen and Kou [CK05]. □

By using the notations (2.69) and (2.70), we substitute (2.71) in (2.63), (2.67), (2.68) so that we obtain closed form solutions for the debt value, the total firm value and the equity value.

$$D(\delta,\delta_B^*,P,C) = \frac{(1-\tau_i)\,(C\,+\,m\,P)}{m+\hat{r}}\,\big[1-f_1^*(\hat{r}+m)\big] + (1-\alpha)\,K\,\delta\,f_2^*(\hat{r}+m,1), \tag{2.72}$$

$$\nu\big(\delta,\delta_B^*(P),C\big) = K\,\delta + (\tau_e-\tau_i)\,\frac{C}{\hat{r}}\,\big[1-f_1^*(\hat{r})\big] - \alpha\,K\,\delta\,f_2^*(\hat{r},1), \tag{2.73}$$

$$\begin{aligned}EQ(\delta,\delta_B^*,P,C) &= K\,\delta + (\tau_e-\tau_i)\,\frac{C}{\hat{r}}\,\big(1-f_1^*(\hat{r})\big) - \alpha\,K\,\delta\,f_2^*(\hat{r},1)\\ &- \frac{(1-\tau_i)\,(C\,+\,m\,P)}{m+\hat{r}}\,\big(1-f_1^*(\hat{r}+m)\big) - (1-\alpha)\,K\,\delta\,f_2^*(\hat{r}+m,1),\end{aligned} \tag{2.74}$$

where $f_1^*(y) = A(y)\,\left(\frac{\delta_B^*}{\delta}\right)^{\beta_{3,y}} + B(y)\,\left(\frac{\delta_B^*}{\delta}\right)^{\beta_{4,y}}$ and in analogous to $f_1^*(y)$ one can define $f_2^*(y)$. Note that the optimal δ_B^* depends on the total principal P, i.e., $\delta_B^* = \delta_B^*(P)$. Therefore, when δ_B^* is plugged into the equation (2.67), the total firm value depends implicitly on P .

2.3.4 Optimal Leverage

The two step optimisation problem is also used in the jump-diffusion case.

The coupon level C is either given exogenously or set by assuming that the initial total debt value is sold at par, which implies C satisfies the following equation

$$D(\delta, \delta_B, P, C) = P, \tag{2.75}$$

where $D(\delta, \delta_B, P, C)$ is given by equation (2.66). Notice that the above equation can be easily numerically solved to find the coupon.

The first step to find the optimal leverage is the choice of the optimal default level, δ_B^* by the equity holders. It maximises the total firm value subject to the limited liability. Therefore, the first step maximisation is a build-in feature of the model.

As already mentioned the optimal δ_B^* depends on P. Plugging it into the total firm value, implies that $\nu(\delta, \delta_B^*(P), C)$ depends on P. The following theorem states the relation between $\nu(\cdot)$ and P.

Theorem 2.3.10. *The total firm value, $\nu(\delta, \delta_B^*)$ given by equation* (2.73) *is concave with respect to P.*

Proof. See Chen and Kou [CK05]. □

Next, we investigate the effect of various parameters on the optimal leverage of a firm. The basic parameters, we shall use in our numerical examples are given as follows.

$\delta_0 = 15,\ \mu = 0.02,\ \sigma = 0.2,\ \lambda = 0.2,\ \xi_1 = 3,\ \xi_2 = 2,\ p, q = 0.5$
$r = 8\%,\ \rho = 8.162\%,\ \alpha = 50\%,\ \tau_c = 25\%,\ \tau_i, \tau_c = 15\%$

Table 2.4: Basic parameters, used in the numerical examples in the Jump-Diffusion case.

Notice that we choose large but infrequent jumps, the risk-free rate is set to 8%, close to the historical average treasury rates in USA during 1973-1998. By following Huang *et. al* [HH03], we choose the coupon rate to be 8.162%, which is the par coupon payments when continuously compounded interest rate is 8%.

In the following table, we examine the total firm value by changing various parameters such as; bankruptcy rate, average debt maturity, diffusion volatility and jump frequency. In particular, it is intersting for us the effect of the jump frequency on the optimal leverage level.

		0.5 years		2 years		5 years	
		$\sigma = 0.2$	$\sigma = 0.4$	$\sigma = 0.2$	$\sigma = 0.4$	$\sigma = 0.2$	$\sigma = 0.4$
$\alpha = 90\%$	$\lambda = 0$	6.36%	1.24 %	15.69 %	3.76%	23.42%	8.73 %
	$\lambda = 0.5$	0.43%	0.086%	2.17%	0.087 %	6.49 %	3.46%
	$\lambda = 1$	0.051%	0.003%	0.65%	0.26 %	3.01 %	1.92%
$\alpha = 75\%$	$\lambda = 0$	10.51%	2.14 %	18.93 %	4.34%	29.45%	10.85 %
	$\lambda = 0.5$	1.28%	1.51%	4.32%	1.73 %	9.83 %	5.76%
	$\lambda = 1$	0.25%	0.068%	1.52%	0.65 %	5.08 %	3.37%
$\alpha = 50\%$	$\lambda = 0$	20.61%	8.3 %	26.80 %	12.88%	39.02%	21.08 %
	$\lambda = 0.5$	5.33%	1.91%	11.31%	6.45 %	19.82 %	13.38%
	$\lambda = 1$	1.71%	0.85%	5.58%	3.44 %	12.59 %	9.62%

Table 2.5: Optimal default ratios for variuos parameters. The other parameters are the same as in Table 2.4.

As a conclusion, firms with higher bankruptcy costs, higher jump frequency, higher diffusion volatility and shorter debt maturity takes less debt. Note that the firm, which has 0.5 year average maturity debt, 0.4 EBIT-process diffusion volatility, 90% bankruptcy costs and in average 0.5 times defaults in 0.5 year, has almost zero leverage level.

Next, let us see in Figure 2.12 the effect of the jump parameter λ on the maximum value and on the optimal leverage of a firm. The jump diffusion term has a significant effect on the optimal leverage of the firm. In the diffusion case it is around 39%, but it decreases to 12% in the jump diffusion case. The maximum value of the levered firm is also bigger in the diffusion case, since the bankruptcy costs dominates tax advantages in the smaller leverage ratio because of the jump risk.

2.3.5 Default Probabilities

Obtaining the probability distribution of the first hitting times of a Brownian motion is a trivial task in the diffusion case, however it is not possible to obtain them in a general

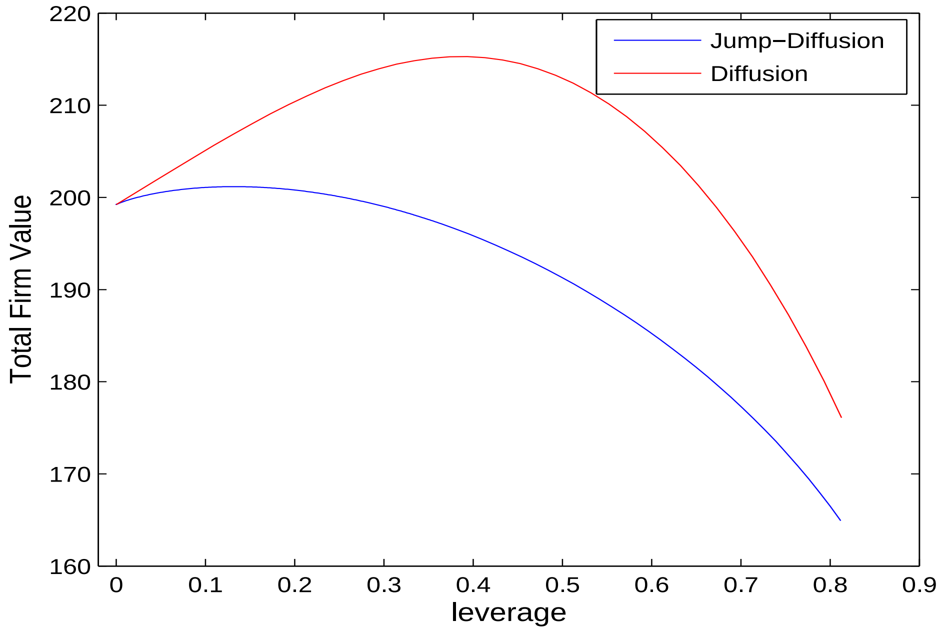


Figure 2.12: $\lambda = 1$, average maturity= 5 years

jump-diffusion setup. Therefore, we restrict ourselves in the case where the jumps are double exponentially distributed, so that the conditional memoryless and independence properties of it enables us to derive at least the Laplace transforms of the first passage time (see Proposition 2.3.4 and Theorem 2.3.5).

Corollary 2.3.11. *Let $\Upsilon(m)$ denote the Laplace transform of the default time, for the parameter $\alpha \in \mathbb{R}^+$. Then*

$$\Upsilon(\alpha) := \int_0^\infty e^{-\alpha t} Q(\tau_B \leq t) dt = \frac{1}{\alpha} \int_0^\infty e^{-\alpha t} dQ(\tau_B \leq t) = \frac{1}{\alpha} \mathbb{E}_Q \left(e^{-\alpha \tau_B} \right).$$

As a consequence, the default probability can be inverted from the Laplace transformation by a numerical algorithm. As in Kou and Wang [KW03], we have chosen the Gaver-Stehfest algorithm, which has the advantage working on real line. Advantages and disadvantages of this algorithm and the following lemma on which the method is based, are described in Abate and Whitt [AW92].

Lemma 2.3.12. *A bounded and real valued function $f(\cdot)$, continuous at t, can be approximated by*

$$f_n^*(t) = \sum_{k=1}^{n} w(k,n) \tilde{f}_k(t),$$

where

$$w(k,n) = \frac{(-1)^{n-k} k^n}{k!\,(n-k)!}, \quad \tilde{f}_n(t) = \frac{\log 2}{t} \frac{(2n)!}{n!\,(n-1)!} \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} \Psi\left(\frac{(n+k)\log 2}{t}\right)$$

and $\Psi(\cdot)$ is the Laplace transform of $f(\cdot)$. The asymptotic behavior of the approximation is

$$f_n^*(t) - f(t) = o\left(n^{-k}\right).$$

Proof. We refer to Stehfest [Ste69] □

In the view of Lemma 2.3.12, we approximate the default probability by

$$Q(\tau_B \leq t) \approx f_n^*(t) = \sum_{k=1}^{n} w(k,n) \tilde{f}_{k+B}(t),$$

where

$$\tilde{f}_n(t) = \frac{\log 2}{t} \frac{(2n)!}{n!(n-1)!} \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} \Upsilon\left(\frac{(n+k)\log 2}{t}\right),$$

and $B \geqslant 0$ is the burning out number as discussed in Kou and Wang [KW03], it is usually equal to 2 or 3.

To test this numerical algorithm, we implement also a Monte Carlo simulation of default probabilities, relying on the Brownian bridge concept. This algorithm can be found in Metwally and Atiya [MA02] or Scherer [Sch05a]. The idea of the algorithm is shortly as follows. The number of jumps are Poisson distributed with the parameter λT. Conditioned on the number of jumps, the jump times are distributed as order statistics $0 < \tau_1 < \ldots < \tau_{N_T} < T$. The process

$$X(t) := \gamma t + \sigma\, W(t) + \sum_{i=1}^{N(t)} Y_i,$$

and its left limits at those times are simulated as follows

$$X(\tau_i^-) - X(\tau_{i-1}) \sim \mathcal{N}(\gamma\Delta\tau_i, \sigma^2\Delta\tau_i),$$
$$X(\tau_i) - X(\tau_i^-) \sim Y_i,$$

where Y_i is distributed with the density given in equation (2.44). If no passage of the process X to the barrier b is observed before and after the jump at time τ_i, calculate the probability of the Brownian bridge connecting $X(\tau_{i-1})$ with $X(\tau_i^-)$ not to fall below the level b by using the following formula[14]

$$\Phi_b^{BB}\left(X(\tau_{i-1}), X(\tau_i^-), \Delta\tau_i\right) = \left(1 - \exp\left(-\frac{2\big(X(\tau_{i-1}) - b\big)\big(X(\tau_i^-) - b\big)}{\sigma^2\Delta\tau_i}\right)\right).$$

This yields the survival probability until time τ_i

$$SP_i = \prod_{j=1}^{i} \Phi^{BB}\left(X(\tau_{j-1}), X(\tau_j^-), \Delta\tau_i\right).$$

If $X(\tau_i) \leq b$ or $X(\tau_i^-) \leq b$ is observed, then the survival probability for this path becomes zero. If it is not observed until time T, then we obtain

$$SP_{N_T} = \prod_{j=1}^{N_T+1} \Phi^{BB}\left(X(\tau_{j-1}), X(\tau_j^-), \Delta\tau_i\right).$$

By repeating this procedure N-times, we obtain the estimate

$$Q(\tau_B > T) \approx \frac{1}{N}\sum_{i=1}^{K} SP^i_{N_T}$$

for the survival probability.

2.3.6 Credit Spreads

The aims of this subsection are to show that in the jump-diffusion case the credit spreads do not converge to zero as maturity converges to zero and the credit spread of the total debt is higher than in the diffusion case.

[14] It can be found in Karatzas and Shreve [KS00].

The next theorem postulates that in the jump-diffusion case we have a positive default rate, i.e. the conditional probability that default occurs before $t+h$, conditioned on no default until time t, tends to zero as h does and the convergence rate is $O(h)$. Compare it with the Corollary 2.2.12 in Chapter 2.

Theorem 2.3.13. *We have*

$$\lim_{h\searrow 0}\frac{1}{h}\,Q\big(\tau_B\le t+h\,|\,\mathcal{F}_t\big)=\lambda\,p_2\left(\frac{\delta_B}{\delta}\right)^{\xi_2}.$$

Therefore, the convergence rate is $O(h)$, as long as $\lambda\,p_2>0$.

Proof. See Scherer [Sch05a] □

Note that Theorem 2.3.13 allows us to write, for an infinitesimal time interval $(0,ds]$,

$$Q\big(\tau_B\in(0,ds]\big)=\lambda\,p_2\left(\frac{\delta_B}{\delta}\right)^{\xi_2}ds. \tag{2.76}$$

In jump-diffusion models, this fact implies non-zero credit spreads as maturity decreases to zero. Let $s(t)$ denote the credit spread defined already in equation (2.41)

$$s(t):=-\frac{\ln(\overline{B}_\Theta(0,t;\rho,1))-\ln(B(0,t;\rho,1)}{t}, \tag{2.77}$$

where $B(0,t;\rho,1)$ is defined in equation (2.40) and $\overline{B}_\Theta(0,t;\rho,1)$ is given by the equation (2.65) as follows

$$\begin{aligned}\overline{B}_\Theta(0,t;\rho,1)=&\mathbb{E}_Q\left(\int_0^{t\wedge\tau_B}(1-\tau_i)\,\rho\,e^{-\hat{r}s}\,ds\right)+\mathbb{E}_Q\left((1-\tau_i)\,1\,e^{-\hat{r}t}\,1_{\{t<\tau_B\}}\right)\\&+\mathbb{E}_Q\left(e^{-\hat{r}t}\,R\,1_{\{t\ge\tau_B\}}\right)\\=&(1-\tau_i)\left(1-\frac{\rho}{\hat{r}}\right)e^{-\hat{r}t}Q\left(t<\tau_B\right)+(1-\tau_i)\frac{\rho}{\hat{r}}\left(1-\mathbb{E}_Q\left(e^{-\hat{r}\tau_B}1_{\{t\ge\tau_B\}}\right)\right)\\&+e^{-\hat{r}t}\frac{(1-\alpha)K}{P}\,\mathbb{E}_Q\left(\delta_{\tau_B}1_{\{t\ge\tau_B\}}\right).\end{aligned}$$

The above defaultable bond price can be approximated by discretising the whole path of the EBIT process and employing a fine-grid Monte Carlo simulation[15]. Notice that the

[15]Notice that the numerical integration, which we employed in diffusion case (see Proposition 2.2.11), can not be employed in the jump-diffusion case, since we don't have the expilicit form of the density function of the first passage time.

default probabilities are estimated either by the Monte Carlo algorithm or the numerical Laplace inversion, explained in Subsection 2.3.5.

The next corollary states that in the jump-diffusion case as $t \downarrow 0$, we obtain non-zero credit spreads.

Corollary 2.3.14. *We have*

$$s(0) = \lim_{t \downarrow 0} s(t) = \lambda\, p_2 \left(\frac{\delta_B}{\delta}\right)^{\xi_2} \left(1 - \frac{R}{(1-\tau_i)} \frac{\xi_2}{\xi_2 + 1}\right),$$

where recovery rate $R = (1-\alpha) K \delta_{\tau_B} / P$.

Proof. For $t \in (0, ds]$, the bond price

$$\overline{B}_\Theta(0,t;\rho,1) = \mathbb{E}_Q \left(\int_0^{t \wedge \tau_B} (1-\tau_i)\, \rho\, e^{-\hat{r}s}\, ds\right) + \mathbb{E}_Q \left((1-\tau_i)\, 1\, e^{-\hat{r}t}\, 1_{\{t < \tau_B\}}\right) + \mathbb{E}_Q \left(e^{-\hat{r}t}\, R\, 1_{\{t \geq \tau_B\}}\right)$$

can be approximated as follows

$$\begin{aligned} \overline{B}_\Theta(0,t;\rho,1) &= (1-\tau_i)\rho \frac{1}{\hat{r}} \left(1 - e^{-\hat{r}t}\right) + (1-\tau_i)\, e^{-\hat{r}t} Q\big(\tau_B \notin (0,t]\big) \\ &\quad + \frac{(1-\alpha)K}{P}\, e^{-\hat{r}t} \mathbb{E}_Q\big(\delta_{\tau_B} 1_{\{\tau_B \in (0,t]\}}\big) + o(t) \\ &= (1-\tau_i)\rho \frac{1}{\hat{r}} \left(1 - e^{-\hat{r}t}\right) + (1-\tau_i)\, e^{-\hat{r}t} Q\big(\tau_B \notin (0,t]\big) \\ &\quad + \frac{(1-\alpha)K}{P}\, e^{-\hat{r}t} \mathbb{E}_Q\big(\delta_{\tau_B} | \tau_B \in (0,t]\big)\, Q\big(\tau_B \in (0,t]\big) + o(t). \end{aligned} \tag{2.78}$$

From Proposition 2.3.4, we have

$$\begin{aligned} \mathbb{E}_Q\big(\delta_{\tau_B} | \tau_B \in (0,t]\big) &= \delta_B \int_{-\infty}^0 e^x \xi_2 e^{\xi_2 x} dx \\ &= \delta_B \frac{\xi_2}{\xi_2 + 1}. \end{aligned} \tag{2.79}$$

Substituting equations (2.79) and (2.76) into equation (2.78), we obtain

$$\begin{aligned} \overline{B}_\Theta(0,t;\rho,1) &= (1-\tau_i)\rho \frac{1}{\hat{r}} \left(1 - e^{-\hat{r}t}\right) + (1-\tau_i)\, e^{-\hat{r}t} \left(1 - \lambda\, p_2 \left(\frac{\delta_B}{\delta}\right)^{\xi_2} t\right) \\ &\quad + \frac{(1-\alpha)K}{P}\, e^{-\hat{r}t} \delta_B \frac{\xi_2}{\xi_2 + 1}\, \lambda\, p_2 \left(\frac{\delta_B}{\delta}\right)^{\xi_2} t + o(t). \end{aligned} \tag{2.80}$$

By substituting equations (2.80) and (2.40) into equation (2.77) as $t \downarrow 0$ and applying l'Hospital's rule, we obtain

$$\lim_{h \downarrow 0} s(t) = \lambda p_2 \left(\frac{\delta_B}{\delta}\right)^{\xi_2} \left(1 - \frac{R}{(1-\tau_i)} \frac{\xi_2}{\xi_2 + 1}\right).$$

□

Figure 2.13 and Figure 2.14 justify the result in Corollary 2.3.14. We observe in both figures that in the jump diffusion case the short term credit spread does not vanish, on the other hand it is zero in the diffusion case. In Figure 2.13, we plot the credit spreads of a firm with respect to its average maturity. It is assumed that the firm's bond maturities are equal to the average maturity of its debt. Whereas in Figure 2.14, we set a constant average maturity and use the fact that the bonds issued by the firm can have maturities different from the average debt maturity. We observe the same effect of the jumps, namely even for short maturities the credit spread is non-zero.

