# Maximal Cohen-Macaulay Modules over Two Cubic Hypersurface Singularities

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# Contents

# Introduction

The present work is concerned with the classification of graded maximal Cohen-Macaulay modules (shortly MCM modules ) over the affine cone of two very special cubic hypersurfaces: the simple node singularity (Newtonknot) and the Fermat surface.

The category of MCM modules has been intensively studied (mostly in 80's) and a long series of interesting results appeared. We note here the one of Greuel-Knörrer ([GK]), Buchweitz-Greuel-Schreyer ([BGS]), Auslander-Reiten ([AR]), Solberg ([S1], [S2]). D.Eisenbud and J.Herzog classified (see [EH]) the homogeneous Cohen–Macaulay rings of finite representation type (that has only a finite number of isomorphism classes of indecomposable graded MCM modules, up to degree shiftings). They are isomorphic to one of the following rings:

1. $k[x_1, x_2, ..., x_n]$,
2. $k[x]/(x^m)$ for some $m > 0$,
3. $k[x_1, x_2, ..., x_n]/(x_1^2 + x_2^2 + ... + x_n^2)$,
4. $k[x_1, x_2]/(x_1^2 x_2 + x_2^3)$,
5. $k[x_1, x_2, x_3]/(x_1x_2, x_2x_3, x_3x_1)$,
6. the scroll of type $(m)$ for some $m$,
7. the scroll of type $(2, 1)$, and
8. $k[x_1, x_2, ..., x_6]/I$, $I$ being the ideal generated by all the $2 \times 2$-minors of the symmetric matrix
$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{pmatrix}$$

Geometrically, the above rings can be described as:

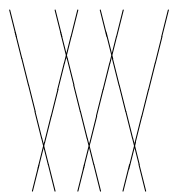
- *Arbitrary dimension*: $\mathbb{P}^n$ and smooth quadric hypersurfaces
- *Dimension* 0: 3 or fewer distinct points in $\mathbb{P}^2$
- *Dimension* 1: A curve of degree $n$ in $\mathbb{P}^n$. ( rational normal curves )
- *Dimension* 2: The rational normal scroll $S(1,2)$ in $\mathbb{P}^4$ and the Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$.

The graded MCM modules over a graded algebra $S = \bigoplus_{n \geq 0} S_n$ generated by $S_1$, with $k = S_0$ algebraically closed field, correspond, by sheafification, to aCM sheaves on the projective cone $Y = \mathrm{Proj} S$ (Theorem of Grothendieck, see for ex. [KL]). We say that over a projective variety $Y$ of dimension $n$, with $\mathcal{O}_Y(1)$ a very ample line bundle, a bundle $\mathcal{E}$ on $Y$ is *Arithmetically Cohen–Macaulay* (shortly aCM) if the graded module $\Gamma_*(\mathcal{E}) = \bigoplus_{k \in \mathbb{Z}} \Gamma(Y, \mathcal{E}(k))$ is graded MCM module (over the affine cone over $Y$) and $H^p(Y, \mathcal{E}(t)) = 0$ for $p \neq 0, n$ and for all $t \in \mathbb{Z}$.

In particular, if $Y$ is a smooth curve, the aCM sheaves are the vector bundles. So, in this case, the problem of the classification of graded MCM modules is equivalent to the classification of the vector bundles on the projective cone.

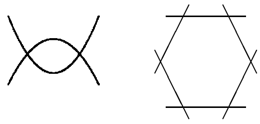
In 2001, G.-M. Greuel and Yu. Drozd (see [DG]) have proved that the category of vector bundles over a reduced curve is:

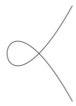
1) finite, for a configuration of projective lines of the type



2) tame, for

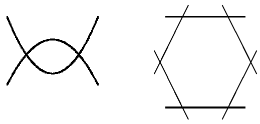
(a) smooth elliptic curve

(b) a configuration of projective lines of the type 

(c) simple node 

3) wild, otherwise.

For projective varieties of dimension 1, the problem of classification of aCM coherent sheaves is quasi-closed by Drozd and Greuel in the above cited paper: long time ago, in 1957, Atiyah (see [At]) has described the category of vector bundles over an elliptic curve and the same done Drozd and Greuel for the other curves with tame category of vector bundles: the simple node singularity and the configuration of projective lines



In the case of the simple node singularity the vector bundles are described as $\mathcal{B}(\mathbf{d}, m, \lambda)$ where:
• $\lambda \in k^*$, continuous parameter
• $m$ natural number
• $\mathbf{d} = d_1, .., d_r$ cycle of integers, $d_1 + .. + d_r$ is the degree of the vector bundle and $rm$ is the rank (see also [B]).

As we can look to MCM modules from two points of view, the algebraic one and the geometric one, if the results from both sides are combined, one can get very useful information. An example, in this sense, is the paper 'Maximal Cohen-Macaulay Modules over the Cone of an Elliptic Curve' (Journal of Algebra, 253, 209-236, 2002) of R. Laza, G. Pfister and D. Popescu, where, using the classification of the vector bundles over a smooth elliptic curve made by Atiyah, there are explicitly described the graded MCM modules over $R = k[y_1, y_2, y_3]/\langle y_1^3 + y_2^3 + y_3^3\rangle$.
They give canonical normal forms for the matrix factorizations of all graded reflexive $R$–modules of rank 1 (see Section 3 in [LPP]) and show effectively

how one can produce the indecomposable graded reflexive $R$–modules of grater rank using SINGULAR (see Section 5 in [LPP]).

Is it possible to do the same in the case of the simple node, using the above description of vector bundles?

The problem seems to be more difficult, since there is one singularity. So, the MCM modules on the affine cone of the simple node (that has a non–isolated singularity) correspond to the coherent aCM sheaves on the simple node, not all locally free. The classification of Drozd and Greuel can provide us at most the locally free MCM modules.

In the first part of the present work, combining methods of commutative algebra and computer algebra with tools from algebraic geometry, we describe explicitly:
• all rank one graded MCM modules;
• all rank two, indecomposable, graded MCM modules
• all rank two locally free MCM modules corresponding to stable vector bundles.

The modules are described using the technique of matrix factorization, introduced by Eisenbud. We remind here the basics.

Let $k$ be an algebraically closed field of characteristic zero and $S$ the polynomial ring $S = k[x_1, ..., x_n]$. Consider $R = S/f$ the hypersurface ring given by some irreducible homogeneous polynomial $f \in S$ of degree $d$ and $M$ a graded MCM module over $R$. As an $S$–module, $M$ is Cohen-Macaulay module, so, by Auslander-Buchsbaum formula, it has a graded minimal resolution of the form

$$0 \longrightarrow \bigoplus_{j=1}^{j=n} S(\beta_j) \xrightarrow{\varphi} \bigoplus_{j=1}^{j=n} S(\alpha_j) \longrightarrow M \longrightarrow 0,$$

with $n$ is the minimal number of generators of $M$ and $\varphi$ the multiplication with a square $n \times n$ matrix $A$ with homogeneous entries. Because of the minimality of the resolution, the entries of $A$ can be either zero or of strictly positive degree.
Tensoring by $R$ we obtain an infinite, graded, 2-periodic resolution of $M$ over $R$, of the form

$$... \xrightarrow{\varphi} \bigoplus_{j=1}^{j=n} R(\alpha_j - d) \xrightarrow{\psi} \bigoplus_{j=1}^{j=n} R(\beta_j) \xrightarrow{\varphi} \bigoplus_{j=1}^{j=n} R(\alpha_j) \longrightarrow M \longrightarrow 0.$$

Again, the map $\psi$ is the multiplication by a square $n \times n$ matrix $A'$ with homogeneous entries. In the paper 'Homological algebra with application to group representations'(1980), D.Eisenbud has proved the remarkable fact that the pair of matrices $(A, A')$ form a *matrix factorization* of the polynomial $f$. This means, $AA' = A'A = fId_n$.

The matrix factorizations proved to be a very useful tool for the study of MCM modules over hypersurface rings.
If $(A, A')$ is a matrix factorization of the polynomial $f$ and $\varphi$ a graded map defined by $A$, $\operatorname{Coker} \varphi$ is a graded MCM module over $R = S/f$.

Of course, two matrix factorizations $(A, A')$ and $(B, B')$ give isomorphic modules if and only if there exist two invertible matrices $U, V$ such that $B = UAV, B' = VA'U$.

The rank of the module $\operatorname{Coker} A$ is precisely the integer $r$ such that $\det A = f^r$. It follows immediately that the minimal number of generators of a MCM $R$–module of rank $r$ is smaller equal $dr$, with $d$ the degree of the polynomial $f$.

The module $\operatorname{Coker} A$ is decomposable if and only if the matrix $A$ decomposes, after some linear transformations.

In the second part of this work it is proved that, over the affine cone of a smooth hypersurface, a rank two graded MCM module is orientable if and only if it has a skew symmetric matrix factorization.

In the subsection 2.2, I prove that, over the affine cone of curves of arithmetic genus 1, the matrix factorizations give information on the stability of the sheafification of graded MCM modules.

Another very useful tool in the classification of MCM modules is given by the extensions of MCM modules. Over hypersurface rings, it is very interesting the following question:
*If $L$ and $F$ are two graded MCM modules with the matrix factorizations $(A, A')$, respectively $(B, B')$, how are looking like the matrix factorizations of the module $E$, if there exists a graded exact sequence:*

$$0 \to L \to E \to F \to 0 \quad ?$$

We answer to this question in the next theorem.

**Theorem 0.1.** *Let $S = k[x_1, ..., x_n]$ and $R = S/f$ a hypersurface ring defined by an irreducible homogeneous polynomial $f$. Consider $L, F$ two graded MCM $R$–modules with the matrix factorizations $(A, A')$, respectively $(B, B')$ and the graded extension $0 \to L \xrightarrow{\alpha} E \xrightarrow{\beta} F \to 0$.*
*Then $E$ is a graded MCM $R$–module and has a matrix factorization $(M, M')$, $M$ of the type $M = \begin{pmatrix} A & D \\ 0 & B \end{pmatrix}$, $D$ a matrix with homogeneous entries in $S$ such that $A' \cdot D \cdot B' = 0$ in $R$.*

*Proof.* Denote $s = \mu(L), t = \mu(F), r_1 =$rank$(L), r_2 =$rank$(F)$.
Consider the following graded diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \bigoplus_{j=1}^{j=s} R(\alpha'_j) & \xrightarrow{i} & \bigoplus_{j=1}^{j=s+t} R(\alpha'_j) & \xrightarrow{\pi} & \bigoplus_{j=s+1}^{j=s+t} R(\alpha'_j) & \longrightarrow & 0 \\
& \downarrow A & & \downarrow & & \downarrow B & & \\
0 \longrightarrow & \bigoplus_{j=1}^{j=s} R(\alpha_j) & \xrightarrow{i} & \bigoplus_{j=1}^{j=s+t} R(\alpha_j) & \xrightarrow{\pi} & \bigoplus_{j=s+1}^{j=s+t} R(\alpha_j) & \longrightarrow & 0 \\
& \downarrow p_A & & \downarrow \delta & & \downarrow p_B & & \\
0 \longrightarrow & \operatorname{Coker} A & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & \operatorname{Coker} B & \longrightarrow & 0
\end{array}$$

where, $p_A, p_B$ are the natural projections, $\pi$ is the projection on the last $t$ variables and $i$ the natural inclusion. Let $\varphi_B : \bigoplus_{j=s+1}^{j=s+t} R(\alpha_j) \to E$ be a graded map such that $\beta \circ \varphi_B = p_B$ and consider $\varphi_A = \alpha \circ p_A$. Define the graded map $\delta$ as the sum of $\varphi_A$ and $\varphi_B$. Then, $\delta$ makes the above diagram commutative. Using Snake Lemma, we get the graded exact sequence

$$0 \longrightarrow \operatorname{Im} A \xrightarrow{i} \operatorname{Ker} \delta \xrightarrow{\pi} \operatorname{Im} B \longrightarrow 0$$

and the surjectivity of $\delta$. Therefore, $\operatorname{Ker} \delta$ is a graded MCM $R$–module of rank $s + t - r_1 - r_2$ such that $E \simeq \bigoplus_{j=1}^{j=s+t} R(\alpha_j)/\operatorname{Ker} \delta$.
So, there exists $(M_1, M'_1)$ a graded matrix factorization of $f$ such that $E \simeq \operatorname{Coker} M'_1$ and such that the graded sequence

$$0 \longrightarrow \operatorname{Im} A \xrightarrow{i} \operatorname{Im} M'_1 \xrightarrow{\pi} \operatorname{Im} B \longrightarrow 0$$

is exact.

Since $\operatorname{Im}\begin{pmatrix} A \\ 0 \end{pmatrix} \subset \operatorname{Im} M_1'$, there exist two invertible matrices $U$ and $V$ such that $UM_1'V = \begin{pmatrix} A & D_1 \\ 0 & B_1 \end{pmatrix}$. Denote $M'' = UM_1'V$. Then $\operatorname{Coker} M''$ is a graded MCM module isomorphic to $E$. Thus, there exists the matrix $B_1'$ with homogeneous entries such that $(B, B_1')$ is a matrix factorization of $f$ and $A'D_1B_1' = 0$ in $R$.
We have therefore a graded commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Im} A & \xrightarrow{i} & \operatorname{Im} M'' & \xrightarrow{\pi} & \operatorname{Im} B_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\wr} & & & & \\ 0 & \longrightarrow & \operatorname{Im} A & \xrightarrow{i} & \operatorname{Im} M_1' & \xrightarrow{\pi} & \operatorname{Im} B & \longrightarrow & 0 \end{array}$$

and an inclusion $\operatorname{Im} B_1 \hookrightarrow \operatorname{Im} B$ of two graded MCM modules of the same rank. This means that they are isomorphic, and therefore, there exist two invertible matrices $U_1$ and $V_1$ such that $B = U_1B_1V_1$. Denote $D = D_1V_1$ and $M = \begin{pmatrix} A & D \\ 0 & B \end{pmatrix}$. Then $\operatorname{Coker} M \simeq \operatorname{Coker} M'' \simeq E$. Evidently, $A'DB' = 0$ in $R$. □

As mentioned before, in the first part of this work, I describe matrix factorizations of all the isomorphism classes of the rank one and two graded, indecomposable MCM modules over the hypersurface $R = k[y_1, y_2, y_3]/(y_1^3 + y_1^2y_3 - y_2^2y_3)$, that is the affine cone over the simple node singularity.

In the first section, it is proven that, up to shiftings, there are three families, parametrized by the points of the curve, of isomorphism classes of graded, rank 1, MCM modules; two of them contain 2–minimally generated modules, the other contains 3–minimally generated modules.

I identify also the graded maps, that, after sheafification, produce the line bundles of degree 0, 1 or -1.

In the second subsection there are described the rank two, graded, indecomposable MCM $R$–modules. In 2.1, I construct an algorithm to produce all rank two locally free MCM modules. In this section, the main role is played by the classification of vector bundles on the simple node, especially by the property of $\mathcal{B}(0,2,1)$ and $\mathcal{B}((0,1),1,\lambda)$ to be uniquely determined through the extensions:

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{B}(0,2,1) \longrightarrow \mathcal{O}_Y \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{B}((0,1),1,\lambda) \longrightarrow \mathcal{B}(1,1,-\lambda) \longrightarrow 0.$$

As a consequence of the above results, in 2.2, I give matrix factorizations of all rank two graded MCM $R$–modules with stable sheafification on the simple node.
In the subsection 2.3 I describe explicitly all the isomorphism classes of indecomposable, non–locally free graded MCM modules of rank two: there are two families parametrized by the regular points of the curve and 22 countable families.

Remark:
It is known that a graded hypersurface has countable CM–representation type iff it is isomorphic to $A_\infty$, that is: $k[x_1, x_2, ..., x_n]/(x_2^2 + ... + x_n^2)$.

But what about Cohen–Macaulay algebras of bigger dimension ?
For rings (respectively projective varieties) of bigger dimension, that are not of finite representation type, one can hope only to classify families of MCM modules (resp. aCM bundles) of some low rank. On Fano threefolds with $\text{Pic}(X) \cong \mathbb{Z}$ and on hypersurfaces of degree up to 6 and dimension at least 3, there are obtained a lot of results regarding the rank two vector bundles, in papers like [AC], [Ma1], [Ma2], [CM1], [CM2], [CM3], [AF], [F1].

In this work, in the second part, we classify all graded, rank two, indecomposable MCM modules over the affine cone of the Fermat surface, that is $k[x_1, x_2, x_3, x_4]/(x_1^3 + x_2^3 + x_3^3 + x_4^3)$.
The graded rank one MCM modules over this algebra were classified in [EP]. There are a finite number, and they correspond to 27 lines, 27 conics respectively 72 twisted cubic curves on the surface. For the completion of the work, I present this classification in the third section.
In the fourth section we classify the orientable rank two MCM modules. The main role here is the description of the orientable modules (over normal rings)

with the help of codimension two Gorenstein ideals, realized by Herzog and Kühl in [HK].
In the case of 6–generated orientable modules, we use the fact that they have skew symmetric matrix factorizations. We prove and use a very interesting property of the Fermat surface: a 6–generated orientable bundle, when restricted to the hyperplane section curve, splits into two line bundles on the curve.( In fact, it is known that the Fermat surface is a very special cubic surface. On it, three lines can meet at a point, while this does not happen on a general nonsingular cubic surface.) Classifying the 6–generated orientable, rank two, MCM modules, we obtain a 5 dimensional moduli space of isomorphism classes of indecomposable modules whose restriction to $V(f, x_4)$ splits into two non-isomorphic modules and a 2-dimensional moduli space of isomorphism classes of indecomposable modules whose restriction to $V(f, x_4)$ splits into two isomorphic modules.
In the last section, we classify the non–orientable rank two MCM modules. We use a similar idea as in the case of orientable, 4–generated modules, but in this case, the ideal is not any more Cohen–Macaulay.
Recently, Daniele Faenzzi, has obtained some very interesting results on the moduli spaces of aCM vector bundles on a cubic surface in $\mathbb{P}^3$ ( see [F2]). Part of them will be used in this work, in order to get some informations on the stability of the orientable modules constructed by us.

I would like to mention that, through all the work, a constant support for the computations is given by the computer algebra system SINGULAR.(see [GPS]) Part of the procedures that I used can be found in [LPP].

There are several interesting issues that leaded me to these computations and would direct the next research:

1. The MCM modules over a non-isolated singularity;

2. The form of the matrix factorizations corresponding to stable vector bundles

3. The behavior of non-orientable modules over a general cubic surface

4. The classification of indecomposable, rank 2, MCM modules over a general cubic surface

In the present work we have constructed examples and tools useful in solving the above problems.

I specify that some results from the first part of this work were published in [Ba1] and [Ba2], and were presented at the NATO Advanced Research Workshop on Computational Commutative and Non–Commutative Algebraic Geometry, Chisinau, June 2004, the Workshop on Cohen–Macaulay Rings and Related Structures, Constanta, April, 2005, and seminars at the University Kaiserslautern. The results from the second part are going to be published in Journal of Algebra, and they were presented to a seminar at the University Essen.

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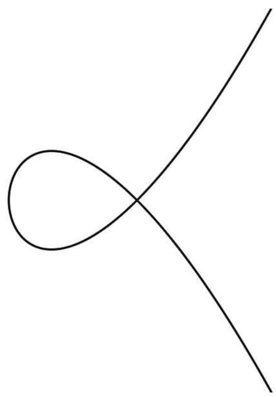
# Part I
# Simple node singularity



In this part of the work, we classify the rank one and rank two, graded MCM over the affine cone of the simple node singularity. We give the matrix factorizations of the rank two stable bundles. Furthermore, we classify the non–locally free, rank one and rank two MCM modules.

# 1 Rank one, graded, MCM modules over the hypersurface $R = k[y_1, y_2, y_3]/(y_1^3 + y_1^2 y_3 - y_2^2 y_3)$

It is known that for any two graded R-modules $M$ and $N$, $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M \otimes_R N}$ and $\widetilde{\mathrm{Hom}_R(M,N)} \cong \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$. If $M$ and $N$ are maximal Cohen-Macaulay $R$-modules, then $\widetilde{M} \cong \widetilde{N}$ if and only if $M \cong N$.

In the classification of the vector bundles on the simple node $Y = \mathrm{Proj}R$ from [B], the line bundles of degree $d \in \mathbb{Z}$ are $\mathcal{B}(d, 1, \lambda)$ with $\lambda \in k^*$ ($\lambda$ run over all regular points of the curve $Y$).
The tensor product of two line bundles is given by:

$$\mathcal{B}(d, 1, \lambda) \otimes \mathcal{B}(d', 1, \lambda') = \mathcal{B}(d + d', 1, \lambda \cdot \lambda').$$

In this section are classified the rank one graded MCM $R$–modules. As mentioned in the Introduction, their minimal number of generators is smaller equal to $e(R)$, the multiplicity of the ring. Therefore, they are two or three minimally generated.

## 1.1 Two–generated graded MCM R–modules

Let $s = (0:0:1)$ be the unique singular point of the curve $V(f) \subset \mathbb{P}^2_k$ and denote $V(f)_{\mathrm{reg}} = V(f) \backslash \{s\}$.
Then $V(f)_{\mathrm{reg}} = \{(\lambda_1 : \lambda_2 : 1), \lambda_1^3 + \lambda_1^2 - \lambda_2^2 = 0, \lambda_1 \neq 0\} \cup \{(0:1:0)\}$.

For any $\lambda = (\lambda_1 : \lambda_2 : 1)$ in $V(f)$ denote:

$$\varphi_\lambda = \begin{pmatrix} y_1 - \lambda_1 y_3 & y_2 y_3 + \lambda_2 y_3^2 \\ y_2 - \lambda_2 y_3 & y_1^2 + (\lambda_1 + 1) y_1 y_3 + (\lambda_1^2 + \lambda_1) y_3^2 \end{pmatrix},$$

$$\psi_\lambda = \begin{pmatrix} y_1^2 + (\lambda_1 + 1) y_1 y_3 + (\lambda_1^2 + \lambda_1) y_3^2 & -(y_2 y_3 + \lambda_2 y_3^2) \\ -(y_2 - \lambda_2 y_3) & y_1 - \lambda_1 y_3 \end{pmatrix}.$$

If $\lambda = (0:1:0)$ let be:

$$\varphi_\lambda = \begin{pmatrix} y_1 + y_3 & y_2^2 \\ y_3 & y_1^2 \end{pmatrix}, \qquad \psi_\lambda = \begin{pmatrix} y_1^2 & -y_2^2 \\ -y_3 & y_1 + y_3 \end{pmatrix}.$$

We consider the following graded maps defined by the above matrices:

1. $\psi_\lambda : R(-2)^2 \longrightarrow R \oplus R(-1), \lambda \in V(f)_{\text{reg}}$;
2. $\varphi_\lambda : R(-2) \oplus R(-3) \longrightarrow R(-1)^2, \lambda \in V(f)_{\text{reg}}$.

**Theorem 1.1.** *For all $\lambda \in V(f)$, $(\varphi_\lambda, \psi_\lambda)$ is a matrix factorization of $f$ and the sets of graded MCM R–modules:*

$$\mathcal{M}_{-1} = \{\operatorname{Coker}\varphi_\lambda \mid \lambda \in V(f)_{\text{reg}}\}, \qquad \mathcal{M}_1 = \{\operatorname{Coker}\psi_\lambda \mid \lambda \in V(f)_{\text{reg}}\}$$

*and*

$$\underline{\mathcal{M}} = \{\operatorname{Coker}\varphi_s, \operatorname{Coker}\psi_s\}$$

*have the following properties:*
*1) Every two-generated non-free graded MCM R-module is isomorphic, up to shifting, with one of the modules from $\mathcal{M}_{-1} \cup \mathcal{M}_1 \cup \underline{\mathcal{M}}$.*
*2) Every two different R-modules from $\mathcal{M}_{-1} \cup \mathcal{M}_1 \cup \underline{\mathcal{M}}$ are not isomorphic.*
*3) All the modules from $\mathcal{M}_{-1} \cup \mathcal{M}_1 \cup \underline{\mathcal{M}}$ have rank 1.*
*4) The modules from $\mathcal{M}_1$ are the syzygies and also the duals of the modules from $\mathcal{M}_{-1}$.*

*Proof.* Clearly $\varphi_\lambda\psi_\lambda = \psi_\lambda\varphi_\lambda = f \cdot \mathbf{1}_2$ for any $\lambda \in V(f)$.

1) Let $M$ be a two-generated non-free graded MCM $R$-module and consider $(\varphi, \psi)$ a graded reduced matrix factorization of it. So $\varphi\psi = \psi\varphi = f \cdot \mathbf{1}_2$ and $\det\varphi \cdot \det\psi = f^2$. Since $f$ is irreducible, we may consider $\det\varphi = \det\psi = f$. Because $\psi$ is the adjoint of $\varphi$, it is sufficient to find

$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$$

such that $\det\varphi = f$ and $\varphi_{11}$ and $\varphi_{21}$ are two linearly independent linear forms. So, applying some elementary transformations on the matrix $\varphi$, we may suppose that:

$$\begin{cases} \varphi_{11} = y_1 - \lambda_1 y_3 \quad \text{and} \quad \varphi_{21} = y_2 - \lambda_2 y_3, \quad \lambda_1, \lambda_2 \in k \\ \qquad\qquad\qquad \text{or} \\ \varphi_{11} = y_1 - \lambda y_2 \quad \text{and} \quad \varphi_{21} = y_3, \qquad\qquad \lambda \in k. \end{cases}$$

Let us consider the first case, when

$$\varphi = \begin{pmatrix} y_1 - \lambda_1 y_3 & \varphi_{12} \\ y_2 - \lambda_2 y_3 & \varphi_{22} \end{pmatrix}$$

with $\varphi_{12}, \varphi_{22}$ two-forms.
Notice that $(\det \varphi)(\lambda_1, \lambda_2, 1) = 0$. Therefore $\lambda = (\lambda_1 : \lambda_2 : 1)$ is a point on the curve $V(f)$. We want to show that $\varphi \sim \varphi_\lambda$.
For this, consider the product $\psi_\lambda \cdot \varphi$ that has the form

$$\psi_\lambda \cdot \varphi = \begin{pmatrix} f & g \\ 0 & f \end{pmatrix}$$

with $g = (\, y_1^2 + (\lambda_1 + 1) y_1 y_3 + (\lambda_1^2 + \lambda_1) y_3^2 \,) \cdot \varphi_{12} - (\, y_2 y_3 + \lambda_2 y_3^2 \,) \cdot \varphi_{22}$. Since $g \cdot (y_1 - \lambda_1 y_3) = \varphi_{12} \cdot f - (y_2 y_3 + \lambda_2 y_3^2) \cdot \det \varphi = f \cdot (\varphi_{12} - y_2 y_3 - \lambda_2 y_3^2)$ and $f$ is irreducible, we can write $g = f \cdot g_1$ with $g_1 \in k[y_1, y_2, y_3]$. Therefore, we have

$$\psi_\lambda \varphi = f \cdot \begin{pmatrix} 1 & g_1 \\ 0 & 1 \end{pmatrix}.$$

Multiplying at left with $\varphi_\lambda$, we obtain

$$f \cdot \varphi = f \cdot \varphi_\lambda \cdot \begin{pmatrix} 1 & g_1 \\ 0 & 1 \end{pmatrix},$$

that means,

$$\varphi = \varphi_\lambda \cdot \begin{pmatrix} 1 & g_1 \\ 0 & 1 \end{pmatrix}.$$

This equality induce the equivalence between $\varphi$ and $\varphi_\lambda$.

The second case ($\varphi_{11} = y_1 - \lambda y_2$ and $\varphi_{21} = y_3; \lambda \in k$) can be treated exactly as above, replacing $\psi_\lambda$ with $\psi_{\lambda_0}$, where $\lambda_0$ denotes the point (0:1:0).

2) Because of the degrees of the entries, no module from $\mathcal{M}_1 \cup \{\text{Coker}\, \psi_s\}$ is isomorphic with a module from $\mathcal{M}_{-1} \cup \{\text{Coker}\, \varphi_s\}$.
Since any two equivalent matrices have the same fitting ideals, for the rest, it is enough to consider the following fitting ideals:

- The modules from $\mathcal{M}_{-1} \cup \mathcal{M}_1$:

$\text{Fitt}_1(\varphi_\lambda) = \text{Fitt}_1(\psi_\lambda) = \langle y_1 - \lambda_1 y_3,\, y_2 - \lambda_2 y_3,\, y_3^2 \rangle, \lambda = (\lambda_1 : \lambda_2 : 1) \in V(f)$
$\text{Fitt}_1(\varphi_{\lambda_0}) = \text{Fitt}_1(\psi_{\lambda_0}) = \langle y_1,\, y_3,\, y_2^2 \rangle, \lambda_0 = (0 : 1 : 0)$.

- The modules from $\underline{\mathcal{M}}$: $\text{Fitt}_1(\varphi_s) = \text{Fitt}_1(\psi_s) = \langle y_1,\, y_2 \rangle$.

3) Follows from Corollary 6.4, [Ei1].

4) By construction, the modules of $\mathcal{M}_1$ are the syzygies of the modules of $\mathcal{M}_{-1}$. Since

$$\varphi_\lambda^t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \psi_\lambda \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\operatorname{Coker} \psi_\lambda \cong (\operatorname{Coker} \varphi_\lambda)^\vee$.

□

**Theorem 1.2.** *Let $Y$ be the projective cone over $R$, that is the simple node singularity.*

1. *The coherent sheaves associated to the modules from $\mathcal{M}_1$ give all the isomorphism classes of line bundles of degree 1 over $Y$ ;*
2. *The coherent sheaves associated to the modules from $\mathcal{M}_{-1}$ give all the isomorphism classes of line bundles of degree -1 over $Y$;*
3. *The coherent sheaves associated to the modules from $\underline{\mathcal{M}}$ are not locally free.*

*Proof.* 3) The only singular point of $V(f)$ is $(0 : 0 : 1)$. To prove that $\operatorname{Coker} \psi_s$ is non–locally free module, by [TJP](1.3.8), it is sufficient to prove that $\operatorname{Fitt}_1(\psi_s)R_{\langle y_1, y_2 \rangle} \neq R_{\langle y_1, y_2 \rangle}$.
We can easily see that $\operatorname{Fitt}_1(\psi_s)R_{\langle y_1, y_2 \rangle} = \langle y_1, y_2 \rangle R_{\langle y_1, y_2 \rangle}$, so, $\operatorname{Coker} \psi_s$ and also its dual, $\operatorname{Coker} \varphi_s$ are non–locally free modules.

1) Any line bundle of degree one on $Y$ has the form $\mathcal{O}_Y(P)$, with $P$ regular point of $Y$. Following the proof of Theorem 3.8 from ([LPP]), we obtain that the graded MCM $R$-module corresponding to $\mathcal{O}_Y(P)$ is a module from $\mathcal{M}_1$, for any regular point $P$ of $Y$.

2) The modules of $\mathcal{M}_{-1}$ are the duals of the modules from $\mathcal{M}_1$, so, they induce the line bundles of degree -1, that are $\mathcal{B}(-1, 1, \lambda)$, $\lambda \in k^*$. □

For any $\lambda = (\lambda_1 : \lambda_2 : 1)$ in $V(f)$ let be:

$$\alpha_\lambda = \begin{pmatrix} 0 & y_1 - \lambda_1 y_3 & y_2 - \lambda_2 y_3 \\ y_1 & y_2 + \lambda_2 y_3 & (\lambda_1^2 + \lambda_1) y_3 \\ y_3 & 0 & -y_1 - (\lambda_1 + 1) y_3 \end{pmatrix}$$

and $\beta_\lambda$ the adjoint of $\alpha_\lambda$.

We define also the graded maps $\alpha_\lambda : R(-2)^3 \longrightarrow R(-1)^3$, given by the matrices $\alpha_\lambda$.

Using the same notations as in the previous section ($s = (0:0:1), \lambda_0 = (0:1:0), V(f)_{\mathrm{reg}} = V(f) \backslash \{s\}$), we have the following:

**Theorem 1.3.** *For all $\lambda = (\lambda_1 : \lambda_2 : 1) \in V(f), (\alpha_\lambda, \beta_\lambda)$ is a matrix factorization of $f$ and the set of three-generated graded MCM R-modules*

$$\mathcal{M}_0 = \{ \operatorname{Coker} \alpha_\lambda \mid \lambda \in V(f)_{\mathrm{reg}} \backslash \{\lambda_0\} \}$$

*has the following properties:*
*1) All the modules from $\mathcal{M}_0$ have rank 1.*
*2) Every two different modules from $\mathcal{M}_0$ are not isomorphic.*
*3) Every three–generated, rank 1, graded MCM R-module is isomorphic with one of the modules from $\mathcal{M}_0$ or to* $\operatorname{Coker} \alpha_s$.

*Proof.* Clearly $\alpha_\lambda \beta_\lambda = \beta_\lambda \alpha_\lambda = f \cdot \mathbf{1}_3$ for any $\lambda = (\lambda_1 : \lambda_2 : 1) \in V(f)$.

1) Since $\det(\alpha_\lambda) = f$, by Corollary 6.4 ([Ei1]), $\operatorname{Coker} \alpha_\lambda$ has rank 1.

2) Suppose that there exist two invertible matrices, $U$ and $V$, with entries in $k$, such that $U\alpha_\lambda = \alpha_\xi V$ for $\lambda, \xi \in V(f)$. With the help of computer (we use Singular[GPS]) we obtain that $\lambda = \xi$:

```
LIB "matrix.lib";
option(redSB);

ring r=0,(y(1..3),u(1..9),v(1..9),a,b,c,d),(c,dp);
ideal I=a3+a2-b2,c3+c2-d2;
qring Q=std(I);
```

Define the matrix A = $\alpha_\lambda$, for $\lambda = (a : b : 1)$ and B = $\alpha_\xi$, for $\xi = (c : d : 1)$.

```
matrix A[3][3]=   0, y(1)-a*y(3),      y(2)-b*y(3),
               y(1), y(2)+b*y(3),      (a2+a)*y(3),
               y(3),           0, -y(1)-(1+a)*y(3);
matrix B=subst(A,a,c,b,d);
```

Investigation the condition $U\alpha_\lambda = \alpha_\xi V$:

```
matrix U[3][3]=u(1..9);
matrix V[3][3]=v(1..9);
matrix C=U*A-B*V;
ideal I=flatten(C);
ideal J=ideal(det(U)-1);
J=J+transpose(coeffs(I,y(1)))[2];
J=J+transpose(coeffs(I,y(2)))[2];
J=J+transpose(coeffs(I,y(3)))[2];
ideal L=std(J);
L;
```

The first two entries of the ideal L are:

```
L[1]=b-d
L[2]=a-c
```

Therefore $a = c$ and $b = d$, that means $\lambda = \xi$.

