Error estimates for quasistatic global elastic correction and linear kinematic hardening material

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Abstract

We consider in this paper the quasistatic boundary problems of linear elasticity and nonlinear elastplasticity with linear kinematic hardening material. We derive expressions and estimates for the difference of the solutions (i.e. the stress, the strain and the displacement) of both models. Further, we study the error between the elastoplastic solution and the solution of a postprocessing method, that corrects the solution of the linear elastic problem in order to approximate the elastoplastic model.

Keywords. Elastic BVP, elastoplastic BVP, variational inequalities, rate-independency, hysteresis, linear kinematic hardening, stop- and play-operator.

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Contents

1 Introduction

In former works, an effective method has been developped that corrects the elastic PDE stress and strain tensors in order to approximate the elastoplastic PDE stress and strain tensors. This method was based on the introduction of certain 'elastic' parameters in order to enter the elastoplastic material constitutive law with the elastic stress (or strain) pointwise.
For details, the reader is referred to Lang, Dressler, Pinnau [14] for theory and [15] for numerical results at practical examples. Further references are [10, 11, 7]. In [14, 15, 7], Jiang's elastoplasticity model [8, 9] has been applied.

In this article, we catch again this correction method, apply it *globally* on the whole deformable body, but we just consider the simplest elastoplastic constitutive material, i.e. linear kinematic hardening, where we have the linear coupling

$$\alpha = B\varepsilon^{pl}$$

between the backstress α and the plastic strain ε^{pl}. We give as well difference estimates between the solutions of the pure elastic (without correction) and the elastoplastic boundary value problem.
The proof of the corresponding error estimates in this article are essentially based on Lipschitz results for the *stop*- and *play*-operators. They can be found in BROKATE, KREJCI [1, 2, 12, 13] and some generalised versions, which we need as well, in [16].

In this whole paper, we will frequently make use of the abbreviations 'e.' (= 'everywhere') and 'a.e.' (= 'almost everywhere').

2 The boundary value problems (E) and (EP)

In this section, we derive estimates for our correction scheme applied to the full quasi-static and rate-independent PDE problem. All the following relations (1), (2) and (3), ..., (6) are assumed to be valid for almost every $t \in [0, T]$, where $T > 0$. Ω is a bounded domain with sufficiently smooth boundary $\partial\Omega = \partial_1\Omega \,\dot\cup\, \partial_2\Omega$, e.g. of class C^1.

2.1 The elastic model (E)

All variabled corresponding to the elastic model (E) are denoted with a small ^{e}-superscript. The *balance equations* in space can be written in differential operator form

$$^e\varepsilon(t) = \mathcal{D}\,^e u(t), \qquad \mathcal{D}^\star\,^e\sigma(t) = F(t). \tag{1}$$

The constitutive material law is *linear Hooke's law*

$$^e\sigma(t) = \mathcal{C}\,^e\varepsilon(t). \tag{2}$$

For each t equations (1) and (2) build up a static elliptic problem, which is well known and understood.

2.2 The elastoplastic model (EP)

We consider the following elastoplastic boundary value problem (EP). The reader may find details in ZEIDLER [18], section 66, or HAN/REDDY [6]. In the latter reference, our formulation corresponds to the so called *dual variational problem*, see chapter 8 therein. Here as well, the equivalence to the *primal variational problem* is proved. The *balance equations*, written in differential operator notation, are

$$\varepsilon(t) = \mathcal{D}u(t), \qquad \mathcal{D}^\star\sigma(t) = F(t). \tag{3}$$

They are identical to (1). *Linear Hooke's law* and the *linear kinematic hardening law*

$$\sigma(t) = \mathcal{C}\varepsilon(t), \qquad \alpha(t) = \mathcal{B}\varepsilon^{pl}(t), \tag{4}$$

together with the additive decomposition of *strain* and *stress*

$$\varepsilon(t) = \varepsilon^{el}(t) + \varepsilon^{pl}(t), \qquad \sigma(t) = \alpha(t) + \beta(t) \tag{5}$$

and the *normality rule*

$$\dot{\varepsilon}^{pl}(t) \in \partial\chi_Z\big(\beta(t)\big) \tag{6}$$

(equivalent to the maximum dissipation principle, see HAN/REDDY, section 4.2) build up the constitutive material law.

2.1 Remark. (a) We apply the same given outer force F to both bodies. We compare in section 4 both the difference of (EP) and (E) and the difference of (EP) and our correction of (E).
(b) We have here the following simplifications in our elastoplastic model.

- **Quasi-staticity**: No dynamic effects are considered. Our body has got density $\varrho = 0$. The model is rate-indepentent, as a reparametrization $\tau = \psi(t)$ with a monotically increasing, absolutely continuous $\psi : [0,T] \to [0,T]$ shows.
- **Linearized geometry**: We neglect all nonlinear terms in the displacement gradients ∇u resp. $\nabla^e u$. The (space-) differential operator $\mathcal{D}$ (and its adjoint) $\mathcal{D}^\star$ are assumed to be linear. □

2.3 Common definitions for both (E) and (EP)

(i) Spaces. We shall make use of the Sobolev and Lebesgue spaces

$$U = W_0^{1,2}\big(\Omega, \partial_1\Omega, \mathbb{R}^3\big), \qquad \Sigma = L^2\big(\Omega, \mathbb{R}_s^{3\times 3}\big).$$

With the scalar products/norms

$$\begin{aligned}
\langle u, v\rangle_U &= \int_\Omega u(x)\cdot v(x) + \nabla u(x) : \nabla v(x)\,\mathrm{d}V(x), & \|u\|_U^2 &= \langle u,u\rangle_U,\\
\langle \varepsilon, \eta\rangle_\Sigma &= \int_\Omega \varepsilon(x) : \eta(x)\,\mathrm{d}V(x), & \|\varepsilon\|_\Sigma^2 &= \langle \varepsilon,\varepsilon\rangle_\Sigma,
\end{aligned}$$

U and Σ become separable Hilbert spaces, see ZEIDLER [18], section 55.3. Korn's inequality, proof in [4], writes

$$\|\mathcal{D}u\|_\Sigma \geq \kappa(\Omega)\|u\|_U, \qquad \kappa(\Omega) > 0. \tag{7}$$

We have

$$\begin{aligned}
U &= \text{space of displacements } {}^{(e)}u, & \Sigma &= \text{space of strains } {}^{(e)}\varepsilon, {}^{(e)}\varepsilon^{el}, {}^{(e)}\varepsilon^{pl},\\
U^\star &= \text{space of outer forces } F, & \Sigma^\star &= \text{space of stresses } {}^{(e)}\sigma, {}^{(e)}\alpha, {}^{(e)}\beta.
\end{aligned}$$

By the Riesz-Fréchet representation theorem, we identify

$$U \simeq U^\star, \qquad \Sigma \simeq \Sigma^\star$$

in the sense of

$$F(u) = \langle F, u\rangle, \qquad \sigma(\varepsilon) = \langle \sigma, \varepsilon\rangle,$$

for $F \in U^\star$, $u \in U$, $\varepsilon \in \Sigma$, $\sigma \in \Sigma^\star$. For the outer force F we can take e.g.

$$\langle F(t), u\rangle = \int_\Omega f(t)\cdot u\,\mathrm{d}V + \int_{\partial_2\Omega} g(t)\cdot u\,\mathrm{d}V \qquad (u \in U) \tag{8}$$

with volume and boundary forces

$$f \in W^{1,2}\big([0,T], L^2(\Omega)\big), \qquad g \in W^{1,2}\big([0,T], L^2(\partial_2\Omega)\big), \tag{9}$$

such that (3) are the weak forms of the Newtonian balance equations

$$\begin{cases} \operatorname{div}\sigma(t) + f(t) &= 0 & \text{in } \Omega \\ \sigma(t)n &= g(t) & \text{on } \partial_2\Omega \end{cases}.$$

The same, of course, holds analogeously for (1) and ${}^e\sigma$. The homogeneous Dirichlet boundary condition is contained in the definition of the space U of admissible displacements, vanishing on $\partial_1\Omega$. The continuity of F can be easily established with Hölder's inequality and the continuous trace-embedding

$$W^{1,2}(\Omega) \hookrightarrow L^2(\partial\Omega), \quad \|u\|_{L^2(\partial\Omega)} \le C(\Omega)\|u\|_{W^{1,2}(\Omega)}.$$

(See Evans, section 5.5, theorem 1.)

(ii) Operators. Hooke's linear elasticity operator and the linear kinematic hardening operator

$$\mathcal{B}, \mathcal{C} \in \mathcal{L}(\Sigma, \Sigma^\star)$$

are assumed to continuous, symmetric and strongly postive with

$$\|\mathcal{B}\varepsilon\|_\Sigma \le \|\mathcal{B}\|\,\|\varepsilon\|_\Sigma, \quad \langle \mathcal{B}\varepsilon, \eta\rangle = \langle \varepsilon, \mathcal{B}\eta\rangle, \quad \langle \mathcal{B}\varepsilon, \varepsilon\rangle_\Sigma \ge \kappa_{\mathcal{B}}\|\varepsilon\|_\Sigma^2 \tag{10}$$

and

$$\|\mathcal{C}\varepsilon\|_\Sigma \le \|\mathcal{C}\|\,\|\varepsilon\|_\Sigma, \quad \langle \mathcal{C}\varepsilon, \eta\rangle = \langle \varepsilon, \mathcal{C}\eta\rangle, \quad \langle \mathcal{C}\varepsilon, \varepsilon\rangle_\Sigma \ge \kappa_{\mathcal{C}}\|\varepsilon\|_\Sigma^2. \tag{11}$$

They are globally acting.