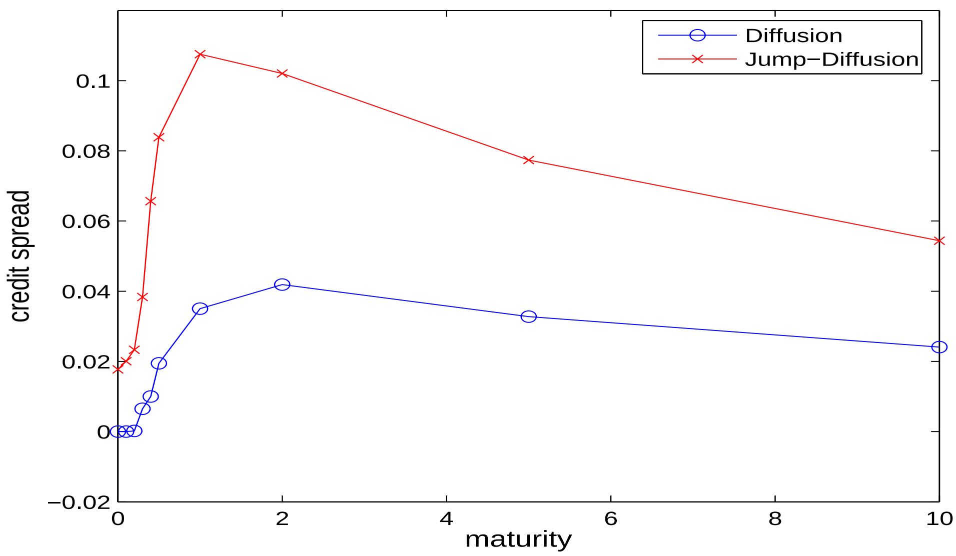


Figure 2.13: Credit spread curve for a bond. The parameters used are the leverage level 70%, other parameters are given in Table 2.4.

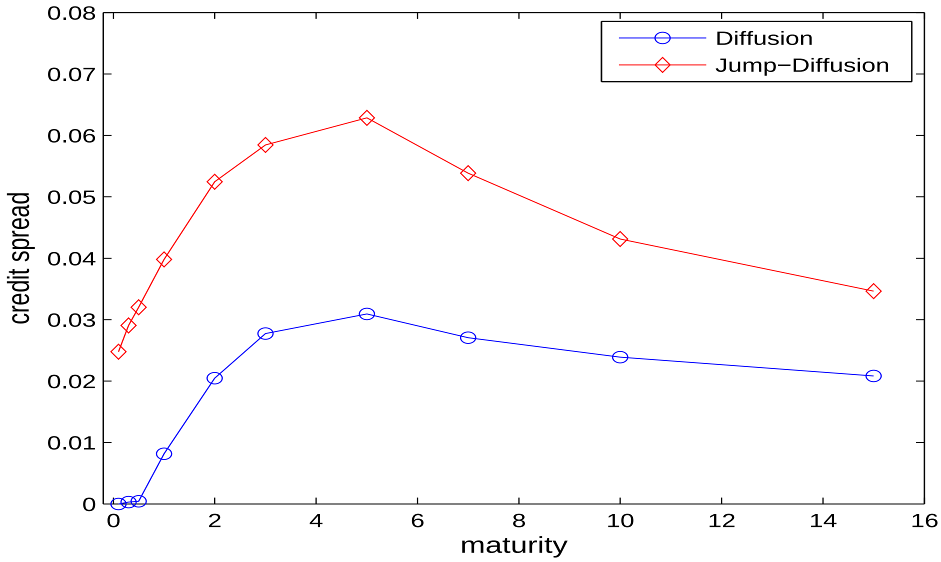


Figure 2.14: Leverage level 70%, average maturity= 5 years, other parameters are the same as in Table 2.4.

One can also observe that the credit spread is an increasing function of bankruptcy costs and jump frequency, see Figures 2.15 and 2.16. Less obvious is, credit spread is a decreasing function for short maturities and an increasing function for long maturities with respect to the diffusion volatility, see Figure 2.17. Since, for short maturities the jump part is mainly responsible for the defaults, the credit spread decreases for increasing diffusion volatility. However, for long maturities the diffusion part is more important to determine the credit spreads.

As in Section 2.2.5, the credit spread of the total debt is defined as $C/P - r$. The coupon C is set such that equation (2.75) is satisfied. Figure 2.18 plots the credit spread of the total debt with respect to the leverage for different average debt maturity by using the base case parameters given in Table 2.4. In Figure 2.19, it is observed that the credit spread of the total debt is higher in the jump-diffusion case than in the diffusion case. This fact also highlights the credit spreads in the double exponential jump diffusion model are different from zero by the influence of jumps.

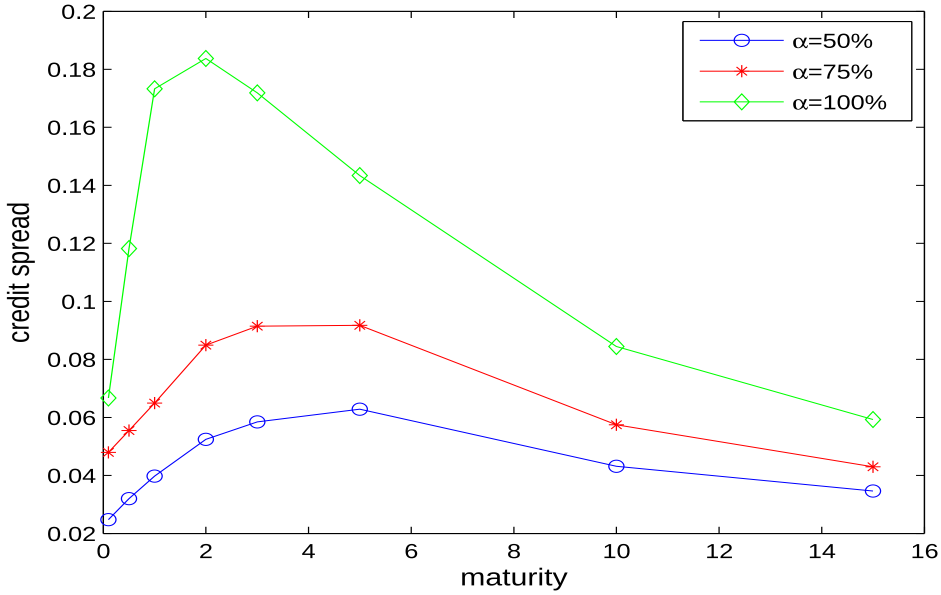


Figure 2.15: Leverage level 70%, average maturity= 5 years and other parameters are the same as in Table 2.4.

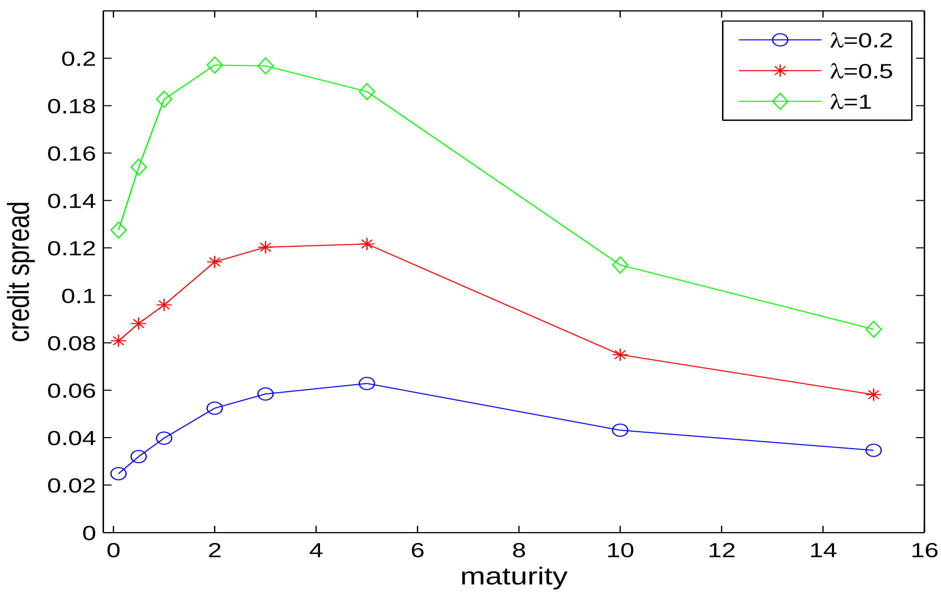


Figure 2.16: Leverage level 70%, average maturity= 5 years and other parameters are the same as in Table 2.4.

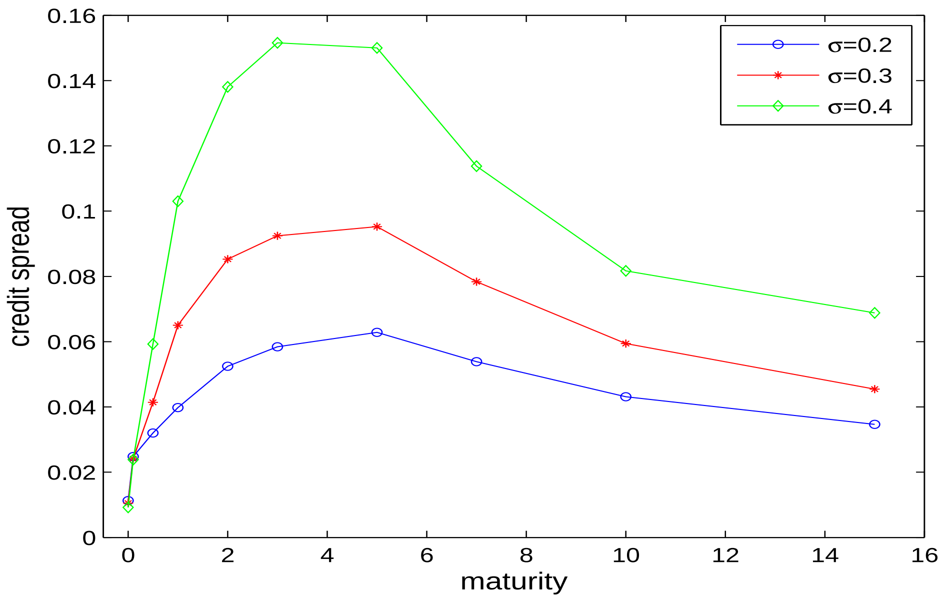


Figure 2.17: Leverage level 70%, average maturity= 5 years and other parameters are the same as in Table 2.4.

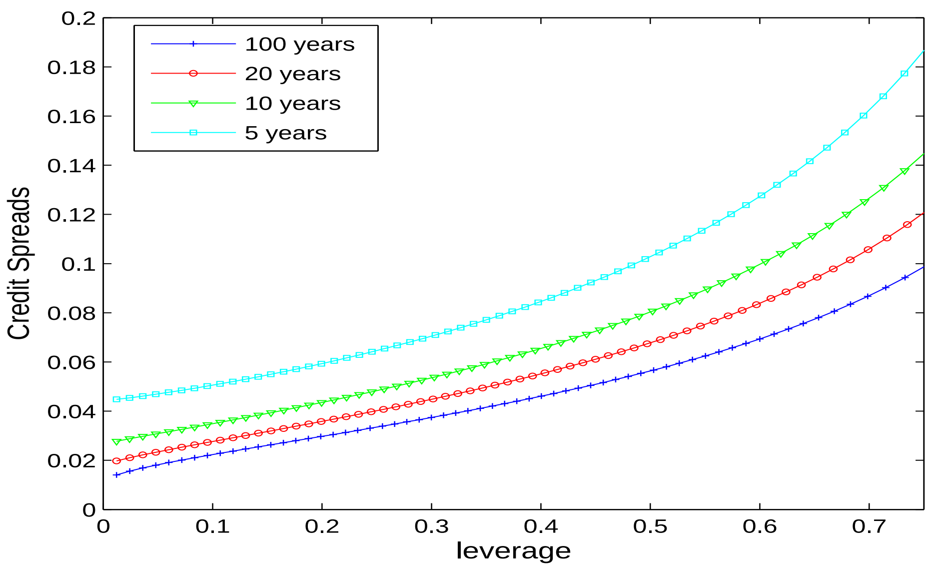


Figure 2.18: Credit spreads versus leverage for different maturities in jump-diffusion case. The coupon is set by solving equation (2.75) and other parameters are the same as in Table 2.4.

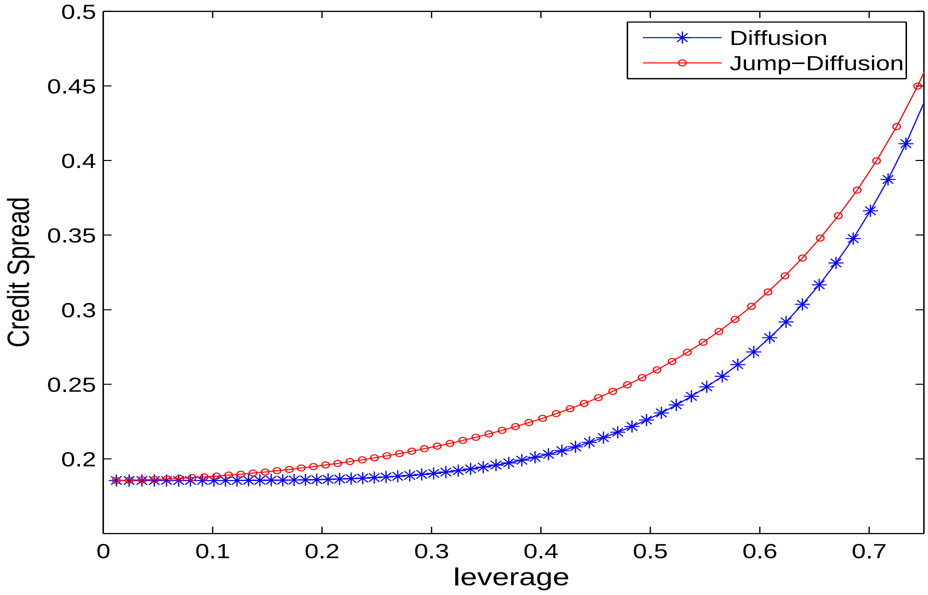


Figure 2.19: Comparison of credit spreads versus leverage in diffusion and jump-diffusion cases for one year average maturity. The coupon is set by solving equation (2.75) and other parameters are the same as in Table 2.4.

Remark 2.3.3. *The model bond prices can be estimated by Monte Carlo simulation and can be calibrated to the market prices of the bonds,* $\overline{B}^{M}_{i,\Theta}$ *issued by the firm. By using the least squares method one can calibrate the parameter set* $\Theta = (\mu, \sigma, \lambda, p, q, \xi_1, \xi_2)$.

$$\Theta = \arg\min_{\Theta} \sum_{i=1}^{N} \left(\overline{B}^{M}_{i,\Theta} - \overline{B}_i \right)^2 .$$

In the jump-diffusion case we have more parameters than in the diffusion case, hence one expects a better fit.

2.4 Conclusion

In Section 2.2, we derive the closed form solutions of the firm CCs in a pure diffusion setup under the assumption that the EBIT value is a non-traded asset. Although this assumption leaves us an additional parameter, μ, to calibrate, it excludes the arbitrage opportunities in the market. Genser [Gen05] consideres a similar setup and estimated the parameters μ and σ of the EBIT process, by using the bond time series and stock time series of the firm in a Kalman filter approach.

In Section 2.3, we model the underlying EBIT value as a jump-diffusion processes. Although we observe that the jump risk has significant effects on the maximum total firm value, the optimal leverage level and the corporate credit spreads of the firm, modeling with jump-diffusion processes is a non-trivial task. Therefore, we restirict ourselves in a special type of a Levy process, namely jump-diffusion process with double exponential jump-heights and take advantage of working with exponentially distributed jump-heights. Hereby, we remark that it is impossible to derive the closed form solutions for the firm CCs with a general Levy process framework.

Chapter 3

An extension of the Libor Market Model with Default Risk

3.1 Introduction

In the interest rate derivative market, the two most commonly traded derivatives are Caps and Swaptions. Therefore, the Cap and Swaption markets are used to calibrate the dynamics of Libor and swap rates, in order to price more complicated interest rate derivatives, underlying on these two quantities. However, the classical continuously compounded short rate and forward rate modeling, induces analytically intractable processes of Libor and swap rates, hence market quoted prices of Caps and Swaptions can not be matched with the theoretical Black's [Bla76] Cap formula and Black's [Bla76] Swaption formula, which are based on the Black and Scholes [BS73] model for stock options and used as the market standard.

In order to be able to match the market prices of Caps and Swaptions with Black's formulas, two theoretical frameworks are introduced in the literature. The first one is the **Libor market model**, introduced by Brace, Gatarek, and Musiela [BGM97] and Miltersen, Sandmann, and Sondermann [MSS97], Jamshidian [Jam97], takes directly observed, discretely compounded Libor forward rates as fundamental variables and models them as martingales under the corresponding forward measures. On the other hand, in the second framework, known as **Swap Market Model**, the swap rates are taken as a fundamental variable and modeled as martingales under the corresponding swap measures, which is introduced by Jamshidian [Jam97]. The martingale modeling of Libor and swap rates enables us to use the Black and Schole's pricing tools to price Caps and Swaptions. Therefore, market models price Caps and Swaptions by using Black's formulas, which enables the calibration of the Libor and swap rates in order to price more structured derivatives.

The functionality of the market models in the interest-rate derivatives world is applied to the credit risk derivatives by Schönbucher [Sch05b]. He introduced the defaultable versions of the Libor Market Model and the Swap Market Model by defining the T_k-survival measure and the default swap measure, which are defaultable versions of the T_k-forward measure and the swap measure respectively. He considered also the pricing of options on credit default swaps and under some restrictive assumptions he derived an option price formula similar to Black's formula.

The plan of this chapter is as follows. First, we shall introduce Schönbucher's model and derive his credit default swap option formula. Then, we shall generalise this formula by relaxing some of his assumptions and derive two alternative credit default swap option formulas. The chapter will be concluded with the comparison of our formulas with Schönbucher's one and with the Monte Carlo simulation[1].

3.2 Default model

Let $(\Omega, (\mathcal{F}_t)_{(t\geq 0)}, \mathbf{Q})$ be a filtered probability space where the filtration satisfies the usual conditions, and $\mathbf{Q}$ is the spot martingale measure. All stochastic processes in the model are adapted to $(\mathcal{F}_t)_{(t\geq 0)}$.

The default time is given by a stopping time τ. Default is triggered by the first jump of a Cox process[2] $N(t)$ with the intensity process $\{\lambda(t)\}_{t\geq 0}$. The survival indicator function

[1] Thanks to Dr.Wenzel for providing the C++ Code.

[2] Our model is in fact independent of having a Cox process triggering the default. We could choose a Poisson process with constant or time dependent intensity. But we stay in the general framework introduced by Schönbucher [Sch05b], who used Cox process assumption only when recovery payoffs are valued, which we will not cover in this thesis.

is denoted by

$$I(t) := 1_{\{\tau>t\}}.$$

Therefore, $I(t)$ is equal to one before default and jumps to zero at the time of default.

In this setup, the survival probability from time t to T is given by

$$\mathbb{E}_{\mathbf{Q}}\Big[\exp\Big(-\int_t^T \lambda(s)\,ds\Big)\Big],$$

and the process $N(t) - \int_0^t \lambda(s)\,ds$ is a martingale.

3.3 Bond prices and basic rates

Suppose that payoffs are paid at discrete dates

$$0 = T_0, T_1, \ldots, T_N.$$

These dates are coupon and repayment dates for bonds, fixing dates for rates and settlement dates for derivatives. The distance between two tenor dates is $\Delta_k := T_{k+1} - T_k$, where $k = 0, \ldots, N-1$ and the function $\kappa(t) := \min\{k : t < T_k\}$ is the next tenor date after t.

Next we introduce notations for the default-free and defaultable bond prices.

Definition 3.3.1.

1. *At time t, default-free zero coupon bond price maturing at T_k is denoted by*

$$B_k(t) = B(t, T_k).$$

2. *The defaultable zero coupon bond price maturing at T_k is denoted by*

$$I(t)\overline{B}_k(t) = \overline{B}(t, T_k).$$

3. *The default-risk factor at time t for maturity T_k is denoted by*

$$D_k(t) = D(t, T_k) = \frac{\overline{B}_k(t)}{B_k(t)}.$$

4. *The money market account at time t is denoted by $M(t)$ and equal to*

$$M(t) = e^{\int_0^t r(s)ds},$$

where r is the risk-free spot interest rate.