3) Let be $M$ a three-generated, rank one, graded MCM $R$-module and $(\varphi, \psi)$ the corresponding graded reduced matrix factorization. We can suppose $\det\varphi = f$ and $\det\psi = f^2$. Since $f \in \langle y_1, y_3\rangle$, by [Ei2], $\varphi$ has generalized zeros. Thus after some elementary transformations,

$$\varphi = \begin{pmatrix} 0 & \varphi_1 & \varphi_2 \\ \varphi_3 & a & b \\ \varphi_4 & c & d \end{pmatrix}$$

$\varphi_1, \varphi_2, \varphi_3, \varphi_4$ and $a, b, c, d$ linear forms, $\{\varphi_1, \varphi_2\}, \{\varphi_3, \varphi_4\}$ linearly independent.
As $f \in \langle\varphi_1, \varphi_2\rangle \cap \langle\varphi_3, \varphi_4\rangle$, we can suppose that $\varphi_1$ and $\varphi_3$ have non-zero coefficient of $y_1$. So, we can choose $\varphi_i, i = \overline{1,4}$ as follows:

$$\begin{cases} \varphi_1 = y_1 - \lambda_1 y_3,\ \varphi_2 = y_2 - \lambda_2 y_3 & \text{or} \quad \varphi_1 = y_1 - \lambda y_2,\ \varphi_2 = y_3 \\ \varphi_3 = y_1 - \xi_1 y_2,\ \varphi_4 = y_2 - \xi_2 y_3 & \text{or} \quad \varphi_3 = y_1 - \xi y_2,\ \varphi_4 = y_3. \end{cases}$$

Since $\det\varphi = f$, the points $(\lambda_1 : \lambda_2 : 1), (\xi_1 : \xi_2 : 1), (\lambda : 1 : 0), (\xi : 1 : 0)$ lay in $V(f)$. Therefore $\lambda = \xi = 0$.
For any $\lambda = (\lambda_1 : \lambda_2 : 1)$ in $V(f)$, we write $\varphi_{1\lambda} = y_1 - \lambda_1 y_3$, $\varphi_{2\lambda} = y_2 - \lambda_2 y_3$ and for $\lambda = (0 : 1 : 0)$ we write $\varphi_{1\lambda} = y_1, \varphi_{2\lambda} = y_3$.
Then $\varphi$ has the form:

$$\varphi = \begin{pmatrix} 0 & \varphi_{1\lambda} & \varphi_{2\lambda} \\ \varphi_{1\xi} & a & b \\ \varphi_{2\xi} & c & d \end{pmatrix}$$

with $a, b, c, d$ linear forms.

Notice that, since $f \notin \langle y_1^2, y_1y_3, y_3^2\rangle$, $\varphi$ can not have the form $\begin{pmatrix} 0 & y_1 & y_3 \\ y_1 & a & b \\ y_3 & c & d \end{pmatrix}$, therefore, it is not possible that $\lambda = \xi = (0 : 1 : 0)$.
To finish the proof, we need two helping results:

**Lemma 1.4.** *Let $M$ be a three-generated, rank one, graded MCM $R$-module and $(\varphi, \psi)$ a matrix factorization of $M$, $\varphi$ having the above form. Then there exists $\lambda' \in V(f)\backslash\{(0 : 1 : 0)\}, a', b', c', d'$ linear forms such that the matrix*

$$\varphi' = \begin{pmatrix} 0 & \varphi_{1\lambda'} & \varphi_{2\lambda'} \\ y_1 & a' & b' \\ y_3 & c' & d' \end{pmatrix}$$

*together with its adjoint matrix $\psi'$ form another matrix factorization $(\varphi', \psi')$ of $M$.*

*Proof.* We have to prove that after some elementary transformation the matrix $\varphi$ will become $\varphi'$, that means, there exist two invertible $3 \times 3$ matrices $U$ and $V$, with entries in $k$, such that $U\varphi' = \varphi V$. For this, it is sufficient to prove that there exist two invertible $3 \times 3$ matrices $U, V$ such that the first column of $U^{-1}\varphi V$ is $\begin{pmatrix} 0 \\ y_1 \\ y_3 \end{pmatrix}$.
Considering $U = (u_{ij})_{1\le i,j\le 3}$ and $V = (v_{ij})_{1\le i,j\le 3}$, the above condition lead to the following system of equations:

$$\begin{cases} \varphi_{1\lambda}v_{21} + \varphi_{2\lambda}v_{31} = y_1u_{12} + y_3u_{13} \\ \varphi_{1\xi}v_{11} + av_{21} + bv_{31} = y_1u_{22} + y_3u_{23} \\ \varphi_{2\xi}v_{11} + cv_{21} + dv_{31} = y_1u_{32} + y_3u_{33}. \end{cases}$$

In particular, $\varphi(0,1,0) \cdot \begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Since $\det(\varphi(0,1,0)) = f(0,1,0) = 0$, we may choose a non-zero solution $(v_{11}, v_{21}, v_{31})$ which can be completed to an invertible matrix $V$ and such that also the corresponding $(u_{12}, u_{13}, u_{22}, u_{23}, u_{32}, u_{33})$ can be completed to an invertible matrix $U$.

□

**Lemma 1.5.** *Let $M$ be a three-generated, rank one, graded MCM $R$-module and $(\varphi, \psi)$ a matrix factorization of $M$, $\varphi$ having the form:*

$$\varphi = \begin{pmatrix} 0 & \varphi_{1\lambda} & \varphi_{2\lambda} \\ y_1 & a & b \\ y_3 & c & d \end{pmatrix}$$

*where $a, b, c, d$ are linear forms. Then $(\alpha_\lambda, \beta_\lambda)$ is another matrix factorization of $M$.*

*Proof.*

Observation:

1. Subtracting the first column, multiplied with some nonzero constant, from the second, respectively the third column, one can annhilate the coefficient of the variable $y_1$ at the entries $a, b$. Similarly, subtracting the first line from the last one, one can "kill" $y_1$ also in $c$.

2. More, since $\lambda \neq (0:1:0)$ we can consider that $b$ is in $\langle y_3 \rangle_k$ .
Indeed, if $b = b_2 y_2 + b_3 y_3$, we subtract the first line multiplied with $b_2$ from the second line (to "kill" $y_2$ in $b$) and then add the first column multiplied with $b_2$ to the second column (to "kill" the new $y_1 b_2$ in $a$).
Therefore, instead of $b$ we can write $by_3$ with $b \in k$.

Consider the following polynomials (2-minors of $\alpha_\lambda$ and $\varphi$):

$\gamma = \begin{vmatrix} y_1 & by_3 \\ y_3 & d \end{vmatrix}$, $\delta = \begin{vmatrix} y_1 & a \\ y_3 & c \end{vmatrix}$, $\bar{\gamma} = \begin{vmatrix} y_1 & (\lambda_1^2 + \lambda_1) y_3 \\ y_3 & -y_1 - (\lambda_1 + 1) y_3 \end{vmatrix}$
and $\bar{\delta} = \begin{vmatrix} y_1 & y_2 + \lambda_2 y_3 \\ y_3 & 0 \end{vmatrix}$.

Since $\det\varphi = \det\alpha_\lambda = f$, we have the following equality:
$\varphi_{1\lambda}(\bar{\gamma} - \gamma) = \varphi_{2\lambda}(\bar{\delta} - \delta)$.        $(*)$

So $\varphi_{1\lambda} \mid \bar{\delta} - \delta$. But $\bar{\delta} - \delta = -c(y_1 - \lambda_1 y_3) - y_3(y_2 + \lambda_2 y_3 + \lambda_1 c - a)$ and $a, c \in \langle y_2, y_3 \rangle_k$. Therefore, $a = y_2 + \lambda_2 y_3 + \lambda_1 c$ and $\bar{\delta} - \delta = -c\varphi_{1\lambda}$. Replacing $\bar{\delta} - \delta$ in $(*)$ we get $\bar{\gamma} - \gamma = -c(y_2 - \lambda_2 y_3)$. But $\bar{\gamma} - \gamma = y_1(-y_1 - (\lambda_1 + 1)y_3) - d) - y_3^2(\lambda_1^2 + \lambda_1 - b)$ and $c \in \langle y_2, y_3 \rangle_k$. Therefore $d = -y_1 - (\lambda_1 + 1)y_3, b = \lambda_1^2 + \lambda_1$ and $c = 0$. This shows that $\varphi \sim \alpha_\lambda$. □

Using Lemma 1.4 and Lemma 1.5 the proof of the theorem 1.3 is finished. □

**Theorem 1.6.** *Let $Y$ be the projective cone over $R$.*

1. *The coherent sheaves associated to the modules from $\mathcal{M}_0$ give all the isomorphism classes of line bundles of degree 0 over $Y$;*
2. *The coherent sheaf associated to* Coker $\alpha_s$ *is not locally free. Together with the sheafifications of the modules from $\underline{\mathcal{M}}$ they give all the isomorphism classes of rank one, non-locally free aCM sheaves.*

*Proof.* By computing $\mathrm{Fitt}_2(\alpha_\lambda)R_{\langle y_1, y_2 \rangle}$ we find that Coker $\alpha_\lambda$ is locally free if and only if $\lambda$ is a regular point of $Y$ (we use again [TJP, Prop 1.3.8]). In the previous subsection we have proved that the modules from $\mathcal{M}_{-1}$ and $\mathcal{M}_1$ have degree –1, respectively 1. So, after some shiftings, they give all line bundles of degree $3k - 1$ and $3k + 1$, with $k \in \mathbb{Z}$. Therefore, the remaining rank one graded MCM $R$–modules (the one from $\mathcal{M}_0$), define the line bundles of degree $3k$.
1) We prove first that the sheafification of Coker $\alpha_\xi$ has degree 0, where $\xi = (-1 : 0 : 1)$.
Let us compute $(\mathrm{Coker}\,\alpha_\xi \otimes \mathrm{Coker}\,\alpha_\xi)^{\vee\vee}$.

```
setring S;
matrix M[3][3]=   0, y(1)-y(3),  y(2),
               y(1),      y(2),     0,
               y(3),         0, -y(1);
tensorCM(M,M);
_[1,1]=0
```

This means that Coker $\alpha_\xi$ is a self dual module, so the matrix $\alpha_\xi$ corresponds to $\mathcal{B}(0, 1, -1)$.
Consider the graded map $\alpha_\xi : R(-2)^3 \longrightarrow R(-1)^3$, that should have the degree a multiple of 3, let it be $3t$.

From the graded exact sequence

$$0 \longrightarrow (\operatorname{Coker} \alpha_\xi)^\vee \longrightarrow R(1)^3 \longrightarrow R(2)^3 \longrightarrow \operatorname{Coker} \alpha_\xi^t \longrightarrow 0$$

we see that the degree of $\operatorname{Coker} \alpha_\xi^t$ is $9 - 3t$. But there exists a graded isomorphism between $\operatorname{Coker} \alpha_\xi$ and $\operatorname{Coker} \alpha_\xi^t \otimes R(-3)$. So, $(9-3t) - 9 = 3t$, that means $t = 0$.
2) Let us now consider $\operatorname{Coker} \alpha_\lambda$, with $\lambda = (a : b : 1)$ a regular point of the simple node. We know its sheafification has degree of the form $3t$.
Consider also the module $\operatorname{Coker} \psi_\xi$, with the corresponding graded map $\psi_\xi : R(-2)^2 \longrightarrow R \oplus R(-1)$. As we have seen in the previous subsection, this module has the degree 1. The line bundle corresponding to $\operatorname{Coker} \psi_\xi$ has the form $\mathcal{B}(1, 1, \mu)$ and the one corresponding to $\operatorname{Coker} \alpha_\lambda$ has the form $\mathcal{B}(3t, 1, \mu')$.
By Serre duality, for any two vector bundles $\mathcal{F}, \mathcal{E}$ on the simple node,

$$\operatorname{Hom}(\mathcal{F}, \mathcal{E}) = \operatorname{Ext}(\mathcal{E}, \mathcal{F}) = \mathrm{H}^1(\mathcal{E}^\vee \otimes \mathcal{F}).$$

Therefore, $\operatorname{Ext}^1(\mathcal{B}(1,1,\mu), \mathcal{B}(3t,1,\mu')) = \mathrm{H}^1(\mathcal{B}(-1,1,\mu^{-1}) \otimes \mathcal{B}(3t,1,\mu')) = \mathrm{H}^1(\mathcal{B}(3t-1,1,\mu^{-1}\mu'))$. From [B], $\dim_k(\mathrm{H}^1(\mathcal{B}(3t-1,1,\mu^{-1}\mu'))) = 1$ if and only if $3t - 1 = -1$, so, if and only if $t = 0$.
We prove that $\dim_k(\operatorname{Ext}^1(\operatorname{Coker} \psi_\xi, \operatorname{Coker} \alpha_\lambda) = 1$, for any $\lambda = (a : b : 1)$ regular point of the simple node. With this we are done.
According to the theorem 0.1, a module $M$ with a graded extension

$$0 \longrightarrow \operatorname{Coker} \alpha_\lambda \longrightarrow M \longrightarrow \operatorname{Coker} \psi_\xi \longrightarrow 0,$$

is the cokernel of a graded map $T : R^5(-2) \longrightarrow R^3(-1) \oplus R \oplus R(-1)$, given by a square $5 \times 5$ matrix of the form $\begin{pmatrix} \alpha_\lambda & D \\ 0 & \psi_\xi \end{pmatrix}$.
$D$ is a $3 \times 2$ matrix of the form $\begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \\ d_5 & d_6 \end{pmatrix}$ with linear entries, such that $\beta_\lambda D \phi_\xi = 0$ in $R$.
We make some linear transformations on $T$, in order to obtain a simple form of $D$. First of all, we can eliminate the variable $y_1$ in all entries of $D$, by subtracting one of the first three columns multiplied with some constants. By subtracting the last line from the first one we eliminate the variable $y_2$ in the entry $d_1$. We add in this way $y_1$ to the entry $d_2$, but we can "kill" it using the second column. In the same way, using the last line and the third

column of $T$, we eliminate $y_2$ also in the entry $d_5$. By subtracting the first column from the last one, we eliminate the variable $y_3$ in $d_6$. We "kill" the new $y_1$ in the entry $d_4$ using the last line.
Let us now study the relation $\beta_\lambda D\phi_\xi = 0$ in $R$. For simplicity, we use for this the computer.

The procedure `condext` returns the ideal given by the coefficients of $y_1$ in the entries of a matrix, after it reduces the entries to the polynomial $y_1^3 + y_1^2 y_3 - y_2^2 y_3$.
This procedure will be used also later, in the subsection 2.3.

```
LIB"matrix.lib";
LIB"homolog.lib";
LIB"linalg.lib";

proc condext(matrix G)
{
  ideal g=flatten(G);
  matrix V;ideal P;
  int k,j;
  P=0;
  for(j=1;j<=size(G);j++)
   { g[j]=reduce(g[j],std(y(1)^3+y(1)^2*y(3)-y(2)^2*y(3)));
     V=coef(g[j],y(1));
     for(k=1;k<=1/2*size(V);k++)
     {
       P=P+V[2,k];
     };
   }
P=interred(P);
return(P);
}
```

We define the ring $R$ and the matrices $\psi_\xi$, $\phi_\xi$, respectively $\alpha_\lambda$.

```
ring R=0,(y(1..3),d(1..6),a,b),(c,dp(3),dp(6),dp(2));
ideal i=y(1)^3+y(1)^2*y(3)-y(2)^2*y(3),a3+a2-b2;
qring S=std(i);

matrix psi[2][2]= y(1)^2,-y(2)*y(3),
```

```
                    -y(2), y(1)+y(3);

matrix phi[2][2]=y(1)+y(3), y(2)*y(3),
                       y(2), y(1)^2;

matrix A[3][3]=0,y(1)-a*y(3),     y(2)-b*y(3),
           y(1),y(2)+b*y(3),     (a2+a)*y(3),
           y(3),         0,-y(1)-(a+1)*y(3);
```

We define the matrix $D$ and put the condition $\beta_\lambda D \phi_\xi = 0$.

```
matrix Aa=adjoint(A);
matrix D[3][2]=d(1..6);
matrix G=Aa*D*phi;
ideal P;
P=condext(G);
P[4]=P[4]/y(3)^2;
P[5]=P[5]/y(3)^2;
P;
```

The ideal of the coefficients is:

```
P[1]=y(2)*d(6)-y(3)*d(3)-y(3)*d(5)*a
P[2]=y(2)*d(2)+y(2)*d(5)+y(3)*d(1)*a+y(3)*d(1)-y(3)*d(5)*b
P[3]=y(2)*d(1)-y(2)*d(4)+y(3)*d(1)*b-y(3)*d(3)-y(3)*d(5)*a^2-
    -y(3)*d(5)*a
P[4]=d(1)*b-d(2)*a^2-d(4)*b-d(5)*a^2-d(6)*b
P[5]=d(1)*a+d(1)-d(2)*b-d(4)*a-d(4)-d(5)*b-d(6)*a-d(6)
```

From P[1], we obtain $y_3|d_6$, but we have already eliminated the variable $y_3$ in $d_6$, so we can suppose that $d_6 = 0$. Under this condition, P[1] implies that $d_3 = -ad_5$.
From P[2] we obtain that $y_2|d_1(a+1)-d_5 b$, but $d_1$ and $d_5$ have no $y_2$. Therefore, $d_1(a+1) - d_5 b = d_2 + d_5 = 0$. Since $a \neq 0$, P[5] implies that $d_4 = d_1$ and P[4] implies $d_5 = d_1 b/a^2$. We know that $d_1$ has the form $d_1 = a_1 y_3$, with $a_1 \in k$.
Since we need that the matrix $T$ do not decomposes (nonzero extension), $a_1/a^2$ can not be zero. Dividing the last two columns by $a_1/a^2$, and multiplying the last two lines with $a_1$, we obtain the matrix:

$$T = \begin{pmatrix} 0 & y_1 - ay_3 & y_2 - by_3 & a^2y_3 & -by_3 \\ y_1 & y_2 + by_3 & (a^2+a)y_3 & -by_3 & a^2y_3 \\ y_3 & 0 & -y_1 - (a+1)y_3 & by_3 & 0 \\ 0 & 0 & 0 & y_1^2 & -y_2y_3 \\ 0 & 0 & 0 & -y_2 & y_1 + y_3 \end{pmatrix}.$$

One can see easily that this matrix do not decomposes. (There are no matrices $U, V$ such that $D = \alpha_\lambda U + V\psi_\xi$)
This proves that $\dim_k(\operatorname{Ext}^1(\operatorname{Coker}\psi_\xi, \operatorname{Coker}\alpha_\lambda) = 1$, and with this we are done. □

**Proposition 1.7.** *Every three-generated, rank 2, indecomposable, graded MCM R-module is isomorphic with one of the modules* $\operatorname{Coker}\beta_\lambda$, $\lambda = (\lambda_1 : \lambda_2 : 1) \in V(f)$.

*Proof.* Let $M$ be a three-generated, rank two, indecomposable, graded MCM $R$-module and $(\varphi, \psi)$ a matrix factorization of $M$. Then $\operatorname{Coker}\psi$ is a three-generated, rank one, graded MCM $R$-module. Therefore, it is isomorphic to one of the modules from $\mathcal{M}_0 \cup \{\operatorname{Coker}\alpha_s\}$ and $\operatorname{Coker}\varphi$ is isomorphic to one of the modules from $\{\operatorname{Coker}\beta_\lambda | \lambda = (\lambda_1 : \lambda_2 : 1), \lambda \in V(f)\}$. □

We present a short overview over the results concerning the rank one graded MCM $R$–modules:

The rank one graded MCM R-modules corresponding to the **line bundles** on $Y$ are given by the cokernel of some graded maps defined by the following matrices, with $(\lambda_1 : \lambda_2 : 1)$ a regular point on $Y$:

$$\begin{pmatrix} y_1 - \lambda_1 y_3 & y_2 y_3 + \lambda_2 y_3^2 \\ y_2 - \lambda_2 y_3 & y_1^2 + (\lambda_1 + 1) y_1 y_3 + (\lambda_1^2 + \lambda_1) y_3^2 \end{pmatrix}, \begin{pmatrix} y_1 + y_3 & y_2^2 \\ y_3 & y_1^2 \end{pmatrix},$$

$$\begin{pmatrix} y_1^2 + (\lambda_1 + 1) y_1 y_3 + (\lambda_1^2 + \lambda_1) y_3^2 & - y_2 y_3 - \lambda_2 y_3^2 \\ -(y_2 - \lambda_2 y_3) & y_1 - \lambda_1 y_3 \end{pmatrix}, \begin{pmatrix} y_1^2 & -y_2^2 \\ -y_3 & y_1 + y_3 \end{pmatrix},$$

$$\begin{pmatrix} 0 & y_1 - \lambda_1 y_3 & y_2 - \lambda_2 y_3 \\ y_1 & y_2 + \lambda_2 y_3 & (\lambda_1^2 + \lambda_1) y_3 \\ y_3 & 0 & -y_1 - (\lambda_1 + 1) y_3 \end{pmatrix}.$$

The rank one graded MCM $R$-modules corresponding to the **non-locally free** aCM sheaves are given by the following set of matrices:

$$\{ \begin{pmatrix} y_1 & y_2 y_3 \\ y_2 & y_1^2 + y_1 y_3 \end{pmatrix}, \begin{pmatrix} y_1^2 + y_1 y_3 & - y_2 y_3 \\ -y_2 & y_1 \end{pmatrix}, \begin{pmatrix} 0 & y_1 & y_2 \\ y_1 & y_2 & 0 \\ y_3 & 0 & -y_1 - y_3 \end{pmatrix} \}.$$

# 2 Rank two, graded, MCM modules over the hypersurface $R = k[y_1, y_2, y_3]/(y_1^3 + y_1^2 y_3 - y_2^2 y_3)$

In the following there are described all isomorphism classes of rank two, indecomposable MCM modules over $R = k[y_1, y_2, y_3]/(f), f = y_1^3 + y_1^2 y_3 - y_2^2 y_3$, using their matrix factorizations.
In the case of locally free MCM modules, it is used the classification of vector bundles over the projective cone $Y = \text{Proj } R$, from [B].
The main "ingredient" is the existence of two extensions that determine uniquely the rank 2 vector bundles $\mathcal{B}(0, 2, 1)$ and $\mathcal{B}((0, 1), 1, \lambda)$; very important is also the SINGULAR procedure for computing the tensor product of two locally free MCM modules ( see [LPP] ).
Furthermore, it is given a characterization of the MCM modules with stable sheafification, over a projective curve with arithmetic genus 1, using a matrix factorization of it. I construct also the matrix factorizations corresponding to stable vector bundles of rank two over the simple node $Y$.
In the last part of this section, there are computed matrix factorizations of the non–locally free, rank two, MCM modules over $R$, using Proposition 0.1. They correspond to the rank two, non–locally free aCM sheaves on the simple node $Y$. There are two families parametrized by the regular points of the curve $Y = V(f)$ and 22 countable families.

## 2.1 The classification of rank two locally free MCM R–modules

According to the classification of the vector bundles on the simple node singularity from [B], there are two types of rank two, indecomposable, vector bundles on Proj$R$:
• $\mathcal{B}(a, 2, \lambda)$, with $a \in \mathbb{Z}$ and $\lambda \in k^*$
•$\mathcal{B}(\mathbf{d}, 1, \lambda)$, with $\mathbf{d}$ a 2-cycle with entries in $\mathbb{Z}$ and $\lambda \in k^*(\mathbf{d} = (a, b), a \neq b)$.

To generate the first type of rank two vector bundles it is sufficient to know the bundle $\mathcal{B}(0, 2, 1)$ and the line bundles, because:

$$\mathcal{B}(a,2,\lambda) \cong \mathcal{B}(a,1,\lambda) \otimes \mathcal{B}(0,2,1).$$

The fact that the bundle $\mathcal{B}(0,2,1)$ is uniquely determined by the exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{B}(0,2,1) \longrightarrow \mathcal{O}_Y \longrightarrow 0, \tag{1}$$

provide a way to determine the graded MCM $R$-module corresponding to it. Using the matrix factorizations of the rank 1 MCM nodules given in the previous section and a SINGULAR–procedure that computes the tensor product of two locally free MCM modules, one can easily construct matrix factorizations for the vector bundles $\mathcal{B}(a,2,\lambda)$.

The second type of rank two vector bundles can be generated by the bundles $\mathcal{B}((0,n),1,\lambda)$ and the line bundles, using the tensor product formula: $\mathcal{B}((a,b),1,\lambda) \otimes \mathcal{B}(c,1,\mu) \cong \mathcal{B}((a+c,b+c),1,\lambda\mu^2)$.

For $\mathbf{d} = (a,b)$ and $\mathbf{e} = (c,d)$ two 2-cycles with entries in $\mathbb{Z}$, we have:

$$\mathcal{B}(\mathbf{d},1,\lambda) \otimes \mathcal{B}(\mathbf{e},1,\mu) \cong \mathcal{B}(\mathbf{f}_1,1,\lambda\cdot\mu) \oplus \mathcal{B}(\mathbf{f}_2,1,\lambda\cdot\mu),$$

where $\mathbf{f}_1 = (a+c, b+d)$ and $\mathbf{f}_2 = (a+d, b+c)$. If $\mathbf{f}_i = (\alpha,\alpha) (i = 1 \text{ or } 2)$, then $\mathcal{B}(\mathbf{f}_i,1,\lambda\cdot\mu)$ splits as: $\mathcal{B}(\mathbf{f}_i,1,\lambda\cdot\mu) = \mathcal{B}(\alpha,1,\sqrt{\lambda\cdot\mu}) \oplus \mathcal{B}(\alpha,1,-\sqrt{\lambda\cdot\mu})$.

Therefore, inductively, we can obtain all $\mathcal{B}((0,n),1,\lambda)$, $n \in \mathbb{N}^*$, if we know the bundles $\mathcal{B}((0,1),1,\lambda)$. By duality, $(\mathcal{B}(\mathbf{d},1,\lambda)^\vee \cong \mathcal{B}(-\mathbf{d},1,\lambda^{-1}))$ we obtain also $\mathcal{B}((0,n),1,\lambda)$ with n negative integer.
The bundles $\mathcal{B}((0,1),1,\lambda)$ are uniquely determined by the existence of the exact sequences

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{B}((0,1),1,\lambda) \longrightarrow \mathcal{B}(1,1,-\lambda) \longrightarrow 0. \tag{2}$$

Using this, I compute the graded MCM $R$-modules corresponding to them. So, inductively, one can obtain all rank two graded indecomposable locally free MCM $R$-modules.

In the sequel we determine the module $M_2$ corresponding to $\mathcal{B}(0,2,1)$.

**Lemma 2.1.** *Let be* $\rho = \begin{pmatrix} y_1^2 + y_1y_3 & -\,y_2 & -\,y_3 & 0 \\ -y_2y_3 & y_1 & 0 & -\,y_3 \\ 0 & 0 & y_1 & y_2 \end{pmatrix}$,

$$\psi = \begin{pmatrix} y_1 & y_2 & y_3 & 0 \\ y_2y_3 & y_1^2+y_1y_3 & 0 & y_3 \\ 0 & 0 & y_1^2+y_1y_3 & -y_2 \\ 0 & 0 & -y_2y_3 & y_1 \end{pmatrix},$$

$\gamma = \begin{pmatrix} 0 & 0 & y_2y_3 & y_1^2+y_1y_3 \end{pmatrix}$ *and* $\varphi = \begin{pmatrix} \rho \\ \gamma \end{pmatrix}$. *Then* $(\psi, \varphi)$ *is a matrix factorization of* $\Omega^1_R(m)$, *where* $m$ *is the unique graded maximal ideal of* $R$, $m = \langle y_1, y_2, y_3\rangle$. *More, the following exact sequence*

$$\xrightarrow{\psi} R(-3) \oplus R(-2)^3 \xrightarrow{\rho} R(-1)^3 \xrightarrow{(y_1,y_2,y_3)} m \longrightarrow 0 \qquad (3)$$

*is a graded minimal free resolution of* $m$. *In particular,* $\Omega^1_R(m)$ *has no free summands.*

*Proof.* Clearly $\varphi\psi = \psi\varphi = f \cdot \mathbf{1}_4$ and the above sequence is a complex.

Let $u_1, u_2, u_3 \in R$ such that $\sum_{i=1}^{3} y_iu_i = 0$. We show that $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ is an element of Im$\rho$. Subtracting multiples of the second and third columns of $\rho$ from $u$ we may suppose that $u_1$ depends only on $y_1$. As the maps are graded, we may suppose that $u$ is graded, so $u_1 = ay_1^s, a \in k, s \in \mathbb{N}$. If $a \neq 0$, $s+1 \geq 3$, so, subtracting from $u$ multiples of $\begin{pmatrix} y_1^2 \\ -y_2y_3 \\ y_1^2 \end{pmatrix} \in$ Im$\rho$, we reduce to the case $u_1 = 0$. Then $y_2u_2 + y_3u_3 = 0$ and, since $\{y_2, y_3\}$ is a regular sequence in $R$, $u$ is a multiple of the fourth column of $\rho$.
To show that $\operatorname{Ker}\rho \subset \operatorname{Im}\psi$, it is enough to prove that $\operatorname{Ker}\rho \subset \operatorname{Ker}\varphi$.
We denote with $\rho_3$ the third row of $\rho$ and choose $\nu$ an element of $\operatorname{Ker}\rho$. Since $y_1\gamma = y_2y_3\rho_3$ and $\rho_3\nu = 0$, $y_1(\gamma\nu) = 0$. So $\gamma\nu = 0$ and $\nu$ is an element of $\operatorname{Ker}\varphi$.
Because no entry of $\varphi$ or $\psi$ is unite, $\Omega^1_R(m)$ has no free summands.

□

**Proposition 2.2.** *There exists a graded exact sequence:*

$$0 \longrightarrow R \xrightarrow{i} \Omega^2_R(m) \otimes R(3) \xrightarrow{\pi} m \longrightarrow 0 \qquad (4)$$

*and* $\Omega^2_R(m) \otimes R(3)$ *corresponds to the bundle* $\mathcal{B}(0, 2, 1)$.

*Proof.* 1) We prove the existence of the exact sequence (4). Define the map $i : R \longrightarrow \Omega^2_R(m) \otimes R(3)$ by $i(1) = \begin{pmatrix} 0 \\ y_3 \\ -y_2 \\ y_1 \end{pmatrix}$ (the fourth column of $\psi$) and let $\pi : \Omega^2_R(m) \otimes R(3) \longrightarrow m$ be the projection on the first component. Since $\Omega^2_R(m) \otimes R(3) = \mathrm{Im}\psi \otimes R(3) \subset R \oplus R(1)^3$, $i$ and $\pi$ are graded morphisms. Clearly, $i$ is injective, $\pi$ is surjective and $\mathrm{Im}\, i \subset \mathrm{Ker}\, \pi$. We prove that $\mathrm{Ker}\, \pi \subset \mathrm{Im} i$.

Let $v = \psi \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ be an element in $\mathrm{Ker}\, \pi$. Then $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathrm{Im}\, \rho$.

Denote by $\psi'$ the $4 \times 3$ matrix obtained from $\psi$ by eliminating the last column. Then $v = \psi' \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} + d \begin{pmatrix} 0 \\ y_3 \\ -y_2 \\ y_1 \end{pmatrix}$. So we have to prove that the columns of $\psi'\rho$ are in $\mathrm{Im} i$.

We write $\psi'\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & y_3 \cdot (-y_2 y_3) & y_3 \cdot (-y_1^2 - y_1 y_3) \\ 0 & 0 & -y_2 \cdot (-y_2 y_3) & -\, y_2 \cdot (-y_1^2 - y_2^2) \\ 0 & 0 & y_1 \cdot (-y_2 y_3) & y_1 \cdot (-y_1^2 - y_2^2) \end{pmatrix}$.

Therefore, $v \in R \cdot \begin{pmatrix} 0 \\ y_3 \\ -y_2 \\ y_1 \end{pmatrix} = \mathrm{Im} i.$

2) Let $M_2$ be the graded $R$-module $\Omega^2_R(m) \otimes R(3)$. We prove that it is indecomposable.

If it would decompose, it would be isomorphic to $\mathrm{Coker}\, \theta$, with $\theta$ of the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, $A$ and $B$ quadratic matrices with determinant equal to $f$. By the Corollary 6.4 from [Ei1], $A$ and $B$ define rank 1, graded MCM $R$-modules, so they are equivalent to one of $\{\varphi_\lambda, \psi_\lambda \mid \lambda \in V(f)\}$.(see the previous sections) Since $\theta \sim \varphi$, their fitting ideals are equal, so $\mathrm{Fitt}_2(\theta) = \mathrm{Fitt}_2(\varphi) = m^2$.

But the elements of degree 2 from $\mathrm{Fitt}_2(\theta)$ are given just by $l_1^A \cdot l_1^B$, $l_1^A \cdot l_2^B$, $l_2^A \cdot l_1^B$, $l_2^A \cdot l_2^B$ where $l_1^A, l_2^A$ respectively $l_1^B, l_2^B$ are the entries of $A$ and $B$ of degree 1. The ideal generated by them is not $m^2$ since $m^2$ is minimally generated by 6 elements. Therefore, $M_2$ is indecomposable.

