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Easy Differentiation and Integration of Homogeneous Harmonic Polynomials

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Easy Differentiation and Integration of Homogeneous Harmonic Polynomials

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Abstract

We will give explicit differentiation and integration rules for homogeneous harmonic polynomial polynomials and spherical harmonics in $\mathbb{R}^3$ with respect to the following differential operators: ∂_1, ∂_2, ∂_3, $x_3\partial_2 - x_2\partial_3$, $x_3\partial_1 - x_1\partial_3$, $x_2\partial_1 - x_1\partial_2$ and $x_1\partial_1 + x_2\partial_2 + x_3\partial_3$.

A numerical application to the problem of determining the geopotential field will be shown.

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1 Introduction

In the field of gravity determination a special kind of boundary value problem respectively ill-posed satellite problem occurs; the data and hence side condition of our PDE are oblique second order derivatives of the gravitational potential. One is able to show [3] that the only first order derivatives D which obey

$$\Delta v = 0 \qquad \Rightarrow \qquad \Delta \mathsf{D} v = 0$$

are linear combinations of the following ones:

$$\begin{aligned}
&\mathsf{D}_{id} = 1 \\
&\mathsf{D}_{x_1} = \partial_1\,, \qquad \mathsf{D}_{x_2} = \partial_2\,, \qquad \mathsf{D}_{x_3} = \partial_3 \\
&\mathsf{D}_{\neg x_1} = x_3\partial_2 - x_2\partial_3\,, \qquad \mathsf{D}_{\neg x_2} = x_3\partial_1 - x_1\partial_3\,, \mathsf{D}_{\neg x_3} = x_2\partial_1 - x_1\partial_2 \\
&\mathsf{D}_r = x_1\partial_1 + x_2\partial_2 + x_3\partial_3\,,
\end{aligned}$$

Additionally one can show that all purely second order differential operators which fulfill the above equation are combinations of the above operators.

Therefore it is interesting to know how one can efficiently compute these derivatives and their inversions when given a basis of homogeneous harmonic polynomials or spherical harmonics, respectively.

2 Preliminaries

We will use the following results of potential theory taken from [2, 6, 8, 10, 11] and therefore want to shortly cite them without proof:

2.1 Harmonics and Kelvin Transform

Definition 2.1 (Harmonics)

Take a smooth regular surface Σ in $\mathbb{R}^3 \cup \{\infty\}$ which divides $\mathbb{R}^3$ in a bounded part Σ_{int} and an unbounded part Σ_{ext}, i.e., $\infty \in \Sigma_{ext}$, where each part is assumed to be path connected. I.e., we have $\mathbb{R}^3 = \Sigma_{int} \cup^* \Sigma \cup^* \Sigma_{ext}$, where $\cup^*$ denotes the disjoint union. Without loss of generality assume $0 \in \Sigma_{int}$. Furthermore Σ should be a smooth surface.
All functions $v : \overline{\Sigma_{int}} \to \mathbb{R}$ which fulfill

$$\begin{aligned}
&v \in C^{(\infty)}(\Sigma_{int}) \cap C^{(0)}(\overline{\Sigma_{int}}) \\
&(\Delta v)|_{\Sigma_{int}} = 0
\end{aligned}$$

constitute the space $Pot(\overline{\Sigma_{int}})$.
All functions $v : \overline{\Sigma_{ext}} \to \mathbb{R}$ which fulfill

$$\begin{aligned}
&v \in C^{(\infty)}(\Sigma_{ext}) \cap C^{(0)}(\overline{\Sigma_{ext}}) \\
&|v(x)| = O(||x||^{-1}) \\
&(\Delta v)|_{\Sigma_{ext}} = 0
\end{aligned}$$

constitute the space $Pot(\overline{\Sigma_{ext}})$.

Definition 2.2 (Kelvin Transform)

Let $\{0\} \subset S \subset \mathbb{R}^3 \cup \{\infty\}$*. Define* $K(S) = \left\{ x \in \mathbb{R}^3 \cup \{\infty\} \,\middle|\, \frac{x}{|x|^2} \in S \right\}$*. Furthermore, if* $v \in Pot(S)$ *define*

$$\breve{v} = K(v)\,(x) = \frac{1}{|x|}\, v\!\left(\frac{x}{|x|^2}\right)$$

The operator K *is called Kelvin Transform.*

Lemma 2.1 (Kelvin Transform)

We have the following two properties (Id *is the identity operator):*

- $K(K(\cdot)) = \mathrm{Id}$
- $K(Pot(S)) = Pot(K(S))$

Please note that in the above lemma S and $K(S)$ have two different surfaces.

2.2 Homogeneous Harmonic Polynomials

Definition 2.3 (Homogeneous Harmonic Polynomials)

A polynomial H_n *is called homogeneous of degree* n *if it fulfills*

$$H_n(x) = ||x||^n H_n\left(\frac{x}{||x||}\right) \qquad \forall x \in \mathbb{R}^3 \setminus \{0\}$$

If furthermore $H_n \in Pot(\overline{\Sigma_{int}})$ *it is called homogeneous harmonic polynomial of degree* n*, the corresponding space of all such polynomials is called* $Pot_n(\overline{\Sigma_{int}})$*.*

Lemma 2.2

Any homogeneous harmonic polynomial H_n *of degree* n *can be represented in the form*

$$H_n(x) = H_n(x_1, x_2, x_3) = \sum_{j=0}^{n} a_{n-j}(x_1, x_2)\, x_3^j$$

where a_k *denotes a homogeneous polynomial of degree* k *in the variables* x_1 *and* x_2*. Furthermore, the* $a_{\square}$ *fulfill the following relation*

$$a_{n-j-2}(x_1, x_2) = -\frac{1}{(j+1)\,(j+2)}\,(\partial_1\partial_1 + \partial_2\partial_2)\, a_{n-j}(x_1, x_2)$$

The dimension of the space of homogeneous harmonic polynomials of degree n *is* $2n + 1 = \dim(Pot_n(\overline{\Sigma_{int}}))$*.*
Any harmonic polynomial can be written in terms of convergent series of homogeneous harmonic polynomials.

Remark

Hence homogeneous harmonic polynomials are fully determined by their $x_1^k x_2^{n-k-i} x_3^i$ part, where $i \in \{0,1\}$ and $0 \le k \le n$.

So the following basis of the space of harmonic functions is well-defined.