Remark 3.3.1.

1. *Note that by the definition, the defaultable zero coupon bond has zero recovery, and hence the influence of the defaults and the predefault bond price are separated, i.e $\overline{B}_k(t)$ need not jump to zero at default, since $I(t)$ already does.*

2. *The default risk factor D separates the influence of the default risk from the standard discounting with default-free interest rate. Schönbucher [Sch05b]shows that $D_k(0)$ is the survival probability until T_k under the respective forward martingale measure $\mathbf{P}_k$ (will be introduced later), that is*

$$\mathbf{P}_k(\tau > T_k) = D_k(0).$$

Next, we give the definitions of the rates that are used in the model. It is assumed that all rates in the model are discretely compounded effective rates. In the following section, we will equip their continuous time $\mathbf{Q}$-dynamics with log-normal diffusion processes.

Definition 3.3.2.

1. *The default-free effective forward rate over the period $[T_k, T_{k+1}]$ as seen from t is*

$$F_k(t) := \frac{1}{\Delta_k}\left(\frac{B_k(t)}{B_{k+1}(t)} - 1\right). \tag{3.1}$$

2. *The defaultable effective forward rate is*

$$\overline{F}_k(t) := \frac{1}{\Delta_k}\left(\frac{\overline{B}_k(t)}{\overline{B}_{k+1}(t)} - 1\right). \tag{3.2}$$

3. *The forward credit spread S_k is the difference between defaultable and default-free effective forward rate, that is*

$$S_k(t) := \overline{F}_k(t) - F_k(t). \tag{3.3}$$

4. The discrete-tenor forward default intensity over $[T_k, T_{k+1}]$ as seen from t is

$$H_k(t) := \frac{1}{\Delta_k}\left(\frac{D_k(t)}{D_{k+1}(t)} - 1\right). \tag{3.4}$$

Remark 3.3.2.

i) A lender would agree at time t to lend with the rate $\overline{F}_k(t)$ to the obligor over the future time interval $[T_k, T_{k+1}]$, conditional on the obligor's survival until T_k.

ii) From the above definitions, we get the following relations

$$B_k(t) = B_{k+1}(t)\big(1 + \Delta_k F_k(t)\big), \tag{3.5}$$

$$S_k(t) = H_k(t)\big(1 + \Delta_k F_k(t)\big). \tag{3.6}$$

3.4 Dynamics of Rates under the Spot Martingale Measure

In this section, we specify the continuous time dynamics of default-free and defaultable forward rates. It is assumed that the default-free forward rates are modeled as geometric Brownian motions (GBM), whose dynamics are given as follows

$$dF_k(t) = F_k(t)\left(\mu_k^F dt + \left(\boldsymbol{\sigma}_\mathbf{k}^\mathbf{F}\right)' \mathbf{dW}(t)\right), \tag{3.7}$$

where $\mathbf{W(t)}$ is a d-dimensional $\mathbf{Q}$-Brownian motion under the martingale measure $\mathbf{Q}$ and $\boldsymbol{\sigma}_\mathbf{k}^\mathbf{F}$ is a d-dimensional, constant, possibly time dependent, default-free forward rate volatility vector, i.e, $(\boldsymbol{\sigma}_\mathbf{k}^\mathbf{F})' = (\sigma_{k,1}^F, \ldots, \sigma_{k,d}^F)$, $(\cdot)'$ stands for the transpose of the vector, the dimension d will be later suitably specified. μ_k^F is the drift coefficient and will be derived after the forward measures[3] are introduced.

Let us assume that the defaultable forward rates are modeled also as GBM

$$d\overline{F}_k(t) = \overline{F}_k(t)\left(\mu_k^{\overline{F}} dt + \left(\boldsymbol{\sigma}_\mathbf{k}^{\overline{\mathbf{F}}}\right)' \mathbf{dW}(t)\right), \tag{3.8}$$

[3]Forward measures will be explained in Section 3.5. Briefly they are equivalent martingale measures to $\mathbf{Q}$, whose numeraires are corresponding $B_k(t)$'s.

where $\mathbf{W}(t)$ is a d-dimensional $\mathbf{Q}$-Brownian motion and $\boldsymbol{\sigma}_{\mathbf{k}}^{\overline{\mathbf{F}}}$ is a d-dimensional, constant, possibly time dependent, defaultable forward rate volatility vector.

If the defaultable forward rates were modeled as in equation (3.8), it could be possible to have $\overline{F}_k < F_k$ i.e arbitrage opportunities. To avoid the arbitrage opportunities the following two ways of specifying the $\mathbf{Q}$-dynamics of the defaultable forward rates can be used

1. The (discrete) forward default intensities H_k are GBM

$$dH_k(t) = H_k(t)\left(\mu_k^H dt + \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' \mathbf{dW}(t)\right). \tag{3.9}$$

2. The forward credit spreads S_k are GBM

$$dS_k(t) = S_k(t)\left(\mu_k^S dt + \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{S}}\right)' \mathbf{dW}(t)\right), \tag{3.10}$$

where $\mathbf{W}(t)$ is a d-dimensional $\mathbf{Q}$-Brownian motion and $\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}$, $\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{S}}$ are d-dimensional constant, possibly time dependent, vectors.

Remark 3.4.1. *Notice that modeling either the (discrete) forward default intensities or the forward credit spreads as log-normal diffusion processes, excludes the arbitrage opportunities. For instance, choosing the dynamic* (3.10) *for forward credit spreads yields*

$$\overline{F}_k(t) = F_k(t) + S_k(t) \;\Rightarrow\; \overline{F}_k(t) \geq F_k(t),$$

since $S_k(t) > 0$*.*

In this work, we choose to model the forward default intensities and assume that the dynamic of H_k is given by equation (3.9).

In the above described general setup, both default-free forward rates and forward default intensities are driven by the same d-dimensional Brownian vector. The number d can be chosen according to the modeling purposes.

Firstly, we set $d = 2$ and consider a **two-factor model**. One factor is for the default-free forward rate and the other one is for the forward default intensity. In other words, we assume that default-free forward rates, defined for each tenor date, are perfectly mutually correlated and also forward default intensities, defined for each tenor date, are perfectly mutually correlated.

As a second choice, we shall set $d = N$ and consider a **multi factor model**, in order to decorrelate forward default intensities for each tenor date, under the assumption of independent default-free forward rates and forward default intensities.

Independent of having a two-factor or multi-factor model, the GBM assumption for H_k's implies that $\boldsymbol{\sigma}_{\mathbf{k}}^{\overline{\mathbf{F}}}$ and $\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{S}}$ are no longer deterministic, hence $\overline{F}_k$ and S_k are not any more GBMs. The next proposition states this fact and introduces the relation between volatilities of H_k, S_k and $\overline{F}_k$.

Proposition 3.4.1. *Assume the dynamics of F_k and H_k are given by equations* (3.7) *and* (3.9) *respectively. Then, the volatility of S_k and $\overline{F}_k$, given in equations* (3.10) *and* (3.8) *are no longer deterministic. Moreover the following relations hold,*

$$\begin{aligned} \boldsymbol{\sigma}_{\mathbf{k}}^{\overline{\mathbf{F}}} &= \frac{1}{\overline{F}_k}\left(F_k\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}} + S_k\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{S}}\right) = \frac{1}{\overline{F}_k}\left((1+\Delta_k F_k)H_k\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}} + (1+\Delta_k H_k)F_k\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}\right), \quad (3.11)\\ \boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{S}} &= \boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}} + \frac{\Delta_k F_k}{1+\Delta_k F_k}\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}. \quad (3.12)\end{aligned}$$

Proof. (The proof can be also found in Krekel and Wenzel [KW06].)

Recall equation (3.3),

$$S_k = \overline{F}_k - F_k,$$

implies

$$d\overline{F}_k = dF_k + dS_k = \left(F_k\mu_k^F + S_k\mu_k^S\right)\, dt + \left(F_k\left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}\right)' + S_k\left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{S}}\right)'\right)\mathbf{dW}(t). \quad (3.13)$$

Comparing the volatility terms in (3.8) and (3.13) yields

$$\overline{F}_k\boldsymbol{\sigma}_{\mathbf{k}}^{\overline{\mathbf{F}}} = F_k\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}} + S_k\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{S}}. \quad (3.14)$$

In order to connect the volatilities of H_k and S_k, we use the definitions in Section 3.3 to see that

$$1 + \Delta_k H_k = \frac{D_k}{D_{k+1}} = \frac{\overline{B}_k}{\overline{B}_{k+1}}\frac{B_{k+1}}{B_k} = \frac{1+\Delta_k\overline{F}_k}{1+\Delta_k F_k} = 1 + \frac{\Delta_k S_k}{1+\Delta_k F_k}.$$

Therefore, we obtain

$$H_k = \frac{S_k}{1+\Delta_k F_k} \tag{3.15}$$

and it follows that

$$\begin{aligned}
\text{diffusion-coeff}\Big[\frac{dH_k}{H_k}\Big] &= \text{diffusion-coeff}\big[d(\log H_k)\big] \\
&= \text{diffusion-coeff}\big[d(\log S_k - \log(1+\Delta_k F_k))\big] \\
&= \text{diffusion-coeff}\Big[\frac{dS_k}{S_k} - \frac{d(1+\Delta_k F_k)}{1+\Delta_k F_k}\Big].
\end{aligned}$$

Substituting the dynamics of S_k and F_k yields

$$\text{diffusion-coeff}\Big[\frac{dH_k}{H_k}\Big] = \left((\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{S}})' - \frac{\Delta_k F_k}{1+\Delta_k F_k}(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}})'\right). \tag{3.16}$$

Comparing the diffusion coefficients in equations (3.9) and (3.16) gives

$$\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}} = \boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{S}} - \frac{\Delta_k F_k}{1+\Delta_k F_k}\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}. \tag{3.17}$$

Finally, substituting (3.15) and (3.17) into (3.14), we obtain

$$\frac{\overline{F}_k \boldsymbol{\sigma}_{\mathbf{k}}^{\overline{\mathbf{F}}}}{1+\Delta_k \overline{F}_k} = \frac{\Delta_k H_k \boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}}{1+\Delta_k H_k} + \frac{\Delta_k F_k \boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}}{1+\Delta_k F_k}. \tag{3.18}$$

Thus,

$$\overline{F}_k \boldsymbol{\sigma}_{\mathbf{k}}^{\overline{\mathbf{F}}} = (1+\Delta_k F_k) H_k \boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}} + (1+\Delta_k H_k) F_k \boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}. \tag{3.19}$$

□

3.4.1 The two factor model

The two-factor model shall be used to derive Schönbuchers's credit default swap option formula[4] price. We model the default-free forward rates F_k and forward intensities H_k as correlated processes, by the following dynamics respectively

$$\begin{aligned}
dF_k(t) &= F_k(t)\left(\mu_k^F dt + (\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}})' \mathbf{dW}(t)\right), \qquad (3.20)\\
dH_k(t) &= H_k(t)\left(\mu_k^H dt + (\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}})' \mathbf{dW}(t)\right), \qquad (3.21)
\end{aligned}$$

[4]See section 3.7.3.

where $\mathbf{W} = \begin{pmatrix} W^F \\ W^H \end{pmatrix}$ is a two dimensional $\mathbf{Q}$-Brownian motion and $\left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}\right)' = (\sigma_{k,1}^F, \sigma_{k,2}^F)$ and $\left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' = (\sigma_{k,1}^H, \sigma_{k,2}^H)$ are volatility vectors. Having the dynamics (3.20) for default-free forward rates implies that these rates, defined for each tenor date, are perfectly mutually correlated. Having the dynamics (3.21) for forward default intensities implies that these intensities, defined for each tenor date, are perfectly mutually correlated.

Next, assuming independence of F_k and H_k implies the volatility vectors $\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}$ of all forward rates F_k are orthogonal to the volatility vectors $\boldsymbol{\sigma}_{\mathbf{l}}^{\mathbf{H}}$ of the default intensities H_l, that is $\left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}\right)' \left(\boldsymbol{\sigma}_{\mathbf{l}}^{\mathbf{H}}\right) = 0$ for all $k, l = 1, 2$. In particular, one can think of $\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}} = (\sigma_{k,1}^F, 0)$ having zero second component and $\boldsymbol{\sigma}_{\mathbf{l}}^{\mathbf{H}} = (0, \sigma_{l,2}^H)$ having zero first component, so that the F_k's are driven by the first Brownian motion, $W^F(t)$ and the H_k's are driven by the second independent Brownian motion, $W^H(t)$. Then, the dynamics of F_k and H_k under the spot martingale measure $\mathbf{Q}$ are given as

$$\begin{aligned} dF_k(t) &= F_k(t)\left(\mu_k^F dt + \sigma_{k,1}^F dW^F(t)\right) \\ dH_k(t) &= H_k(t)\left(\mu_k^H dt + \sigma_{k,2}^H dW^H(t)\right) \end{aligned}$$

From now on, we drop the subindices 1 and 2 in the entries of volatility vector in the independent case

$$dF_k(t) = F_k(t)\left(\mu_k^F dt + \sigma_k^F dW^F(t)\right) \tag{3.22}$$

$$dH_k(t) = H_k(t)\left(\mu_k^H dt + \sigma_k^H dW^H(t)\right), \tag{3.23}$$

where W^F and W^H are independent $\mathbf{Q}$-Brownian motions

3.4.2 The multi factor model

We consider the following multi factor model to generalise Schönbucher's credit default swap option formula[5].

We assume that the default-free forward rates $F_k(t)$ and forward default intensities $H_k(t)$ are independent. Further, we assume that each forward default-free rate is modeled

[5]See section 3.7.4.

by a Brownian motion and each forward default intensity, H_k is modeled as a linear combination of independent N-Brownian motions. In what follows, we have a $N+1$-factor model.

Therefore, the $\mathbf{Q}$- dynamics of F_k and H_k are given by

$$\frac{dF_k}{F_k} = \mu^F_{k,N}\, dt + \sigma^F_k\, dW^F \tag{3.24}$$

$$\frac{dH_k}{H_k} = \mu^H_{k,N}\, dt + \left(\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{k}}\right)' \mathbf{dW^H}, \tag{3.25}$$

where $\left(\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{k}}\right)' = (\sigma^H_{k,0}, \ldots, \sigma^H_{k,N-1})$ is the N-dimensional volatility vector, W^F is one dimensional $\mathbf{Q}$-Brownian motion and $\left(\mathbf{W^H}\right)' = (W^H_0, \ldots, W^H_{N-1})$ is N-dimensional $\mathbf{Q}$ -Brownian vector. The covariance of the returns of $H_i(t)$ and $H_j(t)$ can be calculated as follows,

$$\begin{aligned}
cov\left(\frac{dH_i(t)}{H_i(t)}, \frac{dH_j(t)}{H_j(t)}\right) &= cov\left((\ldots)dt + \sum_{k=0}^{N-1} \sigma^H_{i,k} dW^H_k(t), (\ldots)dt + \sum_{l=0}^{N-1} \sigma^H_{j,l} dW^H_l(t)\right) \\
&= \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \sigma^H_{i,k}\sigma^H_{j,l}\, cov\big(dW^H_{N,k}(t), dW^H_l(t)\big) \\
&= \left(\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{i}}\right)' \boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{j}}\, dt.
\end{aligned}$$

Hence, the covariance matrix ban be written as follows

$$\mathbf{CV} = \begin{pmatrix}
|\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{1}}|^2 & |\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{1}}||\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{2}}|\rho_{12} & \ldots & |\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{1}}||\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{N}}|\rho_{1N} \\
|\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{2}}||\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{1}}|\rho_{21} & |\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{2}}|^2 & \ldots & |\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{2}}||\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{N}}|\rho_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
|\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{N}}||\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{1}}|\rho_{N1} & |\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{N}}||\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{2}}|\rho_{N2} & \ldots & |\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{N}}|^2
\end{pmatrix} \cdot t$$

where ρ_{ij} is the correlation of the returns of H_i and H_j and defined as

$$\rho_{ij} := \frac{\left(\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{i}}\right)' \boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{j}}}{|\boldsymbol{\sigma}_{\mathbf{i}}||\boldsymbol{\sigma}_{\mathbf{j}}|}, \tag{3.26}$$

and $|\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{i}}| = \sqrt{(\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{i}})' \boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{i}}}$ is the Euclidean norm of $\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{i}}$.

Remark 3.4.2. *By Cholesky decomposition, one can assume that the above covariance matrix* $\mathbf{CV}$ *has a lower triangular form, i. e.* $\sigma^H_{k,j} = 0$ *for* $j \geq k$

3.5 Forward and Survival Measures

Three types of probability measures are used in the model; the spot martingale (already mentioned in section 3.2), the T_k-forward and the T_k-survival measure. $M(t)$, $B_k(t)$ and $I(t)\overline{B}_k(t)$ are the corresponding numeraires to these probability measures. The spot martingale measure is a standard technique in financial modeling, see for example Korn and Korn [KK01] . Jamshidian [Jam87] introduced the forward measure and Schönbucher [Sch05b] introduced the survival measure technique.

In this section, we shall define the spot martingale, T_k-forward and T_k-survival measure and show how to change between these three measures by following Schönbucher [Sch05b]. We present the results in a general d-factor model setup, implying that they are valid also in two and multi factor setups.

To motivate the change of measure methodology, Schönbucher [Sch05b] introduced the dynamics of the default-free and defaultable bond prices under a continuous tenor setting, where continuously compounded default-free and defaultable forward rates are used to described the term structure of interest rates

$$f(t,T) = -\frac{\partial}{\partial T}\ln B(t,T) \qquad \overline{f}(t,T) = -\frac{\partial}{\partial T}\ln \overline{B}(t,T)$$

To avoid arbitrage opportunities, the dynamics of defaultable and default-free continuously compounded forward rates and short credit spread must satisfy the following conditions under the spot martingale measure $\mathbf{Q}$

$$\begin{aligned}
d\overline{f}(t,T) &= \boldsymbol{\sigma}^{\overline{\mathbf{f}}}(t,T)'\left(\int_t^T \boldsymbol{\sigma}^{\overline{\mathbf{f}}}(t,s)ds\right)dt + \boldsymbol{\sigma}^{\overline{\mathbf{f}}}(t,T)'\mathbf{dW}(t),\\
df(t,T) &= \boldsymbol{\sigma}^{\mathbf{f}}(t,T)'\left(\int_t^T \boldsymbol{\sigma}^{\mathbf{f}}(t,s)ds\right)dt + \boldsymbol{\sigma}^{\mathbf{f}}(t,T)'\mathbf{dW}(t),\\
\overline{f}(t,t) &= \lambda(t) + f(t,t),
\end{aligned}$$

where $\mathbf{W(t)}$ is a d-dimensional $\mathbf{Q}$-Brownian and $\boldsymbol{\sigma}^{\mathbf{f}}_{\mathbf{k}}$ is a d-dimensional, constant, possibly time dependent, default-free forward rate volatility vector, i.e, $(\boldsymbol{\sigma}^{\mathbf{f}}_{\mathbf{k}})' = (\sigma^f_{k,1},\ldots,\sigma^f_{k,d})$. The proof of these conditions can be found in Schönbucher [Sch05c]. The solutions to the above SDE for the bond prices are

$$
\begin{aligned}
B(t,T) &= B(0,T)\exp\left(\int_0^t r(s) - 0.5\,|\boldsymbol{\alpha}(s,T)|^2 ds - \int_0^t \boldsymbol{\alpha}(s,T)'\mathbf{dW}(s)\right), &(3.27)\\
\overline{B}(t,T) &= \overline{B}(0,T)\exp\left(\int_0^t \lambda(s) + r(s) - 0.5\,|\overline{\boldsymbol{\alpha}}(s,T)|^2 ds - \int_0^t \overline{\boldsymbol{\alpha}}(s,T)'\mathbf{dW}(s)\right), &(3.28)
\end{aligned}
$$

where $\boldsymbol{\alpha}(t,T) := \int_t^T \boldsymbol{\sigma}^{\mathbf{f}}(t,s)ds$, $\overline{\boldsymbol{\alpha}}(t,T) := \int_t^T \boldsymbol{\sigma}^{\overline{\mathbf{f}}}(t,s)ds$ and $|\boldsymbol{\alpha}(t,T)| = \sqrt{\boldsymbol{\alpha}(t,T)'\boldsymbol{\alpha}(t,T)}$, $|\overline{\boldsymbol{\alpha}}(t,T)| = \sqrt{\overline{\boldsymbol{\alpha}}(t,T)'\overline{\boldsymbol{\alpha}}(t,T)}$ are the Euclidean norms of $\boldsymbol{\alpha}(t,T)$, $\overline{\boldsymbol{\alpha}}(t,T)$ respectively.