3) To prove that $M_2$ is locally free, it is sufficient to notice that $\mathrm{Fitt}_2(\varphi)R_{\langle y_1,y_2\rangle} = R_{\langle y_1,y_2\rangle}$ and $\mathrm{Fitt}_3(\varphi) = 0$ (see Proposition 1.3.8, [TJP]).

4) From the exact sequence (4) we get the following exact sequence of vector bundles on $Y = \mathrm{Proj}R$:

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \widetilde{M_2} \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

$\widetilde{M_2}$ is an indecomposable vector bundle of rank 2, so it is isomorphic to $\mathcal{B}(0,2,1)$.

□

Using this result we obtain the following theorem:

**Theorem 2.3.** *The rank two vector bundles of the type $\mathcal{B}(a,2,\lambda)$, with $\lambda \in k^*$ and a integer, are sheafification of the $R$-modules $(M_2 \otimes L)^{\vee\vee} \otimes R(k)$, with $L$ in $\mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_{-1}$ and $k \in \mathbb{Z}$. More, the modules corresponding to*

$$\begin{cases} \mathcal{B}(0,2,\lambda), \lambda \neq 1, & \textit{have 6 generators;} \\ \mathcal{B}(-1,2,\lambda), & \textit{have 4 generators;} \\ \mathcal{B}(1,2,\lambda), & \textit{have 4 generators.} \end{cases}$$

*Proof.* The first statement is a direct consequence of the previous proposition, if we consider the following tensor product:
$\mathcal{B}(a,2,\lambda) \cong \mathcal{B}(a,1,\lambda) \otimes \mathcal{B}(0,2,1)$.
Therefore the modules corresponding to the bundles $\mathcal{B}(0,2,\lambda)$ are $(M_2 \otimes \mathrm{Coker}\,\alpha_\lambda)^{\vee\vee}$ and we can compute their matrix factorizations using the computer. We use the procedures that compute the reflexive hull and the tensor product in the category of locally free Cohen-Macaulay modules from [LPP].

```
  LIB"matrix.lib";
  option(redSB);

proc reflexivHull(matrix M)
{
  module N=mres(transpose(M),3)[3];
  N=prune(transpose(N));
  return(matrix(N));
}
```

```
proc tensorCM(matrix Phi, matrix Psi)
{
   int s=nrows(Phi);
   int q=nrows(Psi);
   matrix A=tensor(unitmat(s),Psi);
   matrix B=tensor(Phi,unitmat(q));
   matrix R=concat(A,B);
   return(reflexivHull(R));
}

proc M2(ideal I)
{
   matrix A=syz(transpose(mres(I,3)[3]));
   return(transpose(A));
}

ring R=0,(y(1..3)),(c,dp);
ideal i=y(1)^3+y(1)^2*y(3)-y(2)^2*y(3);
qring S=std(i);
ideal I=maxideal(1);
matrix C=M2(I);
```

We define the matrix $\alpha_\lambda$, for $\lambda = (a : b : 1)$.

```
ring R1=0,(y(1..3),a,b),(c,dp);
ideal I=y(1)^3+y(1)^2*y(3)-y(2)^2*y(3),a3+a2-b2;
qring S1=std(I);
matrix A[3][3]=    0, y(1)-a*y(3),      y(2)-b*y(3),
                y(1), y(2)+b*y(3),      (a2+a)*y(3),
                y(3),           0, -y(1)-(a+1)*y(3);
matrix C=imap(S,C);
```

We compute the dimension of the matrix corresponding to $(M_2 \otimes \mathrm{Coker}\, \alpha_\lambda)^{\vee\vee}$.

```
nrows(tensorCM(C,A));
6
```

The same, the modules corresponding to the bundles $\mathcal{B}(-1,2,\lambda)$ are $(M_2 \otimes \operatorname{Coker}\varphi_\lambda)^{\vee\vee}$. We compute the size of the matrix corresponding to this module.

```
matrix B[2][2]=y(1)-a*y(3),                    y(2)*y(3)+b*y(3)^2,
               y(2)-b*y(3),y(1)^2+(a+1)*y(1)*y(3)+(a2+a)*y(3)^2;
nrows(tensorCM(C,B));
4
B= y(1)+y(3),y(2)^2,
   y(3),       y(1)^2;
nrows(tensorCM(C,B));
4
```

By duality, it follows also the last statement. □

Let $\xi = (-1:0:1)$ and $\lambda_0 = (0:1:0)$ two regular points on the curve $Y = \operatorname{Proj} R$. We consider the graded maps $\psi_\xi : R(-2)^2 \longrightarrow R \oplus R(-1)$ and $\alpha_{\lambda_0} : R(-2)^3 \longrightarrow R(-1)^3$, given by the matrices with the same name. (see the previous section)

**Lemma 2.4.** *If we denote by $\mu_0$ the regular point on the nodal curve $Y$ such that $\widetilde{\operatorname{Coker}\psi_\xi} \cong \mathcal{B}(1,1,\mu_0)$, then $\mathcal{O}_Y(1) = \mathcal{B}(3,1,-\mu_0^3)$ and the line bundle $\mathcal{B}(1,1,-\mu_0)$ is given by* $\operatorname{Coker}\psi_{\lambda_0}$.

*Proof.* Since the degree of $\operatorname{Coker}\psi_{\lambda_0}$ is 1 (see 1.2 ) we can denote $\widetilde{\operatorname{Coker}\psi_{\lambda_0}}$ by $\mathcal{B}(1,1,\theta)$.
Let us compute, the tensor product $(\operatorname{Coker}\alpha_\xi \otimes \operatorname{Coker}\psi_\xi)^{\vee\vee}$

```
matrix N[2][2]= y(1)^2, -y(2)*y(3),
                -y(2),  y(1)-y(3);
matrix L=tensorCM(M,N);
print(L);
  y(1),            -y(3),
y(2)^2, -y(1)^2+y(1)*y(3)
```

It is easy to see that, after some elementary transformations, the matrix L becomes $\psi_{\lambda_0}$, so we have obtained that $(\operatorname{Coker}\alpha_\xi \otimes \operatorname{Coker}\psi_\xi)^{\vee\vee} \cong \operatorname{Coker}\psi_{\lambda_0}$. By sheafification, it becomes $\mathcal{B}(0,1,-1) \otimes \mathcal{B}(1,1,\theta) \cong \mathcal{B}(1,1,-\mu_0)$, therefore $\theta = \mu_0$.

Let us now compute the reflexive hull of $\operatorname{Coker}\psi_{\lambda_0} \otimes \operatorname{Coker}\psi_{\lambda_0} \otimes \operatorname{Coker}\psi_{\lambda_0}$.

```
matrix L[2][2]= y(1)^2, -(y(1)^2+y(2)^2),
                -y(3),            y(1);
tensorCM(L,tensorCM(L,L));
_[1,1]=0
```

This means that $(\text{Coker}\,\psi_{\lambda_0} \otimes \text{Coker}\,\psi_{\lambda_0} \otimes \text{Coker}\,\psi_{\lambda_0})^\vee \cong R$, so the line bundle $\mathcal{O}_Y(1)$ is $\mathcal{B}(3, 1, -\mu_0^3)$. □

In a similar way, we get the following relations:

**Proposition 2.5.** *Let be* $\lambda_0 = (0 : 1 : 0)$*. For any* $\lambda = (\lambda_1 : \lambda_2 : 1) \in V(f)$*, we have:*

$$\begin{cases} \text{Coker}\,\alpha_{(\lambda_1:\lambda_2:1)} \cong (\text{Coker}\,\varphi_{\lambda_0} \otimes \text{Coker}\,\psi_{(\lambda_1:\lambda_2:1)})^{\vee\vee} \\ \text{Coker}\ \alpha_{(\lambda_1:-\lambda_2:1)} \cong (\text{Coker}\ \ \alpha_{(\lambda_1:\lambda_2:1)})^t \cong (\text{Coker}\,\psi_{\lambda_0} \otimes \text{Coker}\ \varphi_{(\lambda_1:\lambda_2:1)})^{\vee\vee}. \end{cases}$$

For the rest of this section, we fix the notations $\xi$, $\lambda_0$ and $\mu_0$ as in the previous lemma.

Let us now determine the graded MCM modules corresponding to the bundles $\mathcal{B}((0,1), 1, \lambda)$ with $\lambda \in k^*$. Consider the module $\Omega^1_R(M_2) = \text{Coker}\,\psi$, where
$\psi : R(-4)^3 \oplus R(-3) \longrightarrow R(-3) \oplus R(-2)^3$ is given as in lemma 2.1.

**Lemma 2.6.** *Consider* $\lambda$ *a regular point on the curve.*

1. $(\Omega^1_R(M_2) \otimes \textit{Coker}\,\alpha_\lambda)^{\vee\vee}$ *has*

$$\begin{cases} \textit{4 generators}, & \textit{if}\ \lambda = (1:0:1); \\ \textit{3 generators}, & \textit{otherwise}; \end{cases}$$

2. $(\Omega^1_R(M_2) \otimes \textit{Coker}\,\varphi_\lambda)^{\vee\vee}$ *has 5 generators;*

3. $(\Omega^1_R(M_2) \otimes \textit{Coker}\,\psi_\lambda)^{\vee\vee}$ *has 5 generators;*

*Proof.* We define the module $\Omega^1_R(M_2)$ by:

```
matrix D=transpose(syz(C));
```

and use the procedure `tensorCM` as before.

□

We compute the matrix corresponding to the MCM module $(\Omega^1_R(M_2)\otimes \mathrm{Coker}\varphi_\xi)^{\vee\vee}$.

```
matrix D=transpose(syz(C));
matrix B[2][2]=y(1)+y(3), y(2)*y(3),
                 y(2),    y(1)^2;
matrix A=tensorCM(D,B);
```

After some linear transformations, the matrix $A$ it becomes:

$$A=\left(\begin{array}{c|cc} & y_1 & 0\\ \alpha_\xi & 0 & 0\\ & 0 & -y_3\\ \hline 0 & \psi_\xi & \end{array}\right).$$

Remark: This matrix is linear equivalent to the matrix $T$ obtained in the proof of Theorem 1.6, for $a=-1$ and $b=0$.

**Proposition 2.7.** *1. The graded module corresponding to $\mathcal{B}((0,1),1,\mu_0)$ is $(\Omega^1_R(M_2)\otimes Coker\varphi_\xi)^{\vee\vee}\otimes R(2)$.*

*2. $\widetilde{\Omega^1_R(M_2)}=\mathcal{B}((-4,-5),1,\mu_0^{-9})$.*

*Proof.* 1) We have the following graded commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R(-2)^3 & \xrightarrow{i} & R(-2)^5 & \xrightarrow{\pi} & R(-2)^2 & \longrightarrow & 0\\ & & \downarrow{\scriptstyle \alpha_\xi} & & \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle \psi_\xi} & & \\ 0 & \longrightarrow & R(-1)^3 & \xrightarrow{i} & R(-1)^3\oplus R\oplus R(-1) & \xrightarrow{\pi} & R\oplus R(-1) & \longrightarrow & 0, \end{array}$$

where $\pi$ is the projection on the last two components and $i$ is defined as $i(\begin{pmatrix} a\\ b\\ c\end{pmatrix})=\begin{pmatrix} a\\ b\\ c\\ 0\\ 0\end{pmatrix}$.

Using the Snake–Lemma and the surjectivity of the map $\operatorname{Ker} A \xrightarrow{\pi} \operatorname{Ker} \psi_\xi$, we obtain the graded exact sequence

$$0 \longrightarrow \operatorname{Coker} \alpha_\xi \longrightarrow \operatorname{Coker} A \longrightarrow \operatorname{Coker} \psi_\xi \longrightarrow 0.$$

By sheafification, it becomes

$$0 \longrightarrow \mathcal{B}(0,1,-1) \longrightarrow \mathcal{B} \longrightarrow \mathcal{B}(1,1,\mu_0) \longrightarrow 0,$$

where $\mathcal{B}$ is $\widetilde{\operatorname{Coker} A}$ . We tensorise it with the locally free sheaf $\mathcal{B}(0,1,-1)$ and we get:

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{B} \otimes \mathcal{B}(0,1,-1) \longrightarrow \mathcal{B}(1,1,-\mu_0) \longrightarrow 0.$$

Since $\mathcal{B}((0,1),1,\mu_0)$ is uniquely determined by the existence of the exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{B}((0,1),1,\mu_0) \longrightarrow \mathcal{B}(1,1,-\mu_0) \longrightarrow 0,$$

the vector bundle $\mathcal{B} \otimes \mathcal{B}(0,1,-1)$ is isomorphic to $\mathcal{B}((0,1),1,\mu_0)$. But this means that $\mathcal{B}$ is isomorphic to $\mathcal{B}((0,1),1,\mu_0)$.

Consider the map $\varphi_\xi : R(-2) \oplus R(-3) \longrightarrow R(-1)^2$ as in the subsection 1.1, such that the degree of $\operatorname{Coker} \varphi_\xi$ is -1.
From the exact sequence (3) we see that $\deg(\Omega^1_R(M_2))=-9$. Therefore the $R$-module corresponding to $\mathcal{B}((0,1),1,\mu_0)$ is $(\Omega^1_R(M_2) \otimes \operatorname{Coker} \varphi_\xi)^{\vee\vee} \otimes R(2)$. It is the cokernel of the graded map $A : R(-2)^5 \longrightarrow R(-1)^3 \oplus R \oplus R(-1)$, defined by the matrix $A$ (see above).

2) The previous statement implies that
$\widetilde{\Omega^1_R(M_2)} = \mathcal{B}((0,1),1,\mu_0) \otimes (\widetilde{\operatorname{Coker} \varphi_\xi})^\vee \otimes \mathcal{O}_Y(-2)$.(we consider the graded maps given as above)
Using 1.1 and 1.2 we obtain that $(\widetilde{\operatorname{Coker} \varphi_\xi})^\vee = \widetilde{\operatorname{Coker} \psi_\xi} = \mathcal{B}(1,1,\mu_0)$.
Therefore $\widetilde{\Omega^1_R(M_2)} = \mathcal{B}((0,1),1,\mu_0) \otimes \mathcal{B}(1,1,\mu_0) \otimes \mathcal{B}(-6,1,\mu_0^{-6}) =$
$\mathcal{B}((0,1),1,\mu_0) \otimes \mathcal{B}(-5,1,\mu_0^{-5}) = \mathcal{B}((-5,-4),1,\mu_0^{-9})$. □

### 2.2 The rank two, stable vector bundles on ProjR and their matrix factorizations

Let $Y \subset \mathbb{P}^2$ be a rational curve. For a locally free sheaf on $Y$, $\mathcal{E}$, we define $\deg \mathcal{E}$ to be $\deg \mathcal{E} = \chi(\mathcal{E}) + (p_a(Y)-1)\operatorname{rank}(\mathcal{E})$.

The vector bundle $\mathcal{E}$ is called *stable* if for any torsion free quotient $\mathcal{F}$ of $\mathcal{E}$,

$$\frac{\deg\ \mathcal{E}}{\mathrm{rank}\ \mathcal{E}} < \frac{\deg\ \mathcal{F}}{\mathrm{rank}\ \mathcal{F}}.$$

A graded MCM module over the affine cone of $Y$ is called *stable* if the its sheafification is a stable vector bundle on $Y$.

Denote $S = k[y_1, y_2, y_3]$ and let $F$ be the homogeneous polynomial defining the curve $Y$.
Let $M$ be a graded, indecomposable, locally free MCM module over the affine cone of $Y$ and let $\mu$ be the minimal number of generators of M.
Consider $(A, A')$ a graded matrix factorization of $F$ corresponding to $M$.
Set $\mathcal{L}_A$ the vector space

$$\mathcal{L}_A = \{D \in \mathcal{M}_{\mu\times\mu}(S) | A'DA' = 0 \text{ modulo}(F)\}$$

and define the following equivalence on it: two matrices $D$ and $D'$ from $\mathcal{L}_A$ are equivalent ($D \sim D'$) iff there exist two quadratic matrices $U$ and $V$ such that $D - D' = UA - AV$.
Denote $\mathcal{S}_A = \mathcal{L}_A / \sim$.

**Remark 2.8.** *If $(B, B')$ is another matrix factorization of $M$, the vector spaces $\mathcal{S}_A$ and $\mathcal{S}_B$ are isomorphic.*

Indeed, if $U$ and $V$ are the invertible matrices such that $B = UAV$, we construct the vector spaces isomorphism:

$$\theta : \mathcal{S}_B \longrightarrow \mathcal{S}_A,$$

$$\theta(D) = U^{-1}DV^{-1}.$$

**Theorem 2.9.** *Let $Y \subset \mathbb{P}^2$ be a projective curve with arithmetic genus 1 and let $M$ be a graded, indecomposable, locally free MCM module over the affine cone of $Y$. The following statements are equivalent:*

1. *$M$ is a stable module;*
2. *dim $\mathcal{S}_A = 1$ for $(A, A')$ a matrix factorization of $M$;*
3. *dim $\mathcal{S}_A = 1$ for all $(A, A')$ matrix factorizations of $M$.*

*Proof.* A.Caldararu has proved that a vector bundle $\mathcal{E}$ on a curve with arithmetic genus 1 is stable if and only if is simple, that means $\mathrm{Ext}^1(\mathcal{E},\mathcal{E})$=1 (see a proof in [B]) The proof of the theorem follows now immediately from 0.1 and the previous remark.

□

In his PhD work (see [B]), I.Burban has proved:

**Proposition 2.10.** *A bundle $\mathcal{B}(\mathbf{d}, m, \lambda)$ on the simple node $Y$ is stable if and only if the following conditions are fulfilled:*
*1.* $m = 1$
*2.* $|d_i - d_j| \leq 1$ *for any* $1 \leq i, j \leq r$
*3. no possible differences* $\mathbf{d} - \mathbf{d}[t]$ *contains a subsequence of the form* 10...01*,* *where* $\mathbf{d}[t] = d_{t+1}...d_r d_1...d_{t-1}d_t$ *is a shifting of* $\mathbf{d}$*.*

Therefore, the rank two stable vector bundles on $Y$ are of the type $\mathcal{B}((a, a\underline{+}1), 1, \lambda)$ with $a$ integer and $\lambda \in k^*$. Since

$$\mathcal{B}((a, b), 1, \lambda) \otimes \mathcal{B}(c, 1, \mu) \cong \mathcal{B}((a + c, b + c), 1, \lambda\mu^2)$$

($a, b, c$ integers, $\lambda, \mu \in k^*$), if we succeed to compute the module corresponding to one of the rank two stable bundles, we can generate all the other by tensoring it with the modules corresponding to line bundles.

From Proposition 2.7 we see that $\widetilde{\Omega^1_R(M_2)}$ is a stable vector bundle. More, using Lemma 2.6 we can say that $\Omega^1_R(M_2)$, up to shifting, is the only rank two stable $R$–module that is is minimally 4–generated. (Notice that its dual is just the module itself shifted with 3) We choose this vector bundle to be the one generating together with the line bundles all the other stable vector bundles.

**Proposition 2.11.** *There exists only one indecomposable, stable, rank two MCM R-module that is even minimally generated. It is 4–generated, and by tensoring it with rank one MCM modules, it generates all other stable modules of rank two. More, all 3-generated, rank 2, indecomposable, graded MCM R-modules that are locally free, have stable sheafification.*

*Proof.* It follows immediately from Lemma 2.6 and Proposition 1.7. For details, see [Ba2].

□

**Proposition 2.12.** *The isomorphism classes of rank 2, indecomposable, stable vector bundles are given by the following MCM R-modules:*

1. Coker $\psi$, *for* $\psi = \begin{pmatrix} y_1 & y_2 & y_3 & 0 \\ y_2y_3 & y_1^2+y_1y_3 & 0 & y_3 \\ 0 & 0 & y_1^2+y_1y_3 & -y_2 \\ 0 & 0 & -y_2y_3 & y_1 \end{pmatrix}$,

2. Coker $\beta_\lambda$, $\lambda \in V(f)_{\mathrm{reg}} \backslash \{(0:1:0)\}$, *that are 3–generated;*

3. Coker $S_\lambda$, *where*

$$S_\lambda = \left(\begin{array}{c|cc} \alpha_{(a:-b:1)} & \begin{matrix} y_1(1+2a)+y_3(a+a^2) \\ 0 \\ y_2 - by_3 \end{matrix} & \begin{matrix} -y_2+by_3 \\ 0 \\ 0 \end{matrix} \\ \hline 0 & \multicolumn{2}{c}{\psi_{(a:-b:1)}} \end{array}\right)$$

*for* $\lambda = (a:b:1) \in V(f)_{\mathrm{reg}}$
*or*

$$S_\lambda = \left(\begin{array}{c|cc} \alpha_{(0:0:1)} & \begin{matrix} y_3 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ y_3 \\ 0 \end{matrix} \\ \hline 0 & \multicolumn{2}{c}{\psi_{(0:0:1)}} \end{array}\right) \quad \textit{for } \lambda = (0:1:0);$$

4. Coker $S^t_\lambda, \lambda \in V(f)_{\mathrm{reg}}$.

*Proof.* Using the procedure "tensorCM" we compute the matrix corresponding to $(\Omega^1_R(M_2) \otimes \varphi_\lambda)$ for all regular points $\lambda$ on $Y$.

```
ring R=0,(y(1..3)),dp;
ideal i=y(1)^3+y(1)^2*y(3)-y(2)^2*y(3);
qring S=std(i);
ideal m=maxideal(1);
matrix D=mres(m,3)[3];

ring R1=0,(y(1..3),a,b),dp;
ideal I=y(1)^3+y(1)^2*y(3)-y(2)^2*y(3),a3+a2-b2;
```

```
qring S1=std(I);
matrix D=imap(S,D);
matrix M[2][2]=    y(1)-a*y(3), y(2)*y(3)+b*y(3)^2,
                y(2)-b*y(3), y(1)^2+(a+1)*y(1)*y(3)+(a2+a)*y(3)^2;

matrix B=tensorCM(D,M);
B=permcol(B,5,4); B=permcol(B,2,4); B=addrow(B,1,a+1,4);
B=addcol(B,1,a+1,3); B=permrow(B,1,3); B=permrow(B,2,4);
B=addrow(B,1,-y(3),5); B=permrow(B,4,5); B=multrow(B,3,-1);
B=multcol(B,1,-1); B=multcol(B,2,-1); B=multrow(B,5,-1);
B=multrow(B,4,-1); B=addcol(B,1,-b,4); B=addcol(B,1,-b,4);
print(B);
```

This gives the matrices $S_\lambda$. Using Lemma 2.6 we are done. $\square$

## 2.3 The classification of non–locally free, rank 2, MCM R–modules

**Remark 2.13.** *It is known that on a smooth curve any vector bundle of rank $r \geq 2$, say $\mathcal{E}$, fits in an extension*

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$$

*where $\mathcal{F}$ is a vector bundle of rank $r-1$ and $\mathcal{L}$ is a line bundle.*
*Over an isolated curve singularity, any coherent sheaf $\mathcal{C}$ of rank $r$ has an extension of type*

$$0 \to \mathcal{C}_1 \to \mathcal{C} \to \mathcal{C}_2 \to 0$$

*where $\mathcal{C}_1$ and $\mathcal{C}_2$ are coherent sheaves of rank 1, respectively $r-1$. It can happen that $\mathcal{C}_1$ and $\mathcal{C}_2$ are non–locally free but $\mathcal{C}$ is vector bundle.*

The theorem 0.1 describe the extensions of graded MCM modules over a hypersurface ring. This description gives us an algorithm to compute matrix factorizations of the non–locally free, rank two, indecomposable, graded MCM modules over the ring $R$.
Since their minimal number of generators is smaller equal to $2e(R)$, they can be 3, 4, 5 or 6–minimally generated. The one that are three minimally generated, are isomorphic, up to shiftings, to Coker $\beta_s$, as it was proved in Proposition 1.7.

### 2.3.1 6–generated

We will describe in details only the 6-generated ones, the other cases, 4 and 5–generated are explained roughly.

For each $m \in \mathbb{Z}$, $m \geq 1$, define the matrix:

$$\delta_m = \begin{pmatrix} 0 & y_1 & y_2 & 0 & y_3^m & -y_3^m \\ y_1 & y_2 & y_1 & 0 & y_3^m & -y_3^m \\ y_3 & 0 & -y_1 & 0 & 0 & y_3^m \\ 0 & 0 & 0 & 0 & y_1 & y_2 \\ 0 & 0 & 0 & y_1 & y_2 & y_1 \\ 0 & 0 & 0 & y_3 & 0 & -y_1 \end{pmatrix}.$$

**Theorem 2.14.** *There are countably many isomorphism classes of graded, indecomposable, rank two, non–locally free MCM R-modules that are minimally 6–generated.*
*They are cokernel of graded maps defined by the matrices*

$$\{\delta_m, \delta_m^t | m \in \mathbb{Z}, m \geq 1\}.$$

*Proof.* Let $M$ be a graded, indecomposable, rank two, non–locally free MCM $R$-modules with $\mu(M) = 6$. Then, up to a shifting, $M$ fits in a graded extension of the type

$$0 \to \operatorname{Coker} \alpha_s \to M \to \operatorname{Coker} \alpha_\lambda \otimes R(k) \to 0 \tag{5}$$

or in one of the type

$$0 \to \operatorname{Coker} \alpha_\lambda \otimes R(k) \to M \to \operatorname{Coker} \alpha_s \to 0 \tag{6}$$

with $k \in \mathbb{Z}, \lambda = (a : b : 1) \in V(f)$.

In the above graded exact sequences we consider, as in the previous section, the graded maps $\alpha_\lambda : R(-2)^3 \longrightarrow R(-1)^3$, for which $\operatorname{Coker} \alpha_\lambda$ has degree 0. So $\operatorname{Coker} \alpha_\lambda \otimes R(k)$ has degree $3k$.

The modules $M$ with an extension of type (6) are duals of some modules from the first extension. Therefore, it is enough to prove the statement for the modules with an extension of type (5).

By Theorem 0.1, the module $M$ has a matrix factorization $(S, S')$, with $S$ a matrix of the form $S = \begin{pmatrix} \alpha_s & D \\ 0 & \alpha_\lambda \end{pmatrix}$. The matrix $D$ has homogeneous entries and it fulfills the condition $\beta_s \cdot D \cdot \beta_\lambda = 0 \bmod (f)$.

The corresponding graded map $S$, is defined as
$S : R(-2)^3 \oplus R(k-2)^3 \longrightarrow R(-1)^3 \oplus R(k-1)^3$, so, the matrix $D$ should have homogeneous entries of degree $1-k$. If $k \geq 2$, the extension splits, if $k = 1$ the module $M$ decomposes. We need, therefore, to consider only the negative shiftings of $\operatorname{Coker} \alpha_\lambda$.
Denote the entries of $D$ with $d_1, ..., d_9$, so that $D = \begin{pmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \\ d_7 & d_8 & d_9 \end{pmatrix}$ and denote $m = 1-k$. The matrix $S$ has the form:

$$S = \begin{pmatrix} 0 & y_1 & y_2 & d_1 & d_2 & d_3 \\ y_1 & y_2 & 0 & d_4 & d_5 & d_6 \\ y_3 & 0 & -y_1 - y_3 & d_7 & d_8 & d_9 \\ 0 & 0 & 0 & 0 & y_1 - ay_3 & y_2 - by_3 \\ 0 & 0 & 0 & y_1 & y_2 + by_3 & (a^2+a)y_3 \\ 0 & 0 & 0 & y_3 & 0 & -y_1 - (a+1)y_3 \end{pmatrix}.$$

We make some linear transformations, in order to simplify the matrix $S$. (In this way we do not change the module $\operatorname{Coker} S$)
By subtracting the first 3 columns from the last three, (with a corresponding multiplication factor) we can "kill" the variable $y_1$ in all entries of $D$.
We kill $y_2$ in $d_1$ by subtracting again the third column from the fourth one; the new appeared $y_1$ in $d_7$ disappear if we subtract the line 5 from the third one.
Similarly, we can kill also $y_3$ in $d_1$, using the line 6 and column 2. So we can suppose that $d_1 = 0$.
In the same way, we eliminate $y_2$ and $y_3$ from $d_4$, using the column 2 and line 5, respectively line 6 and column 1. So we can suppose also that $d_4 = 0$.
Using the first column and the line 5, one can eliminate $y_3$ in $d_7$. Therefore we consider $d_7 = a_7 y_2^m$, with $a_7$ a constant. Using the first column and the line 4, one can eliminate $y_3$ also in $d_8$. So we can consider that $d_8 = a_8 y_2^m, a_8$ constant.
The same, using the second column and the line 4, we eliminate $y_2$ in $d_5$ and we write $d_5 = a_5 y_3^m, a_5 \in K$.

We check now the condition $\beta_s \cdot D \cdot \beta_\lambda = 0$ to get more informations on the entries of $D$.

```
LIB"matrix.lib";
LIB"homolog.lib";
```

```
LIB"linalg.lib";

proc condext(matrix G)
{
  ideal g=flatten(G);
  matrix V;ideal P;
  int k,j;
  P=0;
  for(j=1;j<=size(G);j++)
   { g[j]=reduce(g[j],std(y(1)^3+y(1)^2*y(3)-y(2)^2*y(3)));
     V=coef(g[j],y(1));
     for(k=1;k<=1/2*size(V);k++)
     {
       P=P+V[2,k];
     };
   }
P=interred(P);
return(P);
}

ring R=0,(y(1..3),d(1..9),a,b),(c,dp(3),dp(9),dp(2));
ideal i=y(1)^3+y(1)^2*y(3)-y(2)^2*y(3),a3+a2-b2;
qring S=std(i);

matrix A[3][3]=0,y(1)-a*y(3),      y(2)-b*y(3),
            y(1),y(2)+b*y(3),      (a2+a)*y(3),
            y(3),          0,-y(1)-(a+1)*y(3);

matrix B=subst(A,a,0,b,0);
matrix Aa=adjoint(A);
matrix Ba=subst(Aa,a,0,b,0);
matrix D[3][3]=d(1..9);
D[1,1]=0;D[2,1]=0;
matrix G=Ba*D*Aa;
ideal P;
P=condext(G);
P;
P[1]=y(2)*y(3)*d(8)+y(3)^2*d(6)+y(3)^2*d(7)*a^2+y(3)^2*d(7)*a-
```

```
     y(3)^2*d(8)*b+y(3)^2*d(9)*a+y(3)^2*d(9)
P[2]=y(2)*y(3)*d(7)*b-y(2)*y(3)*d(8)*a-y(3)^2*d(2)*a^2-
     y(3)^2*d(2)*a+y(3)^2*d(3)*b+y(3)^2*d(5)*b-y(3)^2*d(6)*a
P[3]=y(2)*y(3)*d(7)*a-y(3)^2*d(2)*b+y(3)^2*d(3)*a+
     y(3)^2*d(5)*a+y(3)^2*d(7)*a*b-y(3)^2*d(8)*a^2+
     y(3)^2*d(9)*b
P[4]=y(2)*y(3)*d(2)-y(2)*y(3)*d(9)-y(3)^2*d(2)*b+y(3)^2*d(3)*a
P[5]=y(2)^2*d(8)+y(2)*y(3)*d(2)*a+y(2)*y(3)*d(2)+
     y(2)*y(3)*d(6)-y(3)^2*d(5)*a^2-y(3)^2*d(5)*a+
     y(3)^2*d(6)*b
P[6]=y(2)^2*d(7)+y(2)*y(3)*d(3)+y(2)*y(3)*d(5)-y(3)^2*d(5)*b+
     y(3)^2*d(6)*a
P[7]=y(2)^2*d(2)-y(2)^2*d(9)-y(2)*y(3)*d(2)*b+y(2)*y(3)*d(3)*a
```

Since no $y_3$ appear in $d_7$ and $d_8$, from the conditions P[6] and P[5] we get that $d_7 = d_8 = 0$. After we substitute in the ideal P, $d_7$ and $d_8$ with 0, we divide with $y_3$ or $y_2$ where is possible.

```
P=subst(P,d(7),0,d(8),0);
P[1]=P[1]/y(3)^2; P[2]=P[2]/y(3)^2; P[3]=P[3]/y(3)^2;
P[4]=P[4]/y(3); P[5]=P[5]/y(3); P[6]=P[6]/y(3); P[7]=P[7]/y(2);
P=interred(P);
P;
```

```
P[1]=d(6)+d(9)*a+d(9)
P[2]=d(2)*b-d(3)*a-d(5)*a-d(9)*b
P[3]=d(2)*a^2+d(2)*a-d(3)*b-d(5)*b+d(6)*a
P[4]=y(2)*d(3)+y(2)*d(5)-y(3)*d(5)*b+y(3)*d(6)*a
P[5]=y(2)*d(2)-y(2)*d(9)-y(3)*d(2)*b+y(3)*d(3)*a
```

We apply some more linear transformations to the matrix $S$.
Notice that, if we denote the third column of $S$ with $c_3$ and first column with $c_1$, $-c_3 + (1-a)c_1$ has on the third position $y_1 - ay_3$. So, if we eliminate $y_2$ from $d_9$ subtracting the fourth line from the third one, we can make again zero on the position [3,5] subtracting $-c_3 + (1-a)c_1$ from the column 5. We destroy the new $y_1$ from the position of $d_5$ with the line 4, but it still remains there a term $g \cdot y_3$, with $g$ a polynomial of degree $-k$. If $k = 0$, $g$ is

a constant, so we do not need to make any other transformations. If $k \geq 1$, we eliminate so possible $y_2$ from the new $d_5$ using the second column and the line 4. So, at the end of this transformations, if we denote $d_9 = a_9 y_3^m$, $a_9$ constant and we notice that $d_6 = -d_9(a+1)$ (see P[1]) , $S$ is looking like:

$$S = \begin{pmatrix} 0 & y_1 & y_2 & 0 & d_2 & d_3 \\ y_1 & y_2 & 0 & 0 & a_5 y_3^m & -(a+1)a_9 y_3^m \\ y_3 & 0 & -y_1 - y_3 & 0 & 0 & a_9 y_3^m \\ 0 & 0 & 0 & 0 & y_1 - a y_3 & y_2 - b y_3 \\ 0 & 0 & 0 & y_1 & y_2 + b y_3 & (a^2+a) y_3 \\ 0 & 0 & 0 & y_3 & 0 & -y_1 - (a+1) y_3 \end{pmatrix}.$$

From the condition P[4], we obtain that $d_3$ is divisible with $y_3^m$. But $d_3$ is a homogeneous polynomial of degree $m$, so $d_3 = a_3 y_3^m$, $a_3$ constant. The condition P[4] becomes $y_3(-a^2 a_9 - a a_9 - a_5 b) + y_2(a_3 + a_5) = 0$. So, $a_3 = -a_5$ and $a(a+1)a_9 = -a_5 b$. With this information, the ideal P becomes:

```
P=subst(P,d(3),-d(5));
P;

P[1]=d(6)+d(9)*a+d(9)
P[2]=d(2)*b-d(9)*b
P[3]=d(2)*a^2+d(2)*a+d(6)*a
P[4]=-y(3)*d(5)*b+y(3)*d(6)*a
P[5]=y(2)*d(2)-y(2)*d(9)-y(3)*d(2)*b-y(3)*d(5)*a
```

From the condition P[5], in a similar way as above, one get $d_2 = a_2 y_3^m$, with $a_2$ constant and $y_3(a a_5 + b a_2) + y_2(a_9 - a_2) = 0$. Therefore, $a_2 = a_9$ and $a a_5 + b a_9 = 0$. We write the new form of $S$:

$$S = \begin{pmatrix} 0 & y_1 & y_2 & 0 & a_9 y_3^m & -a_5 y_3^m \\ y_1 & y_2 & 0 & 0 & a_5 y_3^m & -(a+1)a_9 y_3^m \\ y_3 & 0 & -y_1 - y_3 & 0 & 0 & a_9 y_3^m \\ 0 & 0 & 0 & 0 & y_1 - a y_3 & y_2 - b y_3 \\ 0 & 0 & 0 & y_1 & y_2 + b y_3 & (a^2+a) y_3 \\ 0 & 0 & 0 & y_3 & 0 & -y_1 - (a+1) y_3 \end{pmatrix}$$

with $a a_5 + b a_9 = 0$.