2.2 Example. For $\mathcal{B}$, $\mathcal{C}$ we may choose for example

$$\begin{aligned} \langle \mathcal{C}\varepsilon, \eta\rangle &= \int_\Omega \big[C(x)\varepsilon(x)\big] : \eta(x)\,\mathrm{d}V(x) \qquad (\varepsilon, \eta \in \Sigma) \\ \langle \mathcal{B}\varepsilon, \eta\rangle &= \int_\Omega \big[B(x)\varepsilon(x)\big] : \eta(x)\,\mathrm{d}V(x) \qquad (\varepsilon, \eta \in \Sigma) \end{aligned}$$

where all pointwise operators

$$C(x), B(x) \in \mathcal{L}\big(\mathbb{R}_s^{3\times3}, \mathbb{R}_s^{3\times3}\big) \qquad (x \in \Omega)$$

for each x are symmetric and positively definite

$$\big(C(x)\varepsilon\big) : \varepsilon \ge \kappa_C(x)\,\|\varepsilon\|^2, \qquad \big(B(x)\varepsilon\big) : \varepsilon \ge \kappa_B(x)\,\|\varepsilon\|^2 \qquad \big(\varepsilon \in \mathbb{R}_s^{3\times3}\big)$$

with definiteness constants uniformly bounded below in x

$$\kappa_C(x) \ge \kappa_{\mathcal{C}} > 0, \qquad \kappa_B(x) \ge \kappa_{\mathcal{B}} > 0$$

such that (10), (11) are globally satisfied. □

2.3 Remark. (a) For each linear, continuous, symmetric and strongly positve operator $A : \Sigma \to \Sigma^\star$ an equivalent scalar product on $\Sigma \simeq \Sigma^\star$ is defined by

$$\langle \sigma, \tau\rangle_A = \langle A^{-1}\sigma, \tau\rangle, \qquad \|\sigma\|_A^2 = \langle \sigma, \sigma\rangle_A. \tag{12}$$

The inverse operator A^{-1} exists according to Zeidler [17], theorem 26.A. As well, A^{-1} is strongly positive and symmetric.
(b) Let for a linear, continuous operator $A : \Sigma \to \Sigma^\star$ the *hatted* operator $\hat{A}$ be defined by

$$\hat{A} : \Sigma^{[0,T]} \to \big(\Sigma^\star\big)^{[0,T]}, \qquad (\hat{A}\varepsilon)(t) = A(\varepsilon(t)).$$

Then clearly, $\hat{A}$ is linear.

1. There holds
$$\hat{A} : W^{1,q}([0,T],\Sigma) \to W^{1,q}([0,T],\Sigma^\star)$$
with
$$\mathrm{d}_t(\hat{A}\varepsilon)(t) = A(\dot{\varepsilon}(t)),$$
since
$$\begin{aligned} \|\hat{A}\varepsilon\|_{W^{1,q}} &= \left(\int_0^T \|A\varepsilon(\tau)\|^q \mathrm{d}\tau\right)^{1/q} + \left(\int_0^T \|A\dot{\varepsilon}(\tau)\|^q \mathrm{d}\tau\right)^{1/q} \\ &\le \left(\|A\|^q \int_0^T \|\varepsilon(\tau)\|^q \mathrm{d}\tau\right)^{1/q} + \left(\|A\|^q \int_0^T \|\dot{\varepsilon}(\tau)\|^q \mathrm{d}\tau\right)^{1/q} \\ &= \|A\|\,\|\varepsilon\|_{W^{1,q}} \end{aligned}$$
for the standard Sobolev norm and
$$\|\hat{A}\varepsilon\|_{W^{1,q,*}} \quad\le\quad \|A\|\,\|\varepsilon\|_{W^{1,q,*}}$$
for the non-standard Sobolev norm, see e.g. [16, 12, 1]. $\hat{A}$ is continuous with the same operator norm $\|\hat{A}\|_{W^{1,q}} \le \|A\|$. The linear operator
$$\hat{}: \mathcal{L}(\Sigma,\,\Sigma^\star) \to \mathcal{L}\big(W^{1,q}([0,T],\Sigma), W^{1,q}([0,T],\Sigma^\star)\big), \quad A \mapsto \hat{A}$$
is continuous with norm $\|\hat{}\| \le 1$. Similar inequalities hold, if A has a continuous inverse. Here, from
$$\|A\varepsilon\| \ge \kappa_A\|\varepsilon\|$$
there follows
$$\|\hat{A}\varepsilon\|_{W^{1,q}} \ge \kappa_A\|\varepsilon\|_{W^{1,q}}, \qquad \|\hat{A}\varepsilon\|_{W^{1,q,*}} \ge \kappa_A\|\varepsilon\|_{W^{1,q,*}}$$
with the same constant κ_A. In the Hilbert space case $q = 2$, if A is symmetric and strongly positive, we similarly have
$$\begin{aligned} \big\langle \hat{A}\varepsilon, \varepsilon\big\rangle_{W^{1,2}} &= \int_0^T \big\langle A\varepsilon(t), \varepsilon(t)\big\rangle \mathrm{d}t + \int_0^T \big\langle A\dot{\varepsilon}(t), \dot{\varepsilon}(t)\big\rangle \mathrm{d}t \\ &\ge \kappa_A \int_0^T \|\varepsilon(t)\|^2 \mathrm{d}t + \kappa_A \int_0^T \|\dot{\varepsilon}(t)\|^2 \mathrm{d}t \\ &= \kappa_A\|\varepsilon\|^2_{W^{1,2}} \end{aligned}$$
for the standard Sobolev norm and
$$\begin{aligned} \big\langle \hat{A}\varepsilon, \varepsilon\big\rangle_{W^{1,2,*}} &= \big\langle A\varepsilon(0), \varepsilon(0)\big\rangle + \int_0^T \big\langle A\dot{\varepsilon}(t), \dot{\varepsilon}(t)\big\rangle \mathrm{d}t \\ &\ge \kappa_A\|\varepsilon(0)\|^2 + \kappa_A \int_0^T \|\dot{\varepsilon}(t)\| \mathrm{d}t \\ &= \kappa_A\|\varepsilon\|^2_{W^{1,2,*}} \end{aligned}$$
for the non-standard Sobolev norm.

2. Similarly, we have
$$\hat{A} : \big(C([0,T],\Sigma), \|\cdot\|_\infty\big) \to \big(C([0,T],\Sigma^\star), \|\cdot\|_\infty\big).$$
There holds
$$\|\hat{A}\varepsilon\|_\infty \le \|A\|\,\|\varepsilon\|_\infty \qquad \text{and} \qquad \|\hat{A}\varepsilon\|_\infty \ge \kappa_A\,\|\varepsilon\|_\infty,$$
implying $\|\hat{A}\|_\infty \le \|A\|$. The map
$$\hat{}: \mathcal{L}(\Sigma,\,\Sigma^\star) \to \mathcal{L}\big(C([0,T],\Sigma), C([0,T],\Sigma^\star)\big), \quad A \mapsto \hat{A}$$
is linear with norm $\|\hat{}\| \le 1$.

3. Convergence
$$A_n \xrightarrow{n} A \qquad \text{in } \mathcal{L}(\Sigma,\, \Sigma^\star)$$
implies both
$$A_n \xrightarrow{n} A \qquad \text{in } \mathcal{L}\big(C([0,T],\Sigma),\, C([0,T],\Sigma^\star)\big)$$
and
$$A_n \xrightarrow{n} A \qquad \text{in } \mathcal{L}\big(W^{1,q}([0,T],\Sigma),\, W^{1,q}([0,T],\Sigma^\star)\big).$$

Here we always assume $1 \le q < \infty$. In the sequel, we identify A with $\hat{A}$, wherever this is possible. □

The differential operators
$$\mathcal{D} \in \mathcal{L}(U,\Sigma), \qquad \mathcal{D}^\star \in \mathcal{L}(\Sigma^\star, U^\star)$$
are defined by the equations
$$\begin{aligned} \mathcal{D}u &= \frac{1}{2}\big(\nabla u + \nabla u^t\big) \\ \big\langle \mathcal{D}^\star\sigma, v\big\rangle &= \int_\Omega \sigma : \mathcal{D}v \,\mathrm{d}V \quad \left(\overset{P.I.}{=} -\int_\Omega \operatorname{div}\sigma : v \,\mathrm{d}V + \int_{\partial_2\Omega} \sigma n\cdot v\,\mathrm{d}A\right) \end{aligned}$$
for $u, v \in U$, $\sigma \in \Sigma^\star$. The latter relation holds for higher regular σ. Explicitely written, relation $\big\langle \mathcal{D}^\star\sigma, v\big\rangle = \big\langle F(t), v\big\rangle$ with $F(t)$ in (8) becomes in its strong form
$$\int_\Omega \big(f(t) - \operatorname{div}\sigma(t)\big) : v\,\mathrm{d}V + \int_{\partial_2\Omega} \big(\sigma(t)n - g(t)\big)\cdot v\,\mathrm{d}A = 0 \qquad \forall v \in U,$$
which is the *principle of virtual displacements*. The same of course holds for the ^{e} quanitities of the elastic problem (E).

We set further for abbreviation

1. The solution operator for the 'elastic' stress
$$S = \mathcal{C}\mathcal{D}(\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1} \in \mathcal{L}(U^\star, \Sigma^\star).$$
2. Two linear projectors ($P^2 = P$, $Q^2 = Q$)
$$Q = S\mathcal{D}^\star \in \mathcal{L}(\Sigma^\star, \Sigma^\star), \qquad P = I - Q \in \mathcal{L}(\Sigma^\star, \Sigma^\star).$$
3. A regularization of the singular operator $P\mathcal{C}$ (which has non-trivial kernel)
$$\mathcal{R} = P\mathcal{C} + \mathcal{B} \in \mathcal{L}(\Sigma, \Sigma^\star).$$

(iii) Elastic domain. We assume the von-Mises yield criterion. The elastic domain is given by
$$Z = \big\{\beta \in \Sigma^\star : \|\operatorname{dev}\beta(x)\| \le \rho \text{ for almost every } x \text{ in } \Omega\big\}.$$
which is convex and closed, containing the origin 0. The subdifferential
$$\partial\chi_Z : \Sigma^\star \to 2^{\Sigma^\star} = \big\{S :\, S \subseteq \Sigma^\star\big\} \ \big(= \text{ power set of } \Sigma^\star\big)$$
of the indicator function
$$\chi_Z(\beta) = \begin{cases} 0 & \text{if } \beta \in Z \\ \infty & \text{if } \beta \notin Z \end{cases},$$

defined by

$$\partial\chi_Z(\beta) = \big\{\tau \in \Sigma^\star : \chi_Z(*) \geq \chi_Z(\beta) + \langle\tau, * - \beta\rangle \text{ for all } * \in \Sigma^\star\big\}$$

is equal to the normal cone at Z in the point β, see Deimling [3], chapter 8, or Brokate [1]. One might distinguish three cases:

1. We have
$$\partial\chi_Z(\beta) = \{0\},$$
if β lies in the interior of Z (for which we have purely elastic behaviour, the plastic strain does not change: $\dot{\varepsilon}^{pl} = 0$, according to (6)).

   But note that for our Z, we have $\mathrm{Int}(Z) = \emptyset$ with respect to the L^2-norm $\|\cdot\|_{\Sigma^\star}$!

2. We have
$$\partial\chi_Z(\beta) = \big\{\tau \in \Sigma^\star : \langle\tau, \beta - *\rangle \geq 0 \text{ for all } * \in Z\big\},$$
if β is a boundary point of Z (which gives all the directions normal to ∂Z).