Next, we state a general change of measure technique which can be found in Brigo and Mercurio [BM01].

Proposition 3.5.1. *Assume that there exists a numeraire N and a probability measure $\mathbf{Q}^N$, equivalent to $\mathbf{Q}$, such that the price of any traded asset X relative to N is a martingale under $\mathbf{Q}^N$, i.e.,*

$$\frac{X(t)}{N(t)} = \mathbb{E}_{\mathbf{Q}^N}\left(\frac{X(T)}{N(T)}\Big|\mathcal{F}_t\right),\ 0 \le t \le T.$$

Let U be an arbitrary numeraire. Then there exists a probability measure $\mathbf{Q}^U$, equivalent to $\mathbf{Q}$, such that the price of any attainable claim Y normalized by U is a martingale under $\mathbf{Q}^U$, i.e.,

$$\frac{Y(t)}{U(t)} = \mathbb{E}_{\mathbf{Q}^U}\left(\frac{Y(T)}{U(T)}\Big|\mathcal{F}_t\right),\ 0 \le t \le T.$$

Moreover, the Radon-Nikodym derivative defining the measure $\mathbf{Q}^U$ is given by

$$\frac{d\mathbf{Q}^U}{d\mathbf{Q}^N}\Big|_{\mathcal{F}_t} = \frac{U(t)N(0)}{U(0)N(t)}.$$

3.5.1 The Spot Martingale Measure

The spot martingale measure $\mathbf{Q}$ is the probability measure, under which the security prices, discounted by the money market account, become martingales. The money market account, introduced in Definition 3.3.1, is the corresponding numeraire.

The price of a contingent claim at time t with random payoff X at time T_k under the spot martingale measure is given as follows

$$p(t) = \mathbb{E}_{\mathbf{Q}}\left(\frac{M(t)}{M(T_k)}X\Big|\mathcal{F}_t\right).$$

Thus, $p(t)$, normalized with the numeraire $M(t)$, is a $\mathbf{Q}$-martingale.

3.5.2 T_k-Forward Measure

The T_k-forward measure is denoted by $\mathbf{P}_k$. It is used to price payoffs that occur at time T_k. $B_k(t)$ is the corresponding numeraire, i.e., the security prices discounted by $B_k(t)$ are $\mathbf{P}_k$-martingales. It is defined by the Radon-Nikodym density

$$L(t) := \frac{d\mathbf{P}_k}{d\mathbf{Q}}\Big|_{\mathcal{F}_t} = \frac{B_k(t)}{B_k(0)}\frac{M(0)}{M(t)}.$$

By using the equations (3.27) and (3.28), we obtain

$$L(t) = \exp\left(-\int_0^t 0.5\,|\boldsymbol{\alpha}(s,T_k)|^2 ds - \int_0^t \boldsymbol{\alpha}(s,T_k)'\mathbf{dW}(s)\right),$$

which is a strictly positive, $\mathbf{Q}$-martingale with initial value 1. Hence $\mathbf{P}_k$ is an equivalent measure to $\mathbf{Q}$ and from Girsanov's Theorem, the Brownian motion $\mathbf{W_k}(t)$ under the equivalent martingale measure $\mathbf{P}_k$ can be defined as follows

$$\mathbf{dW_k}(t) := \mathbf{dW}(t) + \boldsymbol{\alpha}_\mathbf{k}(t)dt, \tag{3.29}$$

where $\boldsymbol{\alpha}_\mathbf{k}(t) := \boldsymbol{\alpha}(t,T_k)$. For more information about the forward measure technique, we refer to the standard book Brigo and Mercurio [BM01].

Next, we show how to change between two successive forward measures. Let $\mathbf{P}_{k+1}$ be the forward measure defined as above then the forward measure $\mathbf{P}_k$ corresponding to the tenor date T_k is defined by the Radon-Nikodym density

$$L_k(t) := \frac{d\mathbf{P}_k}{d\mathbf{P}_{k+1}}\Big|_{\mathcal{F}_t} = \frac{B_k(t)}{B_k(0)}\frac{B_{k+1}(0)}{B_{k+1}(t)} = \frac{1+\Delta_k F_k(t)}{1+\Delta_k F_k(0)}, \tag{3.30}$$

which implies

$$dL_k(t) = L_k(t)\frac{\Delta_k dF_k(t)}{1+\Delta_k F_k(t)}. \tag{3.31}$$

Under $\mathbf{P}_{k+1}$ the process B_k/B_{k+1} must be a martingale (since under $\mathbf{P}_{k+1}$ security prices, discounted by $B_{k+1}(t)$ are martingales and B_k is one possible security), hence also F_k must be a $\mathbf{P}_{k+1}$-martingale. Therefore, we have

$$\frac{dF_k}{F_k} = \left(\boldsymbol{\sigma}_\mathbf{k}^\mathbf{F}\right)'\mathbf{dW}_{k+1}. \tag{3.32}$$

Substituting equation (3.32) into equation (3.31), we obtain

$$dL_k(t) = L_k(t)\frac{\Delta_k F_k(t)}{1+\Delta_k F_k(t)}\left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}\right)'\mathbf{dW_{k+1}}(t).$$

Let us denote $\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{L}}(t) = \frac{\Delta_k F_k(t)}{1+\Delta_k F_k(t)}\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}$. Then from Ito's formula, we have

$$L_k(t) = \exp\big(-0.5\int_0^t |\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{L}}(s)|^2 ds + \int_0^t \boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{L}}(s)'\mathbf{dW_{k+1}}(s)\big),$$

where $|\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{L}}(t)| = \sqrt{\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{L}}(t)'\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{L}}(t)}$ is the Euclidean norm of $\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{L}}$. $L_k(t)$ is a strictly positive, $\mathbf{P}_{k+1}$-martingale with initial value 1. Therefore, from Girsanov's Theorem, the Brownian motion $\mathbf{W_k}(t)$ under the equivalent martingale measure $\mathbf{P}_k$ is defined as follows,

$$\begin{aligned}\mathbf{dW_k}(t) &= \mathbf{dW_{k+1}}(t) - \boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{L}}(t) \\ &= \mathbf{dW_{k+1}}(t) - \frac{\Delta_k F_k(t)}{1+\Delta_k F_k(t)}\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}} dt. \qquad (3.33)\end{aligned}$$

Remark 3.5.1. *Note that in our setup the forward rate is simply compounded. Therefore, in terms of simply compounded forward rates, the quantities* $\boldsymbol{\alpha}_{\mathbf{k}}(\mathbf{t})$ *are recursively related to each other through the following formula*

$$\boldsymbol{\alpha}_{\mathbf{k+1}}(t) = \boldsymbol{\alpha}_{\mathbf{k}}(t) + \frac{\Delta_k F_k(t)}{1+\Delta_k F_k(t)}\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}. \qquad (3.34)$$

This relation can be easily seen from equations (3.29) *and* (3.33). *Recursively, one can rewrite*

$$\boldsymbol{\alpha}_{\mathbf{k}}(t) = \sum_{j=0}^{k-1}\frac{\Delta_j F_j(t)}{1+\Delta_j F_j(t)}\boldsymbol{\sigma}_{\mathbf{j}}^{\mathbf{F}}, \qquad (3.35)$$

where $\boldsymbol{\sigma}_0^F = 0$, *hence* $\boldsymbol{\alpha}_0(t) = 0$.

3.5.3 T_k-Survival Measure

The T_k-survival measure is denoted by $\overline{\mathbf{P}}_k$. It is used to price defaultable payoffs at T_k. $I(t)\overline{B}_k(t)$ is the corresponding numeraire, i.e., the security prices discounted by $I(t)\overline{B}_k(t)$ are $\overline{\mathbf{P}}_k$-martingales. It is defined by the Radon-Nikodym density

$$\overline{L}(t) := \frac{d\overline{\mathbf{P}}_k}{d\mathbf{Q}}\Big|_{\mathcal{F}_t} = \frac{\overline{B}_k(t)}{\overline{B}_k(0)}\frac{M(0)}{M(t)}I(t).$$

By using equation (3.28) and from the definition of $M(t)$, we obtain

$$\overline{L}(t) = I(t)\exp\left(-\int_0^t 0.5\,|\overline{\boldsymbol{\alpha}}_{\mathbf{k}}(s)|^2 ds - \int_0^t \overline{\boldsymbol{\alpha}}_k(s)'\mathbf{dW}(s)\right),$$

which is a $\mathbf{Q}$-martingale with initial value 1, but it is not strictly positive, since $\overline{L}(t)$ jumps to zero at default. Therefore, $\overline{\mathbf{P}}_k$ is not an equivalent measure to $\mathbf{Q}$, but it is absolutely continuous with respect to $\mathbf{Q}$. Hence, Girsanov's Theorem is still applicable.

Therefore, from Girsanov's theorem, the Brownian motion $\overline{\mathbf{W}}_{\mathbf{k}}(t)$ under the equivalent martingale measure $\overline{\mathbf{P}}_k$ is defined as follows

$$\mathbf{d}\overline{\mathbf{W}}_{\mathbf{k}}(t) := \mathbf{dW}(t) + \overline{\boldsymbol{\alpha}}_{\mathbf{k}}(t)dt. \tag{3.36}$$

Motivation of the survival measure is as follows. Assume that we have a payoff X at T_k contingent on survival. Then, it can be written as $I(T_k)X$ and its present value can be calculated as

$$\mathbb{E}_{\mathbf{Q}}\left(I(T_k)X\frac{M(t)}{M(T_k)}\Big|\mathcal{F}_t\right) = I(t)\overline{B}_k(t)\mathbb{E}_{\overline{\mathbf{P}}_k}\left(X\big|\mathcal{F}_t\right).$$

As one can see, there is no need to model the function I directly. But it is important to know the dynamics of the underlyings F_k and H_k's of the payoff under $\overline{\mathbf{P}}_k$. In the following section, we will derive these dynamics under the survival probability measure.

Since $\overline{F}_{k+1}$ is also a $\mathbf{P}_{k+1}$-martingale, similarly as for the T_k-forward measure case, it follows that for a given $\overline{\mathbf{P}}_{k+1}$-Brownian motion $\overline{\mathbf{W}}_{\mathbf{k+1}}$, a $\overline{\mathbf{P}}_k$-Brownian motion $\overline{\mathbf{W}}_{\mathbf{k}}$ is defined by

$$\mathbf{d}\overline{\mathbf{W}}_{\mathbf{k}}(t) = \mathbf{d}\overline{\mathbf{W}}_{\mathbf{k+1}}(t) - \frac{\Delta_k\overline{F}_k(t)}{1+\Delta_k\overline{F}_k(t)}\boldsymbol{\sigma}_{\mathbf{k}}^{\overline{\mathbf{F}}}dt, \tag{3.37}$$

and $\overline{\boldsymbol{\alpha}}_{\mathbf{k}}(\mathbf{t})$ are recursively related to each other through the following formula

$$\overline{\boldsymbol{\alpha}}_{\mathbf{k+1}}(t) = \overline{\boldsymbol{\alpha}}_{\mathbf{k}}(t) + \frac{\Delta_k\overline{F}_k(t)}{1+\Delta_k\overline{F}_k(t)}\boldsymbol{\sigma}_{\mathbf{k}}^{\overline{\mathbf{F}}}. \tag{3.38}$$

Similarly as in Remark 3.5.1, one can rewrite

$$\overline{\boldsymbol{\alpha}}_{\mathbf{k}}(t) = \sum_{j=0}^{k-1}\frac{\Delta_j\overline{F}_j(t)}{1+\Delta_j\overline{F}_j(t)}\boldsymbol{\sigma}_{\mathbf{j}}^{\overline{\mathbf{F}}}. \tag{3.39}$$

Next, we cover the change of measure from forward to survival measure methodology. Let $\mathbf{P}_k$ be the T_k-forward measure, then the survival measure corresponding to the tenor date T_k is defined by the Radon-Nikodym density,

$$\underline{L}(t) := \frac{d\overline{\mathbf{P}}_k}{d\mathbf{P}_k}\Big|_{\mathcal{F}_t} = \frac{\overline{B}_k(t)}{\overline{B}_k(0)}\frac{B_k(0)}{B_k(t)}I(t) = \frac{D_k(t)}{D_k(0)}I(t).$$

By using equations (3.27) and (3.28), we obtain

$$\underline{L}(t) = I(t)\exp\left(\int_0^t \lambda(s)ds - \int_0^t 0.5\left(|\overline{\boldsymbol{\alpha}}_{\mathbf{k}}(s)|^2 - |\boldsymbol{\alpha}_{\mathbf{k}}(s)|^2\right)ds - \int_0^t \left(\overline{\boldsymbol{\alpha}}_{\mathbf{k}}(s)' - \boldsymbol{\alpha}_{\mathbf{k}}(s)'\right)\mathbf{dW}(s)\right).$$

Let us define $\boldsymbol{\alpha}^{\mathbf{D}}_{\mathbf{k}}(t) := \overline{\boldsymbol{\alpha}}_{\mathbf{k}}(t) - \boldsymbol{\alpha}_{\mathbf{k}}(t)$. Then, by substituting the $\mathbf{Q}$-Brownian motion with $\mathbf{P}_k$-Brownian motion by using equation (3.29), we get

$$\underline{L}(t) = I(t)\exp\left(\int_0^t \lambda(s)ds - \int_0^t 0.5\,|\boldsymbol{\alpha}^{\mathbf{D}}_{\mathbf{k}}(s)|^2 ds - \int_0^t \left(\boldsymbol{\alpha}^{\mathbf{D}}_{\mathbf{k}}(s)\right)'\mathbf{dW}_{\mathbf{k}}(s)\right),$$

which is a $\mathbf{P}_k$-martingale with initial value 1. Note that $-\boldsymbol{\alpha}^{\mathbf{D}}_{\mathbf{k}}(s)$ is the log-volatility of the density process of $\underline{L}(t)$, then from Girsanov's Theorem, the $\overline{\mathbf{P}}_k$-Brownian motion $\overline{\mathbf{W}}_{\mathbf{k}}(t)$ under the equivalent martingale measure can be defined as follows

$$\mathbf{d}\overline{\mathbf{W}}_{\mathbf{k}}(t) := \mathbf{dW}_{\mathbf{k}}(t) + \boldsymbol{\alpha}^{\mathbf{D}}_{\mathbf{k}}\,dt. \tag{3.40}$$

Then, by using equations (3.11), (3.33), (3.37) and (3.40), we obtain the following recursion formula for $\boldsymbol{\alpha}^{\mathbf{D}}_{\mathbf{k}}(t)$

$$\boldsymbol{\alpha}^{\mathbf{D}}_{\mathbf{k+1}}(\mathbf{t}) = \boldsymbol{\alpha}^{\mathbf{D}}_{\mathbf{k}}(t) + \frac{\Delta_k H_k(t)}{1+\Delta_k H_k(t)}\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{k}}. \tag{3.41}$$

Similarly as in Remark 3.5.1, one can rewrite

$$\boldsymbol{\alpha}^{\mathbf{D}}_{\mathbf{k}}(t) = \sum_{j=0}^{k-1}\frac{\Delta_j H_j(t)}{1+\Delta_j H_j(t)}\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{j}}. \tag{3.42}$$

Remark 3.5.2. *The measure $\overline{\mathbf{P}}_k$ attaches zero probability to the default events before T_k i.e.,*

$$\overline{\mathbf{P}}_k = \mathbb{E}_{\mathbf{Q}}\left(\underline{L}(T_k)1_{\{\tau\leq T_k\}}\right)$$

Because it only attaches a positive probability to survival events until T_k, it is called T_k-survival measure.

3.6 Dynamics of the Rates under the Survival Measures

In this section, we will derive the dynamics of default-free and defaultable forward rates by using the change of measures techniques, explained in the previous section. First, we shall state two theorems, whose proofs can also be found in Krekel and Wenzel [KW06], on the dynamics of F_k and H_k under the survival measure $\overline{\mathbf{P}}_i$ by taking into account the general d-factor setup. Then, we shall state corollaries of these theorems by applying them in our two and multi-factor setups.

The first one gives the dynamics of F_k under $\overline{\mathbf{P}}_i$.

Theorem 3.6.1. *Under the log-normal assumption of default-free forward rates, we obtain the dynamics of F_k under the survival measure $\overline{\mathbf{P}}_i$ in the two cases $k+1 \leq i$ and $k+1 > i$ as follows*

$$\frac{dF_k}{F_k} = \left(- \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \boldsymbol{\Sigma}^{\mathbf{H}}_{\mathbf{0,k}} - \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \boldsymbol{\Sigma}^{\mathbf{F}}_{\mathbf{k+1,i}} \right) dt + \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \mathbf{d}\overline{\mathbf{W}}_{\mathbf{i}}, \tag{3.43}$$

if $k+1 \leq i$ and

$$\frac{dF_k}{F_k} = \left(- \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \boldsymbol{\Sigma}^{\mathbf{H}}_{\mathbf{0,k}} + \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \boldsymbol{\Sigma}^{\mathbf{F}}_{\mathbf{i,k+1}} \right) dt + \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \mathbf{d}\overline{\mathbf{W}}_{\mathbf{i}}, \tag{3.44}$$

if $k+1 > i$.

where

$$\boldsymbol{\Sigma}^{\mathbf{F}}_{\mathbf{l,m}} := \sum_{j=l}^{m-1} \frac{\Delta_j F_j}{1+\Delta_j F_j} \boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{j}} \quad \textit{and} \quad \boldsymbol{\Sigma}^{\mathbf{H}}_{\mathbf{l,m}} := \sum_{j=l}^{m-1} \frac{\Delta_j H_j}{1+\Delta_j H_j} \boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{j}}.$$

Proof. Let us start with the default-free forward interest rate. Recall that F_k is a $\mathbf{P}_{k+1}$ martingale,

$$dF_k = F_k \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \mathbf{dW}_{k+1}.$$

By changing the measure from $\mathbf{P}_{k+1}$ to $\overline{\mathbf{P}}_{k+1}$, using equation (3.40) , the dynamics of F_k under $\overline{\mathbf{P}}_{k+1}$ is given by

$$\frac{dF_k}{F_k} = - \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \boldsymbol{\alpha}^{\mathbf{D}}_{\mathbf{k+1}} dt + \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \mathbf{d}\overline{\mathbf{W}}_{k+1}.$$

From equation (3.42), we obtain

$$\boldsymbol{\alpha}^{\mathbf{D}}_{\mathbf{k+1}} = \sum_{j=0}^{k} \frac{\Delta_j H_j(t)}{1+\Delta_j H_j(t)} \boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{j}}.$$

By iteratively changing the measure and using the equation (3.37), one can write the dynamics of $F_k(t)$ under the generic measure $\overline{\mathbf{P}}_i$ as follows.

If $k+1 \leq i$, then

$$\frac{dF_k}{F_k} = \left(- \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \sum_{j=0}^{k-1} \frac{\Delta_j H_j}{1+\Delta_j H_j} \boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{j}} - \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \sum_{j=k+1}^{i-1} \frac{\Delta_j \overline{F}_j}{1+\Delta_j \overline{F}_j} \boldsymbol{\sigma}^{\overline{\mathbf{F}}}_{\mathbf{j}} \right) dt + \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \mathbf{d}\overline{\mathbf{W}}_{\mathbf{i}}.$$

If $k+1 > i$, then

$$\frac{dF_k}{F_k} = \left(- \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \sum_{j=0}^{k-1} \frac{\Delta_j H_j}{1+\Delta_j H_j} \boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{j}} + \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \sum_{j=i}^{k} \frac{\Delta_j \overline{F}_j}{1+\Delta_j \overline{F}_j} \boldsymbol{\sigma}^{\overline{\mathbf{F}}}_{\mathbf{j}} \right) dt + \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \mathbf{d}\overline{\mathbf{W}}_{\mathbf{i}}.$$

By using equation (3.18), we get the result.

If $k+1 \leq i$, then

$$\frac{dF_k}{F_k} = \left(- \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \sum_{j=0}^{i-1} \frac{\Delta_j H_j}{1+\Delta_j H_j} \boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{j}} - \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \sum_{j=k+1}^{i-1} \frac{\Delta_j F_j}{1+\Delta_j F_j} \boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{j}} \right) dt + \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \mathbf{d}\overline{\mathbf{W}}_{\mathbf{i}}.$$

If $k+1 > i$, then

$$\frac{dF_k}{F_k} = \left(- \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \sum_{j=0}^{i-1} \frac{\Delta_j H_j}{1+\Delta_j H_j} \boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{j}} + \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \sum_{j=i}^{k} \frac{\Delta_j F_j}{1+\Delta_j F_j} \boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{j}} \right) dt + \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \mathbf{d}\overline{\mathbf{W}}_{\mathbf{i}}.$$

$\square$

The next theorem gives the dynamics of H_k under $\overline{\mathbf{P}}_i$.