Definition 2.4

Define $H_n \in Pot_n(\overline{\Sigma_{int}})$ *and* $\breve{H}_n = K(H_n)$ *as its Kelvin transformed counterpart.*

Let furthermore ${}^iH_n^k$ *denote the homogeneous harmonic polynomial with the leading term* $x_1^k x_2^{n-k-i} x_3^i$*, where* $n \ge k \ge 0$ *and* $i \in \{0,1\}$*.*

Again denote its Kelvin transformed counterpart by ${}^i\breve{H}_n^k = K\left({}^iH_n^k\right)$*.*

In order to keep our notation simple we will assume that ${}^i\breve{H}_n^k = {}^iH_n^k = 0$ *if* $k < 0$ *or* $k > n - i$*.*

Thus the ${}^iH_n^k$ constitute a $L^2(\Sigma)$-complete basis of $Pot(\overline{\Sigma_{int}})$ and the Kelvin transformed counterparts ${}^i\breve{H}_n^k$ constitute a $L^2(\Sigma)$-complete basis of $Pot(\overline{\Sigma_{ext}})$.

Remark

Neither the ${}^iH_n^k$ are an orthonormal basis for $Pot(\overline{\Sigma_{int}})$ nor the ${}^i\breve{H}_n^k$ for $Pot(\overline{\Sigma_{ext}})$ (with respect to the standard $L^2(\Sigma)$ inner product).

Lemma 2.3

Let H_n *be a homogeneous harmonic polynomial of the degree* n*. The Kelvin transformed spherical harmonic* $K(H_n) = \breve{H}_n$ *is given by*

$$K(H_n)(x) = \frac{1}{|x|^{2n+1}} H_n(x)$$

2.3 Kelvin Transform and Derivatives

Now we want to give a transition formula which allows to convert relations we found for ordinary homogeneous harmonic polynomials H_n to their Kelvin transformed counterpart $\breve{H}_n$.

Lemma 2.4

$$\partial_i \breve{H}_n(x) = \frac{\left(x_1^2 + x_2^2 + x_3^2\right) \partial_i H_n(x) - (2n+1)\, x_i H_n(x)}{|x|^{2n+3}}$$

Proof

$$\begin{aligned}
\partial_i \breve{H}_n(x) &= \partial_i \frac{H_n(x)}{|x|^{2n+1}} \\
&= \frac{|x|^{2n+1} \partial_i H_n(x) - H_n(x)\, \partial_i |x|^{2n+1}}{|x|^{4n+2}} \\
&= \frac{|x|^{2n+1} \partial_i H_n(x) - (2n+1)\, |x|^{2n} \frac{x_i}{|x|} H_n(x)}{|x|^{4n+2}} \\
&= \frac{\left(x_1^2 + x_2^2 + x_3^2\right) \partial_i H_n(x) - (2n+1)\, x_i H_n(x)}{|x|^{2n+3}}
\end{aligned}$$

q.e.d.

3 Homogeneous Harmonic Polynomials

We will strongly use two facts for our computations without pointing out every time when we apply them.

- Applying any of the above differential operators to a harmonic function leaves the function harmonic
- In order to categorize a harmonic polynomial we just need to consider the part consisting of $x_1^k x_2^{n-k-i} x_3^i$, where $n \geq k \geq 0$ and $i \in \{0, 1\}$.

3.1 Differential Operator D_{x_i}

3.1.1 Ordinary Homogeneous Harmonic Polynomials

Lemma 3.1

$$\begin{aligned}
\mathsf{D}_{x_1}\, {}^iH_n^k &= k\, {}^iH_{n-1}^{k-1} \\
\mathsf{D}_{x_2}\, {}^iH_n^k &= (n-k-i)\, {}^iH_{n-1}^{k} \\
\mathsf{D}_{x_3}\, {}^1H_n^k &= {}^0H_{n-1}^{k} \\
\mathsf{D}_{x_3}\, {}^0H_n^k &= -k\,(k-1)\, {}^1H_{n-1}^{k-2} - (n-k)\,(n-k-1)\, {}^1H_{n-1}^{k}
\end{aligned}$$

Proof

In the sequel we will use $p(x)$ and $q(x)$ for arbitrary polynomials in x_1, x_2 and x_3. In particular, $p(x)$ and $q(x)$ are different in every equation. Furthermore we define $\partial_i x_i^0 = x_i^{-1} = 0 = x_i^{-2} = \partial_i \partial_i x_i^0$ for the sake of simpler notation. Hence we get

$$\begin{aligned}
\mathsf{D}_{x_1}\, {}^iH_n^k =& \partial_1\, {}^iH_n^k \\
=& \partial_1 \left(x_1^k x_2^{n-k-i} x_3^i + x_3^2 p(x) \right) \\
=& k x_1^{k-1} x_2^{n-k-i} x_3^i + x_3^2 q(x) \\
=& k\, {}^iH_{n-1}^{k-1}
\end{aligned}$$

$$\begin{aligned}
\mathsf{D}_{x_2}\, {}^iH_n^k =& \partial_2\, {}^iH_n^k \\
=& \partial_2 \left(x_1^k x_2^{n-k-i} x_3^i + x_3^2 p(x) \right) \\
=& (n-k-i)\, x_1^k x_2^{n-k-i-1} x_3^i + x_3^2 q(x) \\
=& (n-k-i)\, {}^iH_{n-1}^{k}
\end{aligned}$$

$$\begin{aligned}
\mathsf{D}_{x_3}\, {}^1H_n^k =& \partial_3\, {}^1H_n^k \\
=& \partial_3 \left(x_1^k x_2^{n-k-1} x_3 + x_3^3 p(x) \right) \\
=& x_1^k x_2^{n-k-1} + x_3^2 q(x) \\
=& {}^0H_{n-1}^{k}
\end{aligned}$$

The only difficult part is while regarding ${}^{0}H_n^k$. We have

$$\begin{aligned}{}^{0}H_n^k =& x_1^k x_2^{n-k} - \frac{1}{2}\left(\partial_1^2 + \partial_2^2\right)\left(x_1^k x_2^{n-k}\right) x_3^2 + \left(p(x)\right) x_3^4 \\ =& x_1^k x_2^{n-k} + \left(p(x)\right) x_3^4 \\ &- \frac{1}{2}\left(k\left(k-1\right) x_1^{k-2} x_2^{n-k} + \left(n-k\right)\left(n-k-1\right) x_1^k x_2^{n-k-2}\right) x_3^2\end{aligned}$$

Thus we have:

$$\begin{aligned}\mathsf{D}_{x_3}\, {}^{0}H_n^k =& \partial_3\, {}^{0}H_n^k \\ =& q(x)\, x_3^3 \\ &- \left(k\left(k-1\right) x_1^{k-2} x_2^{n-k} + \left(n-k\right)\left(n-k-1\right) x_1^k x_2^{n-k-2}\right) x_3 \\ =& -k\left(k-1\right) {}^{1}H_{n-1}^{k-2} - \left(n-k\right)\left(n-k-1\right) {}^{1}H_{n-1}^k\end{aligned}$$

This proves our lemma. q.e.d.