3. We have
$$\partial\chi_Z(\beta) = \emptyset,$$
if β lies in the complement of Z (which is – consequently – forbidden).

It is important to note, that the Lipschitz estimates in [16] do not need the interior of Z being non-empty.

3 Solution of both models

3.1 Solution of elastic model (E)

We are starting with the simpler elastic model. We premultiplying (1) for fixed $t \in [0, T]$ with $\mathcal{C}$

$$^e\sigma(t) \;=\; \mathcal{C}\,^e\varepsilon(t) \;=\; \mathcal{C}\mathcal{D}\,^e u(t),$$

and $\mathcal{D}^\star$

$$F(t) \;=\; \mathcal{D}^\star\,^e\sigma(t) \;=\; \mathcal{D}^\star\mathcal{C}\,^e\varepsilon(t) \;=\; \mathcal{D}^\star\mathcal{C}\mathcal{D}\,^e u(t). \tag{13}$$

The operator

$$\mathcal{D}^\star\mathcal{C}\mathcal{D} \in \mathcal{L}(U, U^\star)$$

is symmetric and – by virtue of Korn – strongly positive

$$\big\langle\mathcal{D}^\star\mathcal{C}\mathcal{D}u, u\big\rangle = \big\langle\mathcal{C}\mathcal{D}u, \mathcal{D}u\big\rangle \overset{(11)}{\geq} \kappa_{\mathcal{C}}\|Du\|^2_\Sigma \overset{(7)}{\geq} \kappa_{\mathcal{C}}\kappa^2\|u\|^2_U,$$

thus invertible, such that we can explicitly and uniquely solve (13) for $^e u$ to obtain for each $t \in [0, T]$

1. the 'elastic' displacement
$$^e u(t) \;=\; (\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}F(t), \tag{14}$$

2. the 'elastic' strain
$$^e\varepsilon(t) \;=\; \mathcal{D}(\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}F(t), \tag{15}$$

3. the 'elastic' stress
$$^e\sigma(t) \;=\; \mathcal{C}\mathcal{D}(\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}F(t) \;=\; SF(t). \tag{16}$$

3.1 Theorem (Elastic solution) *For each given*

$$F \in W^{1,q}\big([0,T],U^\star\big), \qquad (1 \le q < \infty) \tag{17}$$

there exists a uniquely determined triple

$$^e u \in W^{1,q}\big([0,T],U\big), \qquad ^e\varepsilon \in W^{1,q}\big([0,T],\Sigma\big), \qquad ^e\sigma \in W^{1,q}\big([0,T],\Sigma^\star\big),$$

given by (14), ..., (16), satisfying the elastic relations (E).

Proof: In fact (14), ..., (16) hold for each given $F : [0,T] \to U^\star$, as for each fixed t we have a static uniquely solvable elastic problem. If now $F \in W^{1,q}\big([0,T],U^\star\big)$, identifying the operators

$$(\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}, \qquad \mathcal{D}(\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}, \qquad S = \mathcal{C}\mathcal{D}(\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}$$

with their hatted counterparts, cf. remark 2.3, we have

$$\begin{array}{rcl}
\|^e u\|_{W^{1,q}([0,T],U)} & \le & \|(\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}\|\,\|F\|_{W^{1,q}([0,T],U^\star)}, \\
\|^e\varepsilon\|_{W^{1,q}([0,T],\Sigma)}, & \le & \|\mathcal{D}(\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}\|\,\|F\|_{W^{1,q}([0,T],U^\star)}, \\
\|^e\sigma\|_{W^{1,q}([0,T],\Sigma^\star)}, & \le & \|\mathcal{C}\mathcal{D}(\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}\|\,\|F\|_{W^{1,q}([0,T],U^\star)},
\end{array}$$

the right hand sides being finite. ∎

3.2 Solution of elastoplastic problem (EP)

We find that $P\mathcal{C}$ is non-negative and symmetric. (See ZEIDLER [18], proof of theorem 66.A.) Thus $\mathcal{R}$ is strongly positive and symmetric with

$$\big\langle \mathcal{R}\varepsilon,\varepsilon\big\rangle \ge \big\langle P\mathcal{C}\varepsilon,\varepsilon\big\rangle + \big\langle \mathcal{B}\varepsilon,\varepsilon\big\rangle \ge \kappa_{\mathcal{B}}\|\varepsilon\|^2 \tag{18}$$

Consequently the inverse $\mathcal{R}^{-1}$ exists and is strongly positive and symmetric. The same holds for $\mathcal{B}$ and its existing inverse $\mathcal{B}^{-1}$. We define, cf. (12), equivalent scalar products on $\Sigma^\star$ by

$$\big\langle \sigma,\tau\big\rangle_{\mathcal{R}} = \big\langle \mathcal{R}^{-1}\sigma,\tau\big\rangle, \qquad \big\langle \sigma,\tau\big\rangle_{\mathcal{B}} = \big\langle \mathcal{B}^{-1}\sigma,\tau\big\rangle.$$

Let us denote the equivalence constants by

$$c_{\mathcal{B}}\|\cdot\| \le \|\cdot\|_{\mathcal{B}} \le C_{\mathcal{B}}\|\cdot\|, \qquad c_{\mathcal{R}}\|\cdot\| \le \|\cdot\|_{\mathcal{R}} \le C_{\mathcal{R}}\|\cdot\|. \tag{19}$$

Note that due to the equality of topologies on $\Sigma \simeq \Sigma^\star$, which are induced by our new norms, we have $(\Sigma,\langle\cdot,\cdot\rangle_{\mathcal{B}})$, $(\Sigma,\langle\cdot,\cdot\rangle_{\mathcal{R}})$ still separable and Z still closed with respect to both norms. Clearly, 0 remains in Z. We have

$$S\dot{F} - \dot{\beta} \in \partial_{\mathcal{R}}\chi_Z(\beta), \qquad \beta(0) = \beta_0 \in Z \tag{20}$$

and explicit formulas for

$$\begin{array}{rcl}
\varepsilon^{pl} & = & \mathcal{R}^{-1}(SF - \beta), \\
u & = & (\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}\mathcal{D}^\star\big(\beta + (\mathcal{B}+\mathcal{C})\varepsilon^{pl}\big).
\end{array}$$

(See [18], hidden in the proof of theorem 66.A.) This allows us to express all remaining unknowns in terms of

$$\beta(t) \qquad \text{and} \qquad \gamma(t) = \big(SF - \beta\big)(t). \tag{21}$$

We find

1. the plastic strain

$$\varepsilon^{pl}(t) \quad = \quad \mathcal{R}^{-1}\gamma(t), \tag{22}$$

2. the backstress

$$\alpha(t) \quad = \quad \mathcal{B}\mathcal{R}^{-1}\gamma(t), \tag{23}$$

3. the stress

$$\sigma(t) \quad = \quad \mathcal{B}\mathcal{R}^{-1}\gamma(t) + \beta(t), \tag{24}$$

4. the elastic strain

$$\varepsilon^{el}(t) \quad = \quad \mathcal{C}^{-1}\big(\mathcal{B}\mathcal{R}^{-1}\gamma(t) + \beta(t)\big), \tag{25}$$

5. the strain

$$\varepsilon(t) \quad = \quad \big(\mathcal{C}^{-1}\mathcal{B} + I\big)\mathcal{R}^{-1}\gamma(t) + \mathcal{C}^{-1}\beta(t), \tag{26}$$

6. the displacement

$$u(t) \quad = \quad (\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}\mathcal{D}^\star\big(\beta(t) + (\mathcal{B}+\mathcal{C})\mathcal{R}^{-1}\gamma(t)\big). \tag{27}$$

In equation (20),

$$\begin{aligned}
\partial_{\mathcal{R}}\chi_Z(\beta) &= \mathcal{R}\partial\chi_Z(\beta) \\
&= \big\{\mathcal{R}\tau \in \Sigma^\star : \chi_Z(*) \geq \chi_Z(\beta) + \big\langle\tau, * - \beta\big\rangle \text{ for all } * \in Z\big\} \\
&= \big\{\tau \in \Sigma^\star : \chi_Z(*) \geq \chi_Z(\beta) + \big\langle\tau, * - \beta\big\rangle_{\mathcal{R}} \text{ for all } * \in Z\big\}
\end{aligned}$$

is the subdifferential of χ_Z with respect to $\langle\cdot,\cdot\rangle_{\mathcal{R}}$.

3.2 Theorem (Elastoplastic solution) *For each*

$$F \in W^{1,2}\big([0,T], U^\star\big) \qquad \text{and} \qquad \beta_0 \in Z$$

there exists a uniquely determined septuple

$$u \in W^{1,2}\big([0,T], U^\star\big), \quad \varepsilon, \varepsilon^{el}, \varepsilon^{pl} \in W^{1,2}\big([0,T], \Sigma\big), \quad \sigma, \alpha, \beta \in W^{1,2}\big([0,T], \Sigma^\star\big)$$

given by (20), (22), ..., (27), satisfying the elastoplastic relations (EP) with initial condition $\beta(0) = \beta_0$.

Proof: See [18] theorem 66.A, and the preliminaries. ■

3.3 Remark. We have from (22) the following equivalence

$$\varepsilon^{pl}(0) = 0 \qquad \Longleftrightarrow \qquad \beta(0) = {}^{e}\sigma(0) = SF(0) \tag{28}$$

We call the body Ω then *virgin*. □

We can express α, β and γ as stop and play operators with

(A) **the input σ and the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{B}}$**, namely: Rewriting (6) equivalently as

$$\left\{\begin{array}{rcll} \left\langle \mathcal{B}^{-1}\dot\alpha,\, \beta-*\right\rangle = \left\langle \dot\alpha,\, \beta-*\right\rangle_{\mathcal{B}} & \geq & 0 & \text{for all } *\in Z \quad \text{a.e. in } [0,T] \\ \alpha+\beta & = & \sigma & \text{e. in } [0,T] \\ \beta(0) & = & \beta_0 & \end{array}\right. ,$$

the definition of the stop and play operators, see [16] (2), gives us

$$\alpha = \mathcal{P}_{\mathcal{B}}(\sigma,\beta_0), \qquad \beta = \mathcal{S}_{\mathcal{B}}(\sigma,\beta_0). \tag{29}$$

(B) **the input ${}^e\sigma$ and the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{R}}$**, namely: Rewriting (20) equivalently as

$$\left\{\begin{array}{rcll} \left\langle \mathcal{R}^{-1}\dot\gamma,\, \beta-*\right\rangle = \left\langle \dot\gamma,\, \beta-*\right\rangle_{\mathcal{R}} & \geq & 0 & \text{for all } *\in Z \quad \text{a.e. in } [0,T] \\ \gamma+\beta & = & {}^e\sigma & \text{e. in } [0,T] \\ \beta(0) & = & \beta_0 & \end{array}\right. ,$$

the definition of the stop and play operators, see again [16] (2), gives us

$$\gamma = \mathcal{P}_{\mathcal{R}}({}^e\sigma,\beta_0), \qquad \beta = \mathcal{S}_{\mathcal{R}}({}^e\sigma,\beta_0). \tag{30}$$

The following lemma is a generalisation of lemma 2.1 in Krejci [13], section 2.3, for the infinitely dimensional Hilbert space case.
(It can be formulated for any real separable Hilbert space X and any strongly positive and symmteric operators $\mathcal{B}$, $\mathcal{C}\in\mathcal{L}(\Sigma,\Sigma^\star)$.)