Theorem 3.6.2. *Under the log-normal assumption of the forward default intensities, we obtain the dynamics of H_k under the survival measure $\overline{\mathbf{P}}_i$ in the two cases $k+1 \leq i$ and $k+1 > i$ as follows*

$$\frac{dH_k}{H_k} = \left(\frac{\Delta_k F_k}{1+\Delta_k F_k} \frac{1+\Delta_k H_k}{\Delta_k H_k} \left(\boldsymbol{\sigma}^{\mathbf{F}}_{\mathbf{k}}\right)' \boldsymbol{\Sigma}^{\mathbf{H}}_{\mathbf{0,k}} - \left(\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{k}}\right)' \boldsymbol{\Sigma}^{\mathbf{H}}_{\mathbf{k+1,i}} - \left(\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{k}}\right)' \boldsymbol{\Sigma}^{\mathbf{F}}_{\mathbf{k+1,i}} \right) dt + \left(\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{k}}\right)' \mathbf{d}\overline{\mathbf{W}}_{\mathbf{i}}, \tag{3.45}$$

if $k+1 \leq i$ *and*

$$\frac{dH_k}{H_k} = \left(\frac{\Delta_k F_k}{1+\Delta_k F_k} \frac{1+\Delta_k H_k}{\Delta_k H_k} \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}\right)' \boldsymbol{\Sigma}_{\mathbf{0,k}}^{\mathbf{H}} + \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' \boldsymbol{\Sigma}_{\mathbf{k+1,i}}^{\mathbf{H}} + \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' \boldsymbol{\Sigma}_{\mathbf{k+1,i}}^{\mathbf{F}} \right) dt + \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' \mathbf{d}\overline{\mathbf{W}}_{\mathbf{i}}, \tag{3.46}$$

if $k+1 > i$,

where

$$\boldsymbol{\Sigma}_{\mathbf{l,m}}^{\mathbf{F}} := \sum_{j=l}^{m-1} \frac{\Delta_j F_j}{1+\Delta_j F_j} \boldsymbol{\sigma}_{\mathbf{j}}^{\mathbf{F}} \quad \textit{and} \quad \boldsymbol{\Sigma}_{\mathbf{l,m}}^{\mathbf{H}} := \sum_{j=l}^{m-1} \frac{\Delta_j H_j}{1+\Delta_j H_j} \boldsymbol{\sigma}_{\mathbf{j}}^{\mathbf{H}}.$$

Proof. From equation (3.6), we get

$$dS_k(t) = dH_k(t) + \Delta_k H_k dF_k + \Delta_k F_k dH_k + \Delta_k d\langle H_k, F_k \rangle.$$

By using the definition of S_k and rearranging terms, we obtain

$$(1+\Delta_k F_k) dH_k = d\overline{F}_k - dF_k - \Delta_k H_k dF_k - \Delta_k d\langle H_k, F_k \rangle.$$

Let us substitute the $\overline{\mathbf{P}}_{k+1}$-dynamics of $\overline{F}_k$ and F_k

$$\begin{aligned} \frac{d\overline{F}_k}{F_k} &= \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\overline{\mathbf{F}}}\right)' \mathbf{d}\overline{\mathbf{W}}_{k+1} \\ \frac{dF_k}{F_k} &= -\left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}\right)' \boldsymbol{\alpha}_{\mathbf{k+1}}^{\mathbf{D}} dt + \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}\right)' \mathbf{d}\overline{\mathbf{W}}_{k+1} \end{aligned}$$

to the above equation and arrange the terms. Then

$$\begin{aligned} dH_k &= \frac{F_k}{1+\Delta_k F_k} \left((1+\Delta_k H_k) \left(\boldsymbol{\alpha}_{\mathbf{k+1}}^{\mathbf{D}}\right)' - \Delta_k H_k \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' \right) \boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}} dt \\ &\quad + \frac{1}{1+\Delta_k F_k} \left(\left(\boldsymbol{\sigma}_{\mathbf{k}}^{\overline{\mathbf{F}}}\right)' \overline{F}_k - (1+\Delta_k H_k) \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}\right)' F_k \right) \mathbf{d}\overline{\mathbf{W}}_{\mathbf{k+1}}. \end{aligned}$$

From equation (3.19), the diffusion term on the right hand side is equal to $H_k \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)'$. Hence, we obtain

$$dH_k = \frac{F_k}{1+\Delta_k F_k} \left((1+\Delta_k H_k) \left(\boldsymbol{\alpha}_{\mathbf{k+1}}^{\mathbf{D}}\right)' - \Delta_k H_k \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' \right) \boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}} dt + H_k \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' \mathbf{d}\overline{\mathbf{W}}_{\mathbf{k+1}}.$$

By substituting $\boldsymbol{\alpha}_{k+1}^D$ and iteratively changing the measure to $\mathbf{P}_i$, depending on $k+1 \leq i$ or $i < k$. We get the desired results. □

Theorems 3.6.1 and 3.6.2 are stated for general d-factor model. Next, we postulate the direct consequences of these results in our two factor and multi factor setups.

Corollary 3.6.3. *Under the* ***two factor setup****, we obtain the dynamics of F_k and H_k under the survival measure $\overline{\mathbf{P}}_i$ in the two cases $k+1 \leq i$ and $k+1 > i$ as follows*

$$
\begin{aligned}
\frac{dF_k}{F_k} &= \left(-\left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}\right)' \boldsymbol{\Sigma}_{\mathbf{0,k}}^{\mathbf{H}} - \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}\right)' \boldsymbol{\Sigma}_{\mathbf{k+1,i}}^{\mathbf{F}}\right) dt + \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}\right)' \mathbf{d}\overline{\mathbf{W}}_{\mathbf{i}}, \\
\frac{dH_k}{H_k} &= \left(\frac{\Delta_k F_k}{1+\Delta_k F_k} \frac{1+\Delta_k H_k}{\Delta_k H_k} \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}\right)' \boldsymbol{\Sigma}_{\mathbf{0,k}}^{\mathbf{H}} - \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' \boldsymbol{\Sigma}_{\mathbf{k+1,i}}^{\mathbf{H}} - \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' \boldsymbol{\Sigma}_{\mathbf{k+1,i}}^{\mathbf{F}}\right) dt + \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' \mathbf{d}\overline{\mathbf{W}}_{\mathbf{i}},
\end{aligned}
$$

if $k+1 \leq i$ and

$$
\begin{aligned}
\frac{dF_k}{F_k} &= \left(-\left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}\right)' \boldsymbol{\Sigma}_{\mathbf{0,k}}^{\mathbf{H}} + \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}\right)' \boldsymbol{\Sigma}_{\mathbf{i,k+1}}^{\mathbf{F}}\right) dt + \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}\right)' \mathbf{d}\overline{\mathbf{W}}_{\mathbf{i}}, \\
\frac{dH_k}{H_k} &= \left(\frac{\Delta_k F_k}{1+\Delta_k F_k} \frac{1+\Delta_k H_k}{\Delta_k H_k} \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}\right)' \boldsymbol{\Sigma}_{\mathbf{0,k}}^{\mathbf{H}} + \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' \boldsymbol{\Sigma}_{\mathbf{k+1,i}}^{\mathbf{H}} + \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' \boldsymbol{\Sigma}_{\mathbf{k+1,i}}^{\mathbf{F}}\right) dt + \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' \mathbf{d}\overline{\mathbf{W}}_{\mathbf{i}},
\end{aligned}
$$

if $k+1 > i$,

where

$$
\boldsymbol{\Sigma}_{\mathbf{l,m}}^{\mathbf{F}} := \sum_{j=l}^{m-1} \frac{\Delta_j F_j}{1+\Delta_j F_j} \boldsymbol{\sigma}_{\mathbf{j}}^{\mathbf{F}} \quad \text{and} \quad \boldsymbol{\Sigma}_{\mathbf{l,m}}^{\mathbf{H}} := \sum_{j=l}^{m-1} \frac{\Delta_j H_j}{1+\Delta_j H_j} \boldsymbol{\sigma}_{\mathbf{j}}^{\mathbf{H}}.
$$

$\overline{\mathbf{W}}_{\mathbf{i}} = \begin{pmatrix} \overline{W}_i^F \\ \overline{W}_i^H \end{pmatrix}$ *is two dimensional Brownian motion and* $\left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{F}}\right)' = (\sigma_{k,1}^F, \sigma_{k,2}^F)$ *and* $\left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' = (\sigma_{k,1}^H, \sigma_{k,2}^H)$ *are volatility vectors.*

Corollary 3.6.4. *Under the* ***two factor setup with independent default-free forward interest rates*** *F_k* ***and forward default intensities*** *H_k, the dynamics of F_k and H_k simplifies to*

$$
\begin{aligned}
\frac{dF_k}{F_k} &= \Sigma_{k+1,i}^F \sigma_k^F \, dt + \sigma_k^F \, d\overline{W}_i^F, \\
\frac{dH_k}{H_k} &= -\Sigma_{k+1,i}^H \sigma_k^H \, dt + \sigma_k^H d\overline{W}_i^H
\end{aligned}
\tag{3.47}
$$

if $k+1 \leq i$ and

$$
\begin{aligned}
\frac{dF_k}{F_k} &= \Sigma_{i,k+1}^F \sigma_k^F \, dt + \sigma_k^F \, d\overline{W}_i^F, \\
\frac{dH_k}{H_k} &= \Sigma_{i,k+1}^H \sigma_k^H \, dt + \sigma_k^H d\overline{W}_i^H
\end{aligned}
\tag{3.48}
$$

if $k+1 > i$.

Remark 3.6.1. *Note that under the independence assumption, F_k and H_k are $\overline{\mathbf{P}}_{k+1}$ martingales. i.e.,*

$$\begin{aligned} dF_k &= F_k \sigma_k^F \, d\overline{W}_{k+1}^F, \\ dH_k &= H_k \sigma_k^H d\overline{W}_{k+1}^H. \end{aligned}$$

Corollary 3.6.5. *Under the* ***multi factor setup****, we obtain the dynamics of F_k and H_k under the survival measure $\overline{\mathbf{P}}_i$ in the two cases $k+1 \leq i$ and $k+1 > i$ as follows*

$$\begin{aligned} \frac{dF_k}{F_k} &= \Sigma_{k+1,i}^F \sigma_k^F \, dt + \sigma_k^F \, d\overline{W}_i^F, \\ \frac{dH_k}{H_k} &= -\left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' \boldsymbol{\Sigma}_{\mathbf{k+1,i}}^{\mathbf{H}} dt + \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' \mathbf{d}\overline{\mathbf{W}}_{\mathbf{i}}^{\mathbf{H}} \end{aligned} \tag{3.49}$$

if $k+1 \leq i$ and

$$\begin{aligned} \frac{dF_k}{F_k} &= \Sigma_{i,k+1}^F \sigma_k^F \, dt + \sigma_k^F \, d\overline{W}_i^F, \\ \frac{dH_k}{H_k} &= \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' \boldsymbol{\Sigma}_{\mathbf{k+1,i}}^{\mathbf{H}} \, dt + \left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' \mathbf{d}\overline{\mathbf{W}}_{\mathbf{i}}^{\mathbf{H}} \end{aligned} \tag{3.50}$$

if $k+1 > i$,

where $\overline{W}_i^F$ is a one dimensional $\overline{\mathbf{P}}_i$-Brownian motion, $\left(\overline{\mathbf{W}}_{\mathbf{i}}^{\mathbf{H}}\right)' = \left(\overline{W}_{i,0}^H, \ldots, \overline{W}_{i,N-1}^H\right)$ is N-dimensional $\overline{\mathbf{P}}_i$-Brownian vector and $\left(\boldsymbol{\sigma}_{\mathbf{k}}^{\mathbf{H}}\right)' = (\sigma_{k,0}^H, \ldots \sigma_{k,N-1}^H)$ is a N-dimensional volatility vector.

3.7 Basic Credit Derivatives

In this section, we will price the credit default swap and credit default swaption by following Schönbucher [Sch05b] and introduce a more general credit default swaption price formulae by using the multi factor model. Firstly, we need to derive formulas to value arbitrary payoffs. The proofs of the next propositions can be found in Krekel and Wenzel [KW06] and Schönbucher [Sch05b](Proposition 2).

Proposition 3.7.1. *For the the payoffs*[6] *X that can depend on all forward rates $F_0, \ldots, F_{N-1}$ as well as on the default intensities $H_0, \ldots, H_{N-1}$, we have the following formulas for t-time prices, $p(t)$*

1. *The price at time 0 of a contract paying the amount X at T_{k+1}, contingent on no default happening before T_{k+1} is*
$$p(t) = \overline{B}_N(t)\mathbb{E}_{\overline{\mathbf{P}}_N}\left(\frac{X}{\overline{B}_N(T_{k+1})}\Big|\mathcal{F}_t\right) = \overline{B}_{k+1}(t)\mathbb{E}_{\overline{\mathbf{P}}_{k+1}}(X|\mathcal{F}_t).$$

2. *The price at time 0 of a contract paying the amount X at T_{k+1}, contingent on no default happening before T_k is*
$$p(t) = \overline{B}_N(t)\mathbb{E}_{\overline{\mathbf{P}}_N}\left(\frac{X}{\overline{B}_N(T_{k+1})}\big(1+\Delta_k H_k(T_k)\big)\Big|\mathcal{F}_t\right) = \overline{B}_{k+1}(t)\mathbb{E}_{\overline{\mathbf{P}}_{k+1}}\Big(X\big(1+\Delta_k H_k(T_k)\big)\Big|\mathcal{F}_t\Big).$$

3. *The price at time 0 of a contract paying the amount X at T_{k+1}, contingent on a default happening between T_k and T_{k+1} is*
$$p(t) = \overline{B}_N(t)\mathbb{E}_{\overline{\mathbf{P}}_N}\left(\frac{X}{\overline{B}_N(T_{k+1})}\Delta_k H_k(T_k)\Big|\mathcal{F}_t\right) = \overline{B}_{k+1}(t)\mathbb{E}_{\overline{\mathbf{P}}_{k+1}}\big(X\Delta_k H_k(T_k)\big|\mathcal{F}_t\big).$$

The next result is a direct consequence of the Proposition 3.7.1 and shows that if forward default intensities and forward default-free rates are independent, then we can significantly simplify the formulas.

Proposition 3.7.2. *If the payoffs X in the Proposition 3.7.1 depend only on $F_0, \ldots, F_{n-1}$.*

1. *The price at time 0 of a contract paying the amount X at T_{k+1} , contingent on no default happening before T_{k+1} is*
$$p(t) = \overline{B}_{k+1}(t)\mathbb{E}_{\overline{\mathbf{P}}_{k+1}}(X\big|\mathcal{F}_t).$$

2. *The price at time 0 of a contract paying the amount X at T_{k+1}, contingent on no default happening before T_k is*
$$p(t) = \overline{B}_{k+1}(t)\big(1+\Delta_k H_k(t)\big)\mathbb{E}_{\overline{\mathbf{P}}_{k+1}}\big(X\big|\mathcal{F}_t\big).$$

[6] In credit default derivatives, the random payoff, which will be paid at time T_{k+1}, is specified at time T_k, i.e, X, paid at time T_{k+1} is T_k measurable.

3. *The price at time* 0 *of a contract paying the amount* X *at* T_{k+1}, *contingent on a default happening between* T_k *and* T_{k+1} *is*

$$p(t) = \overline{B}_{k+1}(t)\Delta_k H_k(t)\mathbb{E}_{\overline{\mathbf{P}}_{k+1}}\big(X\big|\mathcal{F}_t\big).$$

3.7.1 Credit Default Swaps

A credit default swap (CDS) is a specific kind of counterparty agreement which allows the transfer of the third party credit risk from one party to the other. One party in the swap is a lender (protection seller) and faces credit risk from a third party by paying the loss of the counterparty at default evet, and the counterparty (protection buyer) in the credit default swap agrees to insure this risk in exchange of regular periodic payments (essentially an insurance premium). According to the type of loss payment at the default event, there are two types od CDS contracts: a cash settlement and physical settlement CDS. In a cash settlement CDS, if the third party defaults, the protection seller will have to pay the difference between the reference asset face value and the price of the defaulted asset. Whereas in a physical settlement contract, the protection seller has to buy the defaulted reference asset at its face value, if the third party defaults.

Definition 3.7.1. *A credit default swap consists of two payment legs; the fixed (fee-payment), paid by the protection buyer and the floating (default insurance leg), paid by the protection seller. The payment stream between the fixed leg and the floating leg is described in Table 3.1.*

Buyer	$\rightarrow$	rate $s\,\Delta_i$ at $T_{i+1},\ \forall i = 0,\ldots,N-1$, if no default before T_{i+1}	$\rightarrow$	Seller
A	$\leftarrow$	protection $(1-R)$ at $T_{i+1},\ \forall i = 0,\ldots,N-1$, if default in $[T_i, T_{i+1}]$	$\leftarrow$	B

Table 3.1: Payment stream in a payer credit default swap contract.

where s is called the credit default swap rate (CDS spread)[7]. *R is the recovery rate, which*

[7] s is chosen so that the fixed leg and the floating leg of the CDS contract is equal to zero at time zero. See Definition 3.7.2.

is assumed to be constant.

Therefore, the present value of the fixed leg of a CDS is given by

$$V_{fixed}(0) = s \sum_{i=0}^{N-1} \Delta_i \overline{B}_{i+1}(0).$$

The present value of the i-th payment of the floating leg of the CDS is given by Proposition 3.7.1, part *3.* as follows

$$(1-R)\overline{B}_N(0)\mathbb{E}_{\overline{\mathbf{P}}_N}\left(\frac{\Delta_i H_i(T_i)}{\overline{B}_N(T_{i+1})}\right) = (1-R)\Delta_i \overline{B}_{i+1}(0)\mathbb{E}_{\overline{\mathbf{P}}_{i+1}}\left(H_i(T_i)\right).$$

Therefore, the value of the floating leg of the CDS at time $t=0$ is given by

$$V_{float}(0) = (1-R)\sum_{i=k}^{N-1} \Delta_i \overline{B}_{i+1}(0)\mathbb{E}_{\overline{\mathbf{P}}_{i+1}}\left(H_i(T_i)\right).$$

Definition 3.7.2. *The fair CDS spread s makes the present value of CDS equal to zero, i.e, the fixed leg is equal to floating leg at time $t=0$. Therefore, it is given by*

$$s = (1-R)\frac{\sum_{i=0}^{N-1} \Delta_i \overline{B}_{i+1}(0)\mathbb{E}_{\overline{\mathbf{P}}_{i+1}}\left(H_i(T_i)\right)}{\sum_{i=0}^{N-1} \Delta_i \overline{B}_{i+1}(0)} = (1-R)\sum_{i=0}^{N-1} w_i \Delta_i \mathbb{E}_{\overline{\mathbf{P}}_{i+1}}\big(H_i(T_i)\big), \quad (3.51)$$

where

$$w_i = \frac{\overline{B}_{i+1}(0)}{\sum_{j=0}^{N-1} \Delta_j \overline{B}_{j+1}(0)}.$$

Definition 3.7.3. *A forward start CDS is a CDS, which is contracted at time t, with fee payments and credit protection begining at a later time $T_k > t$. The payment stream between the fixed leg and the floating leg is described in Table 3.2.*

Buyer	$\rightarrow$	rate $s_{k,N}(t)\,\Delta_i$ at T_{i+1}, $\forall i = k,\ldots,N-1$, if no default before T_{i+1}	$\rightarrow$	Seller
A	$\leftarrow$	protection $(1-R)$ at T_{i+1}, $\forall i = k,\ldots,N-1$, if default in $[T_i, T_{i+1}]$	$\leftarrow$	B

Table 3.2: Payment stream in a payer forward credit default swap contract.

where $s_{k,N}(t)$ is called the fair forward CDS spread[8]*, contracted at time t. R is the constant recovery rate.*

Therefore, the value at any time $t \leq T_k$ of the fixed leg of a forward CDS is given by

$$V_{fixed}(t) = s \sum_{i=k}^{N-1} \Delta_i \overline{B}_{i+1}(t).$$

The time t value of the i-th payment of the floating leg of the forward CDS is given by Proposition 3.7.1, part *3*. as follows

$$(1-R)\overline{B}_N(t)\mathbb{E}_{\overline{\mathbf{P}}_N}\left(\frac{\Delta_i H_i(T_i)}{\overline{B}_N(T_{i+1})}\Big|\mathcal{F}_t\right) = (1-R)\Delta_i \overline{B}_{i+1}(t)\mathbb{E}_{\overline{\mathbf{P}}_{i+1}}\left(H_i(T_i)|\mathcal{F}_t\right).$$

Therefore, the time t value of the floating leg of the forward CDS is

$$V_{float}(t) = (1-R)\sum_{i=k}^{N-1} \Delta_i \overline{B}_{i+1}(t)\mathbb{E}_{\overline{\mathbf{P}}_{i+1}}\left(H_i(T_i)|\mathcal{F}_t\right).$$

Definition 3.7.4. *The (k, N)-forward swap rate $\overline{s}_{k,N}$ is the rate of a forward CDS starting at T_k and running for $N-k$ periods, such that the value of the fixed leg equals the value of the floating leg at the begining of the . It is given as*

$$\overline{s}_{k,N}(t) = (1-R)\frac{\sum_{i=k}^{N-1} \Delta_i \overline{B}_{i+1}(t)\mathbb{E}_{\overline{\mathbf{P}}_{i+1}}\left(H_i(T_i)|\mathcal{F}_t\right)}{\sum_{i=k}^{N-1} \Delta_i \overline{B}_{i+1}(t)} = (1-R)\sum_{i=k}^{N-1} \overline{w}_i(t)\Delta_i \mathbb{E}_{\overline{\mathbf{P}}_{i+1}}\big(H_i(T_i)\big|\mathcal{F}_t\big), \tag{3.52}$$

where

$$\overline{w}_i(t) = \frac{\overline{B}_{i+1}(t)}{\sum_{j=k}^{N-1} \Delta_j \overline{B}_{j+1}(t)}. \tag{3.53}$$

In particular the $(k, k+1)$-forward swap rate is

$$\overline{s}_{k,k+1}(t) = (1-R)\mathbb{E}_{\overline{\mathbf{P}}_{k+1}}\big(H_k(T_k)\big|\mathcal{F}_t\big).$$

The main pricing problem of the credit derivatives, underlying on the forward swap rates (e.g. credit derivative swap options) is the conditional expectation term appearing in equation (3.52). The following proposition states that under the independence assumption of default-free forward rate F_k and forward default intensity H_k, the conditional expectations appearing in equation (3.52) vanishes.