Remark

The resulting systems of linear equations are underdetermined because we have

$$\begin{aligned}&\mathsf{D}_{x_1}\, {}^{i}H_n^0 = 0 \\ &\mathsf{D}_{x_2}\, {}^{i}H_n^{n-i} = 0\end{aligned}$$

Of course this fact also holds for D_{x_3} and mixed derivatives of such kind. The most easy way to see that is using a standard basis transformation which maps our differential to D_{x_1}.

3.1.2 Kelvin Transformed Homogeneous Harmonic Polynomials

Lemma 3.2

$$\begin{aligned}\mathsf{D}_{x_1}\, {}^{i}\breve{H}_n^k =& k\, {}^{i}\breve{H}_{n+1}^{k-1} - \left(2n+1-k\right) {}^{i}\breve{H}_{n+1}^{k+1} \\ \mathsf{D}_{x_2}\, {}^{i}\breve{H}_n^k =& -\left(n+k+i+1\right) {}^{i}\breve{H}_{n+1}^{k} + \left(n-k-i\right) {}^{i}\breve{H}_{n+1}^{k+2} \\ \mathsf{D}_{x_3}\, {}^{1}\breve{H}_n^k =& {}^{0}\breve{H}_{n+1}^{k} + {}^{0}\breve{H}_{n+1}^{k+2} \\ \mathsf{D}_{x_3}\, {}^{0}\breve{H}_n^k =& -k\left(k-1\right) {}^{1}\breve{H}_{n+1}^{k-2} - \left(n-k\right)\left(n-k-1\right) {}^{1}\breve{H}_{n-1}^{k+2} \\ &- \left(k\left(k-1\right) + \left(n-k\right)\left(n-k-1\right) + \left(2n+1\right)\right) {}^{1}\breve{H}_{n-1}^{k}\end{aligned}$$

Proof

As stated beforehand we have a formula which allows us to write the derivatives of the Kelvin transformed homogeneous polynomials in terms of the derivative of the original polynomial. Among others we will use this fact strongly.

Again we will denote polynomials we do not need with $p(x)$ and $q(x)$ which are different in each equation.
First we will consider the case $k=0$ for the first equation. Then we get

$$\begin{aligned}
\mathsf{D}_{x_1}\,{}^i\breve{H}_n^0 =& \partial_1\,{}^i\breve{H}_n^0\\
=& \frac{\left(x_1^2+x_2^2+x_3^2\right)\partial_1\,{}^iH_n^0-(2n+1)\,x_1\,{}^iH_n^0}{|x|^{2n+3}}\\
=& \frac{-(2n+1)\,x_1\,{}^iH_n^0}{|x|^{2n+3}}\\
=& -(2n+1)\,\frac{\left(x_1x_2^{n-i}x_3^i+x_3^2p(x)\right)}{|x|^{2n+3}}\\
=& -(2n+1)\,\frac{{}^iH_{n+1}^1}{|x|^{2(n+1)+1}}\\
=& -(2n+1)\,{}^i\breve{H}_{n+1}^1\\
=& k\,{}^i\breve{H}_{n+1}^{k-1}-(2n+1-k)\,{}^i\breve{H}_{n+1}^{k+1}
\end{aligned}$$

Now we will consider the first equation for $k>0$. In particular we have $\partial_1\,{}^iH_n^k\neq 0$

$$\begin{aligned}
\mathsf{D}_{x_1}\,{}^i\breve{H}_n^k =& \partial_1\,{}^i\breve{H}_n^k\\
=& \frac{\left(x_1^2+x_2^2+x_3^2\right)\partial_1\,{}^iH_n^k-(2n+1)\,x_1\,{}^iH_n^k}{|x|^{2n+3}}\\
=& \frac{\left(x_1^2+x_2^2+x_3^2\right)k\,{}^iH_{n-1}^{k-1}-(2n+1)\,x_1\,{}^iH_n^k}{|x|^{2n+3}}\\
=& \frac{kx_1^{k-1}x_2^{n-k+2-i}x_3^i-(2n+1-k)\,x_1^{k+1}x_2^{n-k-i}x_3^i+x_3^2p(x)}{|x|^{2n+3}}\\
=& \frac{k\,{}^iH_{n+1}^{k-1}-(2n+1-k)\,{}^iH_{n+1}^{k+1}}{|x|^{2(n+1)+1}}\\
=& k\,{}^i\breve{H}_{n+1}^{k-1}-(2n+1-k)\,{}^i\breve{H}_{n+1}^{k+1}
\end{aligned}$$

Now we will prove the second equation. First consider the case $k=n-i$:

$$\begin{aligned}
\mathsf{D}_{x_2}\,{}^i\breve{H}_n^{n-i} =& \partial_2\,{}^i\breve{H}_n^{n-i}\\
=& \frac{\left(x_1^2+x_2^2+x_3^2\right)\partial_2\,{}^iH_n^{n-i}-(2n+1)\,x_2\,{}^iH_n^{n-i}}{|x|^{2n+3}}\\
=& \frac{-(2n+1)\,x_2\,{}^iH_n^{n-i}}{|x|^{2n+3}}\\
=& -(2n+1)\,\frac{x_1^{n-i}x_2^1x_3^i+x_3^2p(x)}{|x|^{2n+3}}\\
=& -(2n+1)\,\frac{{}^iH_{n+1}^{n-i}}{|x|^{2(n+1)+1}}\\
=& -(2n+1)\,{}^i\breve{H}_{n+1}^{n-i}\\
=& -(n+k+i+1)\,{}^i\breve{H}_{n+1}^{k}+(n-k-i)\,{}^i\breve{H}_{n+1}^{k+2}
\end{aligned}$$

Now we will assume $k < n - i$. Hence we get:

$$\begin{aligned}
\mathsf{D}_{x_2}\, {}^i\breve{H}_n^k =& \partial_2\, {}^i\breve{H}_n^k \\
=& \frac{\left(x_1^2 + x_2^2 + x_3^2\right) \partial_2\, {}^iH_n^k - (2n+1)\, x_2\, {}^iH_n^k}{|x|^{2n+3}} \\
=& \frac{\left(x_1^2 + x_2^2 + x_3^2\right) (n-k-i)\ {}^iH_{n-1}^k - (2n+1)\, x_2\, {}^iH_n^k}{|x|^{2n+3}} \\
=& \frac{(n-k-i)\, x_1^{k+2} x_2^{n-i-1-k} x_3^i}{|x|^{2n+3}} \\
& \quad - \frac{(n+1+k+i)\, x_1^k x_2^{n+1-k-i} x_3^i + x_3^2 p(x)}{|x|^{2n+3}} \\
=& \frac{(n-k-i)\ {}^iH_{n+1}^{k+2} - (n+k+i+1)\ {}^iH_{n+1}^k}{|x|^{2(n+1)+1}} \\
=& -(n+k+i+1)\ {}^i\breve{H}_{n+1}^k + (n-k-i)\ {}^i\breve{H}_{n+1}^{k+2}
\end{aligned}$$

The next equation will be considered the same way.