We want that the module Coker $S$ is not locally free.
This is equivalent to the condition $\mathrm{Fitt}_2(S) R_{\langle y_1, y_2 \rangle} \neq R_{\langle y_1, y_2 \rangle}$. (see [TJP])

(we denote with $\mathrm{Fitt}_2$ the ideal generated by all 4–minors of the matrix)
To check this, we substitute, in the matrix $S$, the variables $y_1$ and $y_2$ with 0 and the variable $y_3$ with 1. The fitting ideal of the new matrix is zero if and only if $\mathrm{Coker}\, S$ is non–locally free module.

```
matrix A1=subst(A,y(1),0,y(2),0,y(3),1);
matrix B1=subst(B,y(1),0,y(2),0,y(3),1);
matrix D1[3][3]=0,d(9),      -d(5),
                0,d(5),-(a+1)*d(9),
                0,   0,       d(9);
matrix G[6][3]=D1,A1;
matrix S1[6][6]=concat(B1,G);
ideal F=fitting(S1,2);
F;
```

```
F[1]=d(5)*b+d(9)*a^2+d(9)*a
F[2]=d(5)*a+d(9)*b
F[3]=d(5)^2-d(9)^2*a-d(9)^2
```

If $a \neq 0$, $d_5 = -d_9 b/a$. Since $\mathrm{Coker}\, S$ do not decomposes, $a_9 \neq 0$ and can be chosen to be $a$. We obtain in this way the matrix

$$S = \begin{pmatrix} 0 & y_1 & y_2 & 0 & ay_3^m & by_3^m \\ y_1 & y_2 & 0 & 0 & -by_3^m & -(a^2+a)y_3^m \\ y_3 & 0 & -y_1-y_3 & 0 & 0 & ay_3^m \\ 0 & 0 & 0 & 0 & y_1 - ay_3 & y_2 - by_3 \\ 0 & 0 & 0 & y_1 & y_2 + by_3 & (a^2+a)y_3 \\ 0 & 0 & 0 & y_3 & 0 & -y_1-(a+1)y_3 \end{pmatrix}.$$

After some linear transformations, this matrix becomes:

$$S = \begin{pmatrix} 0 & y_1 & y_2 & 0 & y_3^{m-1}(ay_3 - y_1) & y_3^{m-1}(by_3 - y_2) \\ y_1 & y_2 & 0 & 0 & y_3^{m-1}(-by_3 - y_2) & -(a^2+a)y_3^m \\ y_3 & 0 & -y_1-y_3 & 0 & 0 & y_3^{m-1}((a+1)y_3 + y_1) \\ 0 & 0 & 0 & 0 & y_1 - ay_3 & y_2 - by_3 \\ 0 & 0 & 0 & y_1 & y_2 + by_3 & (a^2+a)y_3 \\ 0 & 0 & 0 & y_3 & 0 & -y_1-(a+1)y_3 \end{pmatrix}.$$

As one can easily see, this matrix decomposes, after some evident linear transformations.

If $a = b = 0$, $d_5 = d_9$ or $d_5 = -d_9$ (from F[3]). As before, $a_9 \neq 0$ and can be chosen to be 1. With this choice, the matrix $S$ becomes $\delta_m$ or $\delta_m^t$.

Let us prove now the indecomposability.
Suppose $\delta_m$ decomposes.
Then, there exist the invertible matrices $U$ and $V$, and there exist $\nu_1, \nu_2 \in V(f)$ such that: $\delta_m \cdot U = V \cdot \begin{pmatrix} \alpha_{\nu_1} & 0 \\ 0 & \alpha_{\nu_2} \end{pmatrix}$.
Write $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$ and $V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}$.
Let $j-1$ be the degree of the entries of the matrices $U_1$ and $V_1$ and $j'-1$, the degree of the entries of $U_2$ and $V_2$. Then, the entries of $U_3$ and $V_3$ should have degree $j-m$, the one of $U_4$ and $V_4$ should have degree $j'-m$.
Since $\det U = \det V = 1$, if $m \geq 2$, $U_3 = 0$ (and $V_3 = 0$) or $U_4 = 0$ (and $V_4 = 0$). If $m = 1$, since $\alpha_s U_3 = V_3 \alpha_{\nu_1}$, we get $U_3 = V_3 = 0$ or $\nu_1 = s$ and $U_3 = V_3 = t\mathrm{Id}$, $t \neq 0$. In both cases, (also for $m \geq 2$), we obtain that $\nu_1 = \nu_2 = s$ and that

$$\begin{pmatrix} 0 & y_3^m & -y_3^m \\ 0 & y_3^m & -y_3^m \\ 0 & 0 & y_3^m \end{pmatrix} = V_2 \alpha_s - \alpha_s U_2,$$

that is impossible. (in the right hand-side, the entry [1,2] is in the ideal $\langle y_1, y_2 \rangle$, so, it can not be $-y_3^m$). Therefore, for any $m$, the matrices $\delta_m$ and $\delta_m^t$ are indecomposable.
With a similar proof, one can show that there not exist two invertible matrices $U$ and $V$ such that $U\delta_m = \delta_m^t V$; more, because of degree reason, it is clear that for two different $m_1$ and $m_2$, $\delta_{m_1}$ and $\delta_{m_2}$ do not give isomorphic modules.
This complete the proof of the theorem. □

### 2.3.2 5–generated modules

Denote $v$ a root of -1.
For all $m \in \mathbb{Z}$, $m \geq 1$, we define the matrices:

$$\alpha_{\psi 1}^m = \begin{pmatrix} 0 & y_1 & y_2 & y_3^m & -vy_3^m \\ y_1 & y_2 & 0 & vy_3^m & y_3^m \\ y_3 & 0 & -y_1-y_3 & vy_3^m & 0 \\ 0 & 0 & 0 & y_1^2+y_1y_3 & -y_2y_3 \\ 0 & 0 & 0 & -y_2 & y_1 \end{pmatrix}$$

and

$$\alpha_{\psi 2}^m = \begin{pmatrix} 0 & y_1 & y_2 & y_3^m & vy_3^m \\ y_1 & y_2 & 0 & -vy_3^m & y_3^m \\ y_3 & 0 & -y_1-y_3 & -vy_3^m & 0 \\ 0 & 0 & 0 & y_1^2+y_1y_3 & -y_2y_3 \\ 0 & 0 & 0 & -y_2 & y_1 \end{pmatrix},$$

$$\alpha_{\varphi 1}^m = \begin{pmatrix} 0 & y_1 & y_2 & y_3^m & y_3^{m+1} \\ y_1 & y_2 & 0 & -y_3^m & -y_3^{m+1} \\ y_3 & 0 & -y_1-y_3 & 0 & y_3^{m+1} \\ 0 & 0 & 0 & y_1 & y_2y_3 \\ 0 & 0 & 0 & y_2 & y_1^2+y_1y_3 \end{pmatrix}$$

and

$$\alpha_{\varphi 2}^m = \begin{pmatrix} 0 & y_1 & y_2 & -y_3^m & y_3^{m+1} \\ y_1 & y_2 & 0 & -y_3^m & y_3^{m+1} \\ y_3 & 0 & -y_1-y_3 & 0 & -y_3^{m+1} \\ 0 & 0 & 0 & y_1 & y_2y_3 \\ 0 & 0 & 0 & y_2 & y_1^2+y_1y_3 \end{pmatrix}.$$

Define also:

$$\alpha_{\psi_0} = \begin{pmatrix} 0 & y_1 & y_2 & 0 & 0 \\ y_1 & y_2 & 0 & y_2 & -y_2 \\ y_3 & 0 & -y_1-y_3 & 0 & y_2 \\ 0 & 0 & 0 & y_1^2 & -y_2^2 \\ 0 & 0 & 0 & -y_3 & y_1+y_3 \end{pmatrix}.$$

**Theorem 2.15.** *There are countably many isomorphism classes of graded, indecomposable, rank two, non–locally free MCM R-modules that are minimally 5–generated. They are given by the matrices:*

$$\{\alpha_{\psi 1}^m, \alpha_{\psi 2}^m, \alpha_{\varphi 1}^m, \alpha_{\varphi 2}^m, \alpha_{\psi_0} | m \in \mathbb{Z}, m \geq 1\}$$

*and their transpositions.*

*Proof.* Let $M$ be a graded, indecomposable, rank two, non–locally free MCM $R$-modules with $\mu(M) = 5$. Then, up to a shifting, $M$ or $M^\vee$ fits in one of the following graded extensions:

$$0 \to \operatorname{Coker} \alpha_s \to M \to \operatorname{Coker} \psi_\lambda \otimes R(k) \to 0 \tag{7}$$

$$0 \to \operatorname{Coker} \alpha_\lambda \to M \to \operatorname{Coker} \psi_s \otimes R(k) \to 0 \tag{8}$$

$$0 \to \operatorname{Coker} \alpha_s \to M \to \operatorname{Coker} \varphi_\lambda \otimes R(k) \to 0 \tag{9}$$

$$0 \to \operatorname{Coker} \alpha_\lambda \to M \to \operatorname{Coker} \varphi_s \otimes R(k) \to 0 \tag{10}$$

with $k \in \mathbb{Z}, \lambda \in V(f)$.

As before, we consider the graded maps $\alpha_\lambda : R(-2)^3 \longrightarrow R(-1)^3$, $\psi_\lambda : R(-2)^2 \longrightarrow R \oplus R(-1), \varphi_\lambda : R(-2) \oplus R(-3) \longrightarrow R(-1)^2$.

• Consider first the extension (7) (in the case of the extension (8) the proof and results are identical).

As before, consider $(S, S')$ a matrix factorization of $M$, with $S$ a matrix of the form $S = \left(\begin{smallmatrix} \alpha_s & D \\ 0 & \psi_\lambda \end{smallmatrix}\right)$. The matrix $D$ has homogeneous entries and it fulfills, in the ring $R$, the condition $\beta_s \cdot D \cdot \varphi_\lambda = 0$.
Since the corresponding graded map $S$, is defined as $S : R(-2)^3 \oplus R(k-2)^2 \longrightarrow R(-1)^3 \oplus R(k) \oplus R(k-1)$, the matrix $D$ should have homogeneous entries of degree $1-k$. So, if $k \geq 2$, the extension splits, if $k = 1$ the module $M$ decomposes. We need, therefore, to consider only the negative shiftings of $\operatorname{Coker} \psi_\lambda$.

Denote the entries of $D$ with $d_1, ..., d_6$, so that $D = \left(\begin{smallmatrix} d_1 & d_2 \\ d_3 & d_4 \\ d_5 & d_6 \end{smallmatrix}\right)$ and let $m = 1 - k$.

As in the previous proof, we make some linear transformations to eliminate the variable $y_1$ from all $d_i, i = 1, ..., 6$ and the variable $y_2$ from $d_1, d_3$ and $d_5$.

The extension condition, $\beta_s \cdot D \cdot \varphi_\lambda = 0$, implies that $y_2$ do not appear in $d_2$ and $d_4$ and that $d_6 = 0$. We obtain this making the necessary calculations with Singular:

```
ring R=0,(y(1..3),d(1..6),a,b),(c,dp(3),dp(6),dp(2));
ideal i=y(1)^3+y(1)^2*y(3)-y(2)^2*y(3),a3+a2-b2;
qring S=std(i);

matrix psi[2][2]=
    y(1)^2+(a+1)*y(1)*y(3)+(a2+a)*y(3)^2,-(y(2)*y(3)+b*y(3)^2),
                           -(y(2)-b*y(3)),           y(1)-a*y(3);
```

```
matrix phi[2][2]=
    y(1)-a*y(3),                   y(2)*y(3)+b*y(3)^2,
    y(2)-b*y(3), y(1)^2+(a+1)*y(1)*y(3)+(a2+a)*y(3)^2;

matrix A[3][3]=0,y(1)-a*y(3),     y(2)-b*y(3),
            y(1),y(2)+b*y(3),     (a2+a)*y(3),
            y(3),          0,-y(1)-(a+1)*y(3);

matrix B=subst(A,a,0,b,0); matrix Ba=adjoint(B);
matrix D[3][2]=d(1..6);
matrix G=Ba*D*phi;
ideal P;
P=condext(G);
P;
```

```
P[1]=y(2)*d(6)-y(3)*d(3)-y(3)*d(5)*a-y(3)*d(5)-y(3)*d(6)*b
P[2]=y(2)*d(2)+y(2)*d(5)-y(3)*d(1)*a-y(3)*d(2)*b
P[3]=y(2)*d(1)-y(2)*d(4)+y(3)*d(3)*a+y(3)*d(4)*b
P[4]=y(3)^2*d(1)*b+y(3)^2*d(2)*a^2+y(3)^2*d(2)*a-y(3)^2*
     d(4)*b+y(3)^2*d(5)*a^2+y(3)^2*d(5)*a+y(3)^2*d(6)*a*b
P[5]=y(3)^2*d(1)*a+y(3)^2*d(2)*b-y(3)^2*d(4)*a+y(3)^2*d(5)*b+
     y(3)^2*d(6)*a^2
```

From P[1], we find $d_6 = a_6 y_3^m$, with $a_6$ a constant and $(y_2 - y_3 b)a_6 = y_3(a_3 + a_5 + aa_5)$. Therefore, $a_6 = 0$ and $a_3 = a_5 + aa_5$. The same, from P[2], we have $d_2 = a_2 y_3^m$, $a_2$ constant. More, $a_2 = -a_5$ and $aa_1 = a_5 b$. From P[3], we obtain $d_4 = a_4 y_3^m$ and $a_4 = a_1$, $a_4 \in K$. So the matrix $S$ is looking like:

$$S = \begin{pmatrix} 0 & y_1 & y_2 & a_1 y_3^m & -a_5 y_3^m \\ y_1 & y_2 & 0 & (a_5 + aa_5) y_3^m & a_1 y_3^m \\ y_3 & 0 & -y_1 - y_3 & a_5 y_3^m & 0 \\ 0 & 0 & 0 & y_1^2 + (a+1) y_1 y_3 + (a^2 + a) y_3^2 & -y_2 y_3 - b y_3^2 \\ 0 & 0 & 0 & -y_2 + b y_3 & y_1 - a y_3 \end{pmatrix}$$

with $aa_1 - ba_5 = 0$.

We impose now that the module $\operatorname{Coker} S$ is non-locally free. For this, we should have $\operatorname{Fitt}_2(S) \cdot R_{\langle y_1, y_2 \rangle} \neq R_{\langle y_1, y_2 \rangle}$.

```
matrix A1=subst(psi,y(1),0,y(2),0,y(3),1);
matrix B1=subst(B,y(1),0,y(2),0,y(3),1);
matrix D1[3][2]=d(1),-d(5),d(5)*(-a+1),
                d(1), d(5),          0;
matrix G[5][2]=D1,A1;
matrix S1[5][5]=concat(B1,G);
ideal F=fitting(S1,2);
F;
```

```
F[1]=d(5)*b
F[2]=d(5)*a
F[3]=d(1)*b-d(5)*a^2-d(5)*a
F[4]=d(1)*a-d(5)*b
F[5]=d(1)^2-d(5)^2*a+d(5)^2
```

As before, the indecomposability of Coker $S$ oblige to the conditions $a = b = 0$ and $a_1^2 + a_5^2 = 0$. Therefore, $S$ becomes $\alpha_{\psi 1}^m$ or $\alpha_{\psi 2}^m$. If $\lambda = (0 : 1 : 0)$, with very similar computations, we obtain only one indecomposable extension, Coker $\alpha_{\psi_0}$. The indecomposability follows as in the previous theorem.

• Consider now the extension (9)(the proof and results in the last case are identical with this one). In this case, the corresponding graded map $S$ is defined as $S : R(-2)^3 \oplus R(k-2) \oplus R(k-3) \longrightarrow R(-1)^3 \oplus R(k-1)^2$.
On the first column, the matrix $D$ should have homogeneous entries of degree $1-k$ and on the second column of degree $2-k$. So, if $k \geq 3$, the extension splits, if $k = 2$ the module $M$ decomposes. In the case $k = 1$, the first column of $D$ should be 0. Writing the condition of extension, we obtain that also the second column annihilates, so the module Coker $S$ decomposes.
Consider now $k \leq 0$.
Denote the entries of $D$ with $d_1, ..., d_6$, so that $D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \\ d_5 & d_6 \end{pmatrix}$ and let $m = 1-k$. We make again some linear transformations to eliminate the variable $y_1$ from all $d_1, d_2, d_3, d_5$ and $d_6$ and the variable $y_2$ from $d_5$. More, we eliminate $y_1^2$ from $d_4$ in case that there exists. We write $d_4 = y_1 d_7 + d_8$. Also, in case $m \geq 1$, we can eliminate $y_2 y_3$ from $d_2$ and $d_6$. Subtracting the first column from the the fourth one (multiplied with the right polynomial) we eliminate also $y_3$ in $d_5$, so we can consider $d_5 = 0$. ( the new appeared $y_1$ in $d_3$ is killed using the fourth line) Using the second column one can eliminate $y_1 y_2$ from $d_4$, so $d_7 = a_7 y_3^m$, with $a_7$ constant.

The extension condition $\beta_s \cdot D \cdot \varphi_\lambda = 0$ gives:

```
ring R1=0,(y(1..3),d(1..8),a,b),(c,dp(3),dp(8),dp(2));
ideal i=y(1)^3+y(1)^2*y(3)-y(2)^2*y(3),a3+a2-b2;
qring S1=std(i);
matrix Ba=imap(S,Ba);
matrix psi=imap(S,psi);
matrix D[3][2]=d(1),          d(2),
               d(3),y(1)*d(7)+d(8),
                  0,          d(6);
matrix G=Ba*D*psi;P=interred(P);
ideal P;
P=condext(G);
P[1]=P[1]/y(3); P[2]=P[2]/y(3);  P[3]=P[3]/y(3);
P[4]=P[4]/y(3); P[7]=P[7]/y(3)^2 ;
P=subst(P,d(8),y(3)*d(7)*a+y(3)*d(7)-d(6)*a-d(6));
P=interred(P);
P;
```

```
P[1]=y(3)*d(1)*b+y(3)*d(3)*a+y(3)*d(7)*b+d(2)*a-d(6)*b
P[2]=y(3)*d(1)*a^2+y(3)*d(1)*a+y(3)*d(3)*b+y(3)*d(7)*a^2+
     y(3)*d(7)*a+d(2)*b-d(6)*a^2-d(6)*a
P[3]=y(2)*d(7)*a+y(2)*d(7)+y(3)*d(1)*a*b+y(3)*d(1)*b+
     y(3)*d(3)*a^2+y(3)*d(3)*a+y(3)*d(7)*a*b+y(3)*d(7)*b+
     d(2)*a^2+d(2)*a-d(6)*a*b-d(6)*b
P[4]=y(2)*y(3)*d(3)+y(3)^2*d(3)*b+y(3)^2*d(7)*a^2+
     y(3)^2*d(7)*a+y(2)*d(2)-y(3)*d(6)*a^2-y(3)*d(6)*a
P[5]=y(2)*y(3)*d(1)+y(2)*y(3)*d(7)+y(3)^2*d(1)*b-y(2)*d(6)+
     y(3)*d(2)*a
```

So, $d_4 = y_1 d_7 + y_3(a+1)(d_7 - d_6)$ and $y_3 | d_2$ ( see P[4] ). But we have eliminated $y_2 y_3$ from $d_2$, so $d_2 = a_2 y_3^{m+1}$, where $a_2$ is constant from $K$. The same, P[5]implies that $d_6 = a_6 y_3^{m+1}$, $a_6$ constant. Taking in account the form of $d_7$, P[5] shows that also $d_1 = a_1 y_1^m$, with $a_1$ constant. More, $a_1 + a_7 - a_6 = ba_1 + aa_2$. P[4] gives that also $d_3 = a_3 y_3^m$, $a_3$ constant such that $a_3 = -a_2$. With this information, P becomes:

```
P=subst(P,d(6),y(3)*(d(1)+d(7)),d(2),-y(3)*d(3));
P=interred(P);
P;

P[1]=y(2)*d(7)*a+y(2)*d(7)
P[2]=y(3)^2*d(1)*b-y(3)^2*d(3)*a
P[3]=y(3)^2*d(1)*a^2+y(3)^2*d(1)*a-y(3)^2*d(3)*b
```

Therefore, $d_7 = 0$ and $a_1 b - a_3 a = 0$.

The matrix $S$ is looking like:

$$S = \begin{pmatrix} 0 & y_1 & y_2 & a_1 y_3^m & a_2 y_3^{m+1} \\ y_1 & y_2 & 0 & -a_2 y_3^m & -a_1 y_3^{m+1} \\ y_3 & 0 & -y_1 - y_3 & 0 & a_1 y_3^{m+1} \\ 0 & 0 & 0 & y_1 - a y_3 & y_2 y_3 + b y_3^2 \\ 0 & 0 & 0 & y_2 - b y_3 & y_1^2 + (a+1) y_1 y_3 + (a^2 + a) y_3^2 \end{pmatrix}.$$

We impose now that the module is non-locally free. For this, we should have $\mathrm{Fitt}_2(S) R_{\langle y_1, y_2 \rangle} \neq R_{\langle y_1, y_2 \rangle}$.

```
matrix A1=subst(phi,y(1),0,y(2),0,y(3),1);
matrix B1=subst(B,y(1),0,y(2),0,y(3),1);
matrix D1[3][2]=d(1),d(2),-d(2),
                -d(1),   0, d(1);
matrix G[5][2]=D1,A1;
matrix S1[5][5]=concat(B1,G);
ideal F=fitting(S1,2);
F;

F[1]=d(2)*b^2
F[2]=d(2)*a*b
F[3]=d(2)*a^2-d(2)*b^2
F[4]=d(1)*b+d(2)*a
F[5]=d(1)*a+d(2)*b
F[6]=d(1)^2-d(2)^2
```

If $d_2 = 0$ then $d_1$ should be also zero, so the matrix decomposes. If $d_2 \neq 0$ , then $a = b = 0$ and $d_1 = d_2$ or $d_1 = -d_2$. More, we can choose $d_2 = 1$. So,

the matrix $S$ is equivalent to $\alpha^m_{\varphi 1}$ or $\alpha^m_{\varphi 1}$.
With similar computations, one can prove that there are no indecomposable extensions of this type, if $\lambda = (0:1:0)$.

The indecomposability of the matrices $\alpha^m_{\varphi 1}$ and $\alpha^m_{\varphi 1}$ follows as in the theorem 2.14. $\square$

### 2.3.3 4–generated modules

For all $\lambda = (a : b : 1) \in V(f)$, with $a \neq 0$ we define the matrix:

$$\varphi_{\psi\lambda} = \begin{pmatrix} y_1 & y_2y_3 & -by_3 & ay_3 \\ y_2 & y_1^2 + y_1y_3 & ay_1 + (a^2+a)y_3 & -by_3 \\ 0 & 0 & y_1^2 + (a+1)y_1y_3 + (a^2+a)y_3^2 & -y_2y_3 - by_3^2 \\ 0 & 0 & -y_2 + by_3 & y_1 - ay_3 \end{pmatrix}.$$

If $\lambda = (0:1:0)$ let be:

$$\varphi_{\psi\lambda} = \begin{pmatrix} y_1 & y_2y_3 & 0 & y_2 \\ y_2 & y_1^2 + y_1y_3 & y_1 & 0 \\ 0 & 0 & y_1^2 & -y_2^2 \\ 0 & 0 & -y_3 & y_1 + y_3 \end{pmatrix}.$$

For all $m \in \mathbb{Z}, m \geq 1$ we define:

$$\varphi^m_{\psi 1} = \begin{pmatrix} y_1 & y_2y_3 & y_3^m & y_3^m \\ y_2 & y_1^2 + y_1y_3 & y_1y_3^{m-1} + y_3^m & -y_3^m \\ 0 & 0 & y_1^2 + y_1y_3 & -y_2y_3 \\ 0 & 0 & -y_2 & y_1 \end{pmatrix}$$

and

$$\varphi^m_{\psi 2} = \begin{pmatrix} y_1 & y_2y_3 & -y_3^m & y_3^m \\ y_2 & y_1^2 + y_1y_3 & y_1y_3^{m-1} + y_3^m & y_3^m \\ 0 & 0 & y_1^2 + y_1y_3 & -y_2y_3 \\ 0 & 0 & -y_2 & y_1 \end{pmatrix},$$

$$\psi^m_{\varphi 1} = \begin{pmatrix} y_1^2 + y_1y_3 & -y_2y_3 & y_3^{m+1} & y_1y_3^{m+1} + y_3^{m+2} \\ -y_2 & y_1 & y_3^m & y_3^{m+1} \\ 0 & 0 & y_1 & y_2y_3 \\ 0 & 0 & y_2 & y_1^2 + y_2y_3 \end{pmatrix}$$

and

$$\psi^m_{\varphi 2} = \begin{pmatrix} y_1^2 + y_1y_3 & -y_2y_3 & y_3^{m+1} & -y_1y_3^{m+1} - y_3^{m+2} \\ -y_2 & y_1 & -y_3^m & y_3^{m+1} \\ 0 & 0 & y_1 & y_2y_3 \\ 0 & 0 & y_2 & y_1^2 + y_2y_3 \end{pmatrix},$$

$$\varphi^m_{\varphi 1} = \begin{pmatrix} y_1 & y_2y_3 & y_3^m & -y_3^{m+1} \\ y_2 & y_1^2 + y_1y_3 & y_3^m & -y_1y_3^m - y_3^{m+1} \\ 0 & 0 & y_1 & y_2y_3 \\ 0 & 0 & y_2 & y_1^2 + y_1y_3 \end{pmatrix}$$

and

$$\varphi^m_{\varphi 2} = \begin{pmatrix} y_1 & y_2y_3 & -y_3^m & -y_3^{m+1} \\ y_2 & y_1^2 + y_1y_3 & y_3^m & y_1y_3^m + y_3^{m+1} \\ 0 & 0 & y_1 & y_2y_3 \\ 0 & 0 & y_2 & y_1^2 + y_1y_3 \end{pmatrix}.$$

**Theorem 2.16.** *There are two families of isomorphism classes of graded, indecomposable, rank two, non–locally free MCM R-modules that are minimally 4–generated. One family is countable, given by*

$$\{\varphi^m_{\psi 1}, \varphi^m_{\psi 2}, \psi^m_{\varphi 1}, \psi^m_{\varphi 2}, \varphi^m_{\varphi 1}, \varphi^m_{\varphi 2} | m \in \mathbb{Z}, m \geq 1\}$$

*and their transpositions. The second family is parametrized by the curve $V(f)$:*

$$\{\varphi_{\psi\lambda}, \varphi^t_{\psi\lambda} | \lambda \in V(f)_{reg}\}$$

.

*Proof.* Let $M$ be a graded, indecomposable, rank two, non–locally free MCM $R$-modules with $\mu(M) = 4$. With a similar argumentation as in previous theorems, it is sufficient to consider that, up to a shifting, $M$ or $M^\vee$ fits in one of the following graded extensions:

$$0 \to \operatorname{Coker} \varphi_s \to M \to \operatorname{Coker} \psi_\lambda \otimes R(k) \to 0 \quad (11)$$

$$0 \to \operatorname{Coker} \psi_s \to M \to \operatorname{Coker} \varphi_\lambda \otimes R(k) \to 0 \quad (12)$$

$$0 \to \operatorname{Coker} \varphi_s \to M \to \operatorname{Coker} \varphi_\lambda \otimes R(k) \to 0 \quad (13)$$

with $k \in \mathbb{Z}, \lambda \in V(f)$.

As before, we consider the graded maps $\psi_\lambda : R(-2)^2 \longrightarrow R \oplus R(-1)$, $\varphi_\lambda : R(-2) \oplus R(-3) \longrightarrow R(-1)^2$.

• Suppose $M$ has an extension of type (11).

Therefore, module $M$ has a matrix factorization $(S, S')$, with $S = \begin{pmatrix} \varphi_s & D \\ 0 & \psi_\lambda \end{pmatrix}$.
The matrix $D$ has homogeneous entries of degree $m = 1 - k$.
So, if $k \geq 2$, the extension splits, if $k = 1$ the module $M$ decomposes. We need, therefore, to consider only the negative shiftings of $\operatorname{Coker} \psi_\lambda$. Denote the entries of $D$ with $d_1, ..., d_4$, so that $D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}$.
By linear transformations, we can eliminate the variable $y_1$ in $d_1, d_2, d_4$ and $y_2$ in $d_4$. In case that $m \geq 2$, we can eliminate also $y_2 y_3$ in $d_2$ and $y_1^2$ in $d_3$. We write $d_3 = y_1 d_5 + d_6$ and $d_4 = a_4 y_3^m$.

The extension condition $\beta_s \cdot D \cdot \varphi_\lambda = 0 \bmod (f)$ :

```
setring R;
matrix p=subst(psi,a,0,b,0);
matrix D[2][2]=           d(1),d(2),
                y(1)*d(5)+d(6),d(4);
matrix G=p*D*phi;
ideal P;
P=condext(G);
P=subst(P,d(6),y(3)*d(5)*a+y(3)*d(5));
P=interred(P);
P;

P[1]=y(3)^2*d(5)*b-y(3)*d(1)*a-y(3)*d(2)*b+y(3)*d(4)*a
P[2]=y(3)^2*d(5)*a^2+y(3)^2*d(5)*a+y(2)*d(1)-y(2)*d(4)+
     y(3)*d(4)*b
P[3]=y(2)*y(3)*d(5)-y(2)*d(2)+y(3)*d(1)*a+y(3)*d(2)*b
P[4]=y(2)*y(3)*d(1)-y(2)*y(3)*d(4)+y(3)^2*d(1)*b+
     y(3)^2*d(2)*a^2+y(3)^2*d(2)*a
```

We find out that $d_3 = d_5(y_1 + (a+1)y_3)$. From P[3] we see that $y_3 | d_2$, and since $y_2 y_3$ is eliminated from $d_2$, we can write $d_2 = a_2 y_3^m$. Now, using also P[4], we notice that also $d_1$ has the form $d_1 = a_1 y_3^m$. And again, from P[3], $d_5 = a_5 y_3^m$. Further more, $a_5 = a_2$, $a_4 = a_1$ and $a a_1 + b a_2 = 0$.

We impose now that the module is non-locally free. For this, we should have $\operatorname{Fitt}_2(S) R_{\langle y_1, y_2 \rangle} \neq R_{\langle y_1, y_2 \rangle}$.

```
matrix A1=subst(psi,y(1),0,y(2),0,y(3),1);
matrix B1=subst(phi,a,0,b,0,y(1),0,y(2),0,y(3),1);
matrix D1[2][2]=      d(1),d(2),
```

```
                d(2)*(a+1),d(1);
matrix G[4][2]=D1,A1;
matrix S1[4][4]=concat(B1,G);
ideal F=fitting(S1,2);
F;

F[1]=d(1)*b+d(2)*a^2+d(2)*a
F[2]=d(1)*a+d(2)*b
F[3]=d(1)^2-d(2)^2*a-d(2)^2
```

If $a \neq 0$, then $a_2$ should be nonzero (otherwise the matrix decomposes), and we can choose it to be equal to $a$. Then $a_1 = b$. But, in case $m \geq 2$, the new obtained matrix $S$ , after some simple linear transformations, decomposes. In case $m = 1$ the matrix is:

$$\varphi_{\psi\lambda} = \begin{pmatrix} y_1 & y_2y_3 & -by_3 & ay_3 \\ y_2 & y_1^2 + y_1y_3 & ay_1 + (a^2+a)y_3 & -by_3 \\ 0 & 0 & y_1^2 + (a+1)y_1y_3 + (a^2+a)y_3^2 & -y_2y_3 - by_3^2 \\ 0 & 0 & -y_2 + by_3 & y_1 - ay_3 \end{pmatrix}.$$

In case $a = b = 0$, $a_1 = a_2$ or $a_1 = -a_2$. Choosing $a_2 = 1$, we get the matrices $\varphi_{\psi 1}^m$ and $\varphi_{\psi 2}^m$.