3.4 Lemma. *For the elastoplastic model (EP), there holds*

$$\mathcal{S}_{\mathcal{B}+\mathcal{C}}\big(\sigma+\mathcal{C}\mathcal{B}^{-1}\mathcal{P}_{\mathcal{B}}(\sigma,\beta_0),\beta_0\big) \quad = \quad \mathcal{S}_{\mathcal{B}}(\sigma,\beta_0).$$

Proof: We have according to (29)

$$\alpha = \mathcal{P}_{\mathcal{B}}(\sigma,\,\beta_0), \qquad \beta = \mathcal{S}_{\mathcal{B}}(\sigma,\,\beta_0), \qquad \sigma = \alpha+\beta.$$

We set – $\mathcal{B}+\mathcal{C}$ is again symmetric and strongly positive –

$$p = \mathcal{P}_{\mathcal{B}+\mathcal{C}}(f,\,\beta_0), \qquad s = \mathcal{S}_{\mathcal{B}+\mathcal{C}}(f,\,\beta_0), \qquad f = \sigma+\mathcal{C}\mathcal{B}^{-1}\alpha,$$

and compute

$$f = \sigma+\mathcal{C}\mathcal{B}^{-1}\mathcal{P}_{\mathcal{B}}(\sigma) = (I+\mathcal{C}\mathcal{B}^{-1})\sigma-\mathcal{C}\mathcal{B}^{-1}\beta = (\mathcal{B}+\mathcal{C})\mathcal{B}^{-1}\sigma-\mathcal{C}\mathcal{B}^{-1}\beta.$$

The variational inequalities thus yield

$$\begin{array}{rcrcll} \left\langle \dot\alpha,\,\beta-*\right\rangle_{\mathcal{B}} & = & \left\langle \mathcal{B}^{-1}\dot\alpha,\,\beta-*\right\rangle & \geq & 0 & \text{for all } *\in Z, \\ \left\langle \dot p,\,s-*\right\rangle_{\mathcal{B}+\mathcal{C}} & = & \left\langle (\mathcal{B}+\mathcal{C})^{-1}\dot p,\,s-*\right\rangle & \geq & 0 & \text{for all } *\in Z. \end{array}$$

Chiastically inserting s and β and addition yields

$$\left\langle (\mathcal{B}+\mathcal{C})^{-1}\dot p-\mathcal{B}^{-1}\dot\alpha,\,s-\beta\right\rangle \quad \geq \quad 0. \tag{31}$$

We compute

$$\begin{array}{rcl} \mathcal{B}^{-1} = (\mathcal{B}+\mathcal{C})^{-1}(\mathcal{B}+\mathcal{C})\mathcal{B}^{-1} & = & (\mathcal{B}+\mathcal{C})^{-1}(I+\mathcal{C}\mathcal{B}^{-1}) \\ & = & (\mathcal{B}+\mathcal{C})^{-1}+(\mathcal{B}+\mathcal{C})^{-1}\mathcal{C}\mathcal{B}^{-1}, \end{array}$$

so that the left-hand factor in (31) becomes

$$\begin{aligned}(\mathcal{B}+\mathcal{C})^{-1}\dot{p}-\mathcal{B}^{-1}\dot{\alpha} &= (\mathcal{B}+\mathcal{C})^{-1}\big[(\mathcal{B}+\mathcal{C})\mathcal{B}^{-1}\dot{\sigma}-\mathcal{C}\mathcal{B}^{-1}\dot{\beta}-\dot{s}\big]-\mathcal{B}^{-1}\big[\dot{\sigma}-\dot{\beta}\big]\\ &= [\mathcal{B}^{-1}-(\mathcal{B}+\mathcal{C})^{-1}\mathcal{C}\mathcal{B}^{-1}]\dot{\beta}-(\mathcal{B}+\mathcal{C})^{-1}\dot{s}\\ &= (\mathcal{B}+\mathcal{C})^{-1}(\dot{\beta}-\dot{s}).\end{aligned}$$

Inequality (31) is equivalent to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\beta-s\|^2_{\mathcal{B}+\mathcal{C}}=\big\langle\dot{\beta}-\dot{s},\,\beta-s\big\rangle_{\mathcal{B}+\mathcal{C}}=\big\langle(\mathcal{B}+\mathcal{C})^{-1}(\dot{\beta}-\dot{s}),\,\beta-s\big\rangle\le 0.$$

Integration with the initial value $(\beta-s)(0)=0$ yields $\beta\equiv s$ e. in $[0,T]$. ■

3.5 Theorem. *Let $F\in W^{1,2}([0,T],\Sigma^\star)$ be given with the corresponding solutions*

$$\varepsilon\in W^{1,2}([0,T],\Sigma),\qquad \sigma\in W^{1,2}([0,T],\Sigma^\star)$$

of (EP). Then there holds

$$\sigma=\mathcal{G}(\varepsilon),\qquad \varepsilon=\mathcal{F}(\sigma).$$

The operators

$$\begin{aligned}\mathcal{F}&=\mathcal{C}^{-1}\cdot+\mathcal{B}^{-1}\mathcal{P}_{\mathcal{B}}(\cdot,\beta_0):W^{1,q}\big([0,T],\Sigma^\star\big)\to W^{1,q}\big([0,T],\Sigma\big)\\ \mathcal{G}&=\mathcal{C}(\mathcal{B}+\mathcal{C})^{-1}\big(\mathcal{B}\cdot+\mathcal{S}_{\mathcal{B}+\mathcal{C}}(\mathcal{C}\cdot,\beta_0)\big):W^{1,q}\big([0,T],\Sigma\big)\to W^{1,q}\big([0,T],\Sigma^\star\big)\end{aligned}$$

are turning stress into strain and vice versa, $\mathcal{F}=\mathcal{G}^{-1}$. They are continuous for each $1\le q<\infty$.

Proof: Clearly,

$$\varepsilon=\varepsilon^{el}+\varepsilon^{pl}=\mathcal{C}^{-1}\sigma+\mathcal{B}^{-1}\alpha=\mathcal{C}^{-1}\sigma+\mathcal{B}^{-1}\mathcal{P}_{\mathcal{B}}(\sigma)=\mathcal{F}(\sigma).$$

We have

$$\sigma+\mathcal{C}\mathcal{B}^{-1}\mathcal{P}_{\mathcal{B}}(\sigma)=\mathcal{C}\varepsilon^{el}+\mathcal{C}\mathcal{B}^{-1}\varepsilon^{pl}=\mathcal{C}(\varepsilon^{el}+\mathcal{B}^{-1}\alpha)=\mathcal{C}\varepsilon.$$

From lemma 3.4 we have (with initial memory β_0)

$$\mathcal{S}_{\mathcal{B}+\mathcal{C}}(\mathcal{C}\varepsilon)=\mathcal{S}_{\mathcal{B}}(\sigma)=\beta=\sigma-\alpha=\sigma-\mathcal{B}(\varepsilon-\mathcal{C}^{-1}\sigma)=(\mathcal{B}+\mathcal{C})\mathcal{C}^{-1}\sigma-\mathcal{B}\varepsilon,$$

thus

$$\sigma=\mathcal{C}(\mathcal{B}+\mathcal{C})^{-1}\big(\mathcal{B}\varepsilon+\mathcal{S}_{\mathcal{B}+\mathcal{C}}(\mathcal{C}\varepsilon)\big)=\mathcal{G}(\varepsilon),$$

as stated. The continuity is clear, see [12, 13] or [16]. ■

Estimates for problem (EP). We are now able to gives a *new* proof of the stability results and error estimates of problem (EP) compared to the one in Han and Reddy [6], section 7.3. New is to be understood in the following sense:

1. We use the language of stop ans plays, which is more convenient and concise.
2. In contrast to [6], we consider directly, what they call the dual problem (in the context of convex analysis). This – equivalent – formulation, cf. [6], section 4.2, is the most widespread in both mathematical and engineering literature.
3. We have the case of *mixed* Dirichlet and Neumann boundary conditions. In [6], only homogeneous Dirichlet conditions $u=0$ on whole $\partial\Omega$ are studied, and there is no indication, how to handle the mixed problem, which is *essential* for practice.