[8] It is the CDS spread, making the fixed leg and the floating leg of the forward CDS equal at time t, when the CDS is contracted. See Definition 3.7.4.

Proposition 3.7.3. *Under the assumption of independence of F_k and H_k, $\overline{s}_{k,N}(t)$ is given as follows,*

$$\overline{s}_{k,N}(t) = (1-R)\sum_{i=k}^{N-1}\overline{w}_{i+1}(t)\Delta_i H_i(t). \tag{3.54}$$

In particular the $(k,k+1)$-forward swap rate is

$$\overline{s}_{k,k+1}(t) = (1-R)H_k(t).$$

Proof. Under the assumption of independence of F_k and H_k, we have seen that H_i is in fact a martingale under $\overline{\mathbf{P}}_{i+1}$ (see Remark 3.6.1) and therefore $\mathbb{E}_{\overline{\mathbf{P}}_{i+1}}\left(H_i(T_i)\middle|\mathcal{F}_t\right) = H_i(t)$ and the assertion follows. $\square$

Hence, the independence of F_k and H_k is a crucial assumption, in order to derive closed form solutions.

3.7.2 Credit Default Swaption

A credit default swap option is also known as a credit default swaption. It is an option on a (forward starting) CDS. A credit default swaption gives its holder the right, but not the obligation, to buy (call) or sell (put) protection on a specified reference entity for a specified future time period for a certain spread. The option is knocked out if the reference entity defaults during the life of the option. Most commonly traded CDS options are European style options and in this thesis, we consider only the pricing issue of an European credit default swap call option (CDSwaption).

The payoff function of the CDSwaption Call at T_k is

$$\begin{aligned}
V^{call}_{CDSwaption}(T_k) &= \left(V_{float}(T_k) - V_{fixed}(T_k)\right)^+ \\
&= \left((1-R)\sum_{i=k}^{N-1}\Delta_i\overline{B}_{i+1}(T_k)\mathbb{E}_{\overline{\mathbf{P}}_{i+1}}\left(H_i(T_i)|\mathcal{F}_k\right) - s\sum_{i=k}^{N-1}\Delta_i\overline{B}_{i+1}(t)\right)^+ \\
&= \sum_{i=k}^{N-1}\Delta_i\overline{B}_{i+1}(T_k)\left((1-R)\frac{\sum_{i=k}^{N-1}\Delta_i\overline{B}_{i+1}(T_k)\mathbb{E}_{\overline{\mathbf{P}}_{i+1}}\left(H_i(T_i)|\mathcal{F}_k\right)}{\sum_{i=k}^{N-1}\Delta_i\overline{B}_{i+1}(T_k)} - s\right)^+ \\
&= \sum_{i=k}^{N-1}\Delta_i\overline{B}_{i+1}(T_k)\left(\overline{s}_{k,N}(T_k) - s\right)^+.
\end{aligned}$$

Hence, the value of the CDSwaption at time t is

$$\begin{aligned} V^{call}_{CDSwaption}(t) &= \overline{B}_k(t)\mathbb{E}_{\overline{\mathbf{P}}_k}\left(\sum_{i=k}^{N-1}\Delta_i\overline{B}_{i+1}(T_k)\left(\overline{s}_{k,N}(T_k)-s\right)^+\big|\mathcal{F}_t\right) \\ &= \sum_{i=k}^{N-1}\Delta_i\overline{B}_{i+1}(t)\mathbb{E}_{\overline{\mathbf{P}}_{i+1}}\left(\left(\overline{s}_{k,N}(T_k)-s\right)^+\big|\mathcal{F}_t\right). \end{aligned}$$

The Default Swap Measure

In order to price CDSwaptions, Schönbucher introduced the default swap measure, analog to the swap market measure introduced by Jamshidian [Jam97]. The numeraire asset is taken as

$$X(t) := I(t)\sum_{i=k}^{N-1}\Delta_i\overline{B}_{i+1}(t)$$

for the new probability measure $\overline{\mathbf{P}}^s$. The Radon-Nikodym density of this measure with respect to $\mathbf{Q}$ is given as follows

$$\frac{d\overline{\mathbf{P}}^s}{d\mathbf{Q}}\Big|_{\mathcal{F}_t} = \frac{X(t)}{X(0)}\frac{M(0)}{M(t)}.$$

The measure $\overline{\mathbf{P}}^s$ is associated with the Brownian motion $\overline{W}^s$ and under this measure security prices divided by $X(t)$ are $\overline{\mathbf{P}}^s$-martingales.

3.7.3 Schönbucher's CDSwaption Formula

In this section, we introduce the Schönbucher [Sch05b]'s CDSwaption formula, under several assumptions.

Theorem 3.7.4 (Schönbucher). *Under the assumptions that*

i) we have the two factor model, introduced in Section 3.4.1, where H_k and F_k are independent,

ii) $\sigma_i^H = \sigma^H$, $i = k, \dots, N-1$, i.e., volatilities for different tenor dates are constant and homogeneous, *iii) the weights defined in equation* (3.53) *are constant in the interval* $[T_k, T_N]$, *i.e.,* $\overline{w}_i(t) = \overline{w}_i(0), \ \forall t \in [T_k, T_N]$

the time t price of a CDSwaption with maturity T_k and strike s is given by

$$V^{call}_{CDSwaption}(t) = \sum_{i=k}^{N-1} \Delta_i \overline{B}_{i+1}(t)\big(s_{k,N}(t)\Phi(d_1(t)) - s\,\Phi(d_2(t))\big), \tag{3.55}$$

where

$$d_{1,2}(t) = \frac{\ln(s_{k,N}(t)/s) \pm (\sigma^H)^2(T_k - t)/2}{\sigma^H\sqrt{T_k - t}}.$$

Proof. Without loss of generality we assume that $t = 0$. Therefore, our aim is to calculate the price of the call CDSwaption at time 0, given as follows,

$$V^{call}_{CDSwaption}(0) = \sum_{i=k}^{N-1} \Delta_i \overline{B}_{i+1}(0)\mathbb{E}_{\overline{\mathbf{P}}_{i+1}}\left((\overline{s}_{k,N}(T_k) - s)^+\right). \tag{3.56}$$

First, it follows from the independence assumption of F_k and H_k in $i)$ that

$$\begin{aligned} \overline{s}_{k,N}(T_k) &= (1-R)\sum_{i=k}^{N-1} \overline{w}_i(T_k)\Delta_i \mathbb{E}_{\overline{\mathbf{P}}_{i+1}}\left(H_i(T_i)|\mathcal{F}_k\right) \\ &= (1-R)\sum_{i=k}^{N-1} \overline{w}_i(T_k)\Delta_i H_i(T_k). \end{aligned}$$

Next, from assumption $iii)$, we get

$$\overline{s}_{k,N}(T_k) = \sum_{i=k}^{N-1} c_i H_i(T_k),$$

where

$$c_i = (1-R)\overline{w}_i(0)\Delta_i. \tag{3.57}$$

Therefore, the dynamics of $\overline{s}_{k,N}$ is given as

$$d\overline{s}_{k,N} = \sum_{i=k}^{N-1} c_i dH_i.$$

Let us change the $\overline{\mathbf{P}}_{i+1}$ measure to $\overline{\mathbf{P}}^s$ by using the following Radon-Nikodym density

$$\overline{L}^s(t) = \frac{\overline{\mathbf{P}}^s}{\overline{\mathbf{P}}_{i+1}}\Big|_{\mathcal{F}_t} = \frac{I(t)\sum_{i=k}^{N-1}\Delta_i\overline{B}_{i+1}(t)}{\sum_{i=k}^{N-1}\Delta_i\overline{B}_{i+1}(0)}\frac{\overline{B}_{i+1}(0)}{\overline{B}_{i+1}(t)} = \frac{\overline{w}_i(0)}{\overline{w}_i(t)}. \tag{3.58}$$

Note that from assumption $iii)$ the density process is constant over time and there is no drift correction when changing the measure from $\overline{\mathbf{P}}_{i+1}$ to $\overline{\mathbf{P}}^s$, i.e, under $\overline{\mathbf{P}}^s$ the intensities H_i are driftless,

$$d\overline{s}_{k,N} = \sum_{i=k}^{N-1} c_i H_i \sigma_i^H \, d\overline{W}^s.$$

Finally, from the homogenous volatility assumption in $ii)$, it follows that

$$\frac{d\overline{s}_{k,N}}{\overline{s}_{k,N}} = \sigma^H \, d\overline{W}^s,$$

i.e., $\overline{s}_{k,N}$ is log-normal with volatility σ^H.

Under the measure $\overline{\mathbf{P}}^s$, the present value of the CDSwaption is

$$V^{call}_{CDSwaption}(0) = \sum_{i=k}^{N-1} \Delta_i \overline{B}_{i+1}(0) \mathbb{E}_{\overline{\mathbf{P}}^s} \left(\left(\overline{s}_{k,N}(T_k) - s \right)^+ \right).$$

The result follows straight forward by applying the Black and Scholes derivation to our problem. □

Remark 3.7.1.

1. *If we weaken the assumption $ii)$ and consider time dependent homogeneous volatility, then the volatilities can be treated as follows*

$$\left(\sigma^H\right)^2 (T_k - t) = \int_t^{T_k} \left(\sigma^H(s)\right)^2 \, ds.$$

2. *Price of a credit default swap put option can be easily derived as in Theorem 3.7.4.*

3.7.4 An approximation for the valuation of CDSwaptions using multiple factors and inhomogeneous volatilities

In this subsection, we shall derive a CDSwaption formula by considering the multi-factor setup, introduced in Section 3.4.2 and relaxing the homogenous volatility structure in Schönbucher's formula, see Assumption $ii)$ in Theorem 3.7.4. Therefore, we present a twofold generalization of Schönbucher's CDSwaption formula (3.55).

Theorem 3.7.5 (Levy). *Under the assumptions that*

i) we have the multi factor setup, introduced in Section 3.4.2,

ii) the weights defined in equation (3.53) *are constant in the interval* $[T_k, T_N]$*, i.e.,*$\overline{w}_i(t) = \overline{w}_i(0), \ \forall t \in [T_k, T_N]$

the t*-time price of a CDSwaption with maturity* T_k *and strike* s *is approximated by*

$$V^{call}_{CDSwaption}(t) \approx \sum_{i=k}^{N-1} \Delta_i \overline{B}_{i+1}(t)\big(\overline{s}_{k,N}(t)\Phi(d_1(t)) - s\Phi(d_2(t))\big), \tag{3.59}$$

where

$$d_{1,2}(t) = \frac{\ln(\overline{s}_{k,N}(t)/s) \pm \sigma^2(t)/2}{\sigma(t)},$$

and

$$\sigma^2(t) = \ln\left(\frac{\sum_{i,j=k}^{N-1} c_i c_j H_i(t) H_j(t) \exp\left(\left(\boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{i}}\right)' \boldsymbol{\sigma}^{\mathbf{H}}_{\mathbf{j}}(T_k - t)\right)}{\sum_{i,j=k}^{N-1} c_i c_j H_i(t) H_j(t)}\right), \tag{3.60}$$

$$c_j = (1-R)\overline{w}_i(0)\Delta_i. \tag{3.61}$$

Proof. Without loss of generality, we assume that $t = 0$. The price of a call CDSwaption at time 0 is given by the following formula,

$$V^{call}_{CDSwaption}(0) = \sum_{i=k}^{N-1} \Delta_i \overline{B}_{i+1}(0) \mathbb{E}_{\overline{\mathbf{P}}_{i+1}}\left((\overline{s}_{k,N}(T_k) - s)^+\right). \tag{3.62}$$

From assumption $i)$, the (k, N)-forward spread is given as follows,

$$\begin{aligned}
\overline{s}_{k,N}(T_k) &= (1-R)\sum_{i=k}^{N-1} \overline{w}_i(T_k)\Delta_i \mathbb{E}_{\overline{\mathbf{P}}_{i+1}}\left(H_i(T_i)|\mathcal{F}_k\right) \\
&= (1-R)\sum_{i=k}^{N-1} \overline{w}_i(T_k)\Delta_i H_i(T_k).
\end{aligned}$$

Next, form assumption $ii)$ and by using the notation in equation (3.61), we get

$$\overline{s}_{k,N}(T_k) = \sum_{i=k}^{N-1} c_i H_i(T_k).$$

As a result of the assumption $ii)$ $\overline{s}_{k,N}$ is not log-normal any more under $\overline{\mathbf{P}}^s$, but is a sum of the log-normal intensities H_i, which are driftless under $\overline{\mathbf{P}}^s$. We now approximate $\overline{s}_{k,N}(T_k)$ by a random variable X with the same first and second moment, but whose logarithm is normally distributed with mean μ and variance σ^2

$$\ln(X) \sim N(\mu, \sigma^2).$$

This method is known as moment matching or Levy approximation; see [Lev92].

Since under $\overline{\mathbf{P}}^s$ the forward rate $\overline{s}_{k,N}$ is driftless, we obtain for the first moment of $\overline{s}_{k,N}(T_k)$ the value

$$\mathbb{E}_{\overline{\mathbf{P}}^s}\left(\overline{s}_{k,N}(T_k)\right) = \overline{s}_{k,N}(0).$$

For the second moment, let us first recall that form the independence of F_i and H_i, H_i is a $\mathbf{P}_{i+1}$-martingale, which implies it is also a $\mathbf{P}^s$-martingale (see equation (3.58) and assumption $ii)$),

$$\frac{dH_i}{H_i} = \left(\boldsymbol{\sigma}_{\mathbf{i}}^{\mathbf{H}}\right)' \mathbf{d}\overline{\mathbf{W}}^{\mathbf{s}}.$$

Now, from the above $\overline{\mathbf{P}}^s$-dynamic of H_i, one can easily get

$$\mathbb{E}_{\overline{\mathbf{P}}^s}\left(H_i(T_k)H_j(T_k)\right) = H_i(0)H_j(0)e^{\left(\boldsymbol{\sigma}_{\mathbf{i}}^{\mathbf{H}}\right)'\boldsymbol{\sigma}_{\mathbf{j}}^{\mathbf{H}}T_k},$$

which yields

$$\mathbb{E}_{\overline{\mathbf{P}}^s}\left(\overline{s}_{k,N}(T_k)^2\right) = \sum_{i,j=k}^{N-1} c_i c_j H_i(0)H_j(0)e^{\left(\boldsymbol{\sigma}_{\mathbf{i}}^{\mathbf{H}}\right)'\boldsymbol{\sigma}_{\mathbf{j}}^{\mathbf{H}}T_k}.$$

On the other hand, the first two moments of X are

$$\begin{aligned}
\mathbb{E}_{\overline{\mathbf{P}}^s}(X) &= \exp(\mu + \sigma^2/2),\\
\mathbb{E}_{\overline{\mathbf{P}}^s}\left(X^2\right) &= \exp(2\mu + 2\sigma^2).
\end{aligned}$$

Matching the moments yields

$$\begin{aligned}
\mu &= \ln(\overline{s}_{k,N}(0)) - \frac{1}{2}\sigma^2,\\
\sigma^2 &= \ln\left(\frac{\sum_{i,j=k}^{N-1} c_i c_j H_i(0)H_j(0)\exp\left(\left(\boldsymbol{\sigma}_{\mathbf{i}}^{\mathbf{H}}\right)'\boldsymbol{\sigma}_{\mathbf{j}}^{\mathbf{H}}T_k\right)}{\overline{s}_{k,N}(0)^2}\right)\\
&= \ln\left(\frac{\sum_{i,j=k}^{N-1} c_i c_j H_i(0)H_j(0)\exp\left(\left(\boldsymbol{\sigma}_{\mathbf{i}}^{\mathbf{H}}\right)'\boldsymbol{\sigma}_{\mathbf{j}}^{\mathbf{H}}T_k\right)}{\sum_{i,j=k}^{N-1} c_i c_j H_i(0)H_j(0)}\right).
\end{aligned}$$

Now, replacing $\overline{s}_{k,N}(T_k)$ by X in (3.62) yields

$$V^{call}_{CDSwaption} = \sum_{i=k}^{N-1} \Delta_i \overline{B}_{i+1}(0) \mathbb{E}_{\overline{\mathbf{P}}^s}\left((X-s)^+\right),$$

which easily evaluates to the desired formula using

$$\ln(X) \sim N(\mu, \sigma^2)$$

and the expressions above for μ and σ^2. □

Remark 3.7.2.

1. *Note that under assumptions of the two factor setup and homogeneous H_k- volatilities (i.e., $\sigma_i^H = \sigma^H$), we obtain*

$$\sigma^2(t) = (\sigma^H)^2(T_k - t)$$

and Schönbucher's formula (3.55) *follows.*

2. *Also, using the relation between correlation and volatility vectors* (3.26)*, we can write the value of σ^2 as*

$$\sigma^2(t) = \ln\left(\frac{\sum_{i,j=k}^{N-1} c_i c_j H_i(t) H_j(t) \exp\left(\rho_{ij}|\boldsymbol{\sigma}_\mathbf{i}^\mathbf{H}||\boldsymbol{\sigma}_\mathbf{j}^\mathbf{H}|(T_k - t)\right)}{\sum_{i,j=k}^{N-1} c_i c_j H_i(t) H_j(t)}\right).$$

And if all the intensities are perfectly correlated, i.e., $\rho_{ij} = 1,$, for $i, j = 1, \ldots, N$ we get

$$\sigma^2(t) = \ln\left(\frac{\sum_{i,j=k}^{N-1} c_i c_j H_i(t) H_j(t) \exp\left(|\boldsymbol{\sigma}_\mathbf{i}^\mathbf{H}||\boldsymbol{\sigma}_\mathbf{j}^\mathbf{H}|(T_k - t)\right)}{\sum_{i,j=k}^{N-1} c_i c_j H_i(t) H_j(t)}\right),$$

where the exponential term is nothing but $\exp\left(\sigma_i^H \sigma_j^H (T_k - t)\right)$.