If $\lambda = (0 : 1 : 0)$, with similar calculation, we get only one indecomposable extension:

$$S = \begin{pmatrix} y_1 & y_2y_3 & 0 & y_2 \\ y_2 & y_1^2 + y_1y_3 & y_1 & 0 \\ 0 & 0 & y_1^2 & -y_2^2 \\ 0 & 0 & -y_3 & y_1 + y_3 \end{pmatrix}.$$

As one can easily see, the module $\operatorname{Coker} S$ is non–locally free. Indecomposability follows as in theorem 2.14.

• Consider now that $M$ has an extension of type (12), that means,

$$0 \to \operatorname{Coker} \psi_s \to M \to \operatorname{Coker} \varphi_\lambda \otimes R(k) \to 0.$$

In this case, the corresponding matrix $D$ has homogeneous entry of degree $m = 1 - k$ on the position [2,1], entries of degree $m+1$ on the main diagonal and of degree $m+2$ on the position [1,2]. So, if $k \geq 3$, the extension splits. If $k = 2$ the only nonzero entry of $D$ is on position [1,3] and has degree 1. If $k = 1$ the entry [2,1] should be zero. Writing the condition of extension, one get easily that $D = 0$, so $S$ decomposes.

Let us consider now only the negative shiftings of $\operatorname{Coker} \varphi_\lambda$. Denote the entries of $D$ with $d_1, ..., d_4$, so that $D = \left(\begin{smallmatrix} d_1 & d_2 \\ d_3 & d_4 \end{smallmatrix}\right)$.

By linear transformations, we can eliminate the variable $y_1$ in $d_1, d_3, d_4$ and $y_2$ in $d_4$. In case that $m \geq 0$, we can eliminate also $y_1^2$ in $d_2$. We write $d_2 = y_1 d_5 + d_6$ and $d_4 = a_4 y_3^m$. More, in case that $d_5$ has $y_2 y_3$ we eliminate it using the third line.
The extension condition $\beta_s \cdot D \cdot \varphi_\lambda = 0$ means:

```
matrix p=subst(phi,a,0,b,0);
matrix D[2][2]=d(1),y(1)*d(5)+d(6),
               d(3),          d(4);
matrix G=p*D*psi;
ideal P;
P=condext(G);
P=subst(P,d(6),y(3)*d(5)*a+y(3)*d(5));
P=interred(P);
P;
```

```
P[1]=y(3)^2*d(3)*b-y(3)*d(1)*a+y(3)*d(4)*a-y(3)*d(5)*b
P[2]=y(3)^2*d(3)*a^2+y(3)^2*d(3)*a+y(2)*d(1)-y(2)*d(4)+
     y(3)*d(4)*b
P[3]=y(2)*y(3)*d(3)-y(2)*d(5)+y(3)*d(1)*a+y(3)*d(5)*b
P[4]=y(2)*y(3)*d(1)-y(2)*y(3)*d(4)+y(3)^2*d(1)*b+
     y(3)^2*d(5)*a^2+y(3)^2*d(5)*a
```

We find out that $d_2 = y_1 d_5 + d_5 y_3(a+1)$. From P[3] we see that $y_3 | d_5$, and since $y_2 y_3$ is eliminated from $d_5$, we can write $d_5 = a_5 y_3^{m+1}$. Since $d_4 = a_4 y_3^{m+1}$, from P[4], we get that also $d_1$ has the form $d_1 = a_1 y_3^{m+1}$. And, therefore, from P[1], $d_3 = a_3 y_3^m$. Furthermore, $a_4 = a_1$, $a_5 = a_3$ and $(ba_3 + aa_1 = 0$

We impose now that the module is non-locally free. For this, we should have $\mathrm{Fitt}_2(S) R_{\langle y_1, y_2 \rangle} \neq R_{\langle y_1, y_2 \rangle}$.

```
matrix A1=subst(psi,y(1),0,y(2),0,y(3),1);
matrix B1=subst(phi,a,0,b,0,y(1),0,y(2),0,y(3),1);
matrix D1[2][2]=d(1),d(3)*(a+1),
                d(3),      d(1);
matrix G[4][2]=D1,B1;
matrix S1[4][4]=concat(A1,G);
ideal F=fitting(S1,2);
F;
```

```
F[1]=d(1)*b-d(3)*a^2-d(3)*a
F[2]=d(1)*a-d(3)*b
F[3]=d(1)^2-d(3)^2*a-d(3)^2
```

If $a \neq 0$, $a_3$ should be nonzero, and we can choose it to be $a$. So $a_1 = a_4 = -b$ and $a_3 = a_5 = a$. The matrix $S$ has the form:

$$S = \begin{pmatrix} y_1^2 + y_1y_3 & -y_2y_3 & -by_3^{m+1} & ay_1y_3^{m+1} - (a^2+a)y_3^{m+2} \\ -y_2 & y_1 & ay_3^m & -by_3^{m+1} \\ 0 & 0 & y_1 - ay_3 & y_2y_3 + by_3^2 \\ 0 & 0 & y_2 - by_3 & y_1^2 + (a+1)y_2y_3 + (a^2+a)y_3^2 \end{pmatrix}.$$

As in other cases, if $m \geq 1$, this matrix decomposes, after some linear transformations.

If $a = b = 0$, we need $a_1 = a_3$ or $a_1 = -a_3$. We choose $a_1 = 1$ and in this way we get the matrices $\psi_{\varphi 1}^m$ and $\psi_{\varphi 2}^m$. If $\lambda = (0 : 1 : 0)$, one get no indecomposable extensions of this type.

• We consider now the last case, when $M$ has an extension of type (13). Let $(S, S')$ be a matrix factorization of the module $M$ such that $S = \begin{pmatrix} \varphi_s & D \\ 0 & \varphi_\lambda \end{pmatrix}$. The matrix $D$ has homogeneous entries of degree $m = 1 - k$ on the first column, and of degree $m + 1$ on the second. So, if $k \geq 2$, the extension splits. If $k = 1$ the first column of $D$ should be zero. Writing the condition of extension, one get easily that $D = 0$, so $S$ decomposes.
Let us consider now only the negative shiftings of $\operatorname{Coker} \varphi_\lambda$. Denote the entries of $D$ with $d_1, ..., d_4$, so that $D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}$.
By linear transformations, we can eliminate the variable $y_1$ in $d_1, d_2, d_3$ and $y_2$ in $d_1$. In case that $m \geq 1$, we can eliminate also $y_2y_3$ in $d_2$ and $y_1^2$ in $d_4$. We write $d_4 = y_1d_5 + d_6$ and $d_1 = a_1y_3^m$.

The extension condition $\beta_s \cdot D \cdot \varphi_\lambda = 0$ gives :

```
matrix p=subst(psi,a,0,b,0);
matrix D[2][2]=d(1),d(2),d(3),y(1)*d(5)+d(6);
matrix G=p*D*psi;
ideal P;
P=condext(G);
P=subst(P,d(6),y(3)*d(5)*a+y(3)*d(5));
P=interred(P);
P;
```

```
P[1]=y(2)*d(1)+y(2)*d(5)-y(3)*d(3)*a-y(3)*d(5)*b
P[2]=y(3)^2*d(1)*b+y(3)^2*d(3)*a+y(3)^2*d(5)*b+
     y(3)*d(2)*a
P[3]=y(3)^2*d(1)*a^2+y(3)^2*d(1)*a+y(3)^2*d(3)*b+
     y(3)^2*d(5)*a^2+y(3)^2*d(5)*a+y(3)*d(2)*b
P[4]=y(2)*y(3)*d(3)-y(3)^2*d(1)*a^2-y(3)^2*d(1)*a+
     y(2)*d(2)-y(3)*d(2)*b
```

We find out that $d_3 = y_1 d_5 + d_5 y_3(a+1)$. From P[4] we see that $y_3 | d_2$, and since $y_2 y_3$ is eliminated from $d_2$, we can write $d_2 = a_2 y_3^{m+1}$. Since $d_1 = a_1 y_3^m$, from the same condition, we get that also $d_3$ has the form $d_3 = a_3 y_3^m$. And, therefore, from P[1], $d_5 = a_5 y_3^m$. Furthermore, $a_5 = -a_1$, $a_2 = -a_3$ and $aa_3 - ba_1 = 0$

We impose now that the module is non-locally free. For this, we should have $\mathrm{Fitt}_2(S) R_{\langle y_1, y_2 \rangle} \neq R_{\langle y_1, y_2 \rangle}$.

```
matrix A1=subst(phi,y(1),0,y(2),0,y(3),1);
matrix B1=subst(phi,a,0,b,0,y(1),0,y(2),0,y(3),1);
matrix D1[2][2]=d(1),-d(3), d(3),-d(1)*(a+1);
matrix G[4][2]=D1,A1;
matrix S1[4][4]=concat(B1,G);
ideal F=fitting(S1,2);
F;

F[1]=d(1)*b-d(3)*a
F[2]=d(1)*a^2+d(1)*a-d(3)*b
F[3]=d(1)^2*a+d(1)^2-d(3)^2
```

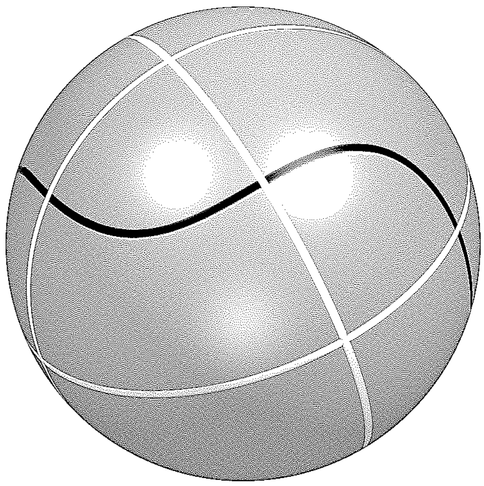
If $a \neq 0$, $a_1$ should be nonzero, and we can choose it to be $a$. So $a_1 = -a_5 = a$ and $a_3 = -a_2 = b$. The matrix $S$ has the form:

$$S = \begin{pmatrix} y_1 & y_2 y_3 & a y_3^m & -b y_3^{m+1} \\ y_2 & y_1^2 + y_1 y_3 & b y_3^m & -a y_1 y_3^m - (a^2+a) y_3^{m+1} \\ 0 & 0 & y_1 - a y_3 & y_2 y_3 + b y_3^2 \\ 0 & 0 & y_2 - b y_3 & y_1^2 + (a+1) y_2 y_3 + (a^2+a) y_3^2 \end{pmatrix}.$$

One can notice immediately, that after some simple linear transformations, $S$ decomposes.
If $a = b = 0$, we need $a_1 = a_3$ or $a_1 = -a_3$. We can choose $a_3 = 1$. In this case, the matrix $S$ becomes one of $\varphi^m_{\varphi 1}$ or $\varphi^m_{\varphi 2}$.
If $\lambda = (0 : 1 : 0)$, one get no indecomposable extensions of this type. The indecomposability of the matrices $\psi^m_{\varphi 1}$, $\psi^m_{\varphi 2}$, $\varphi^m_{\varphi 1}$ and $\varphi^m_{\varphi 2}$ follows as in theorem 2.14.

□

# Part II
# Fermat surface



In this part of the work we would classify the rank two, graded MCM modules on the affine cone of the Fermat surface. In this purpose, we treat separately the orientable and non–orientable modules. With the help of some results from [F2], we give also the matrix factorizations of orientable, 6– and 4–generated stable modules. In the first section we remind the classification of rank one MCM modules, that can be found in [EP] and [F2].

# 3 Rank one, graded, MCM modules over the hypersurface $R = k[x_1, x_2, x_3, x_4]/(x_1^3 + x_2^3 + x_3^3 + x_4^3)$

The aCM line bundles on a smooth cubic surface in $\mathbb{P}^3$ are classified in [F2] by the following theorem:

**Theorem 3.1.** *Let $X = V(F)$ be a smooth hypersurface in $\mathbb{P} = \mathbb{P}^3$ defined by the cubic form $F$ and let $\mathcal{L}$ be a normalized aCM line bundle on $Y$. Then the minimal graded free resolution on $\mathcal{L}$ takes one of the following forms:*

$$0 \to \mathcal{O}_{\mathbb{P}}(-3) \xrightarrow{F=f(\mathcal{L})} \mathcal{O}_{\mathbb{P}} \to \mathcal{L} \to 0 \tag{14}$$

$$0 \to \mathcal{O}_{\mathbb{P}}^2(-2) \xrightarrow{f(\mathcal{L})} \mathcal{O}_{\mathbb{P}}(-1) \oplus \mathcal{O}_{\mathbb{P}} \to \mathcal{L} \to 0 \tag{15}$$

$$0 \to \mathcal{O}_{\mathbb{P}}(-1) \oplus \mathcal{O}_{\mathbb{P}}(-2) \xrightarrow{f(\mathcal{L})} \mathcal{O}_{\mathbb{P}}^2 \to \mathcal{L} \to 0 \tag{16}$$

$$0 \to \mathcal{O}_{\mathbb{P}}^3(-1) \xrightarrow{f(\mathcal{L})} \mathcal{O}_{\mathbb{P}}^3 \to \mathcal{L} \to 0, \tag{17}$$

*where $det(f(\mathcal{L})) = F$.*
*The line bundles with resolutions of type (15), (16) respectively (17) are isomorphic to $\mathcal{O}(L)$, $\mathcal{O}(C)$, respectively $\mathcal{O}(T)$, where $L$ is a line on $X$, $C$ a conic on $X$, and $T$ a twisted cubic on $X$.*

From this theorem one can conclude that there are 27+27+72 isomorphism classes of graded MCM modules of rank one, over a hypersurface ring with smooth projective cone, $K[x_1, x_2, x_3, x_4]/f$, with $f$ an irreducible graded polynomial of degree 3. More, 54 of them are minimally 2–generated, and the other 72 are minimally 3–generated.

In the case of the Fermat surface we have the exact description of this modules (their matrix factorizations), that can be found in [EP].
Let $R = K[x_1, x_2, x_3, x_4]/f$, $f = x_1^3 + x_2^3 + x_3^3 + x_4^3$.
For every $a, b \in K$ such that $a^3 = b^3 = -1$ and for any $\sigma = (i\, j\, s)$ permutation of the set $\{2, 3, 4\}$ with $i < j$, we denote

$$\varphi_\sigma(a, b) = \begin{pmatrix} x_1 - ax_s & -(x_i^2 + bx_ix_j + b^2x_j^2) \\ x_i - bx_j & x_1^2 + ax_1x_s + a^2x_s^2 \end{pmatrix}$$

and

$$\psi_\sigma(a,b) = \begin{pmatrix} x_1^2 + ax_1x_s + a^2x_s^2 & x_i^2 + bx_ix_j + b^2x_j^2 \\ -(x_i - bx_j) & x_1 - ax_s \end{pmatrix}.$$

**Theorem 3.2.** $(\varphi_\sigma(a,b), \psi_\sigma(a,b))$ *is a matrix factorization of* $f = x_1^3 + x_2^3 + x_3^3 + x_4^3$*, for all* $\sigma, a, b$ *as above. The sets:*

$$\mathcal{M} = \{\mathrm{Coker}\,\varphi_\sigma(a,b) \mid a,b,\sigma\}$$

*and*

$$\mathcal{N} = \{\mathrm{Coker}\,\psi_\sigma(a,b) \mid a,b,\sigma\}$$

*have the following properties:*

1. *Every two generated, non–free, graded MCM module is isomorphic with one of the modules of* $\mathcal{M} \cup \mathcal{N}$*.*

2. *Every two different graded MCM modules from* $\mathcal{M} \cup \mathcal{N}$ *are not isomorphic.*

3. *All modules from* $\mathcal{M} \cup \mathcal{N}$ *have rank one.*

From Theorem 3.1 one can see that the matrices $\varphi_\sigma(a,b)$ correspond to the 27 quadrics on the surface and the matrices $\psi_\sigma(a,b)$ correspond to the 27 lines on the surface.

To describe the 3–generated rank one MCM modules, we make the following notations.
For $a,b,c,d,\varepsilon \in K$ such that $a^3 = b^3 = c^3 = d^3 = -1, \varepsilon^3 = 1, \varepsilon \neq 1$, and $bcd = \varepsilon a$, we set

$$\alpha(b,c,d,\varepsilon) = \begin{pmatrix} 0 & x_1 - ax_4 & x_2 - bx_3 \\ x_1 - cx_2 & -b^2x_3 - abc^2\varepsilon^2x_4 & b^2c^2x_3 - abc\varepsilon^2x_4 \\ x_3 - dx_4 & c^2x_2 + bc^2x_3 + acx_4 & -x_1 - cx_2 - ax_4 \end{pmatrix}$$

and

$$\beta(b,c,d,\varepsilon) = \alpha(b,c,d,\varepsilon)^t,$$

that is, the transpose of $\alpha(b,c,d,\varepsilon)$. Then each of the matrices $\alpha(b,c,d,\varepsilon)$ and $\beta(b,c,d,\varepsilon)$ forms with its adjoint, $\alpha(b,c,d,\varepsilon)^*$, respectively $\beta(b,c,d,\varepsilon)^*$, a matrix factorization of $f$.

For $a, b, c \in K$, distinct roots of $-1$, and $\varepsilon$ as above, we set

$$\eta(a,b,c,\varepsilon) = \begin{pmatrix} 0 & x_1 + x_2 & x_3 - ax_4 \\ x_1 + \varepsilon x_2 & -x_3 + cx_4 & 0 \\ x_3 - bx_4 & 0 & -x_1 - \varepsilon^2 x_2 \end{pmatrix}$$

and

$$\vartheta(a,b,c) = \begin{pmatrix} 0 & x_1 + x_3 & x_2 - ax_4 \\ x_1 - a^2 b x_3 & -x_2 + cx_4 & 0 \\ x_2 - bx_4 & 0 & -x_1 + ab^2 x_3 \end{pmatrix}.$$

The matrices $\eta(a,b,c,\varepsilon)$ and $\vartheta(a,b,c)$ form with their adjoint, $\eta(a,b,c,\varepsilon)^*$, respectively $\vartheta(a,b,c)^*$, matrix factorizations of $f$.

**Theorem 3.3 ((3.4) in [EP]).** *Let*

$$\mathcal{P} = \{\operatorname{Coker}\alpha(b,c,d,\varepsilon),\ \operatorname{Coker}\beta(b,c,d,\varepsilon) \mid b,c,d,\varepsilon \in K, \\ b^3 = c^3 = d^3 = -1,\ bcd = \varepsilon a,\ \varepsilon^3 = 1,\ \varepsilon \neq 1\}$$

*and*

$$\mathcal{Q} = \{\operatorname{Coker}\eta(a,b,c,\varepsilon),\ \operatorname{Coker}\vartheta(a,b,c) \mid \varepsilon^3 = 1,\ \varepsilon \neq 1 \\ \text{and } (a,b,c) \text{ is a permutation of the roots of } -1\}.$$

*Then the sets $\mathcal{P}, \mathcal{Q}$ of rank 1, 3–generated, MCM graded $R$–modules have the following properties:*

*(i) Every 3–generated, rank 1, indecomposable, graded MCM $R$–module is isomorphic with one module from $\mathcal{P} \cup \mathcal{Q}$.*

*(ii) If $M = \operatorname{Coker}\alpha(b,c,d,\varepsilon)$ (or $M = \operatorname{Coker}\beta(b,c,d,\varepsilon)$) belongs to $\mathcal{P}$ and $N \in \mathcal{P}$, then $N \simeq M$ if and only if $N = \operatorname{Coker}\alpha(b\varepsilon, c\varepsilon, d\varepsilon, \varepsilon^2)$ (or $N = \operatorname{Coker}\beta(b\varepsilon, c\varepsilon, d\varepsilon, \varepsilon^2)$).*

*(iii) Any two different modules from $\mathcal{Q}$ are not isomorphic.*

*(iv) Any module of $\mathcal{Q}$ is not isomorphic with some module of $\mathcal{P}$.*

Notice that the 3–generated modules from $\mathcal{P} \cup \mathcal{Q}$ correspond to the 72 twisted cubic curves on the Fermat surface.
The map $M \mapsto \Omega^1_R(M)$ is a bijection between the 3–generated, indecomposable, graded, MCM $R$–modules of rank two and the 3–generated, indecomposable, graded, MCM $R$–modules of rank 1. Thus, from the above theorem we obtain the description of the rank 2, 3–generated, indecomposable, graded MCM $R$–modules.

**Theorem 3.4.** *Let*

$$\mathcal{P}^* = \{\operatorname{Coker}\alpha(b,c,d,\varepsilon)^*,\ \operatorname{Coker}\beta(b,c,d,\varepsilon)^* \mid b,c,d,\varepsilon \in K,$$
$$b^3 = c^3 = d^3 = -1, bcd = \varepsilon a,\ \varepsilon^3 = 1, \varepsilon \neq 1\}$$

*and*

$$\mathcal{Q}^* = \{\operatorname{Coker}\eta(a,b,c,\varepsilon)^*,\ \operatorname{Coker}\vartheta(a,b,c)^* \mid \varepsilon^3 = 1,\ \varepsilon \neq 1$$
$$\textit{and}\ \ (a,b,c)\ \ \textit{is a permutation of the roots of}\ -1\ \}.$$

*Then the sets $\mathcal{P}^*, \mathcal{Q}^*$ of rank 2, 3–generated, MCM graded R–modules have the following properties:*

*(i) Every 3–generated, rank 2, indecomposable, graded MCM R–module is isomorphic with one module from $\mathcal{P}^* \cup \mathcal{Q}^*$.*

*(ii) If $M = \operatorname{Coker}\alpha(b,c,d,\varepsilon)^*$ (or $M = \operatorname{Coker}\beta(b,c,d,\varepsilon)^*$) belongs to $\mathcal{P}^*$ and $N \in \mathcal{P}^*$, then $N \simeq M$ if and only if $N = \operatorname{Coker}\alpha(b\varepsilon, c\varepsilon, d\varepsilon, \varepsilon^2)^*$ (or $N = \operatorname{Coker}\beta(b\varepsilon, c\varepsilon, d\varepsilon, \varepsilon^2)^*$).*

*(iii) Any two different modules from $\mathcal{Q}^*$ are not isomorphic.*

*(iv) Any module of $\mathcal{Q}^*$ is not isomorphic with some module of $\mathcal{P}^*$.*

**Corollary 3.5.** *There are* 72 *isomorphism classes of rank 2, indecomposable, graded MCM modules over R with three generators.*

# 4 Rank two, orientable, MCM modules over the hypersurface $R = k[x_1, x_2, x_3, x_4]/(x_1^3 + x_2^3 + x_3^3 + x_4^3)$

Let $R = k[x_1, .., x_n]/f$ be a normal hypersurface ring. A torsion free $R$-module M of rank $m$ is *orientable* if and only if $\det M = 0$, where $\det M$ (called determinant of $M$) is the corresponding class of the bidual $(\wedge^m M)^{**}$ in $Cl(R)$.
J. Herzog and M. Kühl have proved ([HK]) that any graded, indecomposable, orientable MCM $R$–module of rank 2 and can be written as $M \cong \Omega_R^2(I)$, where $I$ is a graded Gorenstein ideal of codimension 2. Two such modules are isomorphic if and only if their corresponding ideals are evenly linked. (We remind that two ideals $I, J$ of $R$ are said to be (directly) *linked* with respect to a regular sequence $\underline{y} = y_1, ...y_g$ in $I \cap J$ if $(\underline{y}) : J = I$ and $(\underline{y}) : I = J$. The ideals $I$ and $J$ are *evenly linked* if there exists a sequence of ideals $I = I_0, I_1, ..., I_l = J$ such that $l$ is even and $I_t$ is directly linked to $I_{t+1}$ for all $i = \overline{0, l-1}$. )
In the same paper it is proved that any graded, indecomposable, orientable MCM $R$–module of rank 2 has an even number of generators and, more, if $2s$ is the minimal number of generators of $M$, than the minimal number of generators of the corresponding Gorenstein ideal $I$ is $2s - 1$. ($!s \leq deg(f)$).
Using this characterization we will describe in the following subsection, the rank two, graded, indecomposable, 4-generated orientable MCM over the cubic hypersurface ring $k[x_1, x_2, x_3, x_4]/(f)$, $f = x_1^3 + x_2^3 + x_3^3 + x_4^3$. Notice that over a cubic hypersurface an orientable MCM module is 4–generated or 6–generated. To describe the 6–generated one, we will use another method, mainly, we prove that they have a skew symmetric matrix factorization. To find all such matrix factorizations, we restrict first the module to the hyperplane section $(x_4 = 0)$.

We use an interesting property of the Fermat surface that the rank two 6–generated orientable bundles, when restricting to the hyperplane section curve splits into two line bundles on the curve (a smooth elliptic curve). We prove in the following that an orientable MCM $R$–module of rank two has a skew symmetric matrix factorization.

Let $\varphi = (a_{ij})_{1\le i,j\le 2s}$ be a generic skew symmetric matrix, that is

$$a_{ii} = 0, a_{ij} = -a_{ji}, \text{ for all } i,j = 1,\dots,2s.$$

Then

$$\det(\varphi) = \mathrm{pf}(\varphi)^2,$$

where $\mathrm{pf}(\varphi)$ denotes the Pfaffian of $\varphi$ (see [Bo1, §5, no. 2] or [BH, (3.4)]). Like determinants, Pfaffians can be developed along a row. Set $\varphi_{ij}$ the matrix obtained from $\varphi$ by deleting the $i^{\text{th}}$ and $j^{\text{th}}$ rows and columns. Then, for all $i = 1,\dots,2s$,

$$\mathrm{pf}(\varphi) = \sum_{\substack{j=1\\ j\neq i}}^{2s} (-1)^{i+j}\sigma(i,j)a_{ij}\,\mathrm{pf}(\varphi_{ij}), \tag{18}$$

where $\sigma(i,j)$ denotes $\mathrm{sign}(j-i)$. Multiplying (18) by $\mathrm{pf}(\varphi)$, we obtain

$$\det(\varphi) = \sum_{j=1}^{2s} a_{ij}b_{ij}, \tag{19}$$

for $b_{ij} = (-1)^{i+j}\sigma(i,j)\,\mathrm{pf}(\varphi_{ij})\,\mathrm{pf}(\varphi)$ when $i \neq j$ and $b_{ii} = 0$. Since $\varphi$ is a generic matrix we see from (19) that $b_{ij}$ is exactly the algebraic complement of $a_{ij}$ and so the transpose matrix $B$ of $(b_{ij})$ is the adjoint matrix of $\varphi$. Set

$$\psi = \frac{1}{\mathrm{pf}(\varphi)}B.$$

Then

$$\varphi\psi = \psi\varphi = \mathrm{pf}(\varphi)\,\mathrm{Id}_{2s},$$

as it is stated also in [[JP], §3].

**Theorem 4.1.** *Let $S = K[x_1,...,x_n]$ and $f \in S$ a polynomial of degree $d$ defining a smooth irreducible hypersurface in $\mathbb{P}^{n-1}$. The cokernel of a homogeneous skew symmetric matrix over $S$ of order $2s \le 2d$, with determinant $f^2$ defines a graded MCM $R$–module $M$ of rank 2, where $R = S/f$.*
*Conversely, each graded indecomposable orientable MCM $R$–module $M$ of rank 2 is the cokernel of a map given by a skew symmetric homogeneous matrix $\varphi$ over $S$, whose determinant is $f^2$.*

*Proof.* Let $\varphi$ a homogeneous skew symmetric matrix over $S$ of order $2s$, $s \le d$, with $\det \varphi = f^2$. Let $\psi$ be given for $\varphi$ as above, that is the $(i,j)$ entry of $\psi$ is $\sigma(i,j)\,\mathrm{pf}(\varphi_{ij})$. Since $\mathrm{pf}(\varphi) = f$, we get

$$\varphi\psi = \psi\varphi = f \cdot \mathrm{Id}_{2s}\,.$$

Therefore $(\varphi, \psi)$ is a matrix factorization of $f$ which defines a graded MCM $R$–module of rank 2.
To prove the converse, consider $M$ a graded indecomposable orientable MCM $R$–module $M$ of rank 2. According to Herzog and Kühl [HK], $M$ must have an even minimal number of generators and it is isomorphic to the second syzygy over $R$ of a Gorenstein ideal $I \subset R$ of codimension 2. More, if $2s$ is the minimal number of generators of $M$, than the minimal number of generators of $I$ is $2s-1$. Using the Buchsbaum–Eisenbud Theorem (see e.g. [BH], (3,4)) there exists an exact sequence

$$0 \longrightarrow S \xrightarrow{d_3} S^{2s-1} \xrightarrow{d_2} S^{2s-1} \xrightarrow{d_1} S \tag{20}$$

such that $d_2$ is a skew symmetric homogeneous matrix, $d_3$ is the dual of $d_1$, $d_3 = d_1^t$, and

$$d_1 = \Big(\mathrm{pf}\big((d_2)_1\big), -\mathrm{pf}\big((d_2)_2\big), \ldots, \mathrm{pf}\big((d_2)_{2s-1}\big)\Big),$$

where $(d_2)_i$ denotes the $(2s-2) \times (2s-2)$ skew symmetric matrix obtained by deleting the $i^{\text{th}}$ row and column of $d_2$. Also $I = J/(f)$, with $J = \mathrm{Im}\, d_1$. Since $f \in J$ there exists $v : S \longrightarrow S^{2s-1}$ such that $d_1 v = f$. It is easy to see from (20) that $I$ has the following minimal resolution over $S$ :

$$0 \longrightarrow S \xrightarrow{\binom{d_3}{0}} S^{2s} \xrightarrow{(d_2,v)} S^{2s-1} \xrightarrow{\tilde{d}_1} I \longrightarrow 0.$$

As in [Ei1], since $fI = 0$, there exists a map $h : S^{2s-1} \to S^{2s}$ such that $(d_2, v)h = -f \cdot \mathrm{Id}_{2s-1}$ and we obtain the following exact sequence

$$R^{2s} \xrightarrow{\left(\bar{h}\middle|\begin{smallmatrix}\bar{d}_3\\ 0\end{smallmatrix}\right)} R^{2s} \xrightarrow{(\bar{d}_2,\bar{v})} R^{2s-1} \xrightarrow{\bar{d}_1} I \longrightarrow 0. \tag{21}$$

On the other hand, $\varphi = \left(\begin{smallmatrix} d_2 & v \\ -v^t & 0 \end{smallmatrix}\right)$ is a skew symmetric homogeneous matrix of order 2s. Let $\psi$ defined as above. By construction, $\psi$ has the form $\left(\begin{smallmatrix} c & d_1^t \\ -d_1 & 0 \end{smallmatrix}\right)$

and so $(d_2, v)\left(\begin{smallmatrix} c \\ -d_1 \end{smallmatrix}\right) = -f \cdot \mathrm{Id}_{2s-1} = pf(\varphi) \cdot \mathrm{Id}_{2s-1}$. Taking $h = \left(\begin{smallmatrix} c \\ -d_1 \end{smallmatrix}\right)$ above, we obtain from (21) the following exact sequence:

$$R^{2s} \xrightarrow{\varphi} R^{2s} \xrightarrow{\psi} R^{2s} \xrightarrow{(\bar{d}_2, \bar{v})} R^{2s-1} \xrightarrow{\bar{d}_1} I \longrightarrow 0,$$

which gives

$$\operatorname{Coker} \varphi \cong \operatorname{Im} \psi = \Omega_R^2(I).$$

We have $pf(\varphi) = -f$ and so $\det(\varphi) = f^2$. Notice that all maps appeared in the proof are graded, but we omitted the necessary shiftings. In [BEPP] one can find a complete proof for the particular case $d = 3, n = 4, s = 3$.

□

The geometric version of this theorem (regarding vector bundles on smooth hypersurfaces) is the following:

If $\epsilon \in \{-1, 1\}$, we say that a matrix $A$ is *$\epsilon$–symmetric* if $A^t = \epsilon A$. Correspondingly we have a notion of $\epsilon$-symmetric duality on a vector bundle.

**Theorem 4.2.** *Let $Y = \boldsymbol{V}(F_Y)$ be a smooth hypersurface of degree $d$ in $\boldsymbol{P}^n$ and let $\mathcal{F}$ be a aCM rank $r$ vector bundle on $Y$. Then the minimal graded free resolution of the sheaf $\mathcal{F}$, extended to zero on $\mathbb{P}^n$, takes the form:*

$$0 \longrightarrow Syz(\mathcal{F}) \xrightarrow{f(\mathcal{F})} Gen(\mathcal{F}) \xrightarrow{p(\mathcal{F})} \mathcal{F} \longrightarrow 0$$

*with $Gen(\mathcal{F}) = \oplus_{i=1}^{s} \mathcal{O}_{\mathbb{P}^n}(a_i)$, $Syz(\mathcal{F}) = \oplus_{i=1}^{s} \mathcal{O}_{\mathbb{P}^n}(b_i)$ and $det(f(\mathcal{F})) = F^r$. Moreover, suppose that there exists an $\epsilon$–symmetric duality $\kappa : \mathcal{F} \otimes \mathcal{F} \to \mathcal{O}_Y(d+t)$. Then we have a natural isomorphism $Syz(\mathcal{F}) \cong Gen(\mathcal{F})^*(t)$, and $f(\mathcal{F}) = \epsilon f(\mathcal{F})^t$. In particular, if $\mathcal{F}$ is a rank two vector bundle, the matrix $f(\mathcal{F})$ is skew symmetric if and only if $\wedge^2(\mathcal{F}) \cong \mathcal{O}_Y(t)$, for some $t \in \mathbb{Z}$.*

Since we shall use it in this section, we recall from [LPP] the description of the rank 1, three-generated, non–free, graded MCM $R_3$–modules, for $R_3 = k[x_1, x_2, x_3]/[(x_1^3 + x_2^3 + x_3^3]$.
First we recall the notations. Let $P_0 = [-1 : 0 : 1] \in V(f_3)$. For each $\lambda = [\lambda_1 : \lambda_2 : 1] \in V(f_3), \lambda \neq P_0$, we set

$$\alpha_\lambda = \begin{pmatrix} 0 & x_1 - \lambda_1 x_3 & x_2 - \lambda_2 x_3 \\ x_1 + x_3 & -x_2 - \lambda_2 x_3 & -w x_3 \\ x_2 & w x_3 & (1 - \lambda_1) x_3 - x_1 \end{pmatrix},$$

where $w = \frac{\lambda_2^2}{\lambda_1+1}$ and, if $\lambda = [\lambda_1 : 1 : 0] \in V(f_3)$, we set

$$\alpha_\lambda = \begin{pmatrix} 0 & x_1 - \lambda_1 x_2 & x_3 \\ x_1 + x_3 & -\lambda_1 x_1 & \lambda_1 x_1 + \lambda_1^2 x_2 \\ x_2 & x_3 - x_1 & -x_1 \end{pmatrix}.$$

Let $\beta_\lambda$ the adjoint matrix of $\alpha_\lambda$.