$$\begin{aligned}
\mathsf{D}_{x_3}\, {}^1\breve{H}_n^k =& \partial_3\, {}^1\breve{H}_n^k \\
=& \frac{\left(x_1^2 + x_2^2 + x_3^2\right) \partial_3\, {}^1H_n^k - (2n+1)\, x_3\, {}^1H_n^k}{|x|^{2n+3}} \\
=& \frac{\left(x_1^2 + x_2^2 + x_3^2\right)\ {}^0H_{n-1}^k - (2n+1)\, x_3\, {}^1H_n^k}{|x|^{2n+3}} \\
=& \frac{\left(x_1^2 + x_2^2 + x_3^2\right) x_1^k x_2^{n-k-1} - (2n+1)\, x_1^k x_2^{n-k-1} x_3^2 + x_3^2 p(x)}{|x|^{2n+3}} \\
=& \frac{x_1^{k+2} x_2^{n-k-1} + x_1^k x_2^{n-k+1} + x_3^2 q(x)}{|x|^{2n+3}} \\
=& \frac{{}^0H_{n+1}^{k+2} + {}^0H_{n+1}^k}{|x|^{2(n+1)+1}} \\
=& {}^0\breve{H}_{n+1}^k + {}^0\breve{H}_{n+1}^{k+2}
\end{aligned}$$

The last equation is also the most complicated one. The special cases $k < 2$ and $k > n - 2$ should be treated separately. However, as it works the same way we will skip this step end immediately introduce the main case.

$$\begin{aligned}
\mathsf{D}_{x_3}\, {}^0\breve{H}_n^k =& \partial_3\, {}^0\breve{H}_n^k \\
=& \frac{\left(x_1^2 + x_2^2 + x_3^2\right) \partial_3\, {}^0H_n^k - (2n+1)\, x_3\, {}^0H_n^k}{|x|^{2n+3}} \\
=& \frac{\left(x_1^2 + x_2^2 + x_3^2\right) \left(-k\,(k-1)\ {}^1H_{n-1}^{k-2}\right)}{|x|^{2n+3}} \\
& \quad - \frac{\left(x_1^2 + x_2^2 + x_3^2\right) \left((n-k)\,(n-k-1)\ {}^1H_{n-1}^k\right)}{|x|^{2n+3}}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{(2n+1)\, x_3\, {}^0H_n^k}{|x|^{2n+3}} \\
=& \frac{-k\,(k-1)\, x_1^{k-2} x_2^{n-k+2} x_3}{|x|^{2n+3}} \\
&- \frac{(2n+1+k\,(k-1)+(n-k)\,(n-k-1))\, x_1^k x_2^{n-k} x_3}{|x|^{2n+3}} \\
&+ \frac{-\,(n-k)\,(n-k-1)\, x_1^{k+2} x_2^{n-k-2} x_3 + x_3^2 q(x)}{|x|^{2n+3}} \\
=& \frac{-k\,(k-1)\ {}^1H_{n+1}^{k-2}}{|x|^{2(n+1)+1}} \\
&+ \frac{-\,(2n+1+k\,(k-1)+(n-k)\,(n-k-1))\ {}^1H_{n+1}^{k}}{|x|^{2(n+1)+1}} \\
&+ \frac{-\,(n-k)\,(n-k-1)\ {}^1H_{n+1}^{k+2}}{|x|^{2(n+1)+1}}
\end{aligned}
$$

$$
\begin{aligned}
=& -k\,(k-1)\ {}^1\breve{H}_{n+1}^{k-2} - (n-k)\,(n-k-1)\ {}^1\breve{H}_{n-1}^{k+2} \\
&- (k\,(k-1)+(n-k)\,(n-k-1)+(2n+1))\ {}^1\breve{H}_{n-1}^{k}
\end{aligned}
$$

This shows our proposition. q.e.d.

Remark

In contrast to the non-Kelvin transformed case we now map from a smaller to a bigger vector space. Obviously the kernel of this map is $\{0\}$ because otherwise we would have a harmonic in $Pot(\overline{\Sigma_{ext}})$ which is non-zero at infinity which would be a contradiction.

3.2 Differential Operator $\mathsf{D}_{\neg x_i}$

3.2.1 Ordinary Spherical Harmonics

Lemma 3.3

$$
\begin{aligned}
\mathsf{D}_{\neg x_1}\, {}^0H_n^k =& k\,(k-1)\ {}^1H_n^{k-2} + (n-k)^2\ {}^1H_n^k \\
\mathsf{D}_{\neg x_1}\, {}^1H_n^k =& -\ {}^0H_n^k \\
\mathsf{D}_{\neg x_2}\, {}^0H_n^k =& k^2\, {}^1H_n^{k-1} + (n-k)\,(n-k-1)\ {}^1H_n^{k+1} \\
\mathsf{D}_{\neg x_2}\, {}^1H_n^k =& -\ {}^0H_n^{k+1} \\
\mathsf{D}_{\neg x_3}\, {}^iH_n^k =& k\, {}^iH_n^{k-1} - (n-k-i)\ {}^iH_n^{k+1}
\end{aligned}
$$

Proof

For this task we may use the relations we got while differentiating with ∂_i. Hence we get:

$$\begin{aligned}
\mathsf{D}_{\neg x_1}\,{}^0H_n^k =& (x_3\partial_2 - x_2\partial_3)\ {}^0H_n^k \\
=& (n-k)\,x_3\,{}^0H_{n-1}^k + k\,(k-1)\,x_2\,{}^1H_{n-1}^{k-2} \\
&\quad + (n-k)\,(n-k-1)\,x_2\,{}^1H_{n-1}^k \\
=& (n-k)\ {}^1H_n^k + k\,(k-1)\ {}^1H_n^{k-2} \\
&\quad + (n-k)\,(n-k-1)\ {}^1H_n^k \\
=& k\,(k-1)\ {}^1H_n^{k-2} + (n-k)^2\ {}^1H_n^k
\end{aligned}$$

$$\begin{aligned}
\mathsf{D}_{\neg x_1}\,{}^1H_n^k =& (x_3\partial_2 - x_2\partial_3)\ {}^1H_n^k \\
=& (n-k-1)\,x_3\,{}^1H_{n-1}^k - x_2\,{}^0H_{n-1}^k \\
=& 0 - {}^0H_n^k \\
=& -\ {}^0H_n^k
\end{aligned}$$

$$\begin{aligned}
\mathsf{D}_{\neg x_2}\,{}^0H_n^k =& (x_3\partial_1 - x_1\partial_3)\ {}^0H_n^k \\
=& k\ x_3\,{}^0H_{n-1}^{k-1} + k\,(k-1)\,x_1\,{}^1H_{n-1}^{k-2} \\
&\quad + (n-k)\,(n-k-1)\,x_1\,{}^1H_{n-1}^k \\
=& k\,{}^1H_n^{k-1} + k\,(k-1)\ {}^1H_n^{k-1} \\
&\quad + (n-k)\,(n-k-1)\ {}^1H_n^{k+1} \\
=& k^2\,{}^1H_n^{k-1} + (n-k)\,(n-k-1)\ {}^1H_n^{k+1}
\end{aligned}$$