3. *Weakening assumption of constant volatilities, the case of time dependent volatilities and correlations can be treated by using*

$$\left(\boldsymbol{\sigma}_\mathbf{i}^\mathbf{H}\right)' \boldsymbol{\sigma}_\mathbf{j}^\mathbf{H}(T_k - t) = \int_t^{T_k} \rho_{ij}(t) |\boldsymbol{\sigma}_\mathbf{i}^\mathbf{H}(\mathbf{t})||\boldsymbol{\sigma}_\mathbf{j}^\mathbf{H}(\mathbf{t})|\, dt.$$

4. *Price of a credit default swap put option can be easily derived as in Theorem 3.7.5.*

One can in fact derive a similar formula using the survival measures $\overline{\mathbf{P}}_i$ instead of the CDS-measure $\overline{\mathbf{P}}^s$. Then the resulting formula involves the drifts $\mu^H_{k,i}$ at time 0. The following Corollary states this fact.

Corollary 3.7.6 (Levy with drifts). *Under the assumptions in Theorem 3.7.5. The time t price of a CDSwaption with maturity T_k and strike s can be approximated as follows*

$$V^{call}_{CDSwaption}(t) \approx \sum_{i=k}^{N-1} \Delta_i \overline{B}_{i+1}(t) \left(M_i(t)\Phi(d^1_i(t)) - s\Phi(d^2_i(t)) \right), \tag{3.63}$$

where

$$\begin{aligned}
d^{1,2}_i(t) &= \frac{\ln(M_i(t)/s) \pm \sigma^2_i(t)/2}{\sigma_i(t)}, \\
M_i(t) &= \sum_{j=k}^{N-1} c_j H_j(t) \exp\left(\mu^H_{j,i+1}(T_k - t)\right), \\
\sigma^2_i(t) &= \ln\left(\frac{\sum_{l,j=k}^{N-1} c_l c_j H_l(t) H_j(t) \exp\left((\mu^H_{l,i+1} + \mu^H_{j,i+1})(T_k - t) + (\boldsymbol{\sigma}_\mathbf{l})' \boldsymbol{\sigma}_\mathbf{j}(T_k - t)\right)}{\sum_{l,j=k}^{N-1} c_l c_j H_l(t) H_j(t) \exp\left(\left(\mu^H_{l,i+1} + \mu^H_{j,i+1}\right)(T_k - t)\right)} \right).
\end{aligned}$$

Proof. The proof is very similar to the one in Theorem 3.7.5. In order to calculate the option price at time 0, we have the following pricing formula

$$V^{call}_{CDSwaption}(0) = \sum_{i=k}^{N-1} \Delta_i \overline{B}_{i+1}(0) \mathbb{E}_{\overline{\mathbf{P}}_{i+1}} \left((\overline{s}_{k,n}(T_k) - s)^+ \right).$$

We again use Levy's approximation for the distribution of $\overline{s}_{k,N}$ by defining a random variable e^X where X is normally distributed with mean μ and variance σ^2 and matching the first two moments of e^X and $\overline{s}_{k,N}$. But this time we consider the moments of the basket value under the measure $\overline{\mathbf{P}}_{i+1}$. Hence, the first two moment of the basket value $\overline{s}_{k,N}$ are

$$\begin{aligned}
M &:= E^{\overline{\mathbf{P}}_{i+1}} \left(\sum_{j=k}^{N-1} c_j H_j(T_k) \right) = \sum_{j=k}^{N-1} c_j H_j(0) e^{\mu^H_{j,i+1} T_k}, \\
V^2 &:= E^{\overline{\mathbf{P}}_{i+1}} \left(\sum_{j=k}^{N-1} c_j H_j(T_k) \right)^2 = \sum_{l,j=k}^{N-1} c_l c_j H_l(0) H_j(0) e^{(\mu^H_{l,i+1} + \mu^H_{j,i+1}) T_k + (\boldsymbol{\sigma}_\mathbf{l})' \boldsymbol{\sigma}_\mathbf{j} T_k}.
\end{aligned}$$

Then, the assertion follows similarly as in the proof of the Theorem 3.7.5. □

3.7.5 Numerical Results

In this subsection, we compare the CDSwaption prices obtained from three formulas; Schönbucher (3.55), Levy first extension (3.59), Levy second extension with drifts (3.63) with the CDSwaption price, approximated by the Monte Carlo simulation.

Next, for our simulation purposes, we some specifications of the volatility of forward default intensities.

Volatility Structure of Forward Default Intensity Rates

We assume that the forward default intensities, $H_k(t)$ have piecewise-constant instantaneous volatilities. In particular, the instantaneous volatility of $H_k(t)$ is constant in each time interval $T_i \leq t \leq T_{i+1}$, with $i = 0, \ldots, k$, i.e., where the rate is alive. On the other hand, the instantaneous volatility of $H_k(t)$ is zero in each time interval $T_i \leq t \leq T_{i+1}$, with $i = k+1, \ldots, N-1$, i.e., where the rate does not exist. This assumption corresponds to the second part of the Remark 3.4.2. Table 3.3 states this volatility structure.

Instantaneous Volatilities	$t \in (T_0, T_1]$	$(T_1, T_2]$	$(T_2, T_3]$	$\ldots$	$(T_{N-2}, T_{N-1}]$
$H_1(t)$	$\sigma^H_{1,0}$	Dead	Dead	$\ldots$	Dead
$H_2(t)$	$\sigma^H_{2,0}$	$\sigma^H_{2,1}$	Dead	$\ldots$	Dead
$\vdots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\vdots$
$H_N(t)$	$\sigma^H_{N,0}$	$\sigma^H_{N,1}$	$\sigma^H_{N,2}$	$\ldots$	$\sigma^H_{N,N-1}$

Table 3.3: Volatility matrix, when the volatility depends on the time and the maturity.

Moreover, we assume that the volatilities depend only on the time to maturity of a forward intensity rate rather than on time and maturity T_k separately. In that case our volatility structure is given in Table 3.4.

Next, we give a parametric analogue of the piecewise-constant volatility structure depending only on the time to maturity and observe different volatility shapes, which we shall also use in our simulations.

$$V(T-t, \gamma_1, \gamma_2, \gamma_3) = \gamma_1 \exp\big(-\gamma_2(T-t)\big)\big(1+\gamma_3(T-t)\big).$$

Instantaneous Volatilities	$t \in (T_0, T_1]$	$(T_1, T_2]$	$(T_2, T_3]$	...	$(T_{N-2}, T_{N-1}]$
$H_1(t)$	V_1	Dead	Dead	...	Dead
$H_2(t)$	V_2	V_1	Dead	...	Dead
$\vdots$	...	...	...	...	$\vdots$
$H_N(t)$	V_N	V_{N-1}	V_{N-2}	...	V_1

Table 3.4: Volatility matrix, when the volatility depends on the time to maturity.

Figure 3.1 plots $V(T-t, \gamma_1, \gamma_2, \gamma_3)$ for

$$\gamma_1 = 0.015, \;\; \gamma_2 = 0.4, \;\; \gamma_3 = 50 \tag{3.64}$$

$$\gamma_1 = 0.015, \;\; \gamma_2 = 0.3, \;\; \gamma_3 = 25 \tag{3.65}$$

$$\gamma_1 = 0.03, \;\; \gamma_2 = 0.6, \;\; \gamma_3 = 25 \tag{3.66}$$

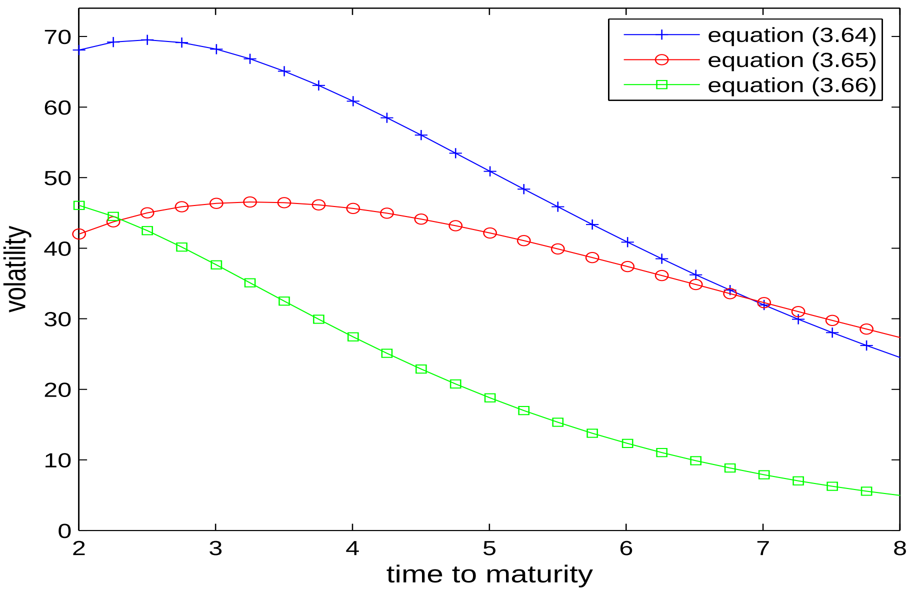


Figure 3.1: Parametric representation of Table 3.4.

In the following three figures, we plot the difference between the Monte Carlo simulation and three formulas with respect to the strike, for the three volatility term structures plotted in Figure 3.1. We plot also the standard deviation of the Monte Carlo simulation. Note that in the implementation of the formulas, the volatilities are assumed to be constant between the tenor dates during the life time of the rates and they depend only on the time to maturity. In what follows, the volatility matrix given in Table 3.4 is fulfilled. Note further that in the implementation of Schönbucher's formula the homogenous volatility is assumed to be the average of the volatilities, represented in Table 3.4.

As we observe in the following three figures, the Monte Carlo price can be taken as the real price, since the standard deviation of the Monte Carlo simulation is very small. The difference between the real price and the prices, generated from our two proposals are smaller than Schönbucher's one. When strike is small (i.e. option is in the money) or strike is big (i.e. option is out of the money), the differences are almost zero. But even when the option is at the money, the performance of our formulas is still better than Schönbucher's formula.

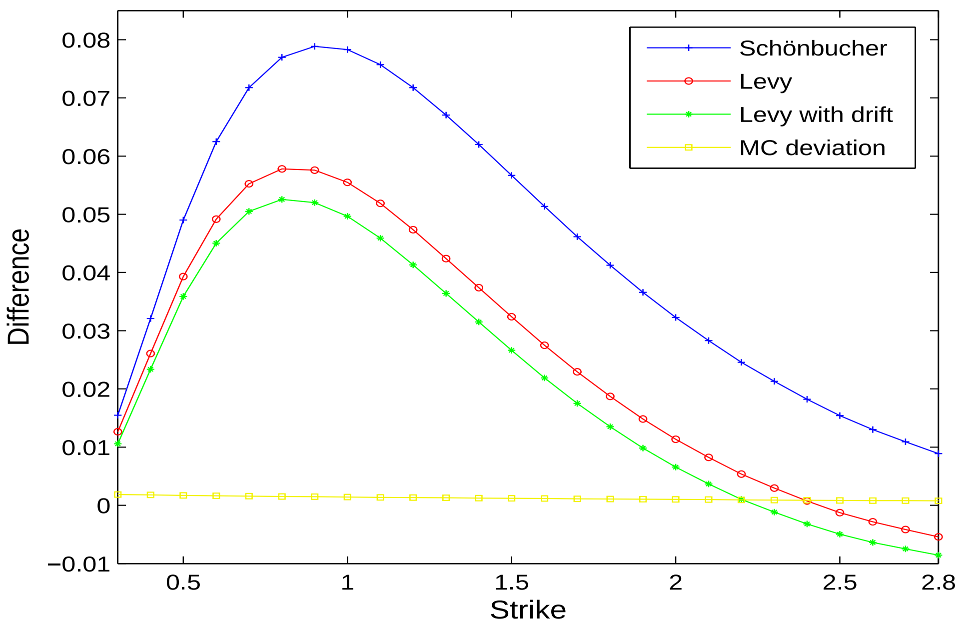


Figure 3.2: Difference between the closed form solution and Monte Carlo results with the first volatility term structure, i.e., $\gamma_1 = 0.015$, $\gamma_2 = 0.4$, $\gamma_3 = 50$.

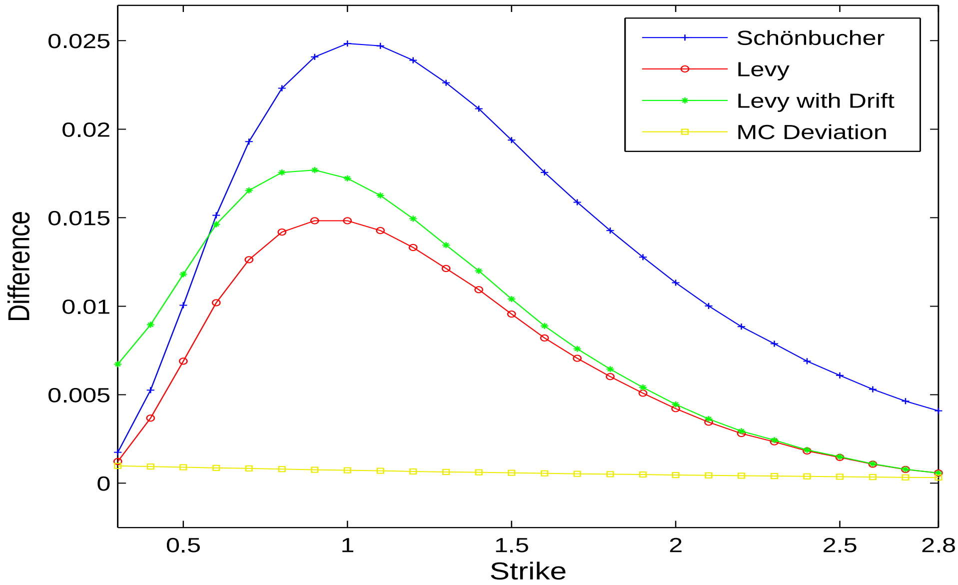


Figure 3.3: Difference between the closed form solutions and Monte Carlo results with the second volatility term structure, i.e., $\gamma_1 = 0.015$, $\gamma_2 = 0.3$, $\gamma_3 = 25$.

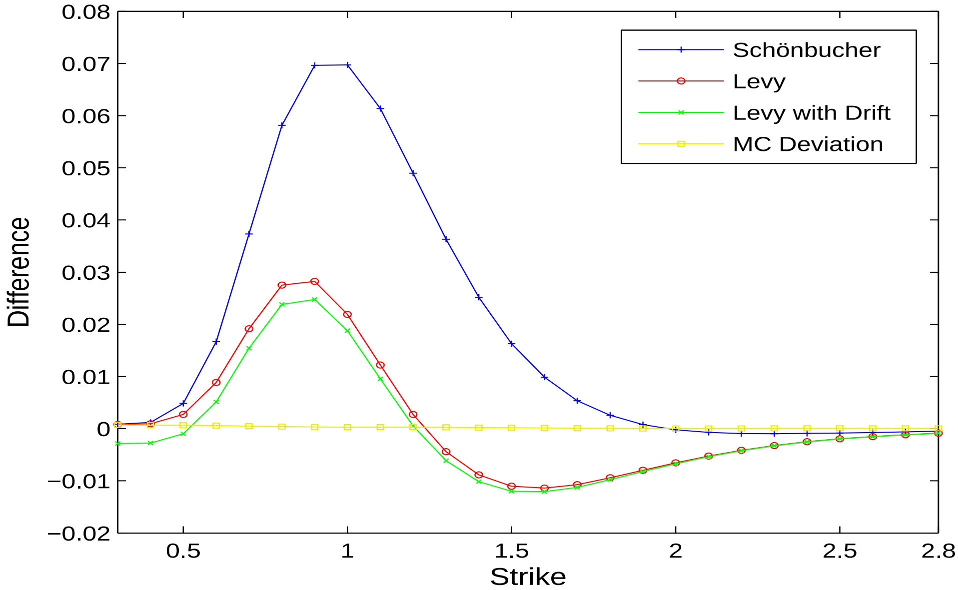


Figure 3.4: Difference between the closed form solutions and Monte Carlo results with the third volatility term structure, i.e., $\gamma_1 = 0.03$, $\gamma_2 = 0.6$, $\gamma_3 = 25$.

3.8 Conclusion

As a conclusion, we observe that Schönbucher's CDSwaption formula suffers under the restrictive assumptions such as; perfectly correlated default intensities and a homogenous default intensity volatility structure. By relaxing these assumptions, we obtain two approximative CDSwaption formulas. Comparing the deviation of our results and the deviation of Schönbucher's results from the Monte Carlo ones, we observe that our formulas are better performing than the Schönbucher's, since they are derived under the same assumptions of the Monte Carlo simulation. The proposed closed formulas can be used as control variables to reduce the variance of the Monte Carlo estimator. An obvious extension of our formula can also consider the multi-factor setup for default-free forward rates. Under the same assumptions as in Theorem 3.7.5, one can easily obtain an approximative pricing formula of the CDSwaption. A less straight forward extension would be to relax the independence assumption of F_k and H_k and consider a general setup, where F_k and H_k are modeled under the multi-factor setup. Although this extension is hard to incorporate in the pricing formula, it is an important further step of the pricing of CDSwaptions. Since the Monte Carlo simulation is extremely time consuming in a general multi-factor setup[9], such an approximated formula could be used to reduce its variance and improve its convergence.

[9] In the general multi-factor setup F_k and H_k are assumed to be dependent.

Appendix A

Modigliani-Miller Results

A.1 Introduction

"How do firms choose their capital structure?" is one of the most important issues in corporate finance. The capital structure is the ratio between the debt (money borrowed by a firm at a fixed interest rate), and the equity (money invested in the firm by shareholders that own the firm, have full possession of its assets and profits).

Modigliani and Miller [MM58] wrote in their seminal paper on the issue of the optimal capital structure. The results, often called the MM-Propositions, have been the subject of controversy for many years. The importance of their paper have been recognized by awarding both authors the Nobel Prize in economics (Modigliani in 1984 and Miller in 1990).

MM-Proposition 1 (see Corollary A.2.1) has become the first step in the capital structure theory and is sometimes called the irrelevance theorem. It states that, as an implication of the no-arbitrage assumption in perfect capital markets, the value of a firm is independent of its capital structure (that is, its debt/equity ratio). In proving this proposition they used an arbitrage argument. The second step was also made by Modigliani and Miller [MM58], but corrected in Modigliani and Miller [MM63], by introducing MM-Proposition 2 (see Corollary A.2.2). It postulates that, if corporate taxes are introduced in the model and it is assumed that the interest payments are tax deductible, then 100% debt financing is optimal. The intuition behind it is as follows. The more debt a firm has, the less taxes it pays, and therefore the more firm value is left for the equity holders and debt holders. This result was extremely puzzling, since in real world one never observes firms with a 100% debt financing. The third step in capital structure theory was first suggested by Baxter (1976), by introducing bankruptcy costs (See Corollary A.2.3). These costs consist of payments that must be made to third parties other than bond or equity holders when the firm goes bankrupt, such as trustee fees, legal fees, costs of reorganisation, etc. These losses, associated with bankruptcy, cause the value of the firm to be less than it would have been otherwise, namely the value based on the expected cash flows from operations. And since the probability of going bankrupt is higher when a firm is financed with more debt, there are costs involved with debt financing. The tradeoff between the tax advantage of debt and bankruptcy costs associated with debt results in an optimal capital structure, the so called balancing theorem.

It is important to note that the MM-Proposition 1 only holds in an ideal, perfect world, which has become known as the MM world.

Assumption A.1.1. *Modigliani and Miller [MM58] assumed that:*

1. *Capital markets are arbitrage-free, perfect and frictionless.*
2. *Firms can lend at the risk-free rate (riskless debt).*
3. *Individuals can also borrow and lend at the risk-free rate.*
4. *There are no costs to bankruptcy.*
5. *Firms only issue two types of claims; risk-free debt and (risky) equity.*
6. *All firms are assumed to be in the same risk class.*
7. *There are no taxes.*
8. *All cash flow streams are perpetuities.*
9. *Corporate insiders and outsiders have the same information.*

10. *Managers always maximize equity holders value and do not expropriate in any way other stake holders of the company (i.e, no agency costs).*

The other papers about the optimal capital structure framework relaxed one or more of these assumptions in order to study the imperfection on the MM results. The driving force behind this theory development is the gap between the theory and the practice. With respect to MM-Proposition 1 and MM-Proposition 2, the gap was immense, since all real world debt-equity ratios vary within a certain range from 60% to 20%.