3.6 Theorem. *There exist constants, such that for two inputs*

$$F_1, F_2 \in W^{1,2}\big([0,T], U^\star\big)$$

and their correspondig solutions

$$\varepsilon_i^{pl}(F_i), \quad \varepsilon_i^{el}(F_i), \quad \varepsilon_i(F_i), \quad \sigma_i(F_i), \quad \alpha_i(F_i), \quad \beta_i(F_i), \quad u_i(F_i) \qquad (i \in \{1,2\})$$

of (EP) there hold the following estimates e. in $[0,T]$

1. *for the strains*

$$\begin{aligned}
\|\Delta\varepsilon^{pl}(t)\|_\Sigma &\leq C^{(0)}_{\varepsilon^{pl}}\|\Delta\beta_0\|_\mathcal{R} + C^{(1)}_{\varepsilon^{pl}}\|S\Delta F(t)\|_\mathcal{R} + C^{(2)}_{\varepsilon^{pl}}\int_0^t \|S\Delta\dot F(\tau)\|_\mathcal{R}\,d\tau \\
\|\Delta\varepsilon^{el}(t)\|_\Sigma &\leq C^{(0)}_{\varepsilon^{el}}\|\Delta\beta_0\|_\mathcal{R} + C^{(1)}_{\varepsilon^{el}}\|S\Delta F(t)\|_\mathcal{R} + C^{(2)}_{\varepsilon^{el}}\int_0^t \|S\Delta\dot F(\tau)\|_\mathcal{R}\,d\tau \\
\|\Delta\varepsilon(t)\|_\Sigma &\leq C^{(0)}_{\varepsilon}\|\Delta\beta_0\|_\mathcal{R} + C^{(1)}_{\varepsilon}\|S\Delta F(t)\|_\mathcal{R} + C^{(2)}_{\varepsilon}\int_0^t \|S\Delta\dot F(\tau)\|_\mathcal{R}\,d\tau,
\end{aligned}$$

2. *for the stresses*

$$\begin{aligned}
\|\Delta\alpha(t)\|_{\Sigma^\star} &\leq C^{(0)}_{\alpha}\|\Delta\beta_0\|_\mathcal{R} + C^{(1)}_{\alpha}\|S\Delta F(t)\|_\mathcal{R} + C^{(2)}_{\alpha}\int_0^t \|S\Delta\dot F(\tau)\|_\mathcal{R}\,d\tau \\
\|\Delta\beta(t)\|_{\Sigma^\star} &\leq C^{(0)}_{\beta}\|\Delta\beta_0\|_\mathcal{R} + C^{(1)}_{\beta}\|S\Delta F(t)\|_\mathcal{R} + C^{(2)}_{\beta}\int_0^t \|S\Delta\dot F(\tau)\|_\mathcal{R}\,d\tau \\
\|\Delta\sigma(t)\|_{\Sigma^\star} &\leq C^{(0)}_{\sigma}\|\Delta\beta_0\|_\mathcal{R} + C^{(1)}_{\sigma}\|S\Delta F(t)\|_\mathcal{R} + C^{(2)}_{\sigma}\int_0^t \|S\Delta\dot F(\tau)\|_\mathcal{R}\,d\tau,
\end{aligned}$$

3. *for the displacements*

$$\|\Delta u(t)\|_U \leq C^{(0)}_{u}\|\Delta\beta_0\|_\mathcal{R} + C^{(1)}_{u}\|S\Delta F(t)\|_\mathcal{R} + C^{(2)}_{u}\int_0^t \|S\Delta\dot F(\tau)\|_\mathcal{R}\,d\tau,$$

where

$$\Delta\cdot \;=\; \cdot|_2^1 \;=\; \cdot_1 - \cdot_2.$$

All the appearing constants are non-negative, some of them may be chosen identical to zero. (All estimates are still valid, if Δ *is cancelled.)*

Proof: We use (21). The explicit formulas (22), ..., (27) and the norm equivalence (18) give us with the definition of the linear operator norm

$$\begin{aligned}
\|\beta(t)\|_{\Sigma^\star} &\leq && C^{(\beta)}_{\beta}\|\beta(t)\|_\mathcal{R} \\
\|\varepsilon^{pl}(t)\|_\Sigma &\overset{(22)}{\leq} && C^{(\gamma)}_{\varepsilon^{pl}}\|\gamma(t)\|_\mathcal{R} \\
\|\alpha(t)\|_{\Sigma^\star} &\overset{(23)}{\leq} && C^{(\gamma)}_{\alpha}\|\gamma(t)\|_\mathcal{R} \\
\|\sigma(t)\|_{\Sigma^\star} &\overset{(24)}{\leq} && C^{(\gamma)}_{\sigma}\|\gamma(t)\|_\mathcal{R} + C^{(\beta)}_{\sigma}\|\beta(t)\|_\mathcal{R} \\
\|\varepsilon^{el}(t)\|_\Sigma &\overset{(25)}{\leq} && C^{(\gamma)}_{\varepsilon^{el}}\|\gamma(t)\|_\mathcal{R} + C^{(\beta)}_{\varepsilon^{el}}\|\beta(t)\|_\mathcal{R} \\
\|\varepsilon(t)\|_\Sigma &\overset{(26)}{\leq} && C^{(\gamma)}_{\varepsilon}\|\gamma(t)\|_\mathcal{R} + C^{(\beta)}_{\varepsilon}\|\beta(t)\|_\mathcal{R} \\
\|u(t)\|_U &\overset{(27)}{\leq} && C^{(\gamma)}_{u}\|\gamma(t)\|_\mathcal{R} + C^{(\beta)}_{u}\|\beta(t)\|_\mathcal{R}
\end{aligned} \tag{32}$$

for each $t \in [0,T]$. Now apply the standard Lipschitz estimates [16] (15) for $\|\beta(t)\|_\mathcal{R}$, [16] (16) for $\|\gamma(t)\|_\mathcal{R}$ and the triangle equality. ∎

3.7 Theorem. *There exist constants, such that for each outer force*

$$F \in W^{1,2}([0,T], U^\star)$$

there holds

$$\|\dot\varepsilon^{pl}\|_\Sigma \le C_{\varepsilon^{pl}}\|S\dot F\|_{\Sigma^\star}, \quad \|\dot\varepsilon^{el}\|_\Sigma \le C_{\varepsilon^{el}}\|S\dot F\|_{\Sigma^\star}, \quad \|\dot\varepsilon\|_\Sigma \le C_\varepsilon\|S\dot F\|_{\Sigma^\star} \tag{33}$$

(for the strains)

$$\|\dot\alpha\|_{\Sigma^\star} \le C_\alpha\|S\dot F\|_{\Sigma^\star}, \quad \|\dot\beta\|_{\Sigma^\star} \le C_\beta\|S\dot F\|_{\Sigma^\star}, \quad \|\dot\sigma\|_{\Sigma^\star} \le C_\sigma\|S\dot F\|_{\Sigma^\star} \tag{34}$$

(for the stresses)

$$\|\dot u\|_U \le C_u\|S\dot F\|_{\Sigma^\star} \tag{35}$$

(for the displacement). All estimates are valid a.e. in $[0,T]$

Proof: We have

$$\|\dot\beta\|_{\mathcal R} \le \|S\dot F\|_{\mathcal R} \le C_{\mathcal R}\|S\dot F\|_{\Sigma^\star}, \quad \|\dot\gamma\|_{\mathcal R} \le \|S\dot F\|_{\mathcal R} \le C_{\mathcal R}\|S\dot F\|_{\Sigma^\star}, \quad \text{a.e. in } [0,T]$$

from (30), (16) and [16] (12), where $C_{\mathcal R}$ from (19). Now apply the explicit expressions (22), ..., (27) and the triangle inequality in order to derive dotted 'equivalents' of (32). ■

4 Difference estimates

In section 4.1, we consider the difference of both models itself, whereas in section 4.2, we study the difference between the elastoplastic solution and the corrected elastic solution, cf. [14, 15].

4.1 Difference of (EP) and (E)

Let us abbreviate

$$\begin{aligned}
\mathcal H_{\sigma,\mathcal R} &= P\mathcal C\mathcal R^{-1}, & \mathcal H_{\sigma,\mathcal B} &= P\mathcal C\mathcal B^{-1},\\
\mathcal H_{\varepsilon,\mathcal R} &= (\mathcal C^{-1}P\mathcal C - I)\mathcal R^{-1}, & \mathcal H_{\varepsilon,\mathcal B} &= (\mathcal C^{-1}P\mathcal C - I)\mathcal B^{-1},\\
\mathcal H_{u,\mathcal R} &= -(\mathcal D^\star\mathcal C\mathcal D)^{-1}\mathcal D^\star\mathcal C\mathcal R^{-1}, & \mathcal H_{u,\mathcal B} &= -(\mathcal D^\star\mathcal C\mathcal D)^{-1}\mathcal D^\star\mathcal C\mathcal B^{-1}.
\end{aligned}$$

We have

$$\mathcal H_{\sigma,\cdot} \in \mathcal L(\Sigma^\star, \Sigma^\star), \qquad \mathcal H_{\varepsilon,\cdot} \in \mathcal L(\Sigma^\star, \Sigma), \qquad \mathcal H_{u,\cdot} \in \mathcal L(\Sigma^\star, U).$$

Let further

$$\|\mathcal H_{\sigma,\mathcal R}\|_{\Sigma^\star,\mathcal R} = \sup_{\sigma\ne 0}\frac{\|\mathcal H_{\sigma,\mathcal R}(\sigma)\|_{\Sigma^\star}}{\|\sigma\|_{\mathcal R}}, \qquad \|\mathcal H_{\sigma,\mathcal B}\|_{\Sigma,\mathcal B} = \sup_{\sigma\ne 0}\frac{\|\mathcal H_{\varepsilon,\mathcal R}(\sigma)\|_{\Sigma}}{\|\sigma\|_{\mathcal B}}, \qquad \ldots$$

the linear operator norms w.r.t.

$$\|\cdot\|_{\mathcal R}, \|\cdot\|_{\mathcal B} \text{ in the preimage space,} \qquad \|\cdot\|_{\Sigma^\star}, \|\cdot\|_{\Sigma}, \|\cdot\|_U \text{ in the image space.}$$

If we compare (E) with (EP), we find easily after a short computation the following remarable theorem:

4.1 Theorem (Model differences I) *The differences of the solutions of models (E) and (EP) are play operators. This means – explicitly written –*

(a) *Difference of stresses*

$$ {}^{e}\sigma - \sigma \;=\; H_{\sigma,\mathcal{R}}\,\mathcal{P}_{\mathcal{R}}({}^{e}\sigma, \beta_0) \;=\; H_{\sigma,\mathcal{B}}\,\mathcal{P}_{\mathcal{B}}(\sigma, \beta_0), \tag{36} $$

(b) *Difference of strains*

$$ {}^{e}\varepsilon - \varepsilon \;=\; H_{\varepsilon,\mathcal{R}}\,\mathcal{P}_{\mathcal{R}}({}^{e}\sigma, \beta_0) \;=\; H_{\varepsilon,\mathcal{B}}\,\mathcal{P}_{\mathcal{B}}(\sigma, \beta_0), \tag{37} $$

(c) *Difference of displacements*

$$ {}^{e}u - u \;=\; H_{u,\mathcal{R}}\,\mathcal{P}_{\mathcal{R}}({}^{e}\sigma, \beta_0) \;=\; H_{u,\mathcal{B}}\,\mathcal{P}_{\mathcal{B}}(\sigma, \beta_0). \tag{38} $$

Proof: We use relations (14), ..., (16) and (22), ..., (27) and

$$ \mathcal{R} - \mathcal{B} = P\mathcal{C}, \qquad \varepsilon^{pl} = \mathcal{R}^{-1}\gamma = \mathcal{B}^{-1}\alpha. $$

Thus first,

$$ {}^{e}\sigma - \sigma \;=\; SF - (\mathcal{B}\mathcal{R}^{-1}\gamma + \beta) = (I - \mathcal{B}\mathcal{R}^{-1})\gamma = (\mathcal{R} - \mathcal{B})\varepsilon^{pl} = P\mathcal{C}\varepsilon^{pl}, $$

second,

$$ {}^{e}\varepsilon - \varepsilon \;=\; \mathcal{C}^{-1}\,{}^{e}\sigma - (\mathcal{C}^{-1}\sigma + \varepsilon^{pl}) = \mathcal{C}^{-1}P\mathcal{C}\varepsilon^{pl} - \varepsilon^{pl} = (\mathcal{C}^{-1}P\mathcal{C} - I)\varepsilon^{pl}, $$

third,

$$ {}^{e}u - u \;=\; (\mathcal{D}^{\star}\mathcal{C}\mathcal{D})^{-1}F - (\mathcal{D}^{\star}\mathcal{C}D)^{-1}\mathcal{D}^{\star}(\sigma + \mathcal{C}\varepsilon^{pl}) = -(\mathcal{D}^{\star}\mathcal{C}D)^{-1}\mathcal{D}^{\star}\mathcal{C}\varepsilon^{pl}. $$

(Note $\mathcal{D}^{\star}\sigma = F$.) This yields the assertion. ■

Thus, we are able to apply the smallness estimates [16] (14) for play in order to derive difference estimates for those differences.