$$\begin{aligned}
\mathsf{D}_{\neg x_2}\,{}^1H_n^k =& (x_3\partial_1 - x_1\partial_3)\ {}^1H_n^k \\
=& k\ x_3\,{}^1H_{n-1}^{k-1} - x_1\,{}^0H_{n-1}^k \\
=& 0 - {}^0H_n^{k+1} \\
=& -\ {}^0H_n^{k+1}
\end{aligned}$$

$$\begin{aligned}
\mathsf{D}_{\neg x_3}\,{}^iH_n^k =& (x_2\partial_1 - x_1\partial_2)\ {}^iH_n^k \\
=& k\ x_2\,{}^iH_{n-1}^{k-1} - (n-k-i)\,x_1\,{}^iH_{n-1}^k \\
=& k\,{}^iH_n^{k-1} - (n-k-i)\ {}^iH_n^{k+1}
\end{aligned}$$

This proves our claim. q.e.d.

Remark

Again, the kernel of this map is obviously non-zero. We can see this particularly easy for the operators $\mathsf{D}_{\neg x_1}$ and $\mathsf{D}_{\neg x_2}$ because we cannot reach ${}^0H_n^n$ and ${}^0H_n^0$ respectively.

Again, a coordinate transformation transfers this result to $\mathsf{D}_{\neg x_3}$.

3.2.2 Kelvin Transformed Spherical Harmonics

Lemma 3.4

$$
\begin{aligned}
\mathsf{D}_{\neg x_1}\, {}^0\breve{H}_n^k &= (n-k)^2\ {}^1\breve{H}_n^k + k\,(k-1)\ {}^1\breve{H}_n^{k-2} \\
\mathsf{D}_{\neg x_1}\, {}^1\breve{H}_n^k &= -\ {}^0\breve{H}_n^k \\
\mathsf{D}_{\neg x_2}\, {}^0\breve{H}_n^k &= k^2\, {}^1\breve{H}_n^{k-1} + (n-k)\,(n-k-1)\ {}^1\breve{H}_n^{k+1} \\
\mathsf{D}_{\neg x_2}\, {}^1\breve{H}_n^k &= -\ {}^0\breve{H}_n^{k+1} \\
\mathsf{D}_{\neg x_3}\, {}^i\breve{H}_n^k &= k\, {}^i\breve{H}_n^{k-1} - (n-k-i)\ {}^i\breve{H}_n^{k+1}
\end{aligned}
$$

Proof

We have

$$
\begin{aligned}
x_j \partial_i \breve{H}_n(x) &= \frac{\left(x_1^2+x_2^2+x_3^2\right) x_j \partial_i H_n - (2n+1)\, x_i x_j H_n}{|x|^{2n+3}} \\
&= \frac{x_j \partial_i H_n(x)}{|x|^{2n+1}} - \frac{(2n+1)\, x_i x_j H_n(x)}{|x|^{2n+3}}
\end{aligned}
$$

And thus

$$
(x_j \partial_i - x_i \partial_j)\, \breve{H}_n(x) = \frac{(x_j \partial_i - x_i \partial_j)\, H_n(x)}{|x|^{2n+1}}
$$

So, using the last lemma yields the above result. q.e.d.

Remark

In particular this means that we are having structurally seen the same kernel in the normal and in the Kelvin transformed case. The kernel of these operators are exactly the rotationally invariant harmonic functions.

3.3 Differential Operator D_r

3.3.1 Ordinary Spherical Harmonics

Lemma 3.5

$$
\mathsf{D}_r\, {}^iH_n^k = n\, {}^iH_n^k
$$

Proof

Again we may use the result of the previous subsections. For $n = 0$ we trivially get the above result. Hence we may assume that $n > 0$.

$$
\begin{aligned}
\mathsf{D}_r\, {}^0H_n^k &= (x_1\partial_1 + x_2\partial_2 + x_3\partial_3)\ {}^0H_n^k \\
&= k\ x_1\, {}^0H_{n-1}^{k-1} + (n-k)\, x_2\, {}^0H_{n-1}^k \\
&\qquad - x_3 \left(k\,(k-1)\ {}^1H_{n-1}^{k-2} + (n-k)\,(n-k-1)\ {}^1H_{n-1}^k\right) \\
&= k\, {}^0H_n^k + (n-k)\ {}^0H_n^k + 0 \\
&= n\, {}^0H_n^k
\end{aligned}
$$

and

$$
\begin{aligned}
\mathsf{D}_r\,{}^1H_n^k =&(x_1\partial_1 + x_2\partial_2 + x_3\partial_3)\;{}^1H_n^k \\
=&k x_1\,{}^1H_{n-1}^{k-1} + (n-k-1)\,x_2\,{}^1H_{n-1}^k + x_3\,{}^0H_{n-1}^k \\
=&k\,{}^1H_n^k + (n-k-1)\;{}^1H_n^k + {}^1H_n^k \\
=&n\,{}^1H_n^k
\end{aligned}
$$

which proves the above proposition. q.e.d.

Remark

In this special case integration is just multiplication by n^{-1} and hence a fast and numerically stable task.

3.3.2 Kelvin Transformed Spherical Harmonics

We have the following lemma:

Lemma 3.6

$$\mathsf{D}_r\,{}^i\breve{H}_n^k = -(n+1)\;{}^i\breve{H}_n^k$$

Proof

Using the lemmas beforehand we get:

$$
\begin{aligned}
\mathsf{D}_r\,{}^i\breve{H}_n^k =&(x_1\partial_1 + x_2\partial_2 + x_3\partial_3)\;{}^i\breve{H}_n^k \\
=&\sum_i \frac{x_i\partial_i\,{}^iH_n^k}{|x|^{2n+1}} - (2n+1)\sum_i \frac{x_i^2\,{}^iH_n^k}{|x|^{2n+3}} \\
=&n\frac{{}^iH_n^k}{|x|^{2n+1}} - (2n+1)\,\frac{{}^iH_n^k}{|x|^{2n+1}} \\
=&-(n+1)\,\frac{{}^iH_n^k}{|x|^{2n+1}} \\
=&-(n+1)\;{}^i\breve{H}_n^k
\end{aligned}
$$

which yields the above result. q.e.d.

Remark

Again integration is just an easy to perform division. But this time all occurring kernels are 0 and hence we do not encounter any problems.

Note that this is the well known solution we already had for our radial derivative case.