**Theorem 4.3 ((3.7) in [LPP]).** *$(\alpha_\lambda, \beta_\lambda)$ is a matrix factorization for all $\lambda \in V(f_3)$, $\lambda \neq P_0$, and the set of 3–generated MCM graded $R_3$–modules,*

$$\mathcal{M}_0 = \{\operatorname{Coker} \alpha_\lambda \mid \lambda \in V(f_3), \lambda \neq P_0\}$$

*has the following properties:*

*(i) All the modules from $\mathcal{M}_0$ have rank 1.*

*(ii) Each two different modules from $\mathcal{M}_0$ are not isomorphic.*

*(iii) Every 3–generated, rank 1, non–free, graded MCM $R_3$–module is isomorphic with one module from $\mathcal{M}_0$.*

### 4.1 Orientable 6–generated MCM modules

Let $S = K[x_1, x_2, x_3, x_4]$, and $R = S/(f)$, $f = x_1^3 + x_2^3 + x_3^3 + x_4^3$.

We have proved that a non–free graded orientable 6–generated MCM $R$–module corresponds to a skew symmetric homogeneous matrix over $S$ of order 6, whose determinant is $f^2$.

Let $\Lambda$ be such a matrix. Notice that $\Lambda$ has linear entries and the matrix $\underline{\Lambda} := \Lambda|_{x_4=0}$, obtained from $\Lambda$ by restricting the entries to $x_4 = 0$, is a homogeneous matrix over $S_3 = K[x_1, x_2, x_3]$, whose determinant is $f_3^2$, where $f_3 = x_1^3 + x_2^3 + x_3^3$. Therefore, Coker$\underline{\Lambda}$ defines a graded rank 2, 6–generated MCM over $R_3 = S_3/(f_3)$. These modules were explicitly described in [LPP].

**Lemma 4.4.** *Let $M$ be a non–free graded orientable 6–generated MCM module over $R$. Then the restriction of $M$ to the curve defined by $f = x_4 = 0$ splits into a direct sum of a 3–generated MCM of rank 1 and its dual.*

*Especially, there exists* $\lambda \in V(f_3) \setminus \{P_0\}$ *and a skew symmetric matrix* $\Gamma \in \mathcal{M}_{6\times 6}(K)$*, such that* $M$ *is the cokernel of a map given by the matrix* $\Lambda = x_4 \cdot \Gamma + \left(\begin{smallmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{smallmatrix}\right)$.

*(The same notations as in [LPP] and in Introduction.)*

*Proof.* Let $\Lambda_1$ be a skew symmetric homogeneous matrix over $S$, corresponding to $M$, and denote $\underline{\Lambda_1} = \Lambda_1|_{x_4=0}$. Suppose that the MCM $S_3$–module corresponding to $\underline{\Lambda_1}$ is indecomposable. Then we can generate it as described in Theorem 4.2 and Lemma 5.4 from [LPP]. Denote with $D$ the matrix which we obtain by this means.

Since $D \sim \underline{\Lambda_1}$, and $\underline{\Lambda_1}$ is skew symmetric, there exist two invertible matrices $U, V \in \mathcal{M}_{6\times 6}(K)$ such that $U \cdot D \cdot V + (U \cdot D \cdot V)^t = 0$. Therefore, there exists $T \in \mathcal{M}_{6\times 6}(K)$ an invertible matrix such that $T \cdot D + (TD)^t = 0$. (Take $T = (V^t)^{-1} \cdot U$.)

With the help of Singular, we find that, in fact, there is no invertible matrix $T$ such that $T \cdot D$ is skew symmetric. Therefore, the module corresponding to $\underline{\Lambda_1}$ should decompose.

First, we generate the matrix D.

```
LIB"matrix.lib";
option(redSB);
proc reflexivHull(matrix M)
{
  module N=mres(transpose(M),3)[3];
  N=prune(transpose(N));
  return(matrix(N));
}

proc tensorCM(matrix Phi, matrix Psi)
{
   int s=nrows(Phi);
   int q=nrows(Psi);
   matrix A=tensor(unitmat(s),Psi);
   matrix B=tensor(Phi,unitmat(q));
   matrix R=concat(A,B,U);
   return(reflexivHull(R));
}
```

```
proc M2(ideal I)
{
   matrix A=syz(transpose(mres(I,3)[3]));
   return(transpose(A));
}

ring R=0,(x(1..3)),(c,dp);
qring S=std(x(1)^3+x(2)^3+x(3)^3);
ideal I=maxideal(1);
matrix C=M2(I);

ring R1=(0,a),(x(1..3),e,b),lp;
ideal I=x(1)^3+x(2)^3+x(3)^3,(a-1)^3+b3+1,e*b+a2-3*a+3,e*a-b2;
qring S1=std(I);

matrix B[3][3]=        0,  x(1)-(a-1)*x(3),         x(2)-b*x(3),
               x(1)+x(3),     -x(2)-x(3)*b,               -x(3)*e,
                    x(2),            x(3)*e,  -x (1)+(-a+2)*x(3);
matrix C=imap(S,C); matrix D=tensorCM(C,B);
```

We check the existence of the invertible matrix T, such that $T \cdot D$ is skew symmetric.

```
ring R2=0,(x(1..3),a,e,b,t(1..36)),dp;
ideal I=x(1)^3+x(2)^3+x(3)^3,(a-1)^3+b3+1,e*b+a2-3*a+3,e*a-b2;
qring S2=std(I);
matrix D=imap(S1,D);
matrix T[6][6]=t(1..36);
matrix A=T*D+transpose(T*D); ideal I=flatten(A);
ideal I1=transpose(coeffs(I,x(1)))[2];
ideal I2=transpose(coeffs(I,x(2)))[2];
ideal I3=transpose(coeffs(I,x(3)))[2];
ideal J=I1+I2+I3+ideal(det(T)-1);
ideal L=std(J);
L;

L[1]=1
```

Therefore, there does not exist such invertible matrix $T$.

This means, that after some linear transformations, $\underline{\Lambda_1}$ decomposes into two matrices of order three and rank 1 with determinant $f_3 = x_1^3 + x_2^3 + x_3^3$, which correspond to two points $\lambda_1, \lambda_2$ in $V(f_3) \setminus \{P_0\}$, $P_0 = [-1 : 0 : 1]$. Let us denote them by $A$ and $B$. We can consider $A = \alpha_{\lambda_1}$, $B = \alpha_{\lambda_2}$.

Since $\underline{\Lambda_1}$ is skew symmetric, there exists an invertible matrix $U \in \mathcal{M}_{6\times 6}(K)$ such that $U \cdot \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ is skew symmetric. Consider $U = \left(\begin{smallmatrix} U_1 & U_2 \\ U_3 & U_4 \end{smallmatrix}\right)$. The skew symmetry condition gives the following equalities:

$$\begin{cases} U_1 \cdot A + (U_1 \cdot A)^t & = & 0 \\ U_4 \cdot B + (U_4 \cdot B)^t & = & 0 \\ U_2 \cdot B + \ A^t \cdot U_3^t & = & 0 \\ U_3 \cdot A + \ B^t \cdot U_2^t & = & 0 \end{cases}$$

So $U_1 \cdot \alpha_{\lambda_1}$ and $U_4 \cdot \alpha_{\lambda_2}$ are skew symmetric, so they have only zeros on the main diagonal. Since the entries of the second and third line and column of $\alpha_{\lambda_1}$ and $\alpha_{\lambda_2}$ are linearly independent, we easily obtain that $U_1 = U_4 = 0$. Therefore, $U_2$ and $U_3$ are invertible matrices and $B = -U_2^{-1} \cdot A^t \cdot U_3^t$.

We have obtained $\underline{\Lambda_1} \sim \begin{pmatrix} \alpha_{\lambda_1} & 0 \\ 0 & \alpha_{\lambda_1}^t \end{pmatrix} \sim \begin{pmatrix} 0 & -\alpha_{\lambda_1}^t \\ \alpha_{\lambda_1} & 0 \end{pmatrix}$.

Therefore, there exists $\Gamma \in \mathcal{M}_{6\times 6}(K)$ skew symmetric, and $\lambda \in V(f_3) \setminus \{P_0\}$ such that $\Lambda_1 \sim \Lambda = x_4 \cdot \Gamma + \begin{pmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{pmatrix}$. We can write $\Gamma = \begin{pmatrix} \Gamma_1 & -\Gamma_2^t \\ \Gamma_2 & \Gamma_3 \end{pmatrix}$, $\Gamma_i \in \mathcal{M}_{3\times 3}(K)$, $i = 1, 2, 3$, $\Gamma_1$ and $\Gamma_3$ skew symmetric. ☐

**Remark 4.5 (Notation).** *For any $\lambda = [a : b : c] \in V(f_3) \setminus \{P_0\}$ there exists a unique point in $V(f_3) \setminus \{P_0\}$ which we denote as $\lambda^t$, such that $\alpha_\lambda^t \sim \alpha_{\lambda^t}$. We find $\lambda^t = [c : b : a]$.*[1]

*For $\lambda = [a : b : 1]$ we denote with $U_\lambda$ and $V_\lambda$ two invertible matrices such that $U_\lambda \cdot \alpha_\lambda^t = \alpha_{\lambda^t} \cdot V_\lambda$.*

*If $a \neq 0$, then we can take $U_\lambda = \begin{pmatrix} b^2 & b(a+1) & -(a+1)^2 \\ -(a+1)^2 & b^2 & -b(a+1) \\ b(a+1) & (a+1)^2 & b^2 \end{pmatrix}$ and $V_\lambda = U_\lambda^t$.*

*If $a = 0$, then we can take $U_\lambda = \begin{pmatrix} -b^2 & -b & 1 \\ -2b & 1 & b^2 \\ 2b^2 & 2b & 1 \end{pmatrix}$ and $V_\lambda = \begin{pmatrix} 1 & -2b & 2b^2 \\ -b & -b^2 & -1 \\ -b & -b^2 & 2 \end{pmatrix}$.*

*Notice that $\lambda^t = \lambda$ for $\lambda = [1 : b : 1] \in V(f_3)$, and $\lambda \neq \lambda^t$ for all other $\lambda \in V(f_3) \setminus \{P_0\}$.*

**Remark 4.6.** *For any $\lambda = [1 : b : 0] \in V(f_3) \setminus \{P_0\}$ and any $\Lambda = x_4 \cdot \Gamma + \begin{pmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{pmatrix}$ skew symmetric with $\det \Lambda = f^2$, we have $\lambda^t = [0 : b : 1]$ in*

[1] If $\lambda$ corresponds to the 3–generated rank 1 MCM $N$, then $\lambda^t$ corresponds to its dual $N^\vee$.

*$V(f_3) \smallsetminus \{P_0\}$ and $\Lambda' = x_4\Gamma' + \begin{pmatrix} 0 & -\alpha^t_{\lambda^t} \\ \alpha_{\lambda^t} & 0 \end{pmatrix}$ skew symmetric with $\det \Lambda' = f^2$ such that $\Lambda \sim \Lambda'$.*

*Indeed, take $\Lambda' = U \cdot \Lambda \cdot U^t$ where $U = \begin{pmatrix} 0 & T_1 \\ T_2 & 0 \end{pmatrix}$, $T_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} b & b^2 & 0 \\ 0 & 1 & -b^2 \\ 2 & 0 & 1 \end{pmatrix}$ and $T_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 & b & b \\ 2 & b^2 & b^2 \\ -2b^2 & 1 & -2 \end{pmatrix}$.*

*Therefore, Coker $\Lambda$ and Coker $\Lambda'$ define two isomorphic MCM modules. This is the reason why we may only consider the case $\lambda = [a : b : 1] \in V(f_3) \smallsetminus \{P_0\}$, from now on.*

**Remark 4.7.** *Consider $\lambda = [a : b : 1] \in V(f_3) \smallsetminus \{P_0\}$ and $\Lambda = x_4 \cdot \Gamma + \begin{pmatrix} 0 & -\alpha^t_{\lambda} \\ \alpha_{\lambda} & 0 \end{pmatrix}$ as in Lemma 4.4. Then there exists $\overline{\Lambda} = x_4 \cdot \overline{\Gamma} + \begin{pmatrix} \alpha_{\lambda} & 0 \\ 0 & \alpha_{\lambda^t} \end{pmatrix}$ with $\det \overline{\Lambda} = f^2$ such that $\overline{\Lambda} \sim \Lambda$.*

*Indeed, consider $\overline{\Lambda} = \begin{pmatrix} 0 & \mathrm{Id} \\ -U_{\lambda} & 0 \end{pmatrix} \cdot \Lambda \cdot \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & V_{\lambda}^{-1} \end{pmatrix}$.*

*We obtain $\overline{\Gamma} = \begin{pmatrix} \Gamma_2 & \Gamma_3 \cdot V_{\lambda}^{-1} \\ -U_{\lambda} \cdot \Gamma_1 & U_{\lambda} \cdot \Gamma_2^t \cdot V_{\lambda}^{-1} \end{pmatrix}$.*

**Lemma 4.8.** *Consider $\Lambda = x_4 \cdot \Gamma + \begin{pmatrix} 0 & -\alpha^t_{\lambda} \\ \alpha_{\lambda} & 0 \end{pmatrix}$ as above. Then the MCM module $M$ corresponding to $\Lambda$ is indecomposable if and only if $\Gamma_1 \neq 0$ or $\Gamma_3 \neq 0$.*

*Proof.* Suppose $M$ is indecomposable. If $\Gamma_1 = \Gamma_3 = 0$, then $\begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \cdot \Lambda = \begin{pmatrix} x_4 \cdot \Gamma_2 + \alpha_{\lambda} & 0 \\ 0 & -x_4 \cdot \Gamma_2^t - \alpha^t_{\lambda} \end{pmatrix}$, so $\Lambda$ decomposes after some linear transformation.

This contradicts the indecomposability of $M =$ Coker $\Lambda$, so we must have $\Gamma_1 \neq 0$ or $\Gamma_3 \neq 0$.

Now, let us suppose $\Gamma_1 \neq 0$ or $\Gamma_3 \neq 0$ and prove that $M$ is indecomposable.

Suppose $M$ decomposes. Then there exists a matrix $\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ equivalent to $\Lambda$ with $T_1, T_2$ two matrices of order three and rank 1, with $\det T_1 = \det T_2 = f$ and $T_1|_{x_4=0} = \alpha_{\lambda_1}$, $T_2|_{x_4=0} = \alpha_{\lambda_2}$, where $\lambda_1, \lambda_2 \in V(f_3) \smallsetminus \{P_0\}$.

Since $\Lambda$ is skew symmetric, after some linear transformations, $\begin{pmatrix} \alpha_{\lambda_1} & 0 \\ 0 & \alpha_{\lambda_2} \end{pmatrix}$ should also become skew symmetric. As we saw in the proof of Lemma 4.4, this gives $\alpha_{\lambda_2} \sim \alpha^t_{\lambda_1}$, so $\lambda_2 = \lambda_1^t$.

Using Remark 4.6, there exist $U, V \in \mathcal{M}_{6\times 6}(K)$ invertible matrices such that $U \cdot \overline{\Lambda} \cdot V = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} = x_4 \cdot \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} + \begin{pmatrix} \alpha_{\lambda_1} & 0 \\ 0 & \alpha_{\lambda_1^t} \end{pmatrix}$.

Therefore,

$$\left\{\begin{array}{lcll} U\cdot\begin{pmatrix}\alpha_\lambda & 0\\ 0 & \alpha_{\lambda^t}\end{pmatrix} & = & \begin{pmatrix}\alpha_{\lambda_1} & 0\\ 0 & \alpha_{\lambda_1^t}\end{pmatrix} \cdot V^{-1} & (1)\\ U\cdot\begin{pmatrix}\Gamma_2 & \Gamma_3\cdot V_\lambda^{-1}\\ -U_\lambda\cdot\Gamma_1 & U_\lambda\cdot\Gamma_2^t\cdot V_\lambda^{-1}\end{pmatrix} & = & \begin{pmatrix}N_1 & 0\\ 0 & N_2\end{pmatrix} \cdot V^{-1} & (2)\,.\end{array}\right.$$

Let us consider $U=\left(\begin{smallmatrix}U_1 & U_2\\ U_3 & U_4\end{smallmatrix}\right)$ and $V^{-1}=\left(\begin{smallmatrix}V_1 & V_2\\ V_3 & V_4\end{smallmatrix}\right)$ with $U_i,V_i\in\mathcal{M}_{3\times 3}(K)$, $i=1,\dots,4$.

The first system of equations gives:

$$\left\{\begin{array}{lcl} U_1\cdot\alpha_\lambda & = & \alpha_{\lambda_1}\cdot V_1\\ U_2\cdot\alpha_{\lambda^t} & = & \alpha_{\lambda_1}\cdot V_2\\ U_3\cdot\alpha_\lambda & = & \alpha_{\lambda_1^t}\cdot V_3\\ U_4\cdot\alpha_{\lambda^t} & = & \alpha_{\lambda_1^t}\cdot V_4\end{array}\right.$$

By comparing the coefficients of $x_1,x_2,x_3$ on the left–hand side and right–hand side of the above equalities, we obtain easily:

$$U_i=V_i=K_i\cdot\mathrm{Id}_3\ \text{ with } K_i\in K,\ i=1,\dots,4\,.$$

Moreover, if $\lambda\neq\lambda_1$, then $K_1=K_4=0$ and if $\lambda\neq\lambda_1^t$, then $K_2=K_3=0$. Since $U$ is invertible, we have $\lambda=\lambda_1$ or $\lambda=\lambda_1^t$.

We know that $\alpha_{\lambda_1}=T_1|_{x_4=0}$ where $T_1$ is a matrix of order three over $S=K[x_1,x_2,x_3,x_4]$ of rank 1 and with determinant $f$. So Coker $T_1$ is a graded 3–generated rank 1 MCM $R$–module. In [EP], all the isomorphism classes of such modules are given explicitly. We obtain $\alpha_{\lambda_1}\sim\alpha|_{x_4=0}$ or $\alpha_{\lambda_1}\sim\alpha^t|_{x_4=0}$ or $\alpha_{\lambda_1}\sim\eta|_{x_4=0}$ or $\alpha_{\lambda_1}\sim\nu|_{x_4=0}$.

Using Singular we check that none of the above matrices is equivalent to $\alpha_{[1:\ell:1]}$, therefore, $\lambda_1\neq\lambda_1^t$.

```
LIB"matrix.lib";
option(redSB);

ring r=0,(x(1..3),l,a,b,c,d,e,v(1..9),u(1..9)),dp;
ideal I=x(1)^3+x(2)^3+x(3)^3,
        l^3+2,
        a3+1,b3+1,c3+1,d3+1,e2+e+1,bcd-e*a;
qring s=std(I);

proc isomorf(matrix X,matrix Y)
```

```
{
   matrix U[3][3]=u(1..9);
   matrix V[3][3]=v(1..9);
   matrix C=U*X-Y*V;
   ideal I=flatten(C);
   ideal I1=transpose(coeffs(I,x(1)))[2];
   ideal I2=transpose(coeffs(I,x(2)))[2];
   ideal I3=transpose(coeffs(I,x(3)))[2];
   ideal J=I1+I2+I3+ideal(det(U)-1,det(V)-1);
   ideal L=std(J);
   return(L);
}
```

We write the matrix $A$ corresponding to the point (1:l:1).

```
matrix A[3][3]=0,            x(1)-x(3),   x(2)-l*x(3),
               x(1)+x(3), -x(2)-l*x(3), -1/2*l^2*x(3),
                    x(2), 1/2*l^2*x(3),         -x(1);
```

We write the matrices corresponding to the rank 1 3–generated MCM modules, restricted to the hyperplane $(x(4) = 0)$.

```
matrix alpha[3][3]=0,      x(1),                       -x(3)*b+x(2),
                 -x(2)*c+x(1),            -x(3)*b^2,  x(3)*b^2*c^2,
                        x(3), x(3)*b*c^2+x(2)*c^2,  -x(2)*c-x(1);

matrix alphat=transpose(alpha);

matrix eta[3][3]=0,x(1)+x(2),      x(3),
                x(1)+e*x(2),     -x(3),                 0,
                       x(3),         0,    -x(1)-e^2*x(2);

matrix nu[3][3]=0,    x(1)+x(3),   x(2),
               x(1)-a^2*b*x(3),  -x(2),                 0,
                          x(2),      0,  -x(1)+a*b^2*x(3);

isomorf(alpha,A); L[1]=1
isomorf(alphat,A); L[1]=1
```

```
isomorf(eta,A); L[1]=1
isomorf(teta,A); L[1]=1
```

If $\lambda = \lambda_1 \neq \lambda_1^t$ as a solution of the system (1), we obtain: $U = V = \left(\begin{smallmatrix} K_1\cdot \mathrm{Id} & 0 \\ 0 & K_4\cdot \mathrm{Id}\end{smallmatrix}\right)$, $K_1 \cdot K_4 \neq 0$.

Replacing $U$ and $V$ in (2), we get: $\begin{cases} K_1 \cdot \Gamma_3 \cdot V_\lambda = 0 \\ K_4 \cdot U_\lambda \cdot \Gamma_1 = 0. \end{cases}$

Since $K_1 \neq 0$, $K_4 \neq 0$ and $U_\lambda, V_\lambda$ are invertible matrices, $\Gamma_1$ and $\Gamma_3$ should be 0–matrices, which is a contradiction to our hypothesis.

If $\lambda = \lambda_1^t \neq \lambda_1$, we obtain, as a solution of (1): $U = V = \left(\begin{smallmatrix} 0 & K_2\cdot \mathrm{Id} \\ K_3\,\mathrm{Id} & 0\end{smallmatrix}\right)$, $K_2 \cdot K_3 \neq 0$.

Replacing $U$ and $V$ in (2), we obtain: $\begin{cases} K_2 \cdot U_\lambda \cdot \Gamma_1 = 0 \\ K_3 \cdot \Gamma_3 \cdot V_\lambda = 0. \end{cases}$

Therefore, we must have again $\Gamma_1 = \Gamma_3 = 0$. □

For each $\lambda = [a : b : 1] \in V(f_3) \setminus \{P_0\}$, we define a family of skew symmetric homogeneous indecomposable matrices of order six over $S = K[x_1, x_2, x_3, x_4]$ with determinant $f^2$:

$$\mathcal{M}_\lambda := \left\{ \Lambda_{(\lambda,\Gamma)} = x_4 \cdot \Gamma + \begin{pmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{pmatrix},\ \det \Lambda_{(\lambda,\Gamma)} = f^2, \right.$$
$$\left. \Gamma = \begin{pmatrix} \Gamma_1 & -\Gamma_2^t \\ \Gamma_2 & \Gamma_3 \end{pmatrix},\quad \Gamma_1, \Gamma_3 \text{ skew symmetric},\ \begin{matrix} \Gamma_1 \neq 0 \\ \text{or}\ \Gamma_3 \neq 0 \end{matrix} \right\}.$$

Notice that, as in the proof of Lemma 4.8, if $\Lambda_{(\lambda,\Gamma)} \sim \Lambda_{(\lambda',\Gamma')}$, then $\lambda' = \lambda$ or $\lambda' = \lambda^t$.

**Lemma 4.9.** *Let $\lambda = [a : b : 1] \in V(f_3) \setminus \{P_0\}$ with $a \neq 1$.*

1. *Inside the family $\mathcal{M}_\lambda$, two matrices, $\Lambda$ and $\Lambda'$, are equivalent if and only if there exists $k \in K^*$ such that $\Lambda' = U_k \cdot \Lambda \cdot U_k^t$, $U_k = \left(\begin{smallmatrix} k\,\mathrm{Id} & 0 \\ 0 & \frac{1}{k}\,\mathrm{Id}\end{smallmatrix}\right)$.*

   *This condition means:* $\begin{cases} \Gamma_2' = \Gamma_2 \\ \Gamma_1' = k^2 \cdot \Gamma_1 \\ \Gamma_3' = \frac{1}{k^2} \cdot \Gamma_3. \end{cases}$

2. *A matrix $\Lambda$ from $\mathcal{M}_\lambda$ is equivalent to a matrix $\Lambda'$ from $\mathcal{M}_{\lambda'}$, $\lambda' \neq \lambda$ if and only if $\lambda' = [1 : b : a]$ and $\Lambda' = U_k \cdot \Lambda \cdot U_k^t$, where $k \in K^*$ and $U_k = \left(\begin{smallmatrix} 0 & k\cdot U_\lambda^{-1} \\ -\frac{1}{k} U_\lambda & 0\end{smallmatrix}\right)$.*

*Proof.* We assume $a \neq 0$. The case $a = 0$ is treated similarly. Two matrices, $\Lambda = \Lambda_{(\lambda,\Gamma)}$ and $\Lambda' = \Lambda_{(\lambda',\Gamma')}$, are equivalent if and only if $\overline{\Lambda}$ and $\overline{\Lambda}'$ are equivalent (see Remark 4.7).

If $U$ and $V$ are two invertible matrices such that $U \cdot \overline{\Lambda} = \overline{\Lambda}' \cdot V$, as in the proof of Lemma 4.8, we obtain

$$U = V = \begin{pmatrix} K_1 \,\mathrm{Id} & K_2 \,\mathrm{Id} \\ K_3 \,\mathrm{Id} & K_4 \,\mathrm{Id} \end{pmatrix} \text{ with } \quad K_1 = K_4 = 0 \text{ if } \lambda \neq \lambda' \text{ and}$$
$$K_2 = K_3 = 0 \text{ if } \lambda' \neq \lambda^t\,.$$

Since $U \cdot \overline{\Lambda} = \overline{\Lambda}' \cdot V$, we have:

$$\begin{pmatrix} 0 & \mathrm{Id} \\ -U_{\lambda'} & 0 \end{pmatrix}^{-1} \cdot U \cdot \begin{pmatrix} 0 & \mathrm{Id} \\ -U_{\lambda} & 0 \end{pmatrix} \cdot \Lambda \cdot \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & V_{\lambda}^{-1} \end{pmatrix} \cdot U^{-1} \cdot \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & V_{\lambda'}^{-1} \end{pmatrix}^{-1} = \Lambda'. \tag{$*$}$$

1. If $\lambda = \lambda'$ then $\lambda' \neq \lambda^t$, so $U = \begin{pmatrix} K_1 \mathrm{Id} & 0 \\ 0 & K_4 \mathrm{Id} \end{pmatrix}$ with $K_1 \neq 0$, $K_4 \neq 0$. So $(*)$ implies: $\begin{pmatrix} K_4 \cdot \mathrm{Id} & 0 \\ 0 & K_1 \mathrm{Id} \end{pmatrix} \cdot \Lambda \cdot \begin{pmatrix} \frac{1}{K_1} \mathrm{Id} & 0 \\ 0 & \frac{1}{K_4} \mathrm{Id} \end{pmatrix} = \Lambda'$. For $k = \sqrt{\frac{K_4}{K_1}}$ and $U_k = \begin{pmatrix} k \,\mathrm{Id} & 0 \\ 0 & \frac{1}{k} \cdot \mathrm{Id} \end{pmatrix}$ we have $\Lambda' = U_k \cdot \Lambda \cdot U_k^t$.

2. If $\lambda' = \lambda^t$ then $\lambda' \neq \lambda$, so $U = \begin{pmatrix} 0 & K_2 \mathrm{Id} \\ K_3 \mathrm{Id} & 0 \end{pmatrix}$, $K_2 \neq 0$, $K_3 \neq 0$. Replacing $U$ in $(*)$ we obtain:

   $$\Lambda' = \begin{pmatrix} 0 & -K_3 U_{\lambda^t}^{-1} \\ -K_2 U_{\lambda} & 0 \end{pmatrix} \cdot \Lambda \cdot \begin{pmatrix} 0 & \frac{1}{K_3} V_{\lambda^t} \\ \frac{1}{K_2} V_{\lambda}^{-1} & 0 \end{pmatrix}.$$

   Since $a \neq 0$ and $a \neq 1$, $V_\lambda = U_\lambda^t$, $\lambda^t = \left[\frac{1}{a} : \frac{1}{b} : 1\right]$, $U_{\lambda^t} = \frac{1}{a^2} \cdot U_\lambda$, $V_{\lambda^t} = \frac{1}{a^2} U_\lambda^t$ (see Remark 4.5).

   So $\Lambda' = \begin{pmatrix} 0 & -K_3 a^2 U_\lambda^{-1} \\ -K_2 U_\lambda & 0 \end{pmatrix} \cdot \Lambda \cdot \begin{pmatrix} 0 & \frac{1}{K_3} \cdot \frac{1}{a^2} \cdot U_\lambda^t \\ \frac{1}{K_2} (U_\lambda^{-1})^t & 0 \end{pmatrix} = \begin{pmatrix} 0 & k U_\lambda^{-1} \\ -\frac{1}{k} U_\lambda & 0 \end{pmatrix} \cdot \Lambda \cdot \begin{pmatrix} 0 & k U_\lambda^{-1} \\ -\frac{1}{k} U_\lambda & 0 \end{pmatrix}^t$, where $k^2 = -a^2 \cdot \frac{K_3}{K_2}$.

□

In a similar way, we can prove the following lemma:

**Lemma 4.10.** *Let* $\lambda = [1 : b : 1] \in V(f_3) \smallsetminus \{P_0\}$.

1. *Inside the family $\mathcal{M}_\lambda$, two matrices $\Lambda$ and $\Lambda'$ are equivalent if and only if $\Lambda' = T \cdot \Lambda \cdot T^t$, where*

   $$T = \left(\begin{smallmatrix} K_4 \cdot \mathrm{Id} & K_3 \cdot U_\lambda^{-1} \\ K_2 \cdot U_\lambda & K_1 \cdot \mathrm{Id} \end{smallmatrix}\right), \ K_1, K_2, K_3, K_4 \in K \text{ such that } K_1K_4 - K_2K_3 = 1\,.$$

2. *No $\lambda \in V(f_3) \smallsetminus \{P_0, [1:b:1]\}$ exists, such that a matrix from $\mathcal{M}_\lambda$ is equivalent to a matrix from $\mathcal{M}_{[1:b:1]}$.*

Now let us see "how large" the family $\mathcal{M}_\lambda$ is for a given $\lambda$ in $V(f_3) \smallsetminus \{P_0\}$. For $\Lambda = \Lambda_{(\lambda,\Gamma)}$ in $\mathcal{M}_\lambda$, we denote:

$$\Gamma_1 = \begin{pmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & a_4 & a_5 \\ -a_4 & 0 & a_6 \\ -a_5 & -a_6 & 0 \end{pmatrix}.$$

The condition $\det \Lambda = f^2$ provides 10 equations in the above 15 parameters. Six of these equations are linear in the entries of $\Gamma_2$ and form a linear system of dimension three.