4.2 Theorem (Model differences II) *There holds e. in* $[0, T]$

(A) *In terms of* ${}^{e}\sigma$ *and* $\gamma = SF - \beta$

$$ \begin{array}{lll} \text{for stresses:} & \|({}^{e}\sigma - \sigma)(t)\|_{\Sigma^\star} & \le\; C_{\mathcal{R}}\|\mathcal{H}_{\sigma,\mathcal{R}}\|_{\Sigma^\star,\mathcal{R}}\,\varphi_\gamma(t), \\ \text{for strains:} & \|({}^{e}\varepsilon - \varepsilon)(t)\|_{\Sigma} & \le\; C_{\mathcal{R}}\|\mathcal{H}_{\varepsilon,\mathcal{R}}\|_{\Sigma,\mathcal{R}}\,\varphi_\gamma(t), \\ \text{for displacements:} & \|({}^{e}u - u)(t)\|_{U} & \le\; C_{\mathcal{R}}\|\mathcal{H}_{u,\mathcal{R}}\|_{U,\mathcal{R}}\,\varphi_\gamma(t), \end{array} $$

where

$$ \varphi_\gamma(t) \;=\; \|\gamma(0)\|_{\Sigma^\star} + \int_0^t \|S\dot{F}(\tau)\|_{\Sigma^\star}\,d\tau. $$

Note that ${}^{e}\sigma = SF$ *is – up to a linear operator – equal to the input (cf. (16)), and that estimates w.r.t. to* F *are easily obtained.*

(B) *In terms of* σ *and* $\alpha = \sigma - \beta$

$$ \begin{array}{lll} \text{for stresses:} & \|({}^{e}\sigma - \sigma)(t)\|_{\Sigma^\star} & \le\; C_{\mathcal{B}}\|\mathcal{H}_{\sigma,\mathcal{B}}\|_{\Sigma^\star,\mathcal{B}}\,\varphi_\alpha(t), \\ \text{for strains:} & \|({}^{e}\varepsilon - \varepsilon)(t)\|_{\Sigma} & \le\; C_{\mathcal{B}}\|\mathcal{H}_{\varepsilon,\mathcal{B}}\|_{\Sigma,\mathcal{B}}\,\varphi_\alpha(t), \\ \text{for displacements:} & \|({}^{e}u - u)(t)\|_{U} & \le\; C_{\mathcal{B}}\|\mathcal{H}_{u,\mathcal{B}}\|_{U,\mathcal{B}}\,\varphi_\alpha(t), \end{array} $$

where

$$ \varphi_\alpha(t) \;=\; \|\alpha(0)\|_{\Sigma^\star} + \int_0^t \|\dot{\sigma}(\tau)\|_{\Sigma^\star}\,d\tau. $$

The constants $C_{\mathcal{R}}$ *and* $C_{\mathcal{B}}$ *are from (19).*

Proof: It is really straight forward with the aid of [16] (14). ■

4.3 Theorem (Model differences III) *There holds a.e. in* $[0,T]$

(A) *In terms of* ${}^e\dot\sigma$

$$\begin{array}{lll} \textit{for stresses:} & \|({}^e\dot\sigma-\dot\sigma)(t)\|^2_{\Sigma^\star} & \le C^2_{\mathcal{R}}\|\mathcal{H}_{\sigma,\mathcal{R}}\|^2_{\Sigma^\star,\mathcal{R}}\|S\dot F(t)\|^2_{\Sigma^\star}, \\ \textit{for strains:} & \|({}^e\dot\varepsilon-\dot\varepsilon)(t)\|^2_{\Sigma} & \le C^2_{\mathcal{R}}\|\mathcal{H}_{\varepsilon,\mathcal{R}}\|^2_{\Sigma,\mathcal{R}}\|S\dot F(t)\|^2_{\Sigma^\star}, \\ \textit{for displacements:} & \|({}^e\dot u-\dot u)(t)\|^2_{U} & \le C^2_{\mathcal{R}}\|\mathcal{H}_{u,\mathcal{R}}\|^2_{U,\mathcal{R}}\|S\dot F(t)\|^2_{\Sigma^\star}, \end{array}$$

(B) *In terms of* $\dot\sigma$

$$\begin{array}{lll} \textit{for stresses:} & \|({}^e\dot\sigma-\dot\sigma)(t)\|^2_{\Sigma^\star} & \le C^2_{\mathcal{B}}\|\mathcal{H}_{\sigma,\mathcal{B}}\|^2_{\Sigma^\star,\mathcal{B}}\|\dot\sigma(t)\|^2_{\Sigma^\star}, \\ \textit{for strains:} & \|({}^e\dot\varepsilon-\dot\varepsilon)(t)\|^2_{\Sigma} & \le C^2_{\mathcal{B}}\|\mathcal{H}_{\varepsilon,\mathcal{B}}\|^2_{\Sigma,\mathcal{B}}\|\dot\sigma(t)\|^2_{\Sigma^\star}, \\ \textit{for displacements:} & \|({}^e\dot u-\dot u)(t)\|^2_{U} & \le C^2_{\mathcal{B}}\|\mathcal{H}_{u,\mathcal{B}}\|^2_{U,\mathcal{B}}\|\dot\sigma(t)\|^2_{\Sigma^\star}, \end{array}$$

where the constants $C_{\mathcal{R}}$ *and* $C_{\mathcal{B}}$ *are from (19).*

Proof: The estimates for the time derivatives follow from [16] (12) for $q=2$ and the dotted versions of relations (36), ..., (38). ■

4.2 Difference of (EP) and correction of (E)

For another linear, continuous, symmetric and strongly positive operator (the 'elastic' parameter)

$${}^e\mathcal{B}\in\mathcal{L}(\Sigma,\,\Sigma^\star),$$

with norm equivalence

$$\langle\sigma,\tau\rangle_{{}^e\mathcal{B}}=\langle{}^e\mathcal{B}^{-1}\sigma,\tau\rangle,\qquad c_{{}^e\mathcal{B}}\|\cdot\|\le\|\cdot\|_{{}^e\mathcal{B}}\le C_{{}^e\mathcal{B}}\|\cdot\|, \tag{39}$$

and a given initial memory ${}^e\beta_0\in Z$, we decompose

(C) **the input** ${}^e\sigma$ **with the scalar product** $\langle\cdot,\cdot\rangle_{{}^e\mathcal{B}}$, namely in the form

$$\left\{\begin{array}{rcll} \langle{}^e\mathcal{B}^{-1}{}^e\dot\alpha,\,{}^e\beta-*\rangle=\langle{}^e\dot\alpha,\,{}^e\beta-*\rangle_{{}^e\mathcal{B}} & \ge & 0 & \text{for all } *\in Z \quad \text{a.e. in } [0,T] \\ {}^e\alpha+{}^e\beta & = & {}^e\sigma & \text{e. in } [0,T] \\ {}^e\beta(0) & = & {}^e\beta_0 & \end{array}\right. ,$$

i.e. with the definition of stop and play, cf. [16] (2),

$${}^e\alpha=\mathcal{P}_{{}^e\mathcal{B}}({}^e\sigma,{}^e\beta_0),\qquad {}^e\beta=\mathcal{S}_{{}^e\mathcal{B}}({}^e\sigma,{}^e\beta_0), \tag{40}$$

and define the *corrected stresses* by

$$\tilde\alpha=\mathcal{B}{}^e\mathcal{B}^{-1}{}^e\alpha,\qquad\tilde\beta={}^e\beta,\qquad\tilde\sigma=\tilde\alpha+\tilde\beta, \tag{41}$$

the *corrected strains* by

$$\tilde\varepsilon^{pl}={}^e\mathcal{B}^{-1}{}^e\alpha,\qquad\tilde\varepsilon^{el}=\mathcal{C}^{-1}\tilde\sigma,\qquad\tilde\varepsilon=\tilde\varepsilon^{el}+\tilde\varepsilon^{pl}, \tag{42}$$

and the *corrected displacements* by

$$\tilde u=(\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}\mathcal{D}^\star\mathcal{C}\,\tilde\varepsilon. \tag{43}$$

Then we arrive at the following error expressions

$$\Delta\alpha = \tilde\alpha-\alpha = \mathcal{B}{}^e\mathcal{B}^{-1}\mathcal{P}_{{}^e\mathcal{B}}(SF,{}^e\beta_0)-\mathcal{B}\mathcal{R}^{-1}\mathcal{P}_{\mathcal{R}}(SF,\beta_0), \tag{44}$$

$$\Delta\beta = \tilde\beta-\beta = \mathcal{S}_{{}^e\mathcal{B}}(SF,{}^e\beta_0)-\mathcal{S}_{\mathcal{R}}(SF,\beta_0). \tag{45}$$

4.4 Example. For ${}^e\mathcal{B}$ we may e.g. choose, cf. (12),

$$\big\langle {}^e\mathcal{B}\varepsilon, \eta \big\rangle = \int_\Omega \big[{}^eB(x)\varepsilon(x)\big] : \eta(x)\,\mathrm{d}V(x) \qquad (\varepsilon, \eta \in \Sigma)$$

with the assumptions on ${}^eB(x)$ that are analogeously to example 2.2. □

4.5 Remark. The choice ${}^e\mathcal{B} = \mathcal{B}$ yields $\tilde{\sigma} = {}^e\sigma$, $\tilde{\varepsilon} = {}^e\varepsilon$, $\tilde{u} = {}^eu$, i.e. the elastic solution. The choice ${}^e\mathcal{B} = \mathcal{R}$ yields $\tilde{\sigma} = \sigma$, $\tilde{\varepsilon} = \varepsilon$, $\tilde{u} = u$, i.e. the elastoplastic solution. □