3.4 Kernel Spaces and other Remarks

Concluding we observe the following stunning facts: The operators considered

- map harmonics to harmonics
- map $Pot(\overline{\Sigma_{int}})$ to $Pot(\overline{\Sigma_{int}})$ and $Pot(\overline{\Sigma_{ext}})$ to $Pot(\overline{\Sigma_{ext}})$.
- obey the degree of the harmonics, i.e., harmonics of the equal degree map to equal degree.

This enables us to consider differentiation and integration as a *finite* dimensional matrix operation.

However there is a drawback. If we want to stay finite dimensional we cannot combine the D_{x_i} derivatives with the other derivative operators because harmonics are not degree invariant under this operation.

This makes it particularly difficult to consider the occurring kernel spaces explicitly. However, we can observe that our linear combination of all the above vector fields incorporates not just the rotational vector fields we are dealing with a finite dimensional (in most cases one or zero dimensional) overall kernel. (E.g., $2 * \mathsf{D}_{id} + \mathsf{D}_r$ has a one dimensional kernel!)

Just if we are dealing with purely rotational invariant vector fields we observe that we have a one dimensional kernel in every degree and hence an infinite dimensional kernel.

These remarks also hold for the next section because spherical harmonics are nothing but an orthonormal system of the harmonic polynomials.

4 Spherical Harmonics

The basis system for the harmonics $Pot(\overline{\Sigma_{ext}})$ we used beforehand is not the standardly used one. Therefore we will introduce the system of spherical harmonics [7]:

Definition 4.1 (Spherical Harmonics)
Define the spherical harmonics Y_n^l in polar coordinates by:

$$\begin{aligned} Y_n^l &= C_n^l P_n^{|l|}(\sin\varphi) \begin{cases} \cos l\lambda & l \geq 0 \\ \sin |l|\lambda & l < 0 \end{cases} \\ &= \varepsilon_l \sqrt{2n+1} \sqrt{\frac{(n-|l|)!}{(n+|l|)!}} P_n^{|l|}(\sin\varphi) \begin{cases} \cos l\lambda & l \geq 0 \\ \sin |l|\lambda & l < 0 \end{cases} \end{aligned}$$

where

$$\varepsilon_l = \begin{cases} 1 & l = 0 \\ \sqrt{2} & \textit{otherwise} \end{cases}$$

and P_n^m is the associated Legendre function fulfilling

$$P_n^m(x) = \frac{(-1)^m}{2^n n!}(1-x^2)^{m/2} \frac{\partial^{n+m}}{\partial x^{n+m}} (x^2-1)^n$$

This particular system fulfills the following properties.

Lemma 4.1
The spherical harmonics Y_n^l constitute a complete orthonormal system under the standard L^2 inner product on the unit sphere.
Furthermore, when continued over the rim of the sphere by multiplication with new radius r^n (for harmonics in $Pot(\overline{\Sigma_{int}})$) and r^{-n-1} (for harmonics in $Pot(\overline{\Sigma_{ext}})$) respectively, these function constitute a basis of $Pot(\overline{\Sigma_{int}})$ and $Pot(\overline{\Sigma_{ext}})$ harmonics, respectively.

We will introduce the necessary conversion formulae between the two systems in use. Afterwards we will use these formulae to determine a direct way to integrate and differentiate the spherical harmonics.

4.1 Basis Change Matrices

The old basis of homogeneous harmonic polynomials was written in Cartesian coordinates which made differentiations more easy. Hence we need the transformation into polar coordinates (y_1, y_2, y_3) by now.

$$y_1 = r\cos\varphi\cos\lambda \qquad y_2 = r\cos\varphi\sin\lambda \qquad y_3 = r\sin\varphi$$

For the following computations we will use the following formulae [1, 4] ($l \geq 0$, $l > 0$ respectively).

$$\cos l\lambda = \sum_{m=0}^{\left[\frac{l}{2}\right]} (-1)^m \binom{l}{2m} \sin^{2m}\lambda \cos^{l-2m}\lambda$$

$$\sin l\lambda = \sum_{m=0}^{\left[\frac{l-1}{2}\right]} (-1)^m \binom{l}{2m+1} \sin^{2m+1}\lambda \cos^{l-2m-1}\lambda$$

and

$$\begin{aligned} P_n(t) =& 2^{-n} \sum_{m=0}^{\left[\frac{n}{2}\right]} (-1)^m \binom{n}{m}\binom{2n-2m}{n} t^{n-2m} \\ P_n^{|l|}(t) =& \left(1-t^2\right)^{l/2} \frac{\partial^l}{\partial t^l} P_n(t) \\ =& 2^{-n}\left(1-t^2\right)^{l/2} \sum_{m=0}^{\left[\frac{n-l}{2}\right]} (-1)^m \binom{n}{m}\binom{2n-2m}{n} \frac{(n-2m)!}{(n-2m-l)!} t^{n-2m-l} \\ =& \frac{l!}{2^n}\left(1-t^2\right)^{l/2} \sum_{m=0}^{\left[\frac{n-l}{2}\right]} (-1)^m \binom{n}{m}\binom{2n-2m}{n}\binom{n-2m}{l} t^{n-2m-l} \end{aligned}$$

For the next computations we will strongly rely on the fact that we know that the Y_n^l are homogeneous harmonic polynomials of degree n. In order to get this degree we will use the property $1 = \cos^2 + \sin^2$.

For reasons of simplicity we will just drop the constant $C_n^l \frac{l!}{2^n}$ right now. (I.e: $\widetilde{Y}_n^l C_n^l \frac{|l|!}{2^n} = Y_n^l$). Thus we get:

$$\widetilde{Y}_n^l = \frac{1}{|l|!} P_n^{|l|}(\sin\varphi) \begin{cases} \cos l\lambda & l \geq 0 \\ \sin|l|\lambda & l < 0 \end{cases}$$

$$= \cos^{|l|}\varphi \sum_{m=0}^{\left[\frac{n-|l|}{2}\right]} (-1)^m \binom{n}{m}\binom{2n-2m}{n}\binom{n-2m}{|l|} \sin^{n-2m-|l|}\varphi$$

$$\begin{cases} \sum_{m=0}^{\left[\frac{l}{2}\right]} (-1)^m \binom{l}{2m} \sin^{2m}\lambda \cos^{l-2m}\lambda & l \geq 0 \\ \sum_{m=0}^{\left[\frac{|l|-1}{2}\right]} (-1)^m \binom{l}{2m+1} \sin^{2m+1}\lambda \cos^{l-2m-1}\lambda & l < 0 \end{cases}$$

$$= \begin{cases} \sum_{m=0}^{\left[\frac{l}{2}\right]} (-1)^m \binom{l}{2m} y_2^{2m} y_1^{l-2m} & l \geq 0 \\ \sum_{m=0}^{\left[\frac{|l|-1}{2}\right]} (-1)^m \binom{|l|}{2m+1} y_2^{2m+1} y_1^{|l|-2m-1} & l < 0 \end{cases}$$