1. If $b = 0$ the solution of this system is:

$$\begin{cases} a_7 = -a_{12} \cdot (a^2 + 1) \\ a_8 = a_{10} = a_{15} = 0 \\ a_9 = a_{11} - a_{13} \\ a_{14} = a^2 \cdot a_{12}\,. \end{cases}$$

2. If $b \neq 0$, the system has the following solution:

$$\begin{cases} a_8 &= -\frac{b}{a+1} a_7 + a_{15} \\ a_9 &= -\frac{a-1}{b(a+1)} a_7 - \frac{a^2}{b^2} \cdot a_{15} \\ a_{10} &= \frac{b}{a+1} \cdot a_7 \\ a_{12} &= -\frac{a^2+3}{(a+1)^2} \cdot a_7 + \frac{b}{a+1} a_{11} + \frac{1-a}{b(a+1)} a_{15} \\ a_{13} &= \frac{a-1}{b(a+1)} \cdot a_7 + a_{11} + \frac{a^2}{b^2} \cdot a_{15} \\ a_{14} &= \frac{2(1-a)}{(a+1)^2} \cdot a_7 - \frac{b}{a+1} \cdot a_{11} + \frac{a-1}{b(a+1)} \cdot a_{15}\,. \end{cases}$$

The other four equations are linear in the entries of $\Gamma_1$ with coefficients in $K[a_4, \ldots, a_{15}]$ and have dimension five:

```
LIB"matrix.lib";
option(redSB);

ring r=0,(x(4),x(1),x(2),x(3),e,a,b,a(1..15)),dp;
ideal ii=a3+b3+1,e*b+a2-a+1,e*a+e-b2;
qring s=std(ii);

matrix B[10][1];
B[1,1]=x(4)*a(1);
B[2,1]=x(4)*a(2);
B[3,1]=-x(4)*a(7);
B[4,1]=-x(4)*a(10)-(x(1)+x(3));
B[5,1]=x(4)*a(3);
B[6,1]=-x(4)*a(8)-(x(1)-a*x(3));
B[7,1]=-x(4)*a(11)+x(2)+b*x(3);
B[8,1]=-x(4)*a(9)-x(2)+b*x(3);
B[9,1]=-x(4)*a(12)+e*x(3);
B[10,1]=x(4)*a(4);

matrix V[1][5];
V[1,1]=-x(4)*a(13)-x(2);
V[1,2]=-x(4)*a(14)-e*x(3);
V[1,3]=-x(4)*a(15)+x(1)+(a-1)*x(3);
V[1,4]=x(4)*a(5); V[1,5]=x(4)*a(6);

poly p1=B[5,1]*B[10,1]-B[6,1]*B[9,1]+B[7,1]*B[8,1];
poly p2=B[2,1]*B[10,1]-B[3,1]*B[9,1]+B[4,1]*B[8,1];
poly p3=B[1,1]*B[10,1]-B[3,1]*B[7,1]+B[4,1]*B[6,1];
poly p4=B[1,1]*B[9,1]-B[2,1]*B[7,1]+B[4,1]*B[5,1];
poly p5=B[1,1]*B[8,1]-B[2,1]*B[6,1]+B[3,1]*B[5,1];

poly g=V[1,1]*p1-V[1,2]*p2+V[1,3]*p3-V[1,4]*p4+V[1,5]*p5;
poly f=x(4)^3+x(1)^3+x(2)^3+x(3)^3; g=g-f;
```

For our skew symmetric matrix the condition $g = f$ is equivalent to $(\det \Lambda = f^2)$.

```
matrix H=coef(g,x(4)*x(1)*x(2)*x(3));
```

```
for(int j=1;j<=13;j++)
{H[1,j]=0;}

ideal I=H; I=interred(I);

I[1]=a(9)-a(11)+a(13) I[2]=a(8)+a(10)-a(15) I[3]=a(7)+a(12)+a(14)
I[4]=a*a(10)-e*a(11)+b*a(12)+2*e*a(13)+2*b*a(14)-2*a*a(15)+a(10)
     +a(15)
I[5]=2*e*a(10)+2*b*a(11)-2*a*a(12)-b*a(13)-a*a(14)-e*a(15)
     +a(12)+2*a(14)
I[6]=a(3)*a(4)-a(2)*a(5)+a(1)*a(6)+a(11)^2+a(10)*a(12)-a(11)*a(13)
     +a(13)^2-a(10)*a(14)-2*a(12)*a(15)-a(14)*a(15)
I[7]=a(1)*a(4)+a(3)*a(5)+a(2)*a(6)-a(10)^2+a(11)*a(12)+a(12)*a(13)
     +2*a(11)*a(14)-a(13)*a(14)+a(10)*a(15)-a(15)^2
I[8]=2*e^2*a(12)+2*a*b*a(12)-3*b^2*a(13)+2*e^2*a(14)-a*b*a(14)
     -3*e*b*a(15)-6*e*a(11)-b*a(12)+12*e*a(13)+2*b*a(14)-6*a*a(15)
I[9]=a(3)*a(5)*a(10)-a(2)*a(6)*a(10)-a(2)*a(5)*a(11)-a(1)*a(6)*a(11)
      +a(1)*a(5)*a(12)+a(3)*a(6)*a(12)-a(2)*a(5)*a(13)
      +2*a(1)*a(6)*a(13)+a(13)^3+a(2)*a(4)*a(14)+a(3)*a(6)*a(14)
      +a(10)*a(11)*a(14)+a(12)^2*a(14)-2*a(10)*a(13)*a(14)
      +a(12)*a(14)^2+a(3)*a(5)*a(15)+2*a(2)*a(6)*a(15)
      +a(11)*a(14)*a(15)-2*a(13)*a(14)*a(15)-a(15)^3-1
I[10]=2*e*a(2)*a(4)-2*e*a(1)*a(5)+2*b*a(2)*a(5)-2*a*a(3)*a(5)
      +2*b*a(1)*a(6)-4*a*a(2)*a(6)-2*b*a(11)^2+2*a*a(11)*a(12)
      +2*e*a(12)^2+5*b*a(11)*a(13)-4*a*a(12)*a(13)-2*b*a(13)^2
      -2*b*a(10)*a(14)-a*a(11)*a(14)+2*a*a(13)*a(14)-2*e*a(14)^2
      +3*e*a(11)*a(15)-6*e*a(13)*a(15)-2*b*a(14)*a(15)+6*a*a(15)^2
      +4*a(3)*a(5)+2*a(2)*a(6)-a(11)*a(12)+2*a(12)*a(13)
      +2*a(11)*a(14)-4*a(13)*a(14)-6*a(15)^2

 ideal J=I[1],I[2],I[3],I[4],I[5],I[8];
```

This is the ideal generated by the linear equations in the entries of $(\Gamma_2)$. We compute its dimension.

```
 ring r1=0,(a(7..15),e,a,b),(c,dp(9),dp(3));
 ideal I=a3+b3+1,e*b+a2-a+1,e*a+e-b2;
 qring s1=std(I);
 ideal J=imap(s,J);
 ideal JJ=std(J);
```

```
 dim(JJ);
4
```

Since we have defined e as a variable, the dimension of our ideal is, in fact, 3. We compute now the dimension of the ideal generated by the remaining 4 equations, $I[6], I[7], I[9], I[10]$, in the quotient ring given by the previous ideal.

```
 ideal J1=I[6],I[7],I[9],I[10];
 ring r2=0,(a(1..15),e,a,b),(c,dp(15),dp(3));
 ideal J=imap(s1,J);
 J=J+(a3+b3+1,e*b+a2-a+1,e*a+e-b2);
 ideal JJ=std(J);
 qring s2=JJ;
 ideal J1=imap(s,J1);
 ideal JJ1=std(J1);
 dim(JJ1);
6
```

As before, in fact, the ideal has dimension is 5.

Let us summarize the results.

Let $M$ be an indecomposable graded rank 2, 6–generated MCM and $\overline{M}$ the restriction of $M$ to the elliptic curve on our surface defined by $f = x_4 = 0$. Then $\overline{M} \cong N_\lambda \oplus N_\lambda^\vee$ for a suitable 3–generated rank 1 MCM $N_\lambda = \operatorname{coker}(\alpha_\lambda)$, $\lambda \in V(f, x_4) \setminus \{[-1:0:1:0]\} \cong V(f_3) \setminus \{[-1:0:1]\} =: C$. If $\lambda = [a:b:c]$ and $\lambda^t := [c:b:a]$, then $N_\lambda^\vee \cong N_{\lambda^t}$, in particular, there exist skew–symmetric $3\times 3$–matrices $\Gamma_1, \Gamma_3$ with constant entries not being zero simultaneously and a $3\times 3$–matrix $\Gamma_2$ such that $M = \operatorname{coker}(\Lambda)$ for $\Lambda = x_4 \begin{pmatrix} \Gamma_1 & -\Gamma_2^t \\ \Gamma_2 & \Gamma_3 \end{pmatrix} + \begin{pmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{pmatrix}$, $\Gamma_1 = \begin{pmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix}$, $\Gamma_3 = \begin{pmatrix} 0 & a_4 & a_5 \\ -a_4 & 0 & a_6 \\ -a_5 & -a_6 & 0 \end{pmatrix}$, $\Gamma_2 = \begin{pmatrix} a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{pmatrix}$ and $\det(\Lambda) = f^2$.

Let $\mathbb{A}^{15}$ be the 15–dimensional affine space with the coordinates $(a_1, \ldots, a_{15})$ and $G$ be the subgroup of $\mathrm{Sl}_2(K)$ generated by the matrices $g_k = \begin{pmatrix} 0 & k \\ -\frac{1}{k} & 0 \end{pmatrix}$, $k \in K \setminus \{0\}$. Consider the action of $G$ on $\mathbb{A}^{15}$: $G \times \mathbb{A}^{15} \to \mathbb{A}^{15}$, $(g_k, \underline{a}) \to \underline{a}' = (k^2 a_1, k^2 a_2, k^2 a_3, \frac{1}{k^2} a_4, \frac{1}{k^2} a_5, \frac{1}{k^2} a_6, a_7, .., a_{15})$. Denote $\mathbb{A} = \mathbb{A}^{15}/G$.

A point $(\lambda; \underline{a}) \in C \times \mathbb{A}$ corresponds to a matrix $\Lambda = x_4 \begin{pmatrix} \Gamma_1 & -\Gamma_2^t \\ \Gamma_2 & \Gamma_3 \end{pmatrix} + \begin{pmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{pmatrix}$. The group $G$ acts on $C \times \mathbb{A}$ in the following way: let $(\lambda; a) \in C \times \mathbb{A}$ correspond to $\Lambda = x_4 \begin{pmatrix} \Gamma_1 & -\Gamma_2^t \\ \Gamma_2 & \Gamma_3 \end{pmatrix} + \begin{pmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{pmatrix}$, then $g_k(\lambda; a) = (\lambda^t; b)$, where $(\lambda^t; b)$ corresponds to $g_k \begin{pmatrix} U_\lambda & 0 \\ 0 & U_\lambda^{-1} \end{pmatrix} \Lambda \begin{pmatrix} U_\lambda^t & 0 \\ 0 & {U^t}_\lambda^{-1} \end{pmatrix} g_k^t$, $U_\lambda$ defined in Remark 4.5.

Let $\mathcal{M} \subseteq C \times \mathbb{A}$ be the $G$–invariant closed subset defined by $\det(\Lambda) = f^2$. Let $\pi : \mathcal{M} \to C$ be the canonical projection. If $\lambda = [1 : b : 1]$ with $b^3 = -2$, then $\mathrm{Sl}_2(K)$ acts on $\pi^{-1}(\lambda)$ via the representation $\mathrm{Sl}_2(K) \to \left\{ \begin{pmatrix} K_1 \mathrm{Id} & K_2 U_\lambda^{-1} \\ K_3 U_\lambda & K_4 \mathrm{Id} \end{pmatrix},\ K_1K_4 - K_2K_3 = 1 \right\}$, $\begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \mapsto \begin{pmatrix} K_1 \mathrm{Id} & K_2 U_\lambda^{-1} \\ K_3 U_\lambda & K_4 \mathrm{Id} \end{pmatrix}$.

**Theorem 4.11.** *1. Every indecomposable graded rank 2, 6–generated MCM is represented by a point in $\mathcal{M}$.*

*2. $\mathcal{M} \smallsetminus \pi^{-1}(\{[1 : b : 1] \mid b^3 = -2\})/G$ is the moduli space of isomorphism classes of indecomposable graded rank 2, 6–generated MCM $M$ such that the restriction to $V(f, x_4)$, $\overline{M} \cong N_\lambda \oplus N_\lambda^\vee$ for $N_\lambda$ being not self–dual. This moduli space is 5–dimensional.*

*3. $Sl_2(K)$ acts on $\pi^{-1}(\{[1 : b : 1] \mid b^3 = -2\})$ and $\pi^{-1}(\{[1 : b : 1] \mid b^3 = -2\})/Sl_2(K)$ is the moduli space of isomorphism classes of indecomposable graded rank 2, 6–generated MCM $M$ such that the restriction to $V(f, x_4)$, $\overline{M} \cong N_\lambda \oplus N_\lambda$ for $N_\lambda$ being self–dual.*

**Remark 4.12.** *It is well known that the ideal defining 5 general points in $\mathbf{P}^3_K$ (this means any four from them are not on a hyperplane) is Gorenstein. Restricting to the 5 general points on the surface $V(f)$ we get a family of Gorenstein ideals whose isomorphism classes of 2-syzygies over $R$ (they are indecomposable, graded, rank 2, 6-generated MCM modules) form a 5-parameter family (see [Mi], [IK]). Here we give an example.*
*Let $[1 : 0 : 0 : -1]$, $[1 : 0 : -1 : 0]$, $[1 : -1 : 0 : 0]$, $[1 : -u : 0 : 0]$, $[1 : -u : 1 : -u]$, $u^2 + u + 1 = 0$ be 5 points in general position on $V(f)$ and $I$ the ideal defined by these points in $R$. $I$ is generated by the following quadratic forms:*
$x_2x_4 + ux_3x_4$, $-ux_2x_3 + ux_3x_4$, $x_1x_4 + x_4^2 - (1-u)x_3x_4$, $u(x_1 + x_3)x_3 + 2x_3x_4$, $-x_3x_4 - x_1^2 + ux_1x_2 - u^2x_2^2 + x_3^2 + x_4^2$.
*The second syzygy of $I$ over $R$ is the cokernel of a skew symmetric matrix $A$ defined by*
$A[1,1] = A[2,2] = A[3,3] = A[4,4] = A[5,5] = A[6,6] = 0,$
$A[1,2] = (-3u - 2)x_3 + (2u - 1)x_4 = -A[2,1],$
$A[1,3] = -ux_1 + (-2u + 1)x_2 + (u + 1)x_3 + ux_4 = -A[3,1],$
$A[1,4] = (u - 2)x_1 - x_2 + (-3u - 4)x_3 + (2u - 1)x_4 = -A[4,1],$
$A[1,5] = (u + 1)x_3 - ux_4 = -A[5,1],$
$A[1,6] = -ux_1 + (u + 1)x_2 + (1/7u + 3/7)x_3 + (-3/7u - 2/7)x_4 = -A[6,1],$
$A[2,3] = (u - 2)x_1 - x_2 + x_3 + (-u + 2)x_4 = -A[3,2],$
$A[2,4] = (3u + 2)x_1 + (2u + 3)x_2 + 4ux_3 + x_4 = -A[4,2],$
$A[2,5] = (-3u - 1)x_3 + (u - 2)x_4 = -A[5,2],$

$A[2,6] = (-u-2)x_1 + (-u+1)x_2 + (-u-1)x_3 + ux_4 = -A[6,2]$,
$A[3,4] = -3x_3 = -A[4,3]$,
$A[3,5] = (u+1)x_3 = -A[5,3]$,
$A[3,6] = (-6/7u - 4/7)x_3 + x_4 = -A[6,3]$,
$A[4,5] = (-3u-1)x_3 = -A[5,4]$,
$A[4,6] = -ux_3 + ux_4 = -A[6,4]$,
$A[5,6] = -x_1 - ux_2 = -A[6,5]$.

*This matrix is equivalent to* $\Lambda_{(\lambda,\Gamma)} = x_4 \cdot \Gamma + \begin{pmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{pmatrix}$ *for* $\lambda = (0 : u+1 : 1)$ *and* $\Gamma$ *given by:*

$$\begin{array}{lll} a_1 = -\frac{4}{3}u - \frac{2}{3}, & a_2 = -u - 1, & a_3 = \frac{2}{3}u + \frac{1}{3}, \\ a_4 = -\frac{1}{4}u + 1, & a_5 = -u - \frac{1}{2}, & a_6 = -\frac{3}{2}u + \frac{3}{4}, \\ a_7 = -\frac{1}{2}, & a_8 = -\frac{1}{2}u, & a_9 = \frac{1}{2}u, \\ a_{10} = -\frac{1}{2}u - \frac{1}{2}, & a_{11} = u + \frac{3}{2}, & a_{12} = u + 1, \\ a_{13} = \frac{1}{2}u + \frac{3}{2}, & a_{14} = -u - \frac{1}{2}, & a_{15} = -u - \frac{1}{2}. \end{array}$$

Similar problem is treated in [F2], where D. Faenzzi describe the moduli spaces of orientable aCM bundles on a smooth cubic surface in $\mathbb{P}^3$. He gives also some informations on the stability of this bundles.

**Theorem 4.13.** *Let $X$ be a smooth cubic surface in $\mathbb{P}^3$ and let $\mathcal{E}$ be an indecomposable aCM rank two vector bundle on $X$. Then the minimal graded free resolution of $\mathcal{E}$ takes one of the following forms:*

*(A)* $0 \to \mathcal{O}^6(-1) \to \mathcal{O}^6 \to 0$;

*(B)* $0 \to \mathcal{O}^4(-1) \oplus \mathcal{O}(-2) \to \mathcal{O}^5 \to 0$;

*(C)* $0 \to \mathcal{O}^5(-2) \to \mathcal{O} \oplus \mathcal{O}^4(-1) \to 0$;

*(D)* $0 \to \mathcal{O}^2(-1) \oplus \mathcal{O}^3(-2) \to \mathcal{O}^4 \to 0$;

*(E)* $0 \to \mathcal{O}(-1) \oplus \mathcal{O}^3(-2) \to \mathcal{O}^3 \oplus \mathcal{O}(-1) \to 0$;

*(F)* $0 \to \mathcal{O}^4(-2) \to \mathcal{O}^2 \oplus \mathcal{O}^2(-1) \to 0$;

*(G)* $0 \to \mathcal{O}^3(-2) \oplus \mathcal{O}(-3) \to \mathcal{O} \oplus \mathcal{O}^3(-1) \to 0$;

*(H)* $0 \to \mathcal{O}^3(-2) \to \mathcal{O}^3 \to 0$.

**Proposition 4.14.** *Let $\mathcal{E}$ be an indecomposable rank 2 aCM bundle on $X$ having resolution $(A)$. Then $c_1(\mathcal{E}) \cong T_1 + T_2, c_2(\mathcal{E}) \cong T_1 \cdot T_2$, where $T_1$ and $T_2$ are twisted cubics contained in $X$ with $3 \leq T_1 \cdot T_2 \leq 5$. Furthermore, the matrix $f(\mathcal{E})$ is skew–symmetric if and only if $T_1 \cdot T_2 = 5$.*

**Proposition 4.15.** *Let $\mathcal{E}$ be indecomposable aCM of rank two, with $f(\mathcal{E})$ is skew–symmetric and resolution $(A)$.*
*The general $\mathcal{E}$ is stable and $MCM^s(2; c_1(\mathcal{E}), c_2(\mathcal{E}))$ (the moduli space of stable rank two aCM bundles with $c_1$ and $c_2$ as Chern classes) is smooth of dimension 5.*

Using this results one can give the matrix factorizations of the rank two, orientable stable modules. We remind, that by stable module, we understand MCM module with stable sheafification on the projective cone.

**Proposition 4.16.** *The matrices from*

$$\bigcup_{\lambda \in V(f)} \mathcal{M}_\lambda$$

*describe all rank two, orientable MCM modules with stable sheafification.*

## 4.2 Orientable 4–generated MCM modules

Let $M$ be a graded, indecomposable, orientable, 4–generated MCM $R$–module of rank 2. We remind that there exists $I$ graded 3–generated Gorenstein ideal in $R$, of codimension 2, such that $M \cong \Omega^2_R(I)$. We can write then $I$ as $I = J/(f)$, with $J \subset S = k[x_1, x_2, x_3, x_4]$ a graded, 3–generated ideal containing $f$, $f \in mJ$ by [HK]. Let $\alpha_1, \alpha_2, \alpha_3$ be a minimal system of homogeneous generators of $J$. Since $\dim S/J = \dim R/I = 1$, it follows that $\alpha_1, \alpha_2, \alpha_3$ is a regular system of elements in $S$.

Let $a, b \in K$ with $a^3 = b^3 = -1$ and $\sigma = (i\ j\ s)$ be a permutation of the set $\{2, 3, 4\}$ with $i < j$. Set

$$\begin{aligned} w_{\sigma 1} &= x_1 - a x_s, & w_{\sigma 2} &= x_i - b x_j, \\ v_{\sigma 1} &= x_1^2 + a x_1 x_s + a^2 x_s, & v_{\sigma 2} &= x_i^2 + b x_i x_j + b^2 x_j^2. \end{aligned}$$

Then we have

$$f = w_{\sigma 1}v_{\sigma 1} + w_{\sigma 2}v_{\sigma 2}.$$

For any $\lambda$ point of the surface $V(f) \subset \mathbb{P}^3$, we set three pairs of polynomials, $(p_{i\lambda}, q_{i\lambda}), 1 \geq i \geq 3$ such that

$$f = \sum_{i=1}^{3} p_{i\lambda}q_{i\lambda}.$$

If $\lambda = [\lambda_1 : \lambda_2 : \lambda_3 : 1]$, set

$$p_{i\lambda} = x_i - \lambda_i x_4, \quad q_{i\lambda} = x_i^2 + \lambda_i x_i x_4 + \lambda_i^2 x_4^2, \quad \text{for} \quad 1 \leq i \leq 3.$$

If $\lambda = [\lambda_1 : \lambda_2 : 1 : 0]$, set

$$\begin{array}{ll} p_{i\lambda} = x_i - \lambda_i x_3, & q_{i\lambda} = x_i^2 + \lambda_i x_i x_3 + \lambda_i^2 x_3^2, \quad \text{for} \quad 1 \leq i \leq 2 \\ p_{3\lambda} = x_4, & q_{3\lambda} = x_4^2. \end{array}$$

If $\lambda = [\lambda_1 : 1 : 0 : 0] \in V(f)$, we set

$$\begin{array}{ll} p_{1\lambda} = x_1 - \lambda_1 x_2, & q_{1\lambda} = x_1^2 + \lambda_1 x_1 x_2 + \lambda_1^2 x_2^2, \\ p_{2\lambda} = x_3, & q_{2\lambda} = x_3^2, \\ p_{3\lambda} = x_4, & q_{3\lambda} = x_4^2. \end{array}$$

Since $f \in (\alpha_1, \alpha_2, \alpha_3)$ and eventually we are interested in $\Omega_R^2(\alpha_1, \alpha_2, \alpha_3)$, we may suppose that either $\alpha_i$ is in the set $\{p_{i\lambda}, q_{i\lambda}\}$ for each $1 \leq i \leq 3$, or $\alpha_i$ is in the set $\{w_{\sigma i}, v_{\sigma i}\}$ for each $1 \leq i \leq 2$ and $\beta = \alpha_3$ is a regular element in $R/(\alpha_1, \alpha_2)$.

**Lemma 4.17.** *Let $M$ be a graded, indecomposable, 4–generated MCM $R$–module of rank 2. Then $M$ is isomorphic to one of the following modules:*

1. $\Omega_R^2(p_{1\lambda}, p_{2\lambda}, p_{3\lambda})$ *or* $\Omega_R^2(q_{1\lambda}, q_{2\lambda}, q_{3\lambda})$, *for some* $\lambda \in V(f)$,
2. $\Omega_R^2(w_{\sigma 1}, v_{\sigma 2}, \beta)$ *or* $\Omega_R^2(w_{\sigma 1}, w_{\sigma 2}, \beta)$ *for some* $a, b, \sigma$ *and* $\beta$ *as above.*

*Proof.* For any $\lambda \in V(f)$ set

$$I_\lambda = (p_{1\lambda}, p_{2\lambda}, p_{3\lambda})$$

and

$$\varphi_\lambda = \begin{pmatrix} 0 & p_{3\lambda} & -p_{2\lambda} & -q_{1\lambda} \\ -p_{3\lambda} & 0 & -p_{1\lambda} & q_{2\lambda} \\ p_{2\lambda} & p_{1\lambda} & 0 & q_{3\lambda} \\ q_{1\lambda} & -q_{2\lambda} & -q_{3\lambda} & 0 \end{pmatrix}, \ \psi_\lambda = \begin{pmatrix} 0 & -q_{3\lambda} & q_{2\lambda} & p_{1\lambda} \\ q_{3\lambda} & 0 & q_{1\lambda} & -p_{2\lambda} \\ -q_{2\lambda} & -q_{1\lambda} & 0 & -p_{3\lambda} \\ -p_{1\lambda} & p_{2\lambda} & p_{3\lambda} & 0 \end{pmatrix}.$$

There exists the following graded exact sequence:

$$R^3(-5)\oplus R(-6) \xrightarrow{\varphi_\lambda} R^4(-4) \xrightarrow{\psi_\lambda} R^3(-2)\oplus R(-3) \xrightarrow{A} R^3(-1) \xrightarrow{\tau} I_\lambda \longrightarrow 0,$$

where $\tau = (-p_{1\lambda}, p_{2\lambda}, p_{3\lambda})$ and $A$ is given by the first three rows of $\varphi_\lambda$. Thus, $\Omega^2(I_\lambda) \cong \mathrm{Coker}(\varphi_\lambda)$ and $(\varphi_\lambda, \psi_\lambda)$ is a matrix factorization of the polynomial $f$, defining the module $\Omega^2(I_\lambda)$. The ideals $I_\lambda$ and $(q_{1\lambda}, q_{2\lambda}, p_{3\lambda})$ belong to the same even linkage class since

$$I_\lambda \sim (q_{1\lambda}, p_{2\lambda}, p_{3\lambda}) \sim (q_{1\lambda}, q_{2\lambda}, p_{3\lambda}).$$

( For the first link we consider the regular sequence $\{p_{1\lambda}q_{1\lambda}, p_{2\lambda}, p_{3\lambda}\}$ and for the second one the sequence $\{q_{1\lambda}, p_{2\lambda}q_{2\lambda}, p_{3\lambda}\}$.) Similarly, one can see that $I_\lambda$ is evenly linked with the ideals $(q_{1\lambda}, p_{2\lambda}, q_{3\lambda})$ and $(p_{1\lambda}, q_{2\lambda}, q_{3\lambda})$. By [HK, Theorem 2.1], we obtain that

$$\begin{aligned}\mathrm{Coker}(\varphi_\lambda) \cong \Omega^2_R(I_\lambda) \cong \Omega^2_R(q_{1\lambda}, q_{2\lambda}, p_{3\lambda}) \cong \Omega^2_R(q_{1\lambda}, p_{2\lambda}, q_{3\lambda}) \\ \cong \Omega^2_R(p_{1\lambda}, q_{2\lambda}, q_{3\lambda}).\end{aligned}$$

Analogously, we see that

$$\begin{aligned}\mathrm{Coker}(\psi_\lambda) \cong \Omega^2_R(q_{1\lambda}, q_{2\lambda}, q_{3\lambda}) \cong \Omega^2_R(p_{1\lambda}, p_{2\lambda}, q_{3\lambda}) \cong \Omega^2_R(p_{1\lambda}, q_{2\lambda}, p_{3\lambda}) \\ \cong \Omega^2_R(q_{1\lambda}, p_{2\lambda}, p_{3\lambda}).\end{aligned}$$

Thus, the case when $\alpha_i$ is one of the forms $\{p_{i\lambda}, q_{i\lambda}\}$ gives (1).

Now let $\sigma, a, b$ as above and $\beta \in S$ which is regular on $R/(w_{\sigma 1}, v_{\sigma 2})$. Set

$$I_{\sigma\beta}(a,b) = (w_{\sigma 1}, v_{\sigma 2}, \beta)$$

and

$$\varphi_{\sigma\beta}(a,b) = \begin{pmatrix} 0 & w_{\sigma 1} & -v_{\sigma 2} & 0 \\ -w_{\sigma 1} & 0 & -\beta & w_{\sigma 2} \\ v_{\sigma 2} & \beta & 0 & v_{\sigma 1} \\ 0 & -w_{\sigma 2} & -v_{\sigma 1} & 0 \end{pmatrix},$$

$$\psi_{\sigma\beta}(a,b) = \begin{pmatrix} 0 & -v_{\sigma 1} & w_{\sigma 2} & \beta \\ v_{\sigma 1} & 0 & 0 & -v_{\sigma 2} \\ -w_{\sigma 2} & 0 & 0 & -w_{\sigma 1} \\ -\beta & v_{\sigma 2} & w_{\sigma 1} & 0 \end{pmatrix}.$$

We have the following exact sequence:

$$R^4 \xrightarrow{\varphi_{\sigma\beta}(a,b)} R^4 \xrightarrow{\psi_{\sigma\beta}(a,b)} R^4 \xrightarrow{B} R^3 \xrightarrow{\tau'} I_{\sigma\beta}(a,b) \longrightarrow 0\,,$$

where $\tau' = (-\beta, v_{\sigma 2}, w_{\sigma 1})$ and $B$ is the matrix given by the first three rows of $\varphi_{\sigma\beta}(a,b)$. Thus,

$$\Omega_R^2\big(I_{\sigma\beta}(a,b)\big) \cong \operatorname{Coker}\big(\varphi_{\sigma\beta}(a,b)\big).$$

As above, we see that

$$\Omega_R^2\big(I_{\sigma\beta}(a,b)\big) \cong \Omega_R^2(w_{\sigma 2}, v_{\sigma 1}, \beta)$$

and

$$\Omega_R^2(w_{\sigma 1}, w_{\sigma 2}, \beta) \cong \Omega_R^2(v_{\sigma 1}, v_{\sigma 2}, \beta) \cong \operatorname{Coker}\big(\psi_{\sigma\beta}(a,b)\big).$$

Thus, the case when $\alpha_i$ is one of the forms $\{w_{\sigma i}, v_{\sigma i}\}$ for $i \leq 2$ gives (2). □

Let

$$\mathcal{M} = \big\{\operatorname{Coker}(\varphi_\lambda), \operatorname{Coker}(\psi_\lambda) \mid \lambda \in V(f)\big\}.$$

For $a, b, \sigma$ as above, set

$$\varphi_\sigma(a,b) = \varphi_{\sigma, x_j x_s}(a,b),\ \ \psi_\sigma(a,b) = \psi_{\sigma, x_j x_s}(a,b),$$

that is, $\beta = x_j x_s$. Let

$$\mathcal{P} = \big\{\operatorname{Coker}\big(\varphi_\sigma(a,b)\big), \operatorname{Coker}(\psi_\sigma(a,b)) \mid a, b, \sigma \text{ as above } \big\}.$$

**Theorem 4.18.** *The set $\mathcal{M} \cup \mathcal{P}$ contains only non–isomorphic, indecomposable, graded, orientable, 4–generated MCM R–modules of rank 2 and every indecomposable, graded, orientable, 4–generated MCM R–module of rank 2 is isomorphic with one module of $\mathcal{M} \cup \mathcal{P}$.*

*Proof.* Applying Lemma 4.17, we must show in the case (2) that $\beta$ can be taken $x_j x_s$. Since $v_{\sigma 1} - w_{\sigma 1}(x_1 + 2ax_s) = 3a^2 x_s^2$, adding in $\varphi_{\sigma\beta}(a,b)$ multiples of the last row to the second one and multiples of the first column to the third one, we may suppose the entry $(2,3)$ of the form $\gamma + x_s\delta$, with $\gamma, \delta$ depending only on $x_j, x_i$. These transformations modify the entries $(2,2), (3,3)$ which are now possibly non–zero. Adding similar multiples of the last column to the second one and multiples of the first row to the third one, we obtain $\varphi_{\sigma,\beta}(a,b)$ of the same type as before but with $\beta = \gamma + x_s\delta$. We may reduce to consider $\delta \notin K$. Indeed, if $\delta \in K$, then, acting on the rows and columns of $\varphi_{\sigma\beta}(a,b)$, we obtain that $M = \operatorname{Coker}(\varphi_{\sigma\beta}(a,b))$ is decomposable or belongs to the set $\mathcal{M}$. Now let $\delta$ be not constant. Similarly, adding in $\varphi_{\sigma\beta}(a,b)$ multiples of the first row to the second one and multiples of the last column to the third one we may suppose that the entry $(2,3)$ has the form $\varepsilon x_j x_s$

with $\varepsilon \in K$. These transformations modify the entries $(2,2),(3,3)$. After similar transformations, we obtain $\varphi_{\sigma\beta}(a,b)$ of the same type as before but with $\beta = \varepsilon x_j x_s$. If $\varepsilon = 0$ we see that $\varphi_{\sigma\beta}(a,b)$ is a direct sum of two $2 \times 2$–matrices, which contradicts the indecomposability of $M = \operatorname{Coker}\bigl(\varphi_{\sigma\beta}(a,b)\bigr)$. So $\varepsilon \neq 0$. Divide the second and the third column of $\varphi_{\sigma\beta}(a,b)$ with $\varepsilon$, and multiply the first and the last row of $\varphi_{\sigma\beta}(a,b)$ with $\varepsilon$. We reduce to the case $\varepsilon = 1$, that is $\beta = x_j x_s$.

Now we show that two different modules from $\mathcal{M} \cup \mathcal{P}$ are not isomorphic. Note that the Fitting ideals of $\varphi_\lambda$ (respectively $\psi_\lambda$) modulo $(x_1,\ldots,x_4)^2$ have the form $(p_{1\lambda},p_{2\lambda},p_{3\lambda})$ and the Fitting ideals of $\varphi_\sigma(a,b)$ (respectively $\psi_\sigma(a,b)$) modulo $(x_1,\ldots,x_4)^2$ have the form $(w_{\sigma 1},w_{\sigma 2})$ and these ideals are all different. Thus,

$$\bigl\{\operatorname{Coker}(\varphi_\lambda) \mid \lambda \in V(f)\bigr\} \cup \bigl\{\operatorname{Coker}\bigl(\varphi_\sigma(a,b)\bigr) \mid \sigma, a, b \text{ as above } \bigr\}$$

contains only non–isomorphic modules (similarly for $\psi$'s). It follows that, if $N, P \in \mathcal{M} \cup \mathcal{P}$ are isomorphic and different, then $N \simeq \Omega^1_R(P)$.

If $N = \operatorname{Coker}(\varphi_\lambda)$, for $\lambda \in V(f)$, then this is not possible since the ideals $(p_{1\lambda},p_{2\lambda},p_{3\lambda})$ and $(q_{1\lambda},q_{2\lambda},q_{3\lambda})$ are not in the same even linkage class. Indeed, by the proof of (1) in Lemma 4.17, $(p_{1\lambda},p_{2\lambda},p_{3\lambda})$ is evenly linked with $(q_{1\lambda},q_{2\lambda},p_{3\lambda})$ and this last ideal is obviously directly linked with $(q_{1\lambda},q_{2\lambda},q_{3\lambda})$. If $N = \operatorname{Coker}\bigl(\varphi_\sigma(a,b)\bigr)$ for some $\sigma, a, b$, and $N \simeq \Omega^1_R(N)$, then the ideals $(w_{\sigma 1}, v_{\sigma 2}, x_j x_s)$ and $(w_{\sigma 1}, w_{\sigma 2}, x_j x_s)$ are evenly linked. But these ideals are directly linked by the regular sequence $\{w_{\sigma 1}, v_{\sigma 2} w_{\sigma 2}, x_j x_s\}$, contradiction!