We set for abbreviation

$$\varphi_F(t) = \int_0^t \|S\dot{F}(\tau)\|_{\Sigma^\star}\mathrm{d}\tau. \tag{46}$$

4.6 Theorem. *For the correction of (E) we have the following error estimation compared to (EP) with respect to the outer force F.*

1. *Estimates for the* ***stresses.*** *There holds for* $\Delta\alpha$

$$\begin{aligned} \|\Delta\alpha(t)\|_{\Sigma^\star} &\le r_1^b\|\Delta\beta(t)\|_{\Sigma^\star} + q_{{}^e\mathcal{B}}h_{\mathcal{B}}\,\varphi_F^{{}^e\alpha}(t), \\ \|\Delta\alpha(t)\|_{\Sigma^\star} &\le {}^e\!b_1^b\|\Delta\beta(t)\|_{\Sigma^\star} + q_{\mathcal{R}}h_{\mathcal{B}}\,\varphi_F^{\gamma}(t). \end{aligned}$$

There holds for $\Delta\beta$

$$\|\Delta\beta(t)\|_{\Sigma^\star} \le q_{\mathcal{R}}\|\Delta\beta(0)\|_{\Sigma^\star} + q_{{}^e\mathcal{B}}c_{\mathcal{R}}^{-2}h\,\varphi_F(t), \tag{47}$$

$$\|\Delta\beta(t)\|_{\Sigma^\star} \le q_{{}^e\mathcal{B}}\|\Delta\beta(0)\|_{\Sigma^\star} + q_{\mathcal{R}}c_{{}^e\mathcal{B}}^{-2}h\,\varphi_F(t). \tag{48}$$

There holds for $\Delta\sigma$

$$\begin{aligned} \|\Delta\sigma(t)\|_{\Sigma^\star} &\le \big(r_1^b + 1\big)\,\|\Delta\beta(t)\|_{\Sigma^\star} + q_{{}^e\mathcal{B}}h_{\mathcal{B}}\,\varphi_F^{{}^e\alpha}(t), \\ \|\Delta\sigma(t)\|_{\Sigma^\star} &\le \big({}^e\!b_1^b + 1\big)\,\|\Delta\beta(t)\|_{\Sigma^\star} + q_{\mathcal{R}}h_{\mathcal{B}}\,\varphi_F^{\gamma}(t). \end{aligned}$$

2. *Estimates for the* ***strains.*** *There holds for* $\Delta\varepsilon^{pl}$

$$\begin{aligned} \|\Delta\varepsilon^{pl}(t)\|_{\Sigma} &\le r_1\|\Delta\beta(t)\|_{\Sigma^\star} + q_{{}^e\mathcal{B}}h\,\varphi_F^{{}^e\alpha}(t), \\ \|\Delta\varepsilon^{pl}(t)\|_{\Sigma} &\le {}^e\!b_1\|\Delta\beta(t)\|_{\Sigma^\star} + q_{\mathcal{R}}h\,\varphi_F^{\gamma}(t). \end{aligned}$$

There holds for $\Delta\varepsilon^{el}$

$$\begin{aligned} \|\Delta\varepsilon^{el}(t)\|_{\Sigma} &\le c_1\big(r_1^b + 1\big)\,\|\Delta\beta(t)\|_{\Sigma^\star} + c_1q_{{}^e\mathcal{B}}h_{\mathcal{B}}\,\varphi_F^{{}^e\alpha}(t), \\ \|\Delta\varepsilon^{el}(t)\|_{\Sigma} &\le c_1\big({}^e\!b_1^b + 1\big)\,\|\Delta\beta(t)\|_{\Sigma^\star} + c_1q_{\mathcal{R}}h_{\mathcal{B}}\,\varphi_F^{\gamma}(t). \end{aligned}$$

There holds for $\Delta\varepsilon$

$$\begin{aligned} \|\Delta\varepsilon(t)\|_{\Sigma} &\le \big(r_1 + c_1(r_1^b + 1)\big)\|\Delta\beta(t)\|_{\Sigma^\star} + q_{{}^e\mathcal{B}}\big(h + c_1h_{\mathcal{B}}\big)\,\varphi_F^{{}^e\alpha}(t), \\ \|\Delta\varepsilon(t)\|_{\Sigma} &\le \big({}^e\!b_1 + c_1({}^e\!b_1^b + 1)\big)\|\Delta\beta(t)\|_{\Sigma^\star} + q_{\mathcal{R}}\big(h + c_1h_{\mathcal{B}}\big)\,\varphi_F^{\gamma}(t). \end{aligned}$$

3. *Estimates for the* ***displacement.*** *There holds for* Δu

$$\begin{aligned} \|\Delta u(t)\|_{U} &\le d\big(r_1 + c_1(r_1^b + 1)\big)\|\Delta\beta(t)\|_{\Sigma^\star} + q_{{}^e\mathcal{B}}d\big(h + c_1h_{\mathcal{B}}\big)\,\varphi_F^{{}^e\alpha}(t), \\ \|\Delta u(t)\|_{U} &\le d\big({}^e\!b_1 + c_1({}^e\!b_1^b + 1)\big)\|\Delta\beta(t)\|_{\Sigma^\star} + q_{\mathcal{R}}d\big(h + c_1h_{\mathcal{B}}\big)\,\varphi_F^{\gamma}(t). \end{aligned}$$

4. *Estimates for* ${}^e\alpha - \gamma$. *There holds*

$${}^e\alpha - \gamma \;=\; \beta - {}^e\beta, \qquad \|({}^e\alpha - \gamma)(t)\|_{\Sigma^\star} = \|\Delta\beta(t)\|_{\Sigma^\star}, \tag{49}$$

so that (47) and (48) yield the same estimates.

Here $\Delta\cdot\ =\ \tilde{\cdot} - \cdot$ *and*

$$\varphi_F^{{}^e\alpha}(t) = \|{}^e\alpha(0)\|_{\Sigma^\star} + \varphi_F(t), \qquad \varphi_F^{\gamma}(t) = \|\gamma(0)\|_{\Sigma^\star} + \varphi_F(t),$$

with the constants

$$q_{\mathcal{R}} = \frac{C_{\mathcal{R}}}{c_{\mathcal{R}}}, \qquad q_{{}^e\mathcal{B}} = \frac{C_{{}^e\mathcal{B}}}{c_{{}^e\mathcal{B}}}, \qquad d = \left\|(\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}\mathcal{D}^\star\mathcal{C}\right\|_{U,\Sigma}.$$

Further

$$c_1 = \left\|\mathcal{C}^{-1}\right\|_{\Sigma,\Sigma^\star}, \qquad {}^e b_1 = \left\|{}^e\mathcal{B}^{-1}\right\|_{\Sigma,\Sigma^\star}, \qquad r_1 = \left\|\mathcal{R}^{-1}\right\|_{\Sigma,\Sigma^\star}$$

and

$$r_1^b = \left\|\mathcal{B}\mathcal{R}^{-1}\right\|_{\Sigma^\star,\Sigma^\star}, \quad {}^e b_1^b = \left\|\mathcal{B}{}^e\mathcal{B}^{-1}\right\|_{\Sigma^\star,\Sigma^\star}.$$

The operator differences

$$h = \left\|{}^e\mathcal{B}^{-1} - \mathcal{R}^{-1}\right\|_{\Sigma,\Sigma^\star}, \qquad h_{\mathcal{B}} = \left\|\mathcal{B}({}^e\mathcal{B}^{-1} - \mathcal{R}^{-1})\right\|_{\Sigma^\star,\Sigma^\star}$$

are 'small'. All estimates are valid e. in $[0, T]$.

Proof: We switch the order, starting with the basic estimate for $\Delta\beta$.

- For the estimates of $\Delta\beta$, we just have to apply corollaries [16] 3.2 (a) and 3.3 with

  (a) the inputs
  $$f_1 = f_2 = SF\big(= {}^e\sigma\big),$$

  (b) the stops
  $$s_1 = {}^e\beta, \quad s_2 = \beta,$$

  (c) the plays
  $$p_1 = {}^e\alpha, \quad p_2 = \gamma,$$

  (d) the initial memories
  $$s_{0,1} = {}^e\beta_0, \quad s_{0,2} = \beta_0,$$

  (e) the scalar products
  $$\big\langle\cdot,\cdot\big\rangle_{A_1} = \big\langle\cdot,\cdot\big\rangle_{{}^e\mathcal{B}}, \quad \big\langle\cdot,\cdot\big\rangle_{A_2} = \big\langle\cdot,\cdot\big\rangle_{\mathcal{R}},$$

  and the estimates (39) and (18). So we easily find (47), (48).

- Assertion (49) is clear, as both ${}^e\alpha - \gamma$ and $\beta - {}^e\beta$ add up to SF.

- There holds
  $$\Delta\alpha = \mathcal{B}\Delta\varepsilon^{pl} = \mathcal{B}\big({}^e\mathcal{B}^{-1}{}^e\alpha - \mathcal{R}^{-1}\gamma\big) \tag{50}$$

  In view of (49), we receive – smuggling appropriate zeros –

  $$\begin{aligned} \|\Delta\alpha(t)\|_{\Sigma^\star} &\leq \|\mathcal{B}\mathcal{R}^{-1}\Delta\beta(t)\|_{\Sigma^\star} + \|\mathcal{B}({}^e\mathcal{B}^{-1} - \mathcal{R}^{-1}){}^e\alpha(t)\|_{\Sigma^\star} && (51)\\ \|\Delta\alpha(t)\|_{\Sigma^\star} &\leq \|\mathcal{B}{}^e\mathcal{B}^{-1}\Delta\beta(t)\|_{\Sigma^\star} + \|\mathcal{B}({}^e\mathcal{B}^{-1} - \mathcal{R}^{-1})\gamma(t)\|_{\Sigma^\star} && (52)\end{aligned}$$

  The second summand in (51) resp. (52) is estimated with [16] (13) and (39) resp. (19) in the usual way.

- For $\Delta\varepsilon^{pl} = {}^e\mathcal{B}^{-1}{}^e\alpha - \mathcal{R}^{-1}\gamma$ like for $\Delta\alpha$, but without $\mathcal{B}$, cf. (50).