$$\sum_{m=0}^{\left[\frac{n-|l|}{2}\right]} (-1)^m \binom{n}{m}\binom{2n-2m}{n}\binom{n-2m}{|l|}$$

$$\sin^{n-2m-|l|}\varphi \left(\sin^2\varphi\cos^2\varphi\right)^m$$

$$= \begin{cases} \sum_{m=0}^{\left[\frac{l}{2}\right]} (-1)^m \binom{l}{2m} y_2^{2m} y_1^{l-2m} & l \geq 0 \\ \sum_{m=0}^{\left[\frac{|l|-1}{2}\right]} (-1)^m \binom{|l|}{2m+1} y_2^{2m+1} y_1^{|l|-2m-1} & l < 0 \end{cases}$$

$$\sum_{m=0}^{\left[\frac{n-|l|}{2}\right]} (-1)^m \binom{n}{m}\binom{2n-2m}{n}\binom{n-2m}{|l|} y_3^{n-2m-|l|}$$

$$\left(y_1^2 + y_2^2 + y_3^2\right)^m$$

We just want to remind the reader that for the definition of the homogeneous harmonic polynomial basis the terms which had a term of x_3 in a power higher than 1 were not relevant. In order to minimize the complexity of the matrices we now choose

$$y_1 := x_1 \qquad y_2 := x_3 \qquad y_3 := x_2$$

Hence we just need the polynomials which contain at most one y_2 in the previous formula. Writing $\widetilde{\widetilde{Y}}_n^l + y_2^2 p(y_1, y_2, y_3) = \widetilde{Y}_n^l$, where $\widetilde{\widetilde{Y}}_n^l$ does not contain any higher power of y_2 and doing the above substitution we get:

$$\widetilde{\widetilde{Y}}_n^l = \begin{cases} x_1^l & l \geq 0 \\ |l| x_1^{|l|-1} x_3 & l < 0 \end{cases}$$

$$\sum_{m=0}^{\left[\frac{n-|l|}{2}\right]} (-1)^m \binom{n}{m}\binom{2n-2m}{n}\binom{n-2m}{|l|} x_2^{n-2m-|l|} \left(x_1^2 + x_2^2\right)^m$$

$$
\begin{aligned}
=& x_1^{|l|-1} \begin{cases} x_1 & l \geq 0 \\ |l| x_3 & l < 0 \end{cases} \\
& \sum_{m=0}^{\left[\frac{n-|l|}{2}\right]} (-1)^m \binom{n}{m}\binom{2n-2m}{n}\binom{n-2m}{|l|} \sum_{k=0}^{m} \binom{m}{k} x_2^{n-|l|-2k} x_1^{2k} \\
=& x_1^{|l|-1} \begin{cases} x_1 & l \geq 0 \\ |l| x_3 & l < 0 \end{cases} \\
& \sum_{m=0}^{\left[\frac{n-|l|}{2}\right]} \sum_{k=0}^{m} (-1)^m \binom{n}{m}\binom{2n-2m}{n}\binom{n-2m}{|l|}\binom{m}{k} x_2^{n-|l|-2k} x_1^{2k} \\
=& x_1^{|l|-1} \begin{cases} x_1 & l \geq 0 \\ |l| x_3 & l < 0 \end{cases} \\
& \sum_{k=0}^{\left[\frac{n-|l|}{2}\right]} x_1^{2k} x_2^{n-|l|-2k} \sum_{m=k}^{\left[\frac{n-|l|}{2}\right]} (-1)^m \binom{n}{m}\binom{2n-2m}{n}\binom{n-2m}{|l|}\binom{m}{k}
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\widetilde{Y}_n^l = & \sum_{k=0}^{[\frac{n-|l|}{2}]} \begin{cases} {}^0H_n^{|l|+2k} & l \geq 0 \\ |l| \, {}^1H_n^{|l|+2k-1} & l < 0 \end{cases} \\
& \sum_{m=k}^{[\frac{n-|l|}{2}]} (-1)^m \binom{n}{m}\binom{2n-2m}{n}\binom{n-2m}{|l|}\binom{m}{k}
\end{aligned}
$$

Thus we get for $l \geq 0$

$$
\begin{aligned}
Y_n^l =& \varepsilon_l \sqrt{2n+1} \sqrt{\frac{(n-l)!}{(n+l)!}} \frac{l!}{2^n} \\
& \sum_{k=0}^{\left[\frac{n-l}{2}\right]} {}^0H_n^{l+2k} \sum_{m=k}^{\left[\frac{n-l}{2}\right]} (-1)^m \binom{n}{m}\binom{2n-2m}{n}\binom{n-2m}{l}\binom{m}{k} \\
=& \varepsilon_l \sqrt{2n+1} \sqrt{\frac{(n-l)!}{(n+l)!}} 2^{-n} \\
& \sum_{k=0}^{\left[\frac{n-l}{2}\right]} {}^0H_n^{l+2k} \sum_{m=k}^{\left[\frac{n-l}{2}\right]} (-1)^m \frac{(2n-2m)!}{(n-m)!\,(n-2m-l)!k!\,(m-k)!}
\end{aligned}
$$

and for $l > 0$

$$
\begin{aligned}
Y_n^{-l} =& \sqrt{2}\sqrt{2n+1} \sqrt{\frac{(n-l)!}{(n+l)!}} \frac{l!}{2^n} l \\
& \sum_{k=0}^{\left[\frac{n-l}{2}\right]} {}^1H_n^{l-1+2k} \sum_{m=k}^{\left[\frac{n-l}{2}\right]} (-1)^m \binom{n}{m}\binom{2n-2m}{n}\binom{n-2m}{l}\binom{m}{k}
\end{aligned}
$$

$$=\sqrt{2}\sqrt{2n+1}\sqrt{\frac{(n-l)!}{(n+l)!}}2^{-n}l$$

$$\sum_{k=0}^{\left[\frac{n-l}{2}\right]} {}^1H_n^{l-1+2k} \sum_{m=k}^{\left[\frac{n-l}{2}\right]} (-1)^m \frac{(2n-2m)!}{(n-m)!\,(n-2m-l)!k!\,(m-k)!}$$

Taking a closer look on the structure of the corresponding basis change matrices we observe that they correspond to triangular matrices when we reorder the Y_n^l according to positive and negative and odd and even l. Correspondingly we need to reorder the ${}^iH_n^k$ according to the value of i and the sign of k.

4.2 Direct Integration

The following differentials are an analogue to the old ones just with the new coordinate system (y_1, y_2, y_3), where again ∂_i differentiates in the y_i direction.