To summarize, in a perfect world - without taxes and bankruptcy costs - the debt-equity ratio is irrelevant for the value of the firm (MM-Proposition 1). When the imperfection of corporate taxes is introduced, 100% debt financing is optimal, i.e. maximizes the value of the firm (MM-Proposition 2). Finally, when also bankruptcy costs are taken into consideration, there is a cost for debt financing and an interior solution for the optimal capital structure emerges; a debt/equity ratio somewhere between 0% and 100% maximizes the value of the firm (MM-Proposition 3). Theoretically, it would also be possible to consider a world with only the imperfection of the bankruptcy costs (and no corporate taxes), in this case 100% equity financing would be optimal. In Figure A.1, all three propositions are illustrated.

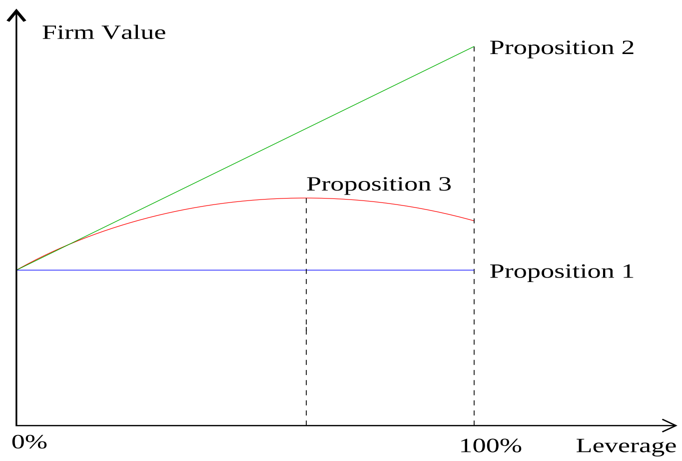


Figure A.1: The optimal leverage ratios of a firm, in the view of MM-Propositions.

A.2 Propositions

In this section, we shall state the MM-Propositions and give the proofs by following Modigliani and Miller [MM58], [MM63].

Consider any firm j and let X_j stand for the Earnings Eefore Interest (EBI) value, generated by the currently owned assets of a given firm in some stated risk class, i.e., its expected profit before deduction of the interest payments. If firm j is financed only by the equity, the market capitalizes the expected returns of the unlevered firm j by ρ_k, k representing a certain risk class

$$V_j^U = \frac{X_j}{\rho_k}.$$

Let D_j denote the debt value of firm j and r be the rate at which the market capitalises the sure streams generated by the debt. Therefore, debt holders continuously receives, $r\,D_j$ amount of the interest payments[1]. EQ_j denotes the equity value of firm j, hence the value of the levered firm, denoted by V_j^L, is equal to $V_j^L = EQ_j + D_j$. Next, we state well-known MM-Proposition 1 and prove it by following the argumentation in Modigliani and Miller [MM58].

Proposition A.2.1 (MM-Propostion 1). *The value of any firm is independent of its capital structure. That is, the equality*

$$V_j^L = EQ_j + D_j = \frac{X_j}{\rho_k}$$

is satisfied for any firm j in risk class k.

Proof. Using the same notation as Modigliani and Miller [MM58], V_1 is the value of an unlevered firm (all-equity) and V_2 is the value of a firm that has some debt in its capital structure (levered firm), but identical in every respect to firm 1. Let us denote the EBI of both firms by X, since they are identical.

$V_1^U = EQ_1$ and the return available for the equity holders of firm 1 equals X.

$V_2^L = EQ_2 + D_2$ and the return available for the equity holders of firm 2 equals $X - rD_2$

[1]Note that all cash flow streams are perpetuities. Therefore, the riskless debt value is equal to the capitalised value of the future cash inflows, i.e., $D_j = \int_0^\infty rD_j\, e^{-rt}dt$.

Now, we consider an investor, holding α proportion of the equities of firm 2. Hence, this investment gives the investor a return of

$$Y_2 = \alpha(X - rD_2). \tag{A.1}$$

Now, we suppose that the investor sells his αEQ_2 worth of firm 2 equities and buys an amount $\alpha(EQ_2 + D_2)$ of the equities of firm 1. He can do so by utilizing the amount αEQ_2, realized from the sale of his initial holding, and borrowing an additional amount, αD_2, on his own account. This gives him a fraction $\alpha(EQ_2+D_2)/EQ_1$ of the equities, and therefore earnings, of firm 1. Taking into account the interest payments on the personal debt, $\alpha r D_2$, the return Y_1 to the investor, will in this case be given by

$$Y_1 = \left(\frac{\alpha(EQ_2 + D_2)}{EQ_1}\right) X - \alpha r D_2 = \alpha \left(\frac{V_2^L}{V_1^U}\right) X - \alpha r D2. \tag{A.2}$$

Since in both cases (($A.1$) and ($A.2$)) the same amount of money has been invested, in equilibrium both investments should give the same return, i.e., $Y_1 = Y_2$. (If not, investors would prefer one of the firm's shares to another, and enjoys the arbitrage opportunity. Comparing now ($A.1$) and ($A.2$), we see that as long as $V_2^L > V_1^U$, we must have $Y_1 > Y_2$. Therefore, equity holders of firm 2 sell their holdings to acquire equities of firm 1, which lowers EQ_2 and hence V_2^L, and thereby raising EQ_1 and thus V_1^U. When $V_2^L < V_1^U$ the same argument works the other way around. Therefore, Modigliani and Miller [MM58] conclude that levered companies cannot command a premium over unlevered companies with identical annual return X, because investors have the opportunity of putting the equivalent leverage into their portfolio directly by borrowing on personal account. The possibility to borrow on personal account is a crucial element in the proof of the theorem and has become known as homemade leverage. □

In Modigliani and Miller [MM58], one of the assumptions in the MM world was already relaxed, by introducing corporate taxes. (A technical correction is made in Modigliani and Miller [MM63]). The important thing about corporate taxes is that interest payments are tax deductible.

Let X be the Earnings Before Interest and Taxes (EBIT), generated by currently owned assets of a given firm in some stated risk class. The corporate tax rate is denoted by τ_c.

Hence, $\tau_c X$ is the after-tax return. Let ρ_k be the rate at which the market capitalizes the expected net returns after taxes of an unlevered firm in a certain risk class k, then we have

$$V^U = \frac{(1-\tau_c)X}{\rho_k}.$$

Let D_j denote the debt value of firm j and r be the rate at which the market capitalizes the sure streams generated by the debt. Therefore, debt holders continuously receives, $r\,D_j$ amount of interest payments, which we denote by G. Then, after tax total return of the levered firm

$$(1-\tau_c)(X-G) + G = (1-\tau_c)X + \tau_c\,G,$$

consists of two components; an uncertain stream $(1-\tau_c)X$, which comes from the equity part, and a sure stream $\tau_c\,G$, which is the interest payment. The market value of the combined stream can be found by capitalising each component separately. Then, we obtain the value of the levered firm with a permanent debt level of D as follows

$$V^L = \frac{(1-\tau_c)X}{\rho_k} + \frac{\tau_c G}{r} = V^U + \tau_c D. \tag{A.3}$$

Modigliani and Miller [MM63] show that if $(A.3)$ does not hold, arbitrage opportunities exists in the market.

Proposition A.2.2 (MM-Propostion 2**).** *When corporate taxes are included, the value of the levered firm is equal to the value of an unlevered firm plus the present value of the tax shields associated by debt*

$$V^L = V^U + \tau_c D,$$

where τ_c is the corporate tax rate. In this way the capital structure that maximizes the value of a firm consists of 100% debt.

Proof. Suppose first that unlevered firms are overvalued, i.e. $V^L - \tau_c\,D < V^U$. Then, an investor holding m dollars of stock in the unlevered firm has the right to get a fraction m/V^U of the return, i.e.

$$Y^U = (1-\tau_c)X\left(\frac{m}{V^U}\right).$$

Consider now an alternative portfolio obtained by investing m dollars as follows. A portion of

$$m\left(\frac{EQ^L}{EQ^L+(1-\tau_c)D}\right)$$

is invested in the stock of the levered firm EQ^L and the remaining portion

$$m\left(\frac{(1-\tau_c)D}{EQ^L+(1-\tau_c)D}\right)$$

is invested in its bonds (= debt). The stock component entitles the investor to a fraction $m\,EQ^L/(EQ^L+(1-\tau_c)D)$ of the net profits of the levered firm, which is equal to

$$(1-\tau_c)(X-R)\left(\frac{m}{EQ^L+(1-\tau_c)D}\right).$$

The holding of bonds yields

$$(1-\tau_c)R\left(\frac{m}{EQ^L+(1-\tau_c)D}\right).$$

Hence, the total return from the alternative portfolio is

$$Y^L=(1-\tau_c)X\left(\frac{m}{EQ^L+(1-\tau_c)D}\right)$$

and this will dominate the uncertain income Y^U, if and only if,

$$EQ^L+(1-\tau_c)D\equiv EQ^L+D-\tau_cD\equiv V^L-\tau_cD<V^U.$$

If V^U exceeds $V^L-\tau_cD$, arbitrage opportunities will appear in the market. Hence, investors would have an incentive to sell shares in the unlevered company and purchase the shares (and bonds) of the levered company. A similar line of reasoning is followed for the other possibility, namely that the market value of the levered firm, $V^L-\tau_cD$, is less than the value of the unlevered firm V^U. Therefore, we obtain

$$V^L=V^U+\tau_cD.$$

$\square$

MM-Proposition 2 is as irrealistic as MM-Proposition 1. There exists not a single firm, which is voluntarily financed with 100% debt. Therefore, the disadvantages of debt are considered, in order to come up with a realistic optimal capital structure. The first idea was the relaxation of the risk free debt assumption. In real life, when leverage increases debt becomes more risky, since the probability that the firm goes bankrupt increases and the debt holders demand a higher yield to bear the high default risk. As a consequence, the value of the firm declines. However, the introduction of the risky debt does not change the MM-Propositions, since it has no impact on the value of the firm. Stiglitz [Sti69] first proved this result. Therefore, the introduction of the risky debt cannot, by itself, be used to explain the existence of an optimal capital structure with a debt-equity ratio between 0% and 100%. However, when bankruptcy costs are taken into account, the value of the firm in bankruptcy is reduced by the fact that payments must be made to third parties other than bond and shareholders. The costs of the bankruptcy are deducted from the net asset value of the bankrupt firm and from the proceeds that should go to bondholders. Consequently, these losses associated with bankruptcy may cause the value of the firm in bankruptcy to be less than the discounted value of the expected cash flows from operations. This fact can be used to explain the existence of an interior optimal capital structure and in this thesis it is called as the third Mogidliani and Miller Proposition. Baxter and Nevins [BN67] suggested for the first time the existence of an internal optimal capital structure, based on bankruptcy costs.

Proposition A.2.3 (MM-Propostion 3)**.** *When corporate taxes and bankruptcy costs are included, the total firm value is the asset value (the unlevered firm value) plus the net effect of debt issuance namely the difference between the tax advantages and the bankruptcy costs.*

Bibliography

[AW92] J. Abate and W. Whitt. The fourier-series method for inverting transforms of probability distributions. *Queueing System*, 10:5–88, 1992.

[BC76] F. Black and J. C. Cox. Valuing corporate securities: Some effects of bond indenture provisions. *Journal of Finance*, 31, 1976.

[BE95] P. Borwein and T. Erdèlyi. *Polynomials and Polynomial Inequalities*. Springer, 1995.

[BGM97] G. Brace, D. Gatarek, and M. and Musiela. The market model of interest rate dynamics. *Mathematical Finace*, 7(2):127–155, 1997.

[Bjö98] T. Björk. *Arbitrage Theory in Continuous Time*. Oxford, 1 edition, 1998.

[Bla76] F. Black. The pricing of commodity products. *Journal of Financial Economics*, 3:167–179, 1976.

[BM01] D. Brigo and F. Mercurio. *Interest Rate Models Theory and Practice*. Springer, 2001.

[BN67] N. Baxter and D. Nevins. Leverage, risk of ruin and the cost of capital. *Journal of Finance*, 22:395–403, September 1967.

[Bre81] P. Bremaud. *Point Processes and Queues: Martingale Dynamics*. Springer, 1981.

[BS73] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81, 1973.

[BS78] M. Brennan and S. Schwartz. Corporate income taxes, valuation and problem of optimal capital structure. *Journal of Bussiness*, 51:103–114, 1978.

[CDG01] P. Collin-Duffrense and R. S. Goldstein. Do credit spreads reflect stationary leverage ratios? *Journal of Finance*, 56:1929, 2001.

[CFLM01] P. O. Christensen, R. C. Flor, D. Lando, and R. Miltersen. Dynamic capital structure with callable debt and debt renegotiations. March 2001.

[CK05] N. Chen and S. Kou. Credit spreads, optimal capital structure, and implied volatility with endogenous default and jump risk. July 2005.

[Cra87] J. Crank. *Free and moving boundary problems.* Oxford, 1987.

[CRR79] J. Cox, S. Ross, and M. Rubinstein. *Option Pricing: A Simplified Approach.* Journal of Financial Economics, September 1979.

[CT04] R. Cont and P. Tankov. *Financial Modelling with Jump Processes.* Chapman & Hall/CRC, 2004.

[DL01] D. Duffie and D. Lando. The term structure of credit spreads with incomplete accounting information. *Econometrica*, 69:633–664, 2001.

[DP93] A. Dixit and R. Pindyck. *Investment under Uncertainty.* Princeton University Press, 1993.

[DS99] D. Duffie and K. J. Singleton. Modeling term structures of defaultable bonds. *Review of Financial Studies*, 12:687–720, 1999.

[ER02] J. Ericsson and J. Reneby. A note on contingent claim pricing with non-traded assets. June 2002.

[ER06] J. Ericsson and J. Reneby. Can structural models price default risk? evidence from bond and credit markets. April 2006.

[FHZ89] E. Fischer, R. Heinkel, and J. Zechner. Dynamic capital structure choice. *Journal of Finance*, 44:19–40, 1989.

[Gen05] M. Genser. *A Structural Framework for the Pricing of Corporate Securities: Economis and Emprical Issues.* PhD thesis, Universtiy of St. Gallen, 2005.

[GNL98] R. Goldstein, J. Nengjiu, and H. E. Leland. An ebit based model of dynamic capital structure. *Working Paper*, 1998.

[HH03] J. Huang and M. Huang. How much of the corporate-treasury yield is due to credit risk? May 2003.

[HK79] J Harrison and D. Kreps. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory*, 20(3):381–403, 1979.

[HP] J Harrison and S. Pilska. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications*, 11:215–260.

[HR02] B. Hilberink and L.C.G. Rogers. Optimal capital structure and endogenous default. *Finance and Stochastic*, 6:237–263, 2002.

[Jam87] F. Jamshidian. Pricing of contingent claims in the one factor term structure model. 1987.

[Jam97] F. Jamshidian. Libor and swap market models and measures. *Finance and Stochastic*, 1:293–330, 1997.

[JLT97] R. Jarrow, D. Lando, and S. M. Turnbull. A markov model for the term structure of credit risk spreads. *Review of Financial Studies*, 10(2):481–523, 1997.

[JP04] R. Jarrow and P. Protter. Structural versus reduced form models: A new information based perspective. *Journal of Investment Management*, 2:34–43, 2004.

[JS00] J. Jacod and A. N. Shiryaev. *Limit Theorems for Stochastic Processes.* Springer, 2 edition, 2000.

[JT95] R. Jarrow and S. M. Turnbull. Pricing derivatives on financial securities subject to credit risk. *Journal of Finance*, 50:53–85, 1995.

[KK01] E. Korn and R. Korn. *Option Pricing and Portfolio Optimisation-Modern methods of financial mathematics*. AMS, 2001.

[KMM84] A. Kane, A. Marcus, and R. McDonald. How big is the tax advantege to debt? *Journal of Finance*, 39:841–852, 1984.

[KPW05] S. G. Kou, G. Petrella, and H. Wang. Pricing path-dependent options with jump risk via laplace transforms. 2005.

[KS00] I. Karatsaz and S. Shreve. *Brownian Motion and Stochastic Calculus*. Springer, 2000.

[KW03] S. G. Kou and H. Wang. First passage times of a jump diffusion process. *Advanced Applied Probabilty*, 35:504, 2003.

[KW06] M. Krekel and J. Wenzel. A unified approach to cdswaption and cmcds valuation. Technical report, Fraunhofer ITWM, Kaiserslautern, 2006.

[Lan94] D. Lando. On cox processes and credit risky bonds. 1994.

[Lel94] H. E. Leland. Corporate debt value, bond covenants, and optimal capital structure. *Journal of Finance*, 49:1213, 1994.

[Lel95] H. E. Leland. Bond prices, yield spreads, and the optimal capital structure with default risk. 1995.

[Lel98] H. E. Leland. Agency costs, risk management, and capital structure. *Journal of Finance*, 53:1213, 1998.

[Lev92] E. Levy. Pricing european average rate currency options. *Journal of International Money and Finance*, 11:474, 1992.

[LS95] F. Longstaff and E. Schwartz. Valuing risky debt: A new approach. *Journal of Finance*, 50:789, 1995.

[LT96] H. Leland and K. B. Toft. Optimal capital structure, endegenous bankruptcy, and the term structure of credit spreads. *Journal of Finance*, 51:987, 1996.

[MA02] S. Metwally and A. Atiya. Using brownian bridge for fast simulation of jump-diffusion process and barrier options. *The journal of derivatives*, 10(4354), 2002.

[MBP97] P. Mella-Barral and W. Perraudin. Strategic debt service. *Journal of Finance*, 52:531–556, 1997.

[Mer74] R. C. Merton. On the pricing of corporate debt. *Journal of Finance*, 29, 1974.

[MM58] F. Modigliani and M. H. Miller. The cost of capital, corporation finance, and the theorey of investment. *The American Economic Review*, 48(3):261–297, June 1958.

[MM63] F. Modigliani and M. H. Miller. Corporate income taxes and the cost of capital. *The American Economic Review*, 53(3):433–443, June 1963.

[MSS97] K. R. Miltersen, K. Sandmann, and D. Sondermann. Closed form solutions for term structure derivatives with log-normal interest rates. *Journal of Finance*, 52(1):409–430, 1997.

[MÜ98] D. B. Madan and H. Ünal. Pricing the risk of default. *Review of Derivatives Research*, 2:121–160, 1998.

[PTVF02] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery. *Numerical Recipes in C++*. Cambridge University Press, 2002.

[Sam60] P. A. Samuelson. Appendix: A free boundary problem for the heat equation arising from a problem of mathematical finance, 1960.

[Sch00] P. Schönbucher. *Credit Risk Modeling and Credit Derivatives.* PhD thesis, Rheinischen Friedrich Wilhems Universität Bonn, 2000.

[Sch05a] M. Scherer. A structural credit-risk model based on a jump diffusion. 2005.

[Sch05b] P. Schönbucher. A libor market model with default risk. *Bonn Econ Discussion Papers*, 2005.

[Sch05c] P. Schönbucher. Term structure modelling of defaultable bonds. *Bonn Econ Discussion Papers*, 2005.

[Ste69] H. Stehfest. Algorithm 368. numerical inversion of laplace transforms. *Comm. ACM*, page 479, July 1969.

[Sti69] J. Stiglitz. A re-examination of the modigliani-miller theorem. *The American Economic Review*, 59(5):784–793, 1969.

[SW79] C. Smith and J. Warner. On financial contracting: An analysis of bond covenants. *Journal of Financial Economics*, 7:117–161, 1979.

[SW89] O. Sarig and A. Warga. Some emprical estimates of the risk structure of interest rates. 44:1351–1360, 1989. Journal of Finance.

[Wil06] P. Wilmott. *Paul Wilmott on Quantitative Finance.* John Wiley and Sons, 2 edition, 2006.

[Zho01] C. Zhou. The term structure of credit spreads with jump risk. *Journal of Banking and Finance*, 25:2015–2040, 2001.

Scientific Career

1991 – 1996	Atatürk Anatolian High School, Ankara
1996 – 2000	Middle East Technical University, Ankara Faculty of Art and Science, Department of Mathematics, Graduation: Bachelor in Mathematics
2000 – 2002	Technical University of Kaiserslautern, Department of Mathematics, Graduation: Master of Science in Financial Mathematics
2002 – 2005	Research Assistant in Technical University of Kaiserslautern, Department of Mathematics
since October 2005	Scientific researcher at the Fraunhofer ITWM, Kaiserslautern, Department of Financial Mathematics