- Estimates for the remaining corrected quantities can straightforwardly by derived via
$$\Delta\sigma = \Delta\alpha + \Delta\beta$$
and
$$\Delta\varepsilon^{el} = \mathcal{C}^{-1}\Delta\sigma, \qquad \Delta\varepsilon = \Delta\varepsilon^{el} + \Delta\varepsilon^{pl}$$
and
$$\Delta u = (\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}\mathcal{D}^\star\mathcal{C}\,\Delta\varepsilon,$$
using the estimates for $\Delta\beta$, $\Delta\alpha$ and $\Delta\varepsilon^{pl}$.

Everything is proved. ■

4.7 Corollary. *If the initial values are chosen in a consistent manner, i.e. $\beta_0 = {}^e\beta_0$, and the material is virgin, i.e. $\varepsilon^{pl}(0) = 0$, there exist positive constants – dependencies in brackets – with*

1. *Estimates for the* ***stresses.*** *There holds*
$$\begin{aligned}\|\Delta\alpha(t)\|_{\Sigma^\star} &\leq \big(c_\alpha({}^e\mathcal{B},\mathcal{R},\mathcal{B})h + k_\alpha({}^e\mathcal{B},\mathcal{R})h_{\mathcal{B}}\big)\,\varphi_F(t),\\ \|\Delta\beta(t)\|_{\Sigma^\star} &\leq c_\beta({}^e\mathcal{B},\mathcal{R})h\,\varphi_F(t),\\ \|\Delta\sigma(t)\|_{\Sigma^\star} &\leq \big(c_\sigma({}^e\mathcal{B},\mathcal{R},\mathcal{B})h + k_\sigma({}^e\mathcal{B},\mathcal{R})h_{\mathcal{B}}\big)\,\varphi_F(t).\end{aligned}$$
2. *Estimates for the* ***strains.*** *There holds*
$$\begin{aligned}\|\Delta\varepsilon^{pl}(t)\|_{\Sigma} &\leq c_{\varepsilon^{pl}}({}^e\mathcal{B},\mathcal{R})h\,\varphi_F(t),\\ \|\Delta\varepsilon^{el}(t)\|_{\Sigma} &\leq \big(c_{\varepsilon^{el}}({}^e\mathcal{B},\mathcal{R},\mathcal{B},\mathcal{C})h + k_{\varepsilon^{el}}({}^e\mathcal{B},\mathcal{R},\mathcal{C})h_{\mathcal{B}}\big)\,\varphi_F(t),\\ \|\Delta\varepsilon(t)\|_{\Sigma} &\leq \big(c_\varepsilon({}^e\mathcal{B},\mathcal{R},\mathcal{B},\mathcal{C})h + k_\varepsilon({}^e\mathcal{B},\mathcal{R},\mathcal{C})h_{\mathcal{B}}\big)\,\varphi_F(t).\end{aligned}$$
3. *Estimates for the* ***displacement.*** *There holds*
$$\|\Delta u(t)\|_U \leq \big(c_u({}^e\mathcal{B},\mathcal{R},\mathcal{B},\mathcal{C},\mathcal{D})h + k_u({}^e\mathcal{B},\mathcal{R},\mathcal{C},\mathcal{D})h_{\mathcal{B}}\big)\,\varphi_F(t).$$
4. *Estimates for ${}^e\alpha - \gamma$. There holds*
$$\|({}^e\alpha - \gamma)(t)\|_{\Sigma^\star} \leq c_\beta({}^e\mathcal{B},\mathcal{R})h\,\varphi_F(t)$$

Here again $\Delta\cdot = \tilde{\cdot} - \cdot$ and $\varphi_F(t)$ from (46). All estimates are valid e. in $[0,T]$.

Proof: We have
$$\beta_0 = {}^e\beta_0 \qquad \big(\Longrightarrow \Delta\beta(0) = 0\big)$$
and
$$\varepsilon^{pl}(0) = 0 \qquad \Big(\overset{(28)}{\Longrightarrow} \gamma(0) \overset{(49)}{=} {}^e\alpha(0) = 0\Big).$$
Therefore $\varphi_F \equiv \varphi_F^\gamma \equiv \varphi_F^{{}^e\alpha}$. From (47), (48) we infer – arithmetically averaging –
$$\|\Delta\beta(t)\|_{\Sigma^\star} \leq \frac{1}{2}\Big(q_{\mathcal{R}}c_{{}^e\mathcal{B}}^{-2} + q_{{}^e\mathcal{B}}c_{\mathcal{R}}^{-2}\Big)h\,\varphi_F(t) \;=:\; c_\beta({}^e\mathcal{B},\mathcal{R})h\,\varphi_F(t)$$
Inserting this into the other error expressions of theorem 4.6, appropriate reaggang-ing and taking the average of each pair finally gives the desired results. ■

We have a counterpart to theorems 3.7 and 4.3. In fact – as the choice ${}^eB = R$ shows – it generalises 4.3. The fact that $SF = {}^e\sigma$ is *common* input into (30), (40) is essential.

4.8 Theorem. *There exist constants, such that for each outer force*

$$F \in W^{1,2}([0,T], U^\star)$$

there holds

$$\|{}^e\dot{\alpha}\|_{\Sigma^\star} \le C_{{}^e\alpha}\|S\dot{F}\|_{\Sigma^\star}, \qquad \|{}^e\dot{\beta}\|_{\Sigma^\star} \le C_{{}^e\beta}\|S\dot{F}\|_{\Sigma^\star}, \tag{53}$$

(for the 'elastic' quantities)

$$\begin{array}{rclrcl} \|d_t\tilde{\varepsilon}^{pl}\|_\Sigma & \le & C_{\varepsilon^{pl}}\|S\dot{F}\|_{\Sigma^\star}, & \|\Delta\dot{\varepsilon}^{pl}\|_\Sigma & \le & k_{\varepsilon^{pl}}\|S\dot{F}\|_{\Sigma^\star}, \\ \|d_t\tilde{\varepsilon}^{el}\|_\Sigma & \le & C_{\varepsilon^{el}}\|S\dot{F}\|_{\Sigma^\star}, & \|\Delta\dot{\varepsilon}^{el}\|_\Sigma & \le & k_{\varepsilon^{el}}\|S\dot{F}\|_{\Sigma^\star}, \\ \|d_t\tilde{\varepsilon}\|_\Sigma & \le & C_{\varepsilon}\|S\dot{F}\|_{\Sigma^\star}, & \|\Delta\dot{\varepsilon}\|_\Sigma & \le & k_{\varepsilon}\|S\dot{F}\|_{\Sigma^\star}, \end{array} \tag{54}$$

(for the corrected strains)

$$\begin{array}{rclrcl} \|d_t\tilde{\alpha}\|_{\Sigma^\star} & \le & C_{\alpha}\|S\dot{F}\|_{\Sigma^\star}, & \|\Delta\dot{\alpha}\|_\Sigma & \le & k_{\alpha}\|S\dot{F}\|_{\Sigma^\star}, \\ \|d_t\tilde{\beta}\|_{\Sigma^\star} & \le & C_{\beta}\|S\dot{F}\|_{\Sigma^\star}, & \|\Delta\dot{\beta}\|_\Sigma & \le & k_{\beta}\|S\dot{F}\|_{\Sigma^\star}, \\ \|d_t\tilde{\sigma}\|_{\Sigma^\star} & \le & C_{\sigma}\|S\dot{F}\|_{\Sigma^\star}, & \|\Delta\dot{\sigma}\|_\Sigma & \le & k_{\sigma}\|S\dot{F}\|_{\Sigma^\star}, \end{array} \tag{55}$$

(for the corrected stresses)

$$\|d_t\tilde{u}\|_U \;\le\; C_u\|S\dot{F}\|_{\Sigma^\star}, \qquad \|\Delta\dot{u}\|_\Sigma \;\le\; k_u\|S\dot{F}\|_{\Sigma^\star} \tag{56}$$

(for the corrected displacement). All estimates are valid a.e. in $[0,T]$.

Proof: We have

$$\|{}^e\dot{\beta}\|_{{}^e\mathcal{B}} \le \|S\dot{F}\|_{{}^e\mathcal{B}} \le C_{{}^e\mathcal{B}}\|S\dot{F}\|_{\Sigma^\star}, \quad \|{}^e\dot{\alpha}\|_{{}^e\mathcal{B}} \le \|S\dot{F}\|_{{}^e\mathcal{B}} \le C_{{}^e\mathcal{B}}\|S\dot{F}\|_{\Sigma^\star} \quad \text{a.e. in } [0,T]$$

from (40), (16) and [16] (12), where $C_{{}^e\mathcal{B}}$ from (39) in order to receive (53). Now apply relations (41), ..., (43) and the triangle inequality to obtain all the left-hand side estimates in (54), ..., (56).
The right-hand side estimates follow from the left-hand ones with the aid of relations (33), ..., (35) and the 'abused' triangle inequality $\|\cdot - \cdot\| \le \|\cdot\| + \|\cdot\|$. ∎

4.9 Corollary (Model differences IV) *If the initial values are chosen in a consistent manner and the material is virgin there exist positive constants – dependencies in brackets – with*

$$\begin{array}{rcl} \|({}^e\sigma - \sigma)(t)\|_{\Sigma^\star} & \le & \big(c_\sigma(\mathcal{R},\mathcal{B})h + k_\sigma(\mathcal{B},\mathcal{R})h_\mathcal{B}\big)\,\varphi_F(t). \\ \|({}^e\varepsilon - \varepsilon)(t)\|_{\Sigma} & \le & \big(c_\varepsilon(\mathcal{R},\mathcal{B},\mathcal{C})h + k_\varepsilon(\mathcal{B},\mathcal{R},\mathcal{C})h_\mathcal{B}\big)\,\varphi_F(t). \\ \|({}^e u - u)(t)\|_{U} & \le & \big(c_u(\mathcal{R},\mathcal{B},\mathcal{C},\mathcal{D})h + k_u(\mathcal{B},\mathcal{R},\mathcal{C},\mathcal{D})h_\mathcal{B}\big)\,\varphi_F(t). \end{array}$$

Here again $\varphi_F(t)$ *from (46) and*

$$h = \|\mathcal{B}^{-1} - \mathcal{R}^{-1}\|_{\Sigma,\Sigma^\star}, \qquad h_\mathcal{B} = \|I - \mathcal{B}\mathcal{R}^{-1}\|_{\Sigma^\star,\Sigma^\star}.$$

All estimates are valid e. in $[0,T]$.

Proof: Let ${}^e\mathcal{B} = \mathcal{B}$ in theorems 4.6 and 4.7. ∎

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