Conjecture

In order to make the formulae simpler we will denote (for $|l| > 1$):

$$l_+ = \begin{cases} l+1 & \textit{if} \quad l > 1 \\ l-1 & \textit{if} \quad l < -2 \end{cases} \qquad l_- = \begin{cases} l-1 & \textit{if} \quad l > 1 \\ l+1 & \textit{if} \quad l < -1 \end{cases}$$

Denote the sign of $-l$ by l_s. Due to easier notation all spherical harmonics with impossible coefficients are assumed to be zero. Sometimes the cases $|l| \leq 1$ are displayed separately. The following formulae for general l is then just holding for $|l| \geq 2$.

Differential Operator $\mathsf{D}_r = y_1\partial_1 + y_2\partial_2 + y_3\partial_3$

$$\mathsf{D}_r Y_n^l = -(n+1) \cdot Y_n^l$$

Differential Operator $\mathsf{D}_{y_1} = \partial_1$

$$\begin{aligned}
\mathsf{D}_{y_1} Y_n^0 &= -\sqrt{\frac{2n+1}{2n+3}}\sqrt{\frac{(n+1)\,(n+2)}{2}} \cdot Y_{n+1}^1 \\
\mathsf{D}_{y_1} Y_n^{-1} &= -\sqrt{\frac{2n+1}{2n+3}}\sqrt{\frac{(n+2)\,(n+3)}{4}} \cdot Y_{n+1}^{-2} \\
\mathsf{D}_{y_1} Y_n^1 &= +\sqrt{\frac{2n+1}{2n+3}}\sqrt{\frac{n\,(n+1)}{2}} \cdot Y_{n+1}^0 \\
&\quad -\sqrt{\frac{2n+1}{2n+3}}\sqrt{\frac{(n+2)\,(n+3)}{4}} \cdot Y_{n+1}^2 \\
\mathsf{D}_{y_1} Y_n^l &= +\sqrt{\frac{2n+1}{2n+3}}\sqrt{\frac{(n-|l|+1)\,(n-|l|+2)}{4}} \cdot Y_{n+1}^{l_-} \\
&\quad -\sqrt{\frac{2n+1}{2n+3}}\sqrt{\frac{(n+|l|+1)\,(n+|l|+2)}{4}} \cdot Y_{n+1}^{l_+}
\end{aligned}$$

Differential Operator $\mathsf{D}_{y_2} = \partial_2$

$$\begin{aligned}
\mathsf{D}_{y_2} Y_n^0 &= -\sqrt{\frac{2n+1}{2n+3}}\sqrt{\frac{(n+1)(n+2)}{2}} \cdot Y_{n+1}^{-1} \\
\mathsf{D}_{y_2} Y_n^1 &= -\sqrt{\frac{2n+1}{2n+3}}\sqrt{\frac{(n+2)(n+3)}{4}} \cdot Y_{n+1}^{-2} \\
\mathsf{D}_{y_2} Y_n^{-1} &= +\sqrt{\frac{2n+1}{2n+3}}\sqrt{\frac{n(n+1)}{2}} \cdot Y_{n+1}^{0} \\
&\quad +\sqrt{\frac{2n+1}{2n+3}}\sqrt{\frac{(n+2)(n+3)}{4}} \cdot Y_{n+1}^{2} \\
\mathsf{D}_{y_2} Y_n^l &= l_s\sqrt{\frac{2n+1}{2n+3}}\sqrt{\frac{(n-|l|+1)(n-|l|+2)}{4}} \cdot Y_{n+1}^{-l_-} \quad + \\
&\quad l_s\sqrt{\frac{2n+1}{2n+3}}\sqrt{\frac{(n+|l|+1)(n+|l|+2)}{4}} \cdot Y_{n+1}^{-l_+}
\end{aligned}$$

Differential Operator $\mathsf{D}_{y_3} = \partial_3$

$$\mathsf{D}_{y_3} Y_n^l = -\sqrt{\frac{2n+1}{2n+3}}\sqrt{(n+1-l)(n+1+l)} \cdot Y_n^l$$

Differential Operator $\mathsf{D}_{\neg y_1} = y_3\partial_2 - y_2\partial_3$

$$\begin{aligned}
\mathsf{D}_{\neg y_1} Y_n^0 &= -\sqrt{\frac{n(n+1)}{2}} \cdot Y_n^{-1} \\
\mathsf{D}_{\neg y_1} Y_n^1 &= -\sqrt{\frac{(n-1)(n+2)}{4}} \cdot Y_n^{-2} \\
\mathsf{D}_{\neg y_1} Y_n^{-1} &= +\sqrt{\frac{n(n+1)}{2}} \cdot Y_n^{0} \\
&\quad +\sqrt{\frac{(n-1)(n+2)}{4}} \cdot Y_n^{2} \\
\mathsf{D}_{\neg y_1} Y_n^l &= l_s\sqrt{\frac{(n-|l|+1)(n+|l|)}{4}} \cdot Y_n^{-l_-} + \\
&\quad l_s\sqrt{\frac{(n-|l|)(n+|l|+1)}{4}} \cdot Y_{n+1}^{-l_+}
\end{aligned}$$

Differential Operator $\mathsf{D}_{\neg y_2} = y_3\partial_1 - y_1\partial_3$

$$\begin{aligned}
\mathsf{D}_{\neg y_2} Y_n^0 &= -\sqrt{\frac{n(n+1)}{2}} \cdot Y_n^{1} \\
\mathsf{D}_{\neg y_2} Y_n^{-1} &= -\sqrt{\frac{(n-1)(n+2)}{4}} \cdot Y_n^{-2} \\
\mathsf{D}_{\neg y_2} Y_n^1 &= +\sqrt{\frac{n(n+1)}{2}} \cdot Y_n^{0} \\
&\quad -\sqrt{\frac{(n-1)(n+2)}{4}} \cdot Y_n^{2}
\end{aligned}$$

$$\mathsf{D}_{\neg y_2} Y_n^l = + \sqrt{\frac{(n-|l|+1)(n+|l|)}{4}} \cdot Y_n^{l-} - \sqrt{\frac{(n-|l|)(n+|l|+1)}{4}} \cdot Y_{n+1}^{l+}$$

Differential Operator $\mathsf{D}_{\neg y_3} = y_2 \partial_1 - y_1 \partial_2$

$$\mathsf{D}_{\neg y_3} Y_n^l = l \cdot Y_n^{-l}$$

The proof consists of simple, but very lengthy calculations using the differentiation formulae for the homogeneous harmonic polynomials and the conversion formulae of the spherical harmonics to homogeneous harmonic polynomials we have obtained beforehand.

Furthermore, for the $\mathsf{D}_{\neg y_i}$ we can find these differentials in books about quantum mechanics [5, 9], where $\mathsf{D}_{\neg y_i}$ can be interpreted as an angular momentum. The result for D_r is well known and can be found e.g., in [6].

5 Numerics

Integration and differentiation using the above shown band-matrices is not just theoretically exact but also extremely fast and stable. For a particular example please have a look in [3] where we used this for a particular example from satellite geodesy.

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