It remains to show that $\mathcal{M} \cup \mathcal{P}$ contains only indecomposable modules. If $N \in \mathcal{M}$, let us say $N = \operatorname{Coker}(\varphi_\lambda)$ for $\lambda = [\lambda_1 : \lambda_2 : \lambda_3 : 1]$, we see that $N/x_4N$ is exactly the module corresponding to the matrix

$$\begin{pmatrix} 0 & x_3 & -x_2 & -x_1^2 \\ -x_3 & 0 & -x_1 & x_2^2 \\ x_2 & x_1 & 0 & x_3^2 \\ x_1^2 & -x_2^2 & -x_3^2 & 0 \end{pmatrix}$$

whose cokernel is the special module $M_2$ (see [LPP] for the special module of rank 2 which corresponds to the special bundle from Atiyah classification). Thus, $N/x_4N$ is indecomposable and, by Nakayama's Lemma, $N$ is indecomposable. Now let $N \in \mathcal{P}$, $N = \operatorname{Coker}\bigl(\psi_\sigma(a,b)\bigr)$. By the permutation of the rows and the columns of $\psi_\sigma(a,b)$, we may suppose that it has the form:

$$\begin{pmatrix} w_{\sigma 1} & -v_{\sigma 2} & x_j x_s & 0 \\ w_{\sigma 2} & v_{\sigma 1} & 0 & x_j x_s \\ 0 & 0 & v_{\sigma 1} & v_{\sigma 2} \\ 0 & 0 & -w_{\sigma 2} & w_{\sigma 1} \end{pmatrix}.$$

Suppose $N$ is decomposable. Then $\psi_\sigma(a,b)$ is equivalent with a direct sum of two matrices of order 2 say $A_1$, $A_2$. Let $B_1$, $B_2$ be the submatrices of the $\psi_\sigma(a,b)$ given by the first two lines and columns, respectively the last two lines and columns. Certainly $A_1, A_2, B_1, B_2$ define some maximal Cohen-Macaulay modules of rank one $N_1, N_2, T_1, T_2$, and due to the particular form of $\psi_\sigma(a,b)$ we have the following exact sequence

$$0 \to T_1 \to N_1 \oplus N_2 = N \to T_2 \to 0.$$

Note that $\psi_\sigma(a,b)$ is modulo $x_j$ or $x_s$ the sum of $B_1, B_2$. Thus $T_i/x_jT_i \cong N_i/x_jN_i$ for $i = 1,2$ and similarly for $x_s$. Since we have the whole description of rank one maximal Cohen-Macaulay modules we can see that $A_i$ is equivalent with $B_i$ modulo $x_j$ and modulo $x_s$ only when $A_i$ is equivalent with $B_i$. Thus $T_i \cong N_i$ for $i = 1,2$ and so $N \cong T_1 \oplus T_2$. By a subtle result of Miyata ( [Mi] ) this happens only if the above exact sequence splits. This means that there exist two matrices $A, B$ of order two such that

$$x_j x_s \cdot \mathrm{Id}_2 = \begin{pmatrix} w_{\sigma 1} & -v_{\sigma 2} \\ w_{\sigma 2} & v_{\sigma 1} \end{pmatrix} A + B \begin{pmatrix} v_{\sigma 1} & v_{\sigma 2} \\ -w_{\sigma 2} & w_{\sigma 1} \end{pmatrix},$$

which is impossible. □

**Proposition 4.19.** *The aCM bundles corresponding to*

**Remarks 4.20.** 1. *There exists a bijection between*

$$\mathcal{P}_1 = \{\mathrm{Coker}(\varphi_\sigma(a,b)) \mid \sigma, a, b\}$$

*and the 2–generated, non–free, MCM R–modules, which remind us of Atiyah's classification. Thus, $\mathcal{P}_1$ contains 27 modules corresponding to 27 lines and 27 pencils of conics of $V(f)$.*
*Similarly, $\mathcal{P}_2 = \{\mathrm{Coker}(\psi_\sigma(a,b)) \mid \sigma, a, b\}$ contains 27 modules.*

2. *$\mathcal{M}$ is a kind of "blowing up" of $M_2, \Omega^1_R(M_2)$ from [LPP] (see the proof of Theorem 4.18). Note also that $\mathcal{M}$ consists of two classes of modules parameterized by the points of $V(f)$, which is also in Atiyah's idea.*

3. *The matrices $\varphi$ defining the modules of $\mathcal{M} \cup \mathcal{P}$ are skew symmetric as our Theorem 4.1 predicted.*

**Lemma 4.21.** *(With the notations from Theorem* 4.13*)*

1. *The aCM bundles corresponding to the modules*

$$\left\{\operatorname{Coker}\psi_\lambda, \operatorname{Coker}\psi_\sigma(a,b) \mid \lambda \in V(f), \sigma, a, b\right\}$$

*have a minimal resolution of type (E);*

2. *The aCM bundles corresponding to the MCM modules*

$$\left\{\operatorname{Coker}\varphi_\lambda, \operatorname{Coker}\varphi_\sigma(a,b) \mid \lambda \in V(f), \sigma, a, b\right\}$$

*have a minimal resolution of type (G);*

3. *The modules $\left\{\operatorname{Coker}\varphi_\sigma(a,b), \operatorname{Coker}\psi_\sigma(a,b) \mid \lambda, \sigma, a, b\right\}$ are extensions of rank one MCM modules, that correspond to 27 lines (conics).*

4. *For $\beta$ linear form, $\left\{\operatorname{Coker}\varphi_{\sigma\beta}(a,b), \operatorname{Coker}\psi_{\sigma\beta}(a,b) \mid \lambda \in V(f), \sigma, a, b\right\}$ are also extensions, but they are isomorphic to modules from $\mathcal{M}$.*

*Proof.* The results follow straightforeward, by simple computations.
For 3), we write $\psi_\sigma(a,b)$ as in the proof of indecomposability (Theorem 4.18). We get that the corresponding aCM bundle, $\mathcal{E}$ fits in one extension of the type:

$$0 \to \mathcal{O}_X(C) \to \mathcal{E} \to \mathcal{O}_X(L) \to 0,$$

where $C$ is a conic and $L$ a line on the surface $X = \operatorname{Proj} R$. Similarly, if $\mathcal{F}$ denotes the corresponding aCM bundle of some $\varphi_\sigma(a,b)$, then we have an extension of type:

$$0 \to \mathcal{O}_X(L+H) \to \mathcal{F} \to \mathcal{O}_X(C) \to 0,$$

for $H = \mathcal{O}_X(1)$. □

**Theorem 4.22.** *1. For any $\lambda \in X$, $X = Proj R$, the module* $\operatorname{Coker}\psi_\lambda$*, corresponds to a stable aCM bundle. More, all stable, indecomposable aCM bundles with a resolution of type (E) have a matrix factorization of type $\psi_\lambda$.*

2. *The aCM bundles corresponding to the modules*

$$\left\{\operatorname{Coker}\varphi_\lambda, \operatorname{Coker}\varphi_\sigma(a,b) \mid \lambda \in V(f), \sigma, a, b\right\}$$

*are semistable.*

The theorem follows from the previous lemma, using some actual results of D. Faenzzi:

**Proposition 4.23.** *( see [F2] ) Let $\mathcal{E}$ be an indecomposable rank two aCM bundle on a smooth cubic hypersurface $X$ in $\mathbb{P}^3$.*

1. *If $\mathcal{E}$ has a minimal resolution of type (E), then $c_1(\mathcal{E}) \equiv H$ and $c_2(\mathcal{E}) = 2$. ( $H = \mathcal{O}_X(1)$)*

2. *If it has a resolution of type (G), then $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = 1$.*

3. *If $\mathcal{E}$ is a general bundle with resolution of type (E), then $\mathcal{E}$ is stable. The moduli space of stable rank 2 aCM bundles with $c_1(\mathcal{E}) \equiv H$ and $c_2(\mathcal{E}) = 2$, denoted $MCM^s(2; H, 2)$ is birational to $X$.*

4. *If $\mathcal{E}$ has resolution of type (G), then $\mathcal{E}$ is semistable with $h^0(\mathcal{E}nd(\mathcal{E})) = 2$ and the moduli space of semistable rank 2 aCM bundles with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 1$, denoted $MCM^{ss}(2; 0, 1)$ is birational to $X$, while . $MCM^s(2; 0, 1)$ is empty.*

# 5 Rank two, non – orientable, MCM modules over the hypersurface $R = k[x_1, x_2, x_3, x_4]/(x_1^3 + x_2^3 + x_3^3 + x_4^3)$

Let $M$ be a graded non–orientable, rank 2, MCM $R$–module, without free direct summands. We should like to express $M$ as a 2–syzygy of an ideal $I$, $M \cong \Omega_R^2(I)$, with $\mu(M) = \mu(I) + 1$ (this is known in orientable case by [HK], see here Section 3).

The following proposition can be found in [Bo1, Korollar 2].

**Proposition 5.1.** *Let $(A, m)$ be a Noetherian normal local domain with $\dim A \geq 2$ and $N$ a finite torsion–free $A$–module. Then there exists a finite free submodule $F \subset N$ such that $N/F$ is isomorphic with an ideal of $A$ and the canonical map $F/mF \to N/mN$ is injective.*

Applying Proposition 5.1, we obtain the following exact sequence:

$$0 \to R \longrightarrow M \longrightarrow I \longrightarrow 0 \tag{22}$$

for an ideal $I \subset R$, which induces an exact sequence

$$0 \longrightarrow K = R/m \longrightarrow M/mM \longrightarrow I/mI \longrightarrow 0.$$

Thus $\mu(M) = \mu(I) + 1$.

As we know in the orientable case to obtain MCM $R$–modules of rank 2 we must choose $I$ such that $\mathrm{Ext}_R^1(I, R)$ is a cyclic $R$–module or, more precisely, such that $R/I$ is Gorenstein. In the non–orientable case one can also show that $\mathrm{Ext}_R^1(I, R)$ must be a cyclic $R$–module, but this is not very helpful since it is hard to check this condition for arbitrary $I$. Below we shall state an easier condition.

Let $J \subset S = K[X_1, \ldots, X_4]$ be an ideal such that $f \in mJ$ and $I = J/(f)$.

**Lemma 5.2.** *Let*

$$0 \longrightarrow S^{s_3} \xrightarrow{d_3} S^{s_2} \xrightarrow{d_2} S^{s_1} \xrightarrow{d_1} J \longrightarrow 0$$

*be a minimal free $S$–resolution of an ideal $J$ with depth $S/J = 1$.*

*If* $\mathrm{rank}\, \Omega_R^2\big(J/(f)\big) = 2$ *and* $\mu\Big(\Omega_R^2\big(J/(f)\big)\Big) = \mu(I) + 1$ *then* $s_1 = s_2 \leq 5$ *and* $s_3 = 1$.

*Proof.* As in the proof of Theorem 4.1, we obtain a minimal free resolution of $I = J/(f)$ over $S$ in the following way:

Let $v : S \to S^{s_1}$ be an $S$–linear map such that $jd_1v = f\,\mathrm{Id}_S$, where $j : J \to S$ is the inclusion. Let $\tilde{d}_1$ be the composite map $S^{s_1} \xrightarrow{d_1} J \to J/(f) = I$. Then the following sequence

$$0 \longrightarrow S^{s_3} \xrightarrow{\binom{d_3}{0}} S^{s_2+1} \xrightarrow{(d_2,v)} S^{s_1} \xrightarrow{\tilde{d}_1} I \longrightarrow 0$$

is exact and forms a minimal free resolution of $I$ over $S$. Since

$$f \cdot S^{s_1} \subset \mathrm{Im}(d_2, v),$$

there exists an $S$–linear map $h : S^{s_1} \to S^{s_2+1}$ such that

$$(d_2, v)h = f\,\mathrm{Id}_{S^{s_1}}$$

and we obtain the following exact sequence

$$R^{s_3+s_1} \xrightarrow{\left(\bar{h}\middle|\begin{smallmatrix}\bar{d}_3\\0\end{smallmatrix}\right)} R^{s_2+1} \xrightarrow{(\bar{d}_2,\bar{v})} R^{s_1} \xrightarrow{\bar{d}_1} I \longrightarrow 0,$$

which is part of a minimal free $R$–resolution of $I$. Thus, $M = \Omega_R^2(I)$ is the image of the first map above and so $s_1 + s_3 = s_2 + 1 = s_1 + 1$ because $\mu(M) = \mu\big(\Omega_R^1(M)\big) = \mu(I) + 1$ by hypothesis. It follows that $s_3 = 1$, $s_1 = s_2$. As $\mu(M) \le 3\,\mathrm{rank}_R M = 6$ we obtain $s_1 \le 5$. □

Let $\det N$ be the corresponding class of the bidual $(\wedge^n N)^{**}$, $n = \mathrm{rank}\, N$, in $Cl(R)$ for a torsion free $R$–module $N$. Since det is an additive function, we obtain $\det(M) = 0$ if and only if $\det(I) = 0$. Thus, $M$ is non–orientable if and only if $I$ is non–orientable, that is, $\mathrm{codim}(J) \le 1$ for all ideals $J \subset R$ isomorphic with $I$, according to [HK]. Since $M$ has rank 2, we obtain $\mathrm{codim}(I) = 1$. Thus, $\dim R/I = 2$ and, from (22), we obtain $\mathrm{depth}\, R/I = 1$, that is, $R/I$ is not Cohen–Macaulay. Also from (22) we obtain $\Omega_R^2(M) \simeq \Omega_R^2(I)$ and so $M \simeq \Omega_R^2(I)$.

**Proposition 5.3.** *Each graded, non–orientable, rank 2, s–generated MCM R–module is the second syzygy $\Omega_R^2(I)$ of an $(s-1)$–generated graded ideal $I \subset R$ with* $\mathrm{depth}\, R/I = 1$ *and* $\dim R/I = 2$.

As in Section 3, let $u, a, b \in K$, with

$$a^3 = b^3 = -1,\ u^2 + u + 1 = 0,$$

$\sigma = (i\ j\ s)$ be a permutation of the set $\{2, 3, 4\}$ with $i < j$ and set

$$\begin{array}{rclrcl} w_{\sigma 1} & = & x_1 - a x_s, & w_{\sigma 2} & = & x_i - b x_j, \\ v_{\sigma 1} & = & x_1^2 + a x_1 x_s + a^2 x_s^2, & v_{\sigma 2} & = & x_i^2 + b x_i x_j + b^2 x_j^2. \end{array}$$

We have

$$v_{\sigma 1} = v'_{\sigma 1} v''_{\sigma 1},\ v_{\sigma 2} = v'_{\sigma 2} v''_{\sigma 2}$$

for

$$\begin{array}{rclrcl} v'_{\sigma 1} & = & x_1 - u a x_s, & v''_{\sigma 1} & = & x_1 + (1+u) a x_s, \\ v'_{\sigma 2} & = & x_i - u b x_j, & v''_{\sigma 2} & = & x_i + (1+u) b x_j. \end{array}$$

Set

$$\begin{array}{rcl} I_{1\sigma}(a,b,u) & = & (x_s v'_{\sigma 2}, v_{\sigma 2}, w_{\sigma 1}), \\ I_{2\sigma}(a,b,u) & = & (x_j v''_{\sigma 1}, v_{\sigma 1}, w_{\sigma 2}), \\ I_{3\sigma}(a,b,u) & = & (x_s v''_{\sigma 2}, v_{\sigma 2}, v_{\sigma 1}), \\ I_{4\sigma}(a,b,u) & = & (x_j v'_{\sigma 1}, v_{\sigma 1}, v_{\sigma 2}), \\ I_{5\sigma}(a,b,u) & = & (x_s v''_{\sigma 2}, v_{\sigma 2}, w_{\sigma 1}), \\ I_{6\sigma}(a,b,u) & = & (x_j v'_{\sigma 1}, v_{\sigma 1}, w_{\sigma 2}), \\ I_{7\sigma}(a,b,u) & = & (x_s v'_{\sigma 2}, v_{\sigma 2}, v_{\sigma 1}), \\ I_{8\sigma}(a,b,u) & = & (x_j v''_{\sigma 1}, v_{\sigma 2}, v_{\sigma 1}) \end{array}$$

Set

$$\varphi_{1\sigma}(a,b,u) = \begin{pmatrix} w_{\sigma 1} & -w_{\sigma 2} & 0 & x_s \\ v_{\sigma 2} & v_{\sigma 1} & x_s v'_{\sigma 2} & 0 \\ 0 & 0 & w_{\sigma 1} & -v''_{\sigma 2} \\ 0 & 0 & -w_{\sigma 2} v'_{\sigma 2} & -v_{\sigma 1} \end{pmatrix},$$

$$\varphi_{2\sigma}(a,b,u) = \begin{pmatrix} w_{\sigma 1} & -w_{\sigma 2} & 0 & -x_j \\ v_{\sigma 2} & v_{\sigma 1} & x_j v''_{\sigma 1} & 0 \\ 0 & 0 & w_{\sigma 2} & -v'_{\sigma 1} \\ 0 & 0 & -w_{\sigma 1} v''_{\sigma 1} & -v_{\sigma 2} \end{pmatrix},$$

$$\psi_{3\sigma}(a,b,u) = \begin{pmatrix} w_{\sigma 1} & -w_{\sigma 2} & -x_s & 0 \\ v_{\sigma 2} & v_{\sigma 1} & 0 & -x_s v''_{\sigma 2} \\ 0 & 0 & -v'_{\sigma 2} & w_{\sigma 1} \\ 0 & 0 & -v_{\sigma 1} & -w_{\sigma 2} v''_{\sigma 2} \end{pmatrix},$$

$$\psi_{4\sigma}(a,b,u) = \begin{pmatrix} w_{\sigma 1} & -w_{\sigma 2} & x_j & 0 \\ v_{\sigma 1} & v_{\sigma 2} & 0 & -x_j v'_{\sigma 1} \\ 0 & 0 & -v''_{\sigma 1} & w_{\sigma 2} \\ 0 & 0 & -v_{\sigma 2} & -w_{\sigma 1} v'_{\sigma 1} \end{pmatrix},$$

$$\varphi_{3\sigma}(a,b,u) = \begin{pmatrix} v_{\sigma 1} & w_{\sigma 2} & 0 & -x_s \\ -v_{\sigma 2} & w_{\sigma 1} & x_s v''_{\sigma 2} & 0 \\ 0 & 0 & -w_{\sigma 2} v''_{\sigma 2} & -w_{\sigma 1} \\ 0 & 0 & v_{\sigma 1} & -v'_{\sigma 2} \end{pmatrix},$$

$$\varphi_{4\sigma}(a,b,u) = \begin{pmatrix} v_{\sigma 1} & w_{\sigma 2} & x_j v'_{\sigma 1} & 0 \\ -v_{\sigma 2} & w_{\sigma 1} & 0 & -x_j \\ 0 & 0 & -w_{\sigma 1} v'_{\sigma 1} & -w_{\sigma 2} \\ 0 & 0 & v_{\sigma 2} & 0 - v''_{\sigma 1} \end{pmatrix},$$

$$\psi_{1\sigma}(a,b,u) = \begin{pmatrix} v_{\sigma 1} & w_{\sigma 2} & 0 & x_s \\ -v_{\sigma 2} & w_{\sigma 1} & -x_s v'_{\sigma 2} & 0 \\ 0 & 0 & v_{\sigma 1} & -v''_{\sigma 2} \\ 0 & 0 & -w_{\sigma 2} v'_{\sigma 2} & -w_{\sigma 1} \end{pmatrix},$$

$$\psi_{2\sigma}(a,b,u) = \begin{pmatrix} v_{\sigma 1} & w_{\sigma 2} & 0 & x_j \\ -v_{\sigma 2} & w_{\sigma 1} & -x_j v''_{\sigma 1} & 0 \\ 0 & 0 & v_{\sigma 2} & -v'_{\sigma 1} \\ 0 & 0 & -w_{\sigma 1} v''_{\sigma 1} & -w_{\sigma 2} \end{pmatrix}.$$

**Theorem 5.4.** *1. For each $1 \le t \le 4$, the pair $\big(\varphi_{t\sigma}(a,b,u), \psi_{t\sigma}(a,b,u)\big)$ forms a matrix factorization of $\Omega_R^2\big(I_{t\sigma}(a,b,u)\big)$.*

*2. The set*

$$\mathcal{N} = \big\{\mathrm{Coker}\big(\varphi_{t\sigma}(a,b,u)\big),\ \mathrm{Coker}\big(\psi_{t\sigma}(a,b,u)\big) \mid 1 \le t \le 4,\ \sigma, a, b, u\big\}$$

*contains only graded, indecomposable, non–orientable, 4–generated MCM R–modules of rank 2.*

*3. Every indecomposable, graded, non–orientable, 4–generated MCM module over R of rank 2 is isomorphic with one module of $\mathcal{N}$.*

4. *The modules of $\mathcal{N}$ are pairwise isomorphic. In particular, there exist 216 isomorphism classes of indecomposable, graded, non–orientable, 4–generated MCM modules over R of rank 2.*

*Proof.* (1) It is easy to check that

$$\varphi_{t\sigma}(a,b,u)\cdot\psi_{t\sigma}(a,b,u)=f\cdot \mathrm{Id}_4$$

and the following sequence is exact:

$$R(-6)^4 \xrightarrow{\varphi_{1\sigma}(a,b,u)} R(-5)^2\otimes R(-4)^2 \xrightarrow{\psi_{1\sigma}(a,b,u)} R(-3)^4 \xrightarrow{A_1} R(-2)^2\otimes R(-1) \longrightarrow I_{1\sigma}(a,b,u)\longrightarrow 0,$$

where $A_1$ is the $3\times 4-$matrix formed by the first three rows of $\varphi_{1\sigma}(a,b,u)$. Thus, (1) holds for $t=1$, the other cases being similar.

(2) Clearly $I_{1\sigma}(a,b,u)\subset (v'_{\sigma 2},w_{\sigma 1})$ and so $\dim R/I_{1\sigma}(a,b,u)=2$. As $x_s$ is zero–divisor in $R/I_{1\sigma}(a,b,u)$ we see that $\operatorname{depth} R/I_{1\sigma}(a,b,u)=1$ and, by Proposition 5.3, $\Omega^2_R(I)$ is non–orientable, 4–generated of rank 2. Note that the module $\operatorname{Coker}\big(\varphi_{1\sigma}(a,b,u)\big)$, as in the last part of the proof of Theorem 4.18, is indecomposable because there exist no two matrices $A,B$ of order two such that

$$\begin{pmatrix} 0 & x_s\\ x_s v'_{\sigma 2} & 0\end{pmatrix}=\begin{pmatrix} w_{\sigma 1} & -w_{\sigma 2}\\ v_{\sigma 2} & v_{\sigma 1}\end{pmatrix}A+B\begin{pmatrix} w_{\sigma 1} & -v''_{\sigma 2}\\ -w_{\sigma 2}v'_{\sigma 2} & -v_{\sigma 1}\end{pmatrix}.$$

Similarly, the cases $t>1$ follows.

(3) Now let $M$ be an indecomposable, graded, non–orientable, 4–generated MCM $R$–module of rank 2. By Proposition 5.3, there exists a graded ideal $I\subset R$ with $\dim R/I=2$, $\operatorname{depth} R/I=1$, which is 3–generated and such that $M\simeq\Omega^2_R(I)$. Then $I=J/(f)$ with $J\subset S=K[x_1,x_2,x_3,x_4]$ is a 3–generated ideal containing $f$. We have still $f\in mJ$, though we are not in the orientable case (see $[EP_1]$ for details). Let $\alpha_1,\alpha_2,\alpha_3$ be a minimal system of homogeneous generators of $J$. If $f$ does not belong to the ideal generated by two $\alpha_t$, then, as in Section 3, $f=\sum_{t=1}^3 p_tq_t$ and, after a renumbering, we may suppose that $\alpha_t$ is necessarily either $p_t$ or $q_t$, for all $1\le t\le 3$. Then $\alpha_1,\alpha_2,\alpha_3$ is a regular system of elements in $S$ and so $R/I=S/J$ is Cohen–Macaulay which is false.

Thus, we may suppose $f\in(\alpha_1,\alpha_2)$. Then there exist $a,b\in K$ with $a^3=b^3=-1$, and $\sigma=(i\ j\ s)$ a permutation of the set $\sigma=\{2,3,4\}$, $i<j$, such that $\alpha_t$ is necessarily either $w_{\sigma t}$ or $v_{\sigma t}$, for $t=1,2$. If $\alpha_1=w_{\sigma 1},\alpha_2=$

$w_{\sigma 2}$, then $R/(\alpha_1, \alpha_2)$ is a domain and $\alpha_1, \alpha_2, \alpha_3$ must be a regular system of elements in $S$ and so, again, $R/I = S/J$ is Cohen–Macaulay, contradiction!

We have the following cases:

**Case I:** $\alpha_1 = w_{\sigma 1}$

Then $\alpha_2$ must be $v_{\sigma 2}$ and we have

$$(\alpha_1, \alpha_2) = (v'_{\sigma 2}, w_{\sigma 1}) \cap (v''_{\sigma 2}, w_{\sigma 1}).$$

It follows that a zero–divisor of $R/(\alpha_1, \alpha_2)$ must be either in $(v'_{\sigma 2}, w_{\sigma 1})$ or in $(v''_{\sigma 2}, w_{\sigma 1})$. As we know, $\alpha_3$ is a zero–divisor in $R/(\alpha_1, \alpha_2)$ and so $\alpha_3 \in (v'_{\sigma 2}, w_{\sigma 1})$ or $\alpha_3 \in (v''_{\sigma 2}, w_{\sigma 1})$.

**I(a)** Suppose

$$\alpha_3 \in (v'_{\sigma 2}, w_{\sigma 1}).$$

Subtracting from $\alpha_3$ a multiple of $w_{\sigma 1}$, we may take $\alpha_3 = v'_{\sigma 2}\beta$ for a form $\beta$ of $S$. Note that the matrices

$$\varphi = \begin{pmatrix} 0 & w_{\sigma 1} & -v''_{\sigma 2} & 0 \\ -w_{\sigma 1} & 0 & -\beta & w_{\sigma 2} \\ v_{\sigma 2} & \beta v'_{\sigma 2} & 0 & v_{\sigma 1} \\ 0 & -w_{\sigma 2} v'_{\sigma 2} & -v_{\sigma 1} & 0 \end{pmatrix},$$

$$\psi = \begin{pmatrix} 0 & -v_{\sigma 1} & w_{\sigma 2} & \beta \\ v_{\sigma 1} & 0 & 0 & -v''_{\sigma 2} \\ -w_{\sigma 2} v'_{\sigma 2} & 0 & 0 & -w_{\sigma 1} \\ -\beta v'_{\sigma 2} & v_{\sigma 2} & w_{\sigma 1} & 0 \end{pmatrix}$$

give the following exact sequence:

$$\longrightarrow R^4 \xrightarrow{\varphi} R^4 \xrightarrow{\psi} R^4 \xrightarrow{B_1} R^3 \longrightarrow I \longrightarrow 0,$$

where $B_1$ is given by the first three rows of $\varphi$. Thus. $(\varphi, \psi)$ is a matrix factorization of $\Omega^2_R(I) \simeq M$. Adding in $\varphi$ multiples of the first row to the second one and adding multiples of the fourth column to the third one, we may suppose that the entry $(2,3)$ of $\varphi$ depends only on $x_1, x_s$. These transformations modify also the entries $(2,2)$ and $(3,3)$, which are now not zero. Adding similar multiples of the first column to the second one and of the fourth row to the third one, we obtain $\varphi$ of the same type as before but with $\beta$ depending only on $x_1, x_s$. Since $v_{\sigma 1} - w_{\sigma 1}(x_1 + 2ax_s) = 3ax_s^2$, adding in $\varphi$ multiples of the first column to the third one and multiples of the fourth row to the second row, we may suppose that the entry $(2,3)$ has the form

$\lambda x_s$ for some $\lambda \in K$. These transformations modify also the entries $(3,3)$ and $(2,2)$, which are now not zero. Adding similar multiples of the first row to the third one and of the fourth column to the second column, we obtain $\varphi$ of the same type as before but with $\beta = \lambda x_s$. If $\lambda = 0$, then, clearly, $\varphi$ is the direct sum of two 2–matrices which contradicts that $M$ is indecomposable. So. $\lambda \neq 0$. Now we divide the second and the third column of $\varphi$ by $\lambda$ and multiply the first and the fourth row by $\lambda$. The new $\varphi$ is as before but with $\lambda = 1$, that is $\varphi = \varphi_{1\sigma}(a,b,u)$.

**I(b)** Suppose

$$\alpha_3 \in (v''_{\sigma 2}, w_{\sigma 1}).$$

Then we may take $\alpha_3 = v''_{\sigma 2}\beta$, for a form $\beta$. With a similar proof as above, we obtain $M \simeq \operatorname{Coker}\big(\psi_{3\sigma}(a,b,u)\big)$.

**Case II:** $\alpha_2 = w_{\sigma 2}$**.**

Then $\alpha_1 = v_{\sigma 1}$. It follows that $(\alpha_1, \alpha_2) = (v'_{\sigma 1}, w_{\sigma 2}) \cap (v''_{\sigma 1}, w_{\sigma 2})$. We have the following two subcases:

**II(a)** $\alpha_3 \in (v'_{\sigma 1}, w_{\sigma 2})$. We may suppose $\alpha_3 = v'_{\sigma 1}\beta$, for a form $\beta$ and we obtain that $M \simeq \operatorname{Coker}\big(\psi_{4\sigma}(a,b,u)\big)$.

**II(b)** $\alpha_3 \in (v''_{\sigma 1}, w_{\sigma 2})$. In this subcase we may take $\alpha_3 = v''_{\sigma 1}\beta$, for a form $\beta$ and we obtain that $M \simeq \operatorname{Coker}\big(\varphi_{2\sigma}(a,b,u)\big)$.

**Case III:** $\alpha_1 = v_{\sigma 1}, \alpha_2 = v_{\sigma 2}$**.**

Then $(\alpha_1, \alpha_2) = (v'_{\sigma 1}, v'_{\sigma 2}) \cap (v'_{\sigma 1}, v''_{\sigma 2}) \cap (v''_{\sigma 1}, v'_{\sigma 2}) \cap (v''_{\sigma 1}, v''_{\sigma 2})$. We proceed as in the above cases, taking $\alpha_3$ from one prime ideal of the above decomposition of $\big(\alpha_1, \alpha_2)\big)$, let us say $\alpha_3 \in (v'_{\sigma 1}, v'_{\sigma 2})$, that is $\alpha_3 = v'_{\sigma 1}\beta + v'_{\sigma 2}\gamma$ for some $\beta, \gamma \in S$. Suppose that one cannot reduce the problem to the case $\beta = 0$ or $\gamma = 0$, this implies, for example, that $v'_{\sigma 1}$ does not divide $\gamma$ and $v'_{\sigma 2}$ does not divide $\beta$. Then $\Omega^1_S\big((\alpha_1, \alpha_2, \alpha_3)\big) \subset S^3$ contains the columns of the following matrix

$$\begin{pmatrix} v_{\sigma 2} & \alpha_3 & 0 & v''_{\sigma 2}\beta \\ -v_{\sigma 1} & 0 & \alpha_3 & v''_{\sigma 1}\gamma \\ 0 & -v_{\sigma 1} & -v_{\sigma 2} & -v''_{\sigma 1}v''_{\sigma 2} \end{pmatrix}$$

and we can see that $\mu(\Omega^1_S\big((\alpha_1, \alpha_2, \alpha_3)\big) \geq 4$, which contradicts Lemma 5.2. Thus we may suppose, let us say $\alpha_3 = v'_{\sigma 1}\beta$, where $\beta$ is not a multiple of $v''_{\sigma 1}$. Now we may proceed as in the above cases and we obtain, in order, $M \simeq \operatorname{Coker}\big(\varphi_{4\sigma}(a,b,u)\big)$, $M \simeq \operatorname{Coker}\big(\varphi_{3\sigma}(a,b,u)\big)$, $M \simeq \operatorname{Coker}\big(\psi_{1\sigma}(a,b,u)\big)$, and $M \simeq \operatorname{Coker}\big(\psi_{2\sigma}(a,b,u)\big)$.

(4) The matrices

$$\varphi_{t\sigma}(a,b,u),\ \psi_{t\sigma}(a,b,u), 1 \leq t \leq 4,\ \sigma, a, b, u,$$

are equivalent in pairs. Namely:

$$\varphi_{1\sigma}(a,b,u) \sim \psi_{3\sigma}(a,b,u^2),\ \ \varphi_{2\sigma}(a,b,u) \sim \psi_{4\sigma}(a,b,u^2),$$

$$\varphi_{3\sigma}(a,b,u) \sim \psi_{1\sigma}(a,b,u^2),\ \ \varphi_{4\sigma}(a,b,u) \sim \psi_{2\sigma}(a,b,u^2).$$

We shall prove that the matrices of the set

$$\mathcal{N}' = \{\varphi_{t\sigma}(a,b,u),\ |\ 1 \leq t \leq 4,\ \sigma, a, b, u\}$$

are pairwise non–equivalent. We shall consider the matrices which are obtained from the matrices of $\mathcal{N}'$, reducing their entries modulo $\mathfrak{m}^2$. If $A, B \in \mathcal{N}'$ are equivalent, then there exist $P, Q$, two invertible $4 \times 4$–matrices with the entries in $K[x_1, x_2, x_3, x_4]$ such that $PA = BQ$. Let $\widetilde{A}$ and $\widetilde{B}$ be the matrices obtained from $A$, respectively $B$, by reducing modulo $\mathfrak{m}^2$ their entries. From the equality $PA = BQ$, we obtain that there exist two invertible scalar matrices $\widetilde{P}, \widetilde{Q} \in \mathcal{M}_4(K)$ such that $\widetilde{P}\widetilde{A} = \widetilde{B}\widetilde{Q}$. This means that the matrices $\widetilde{A}, \widetilde{B}$ are also equivalent by some scalar invertible matrices. We construct the "reduced" matrices $\tilde{\varphi}_{t\sigma}(a,b,u)$ and for all $t$. We see that the matrices $\tilde{\varphi}_{1\sigma}(a,b,u), \tilde{\varphi}_{2\sigma}(a,b,u)$, have the entries of the rows 2 and 4 zero and the rest of the matrices have the entries of the columns 1 and 3 zero. First, we choose two matrices $\widetilde{A}, \widetilde{B}$, one of them with the rows 2 and 4 zero and the other with the columns 1 and 3 zero. Suppose that $\widetilde{A} \sim \widetilde{B}$. It results that there are two invertible scalar $4 \times 4$–matrices $U, V$ such that

$$\widetilde{A}U = V\widetilde{B}.$$

From this equality we obtain that the rows 2 and 4 in the matrix $V\widetilde{B}$ are zero. Looking at the two possibilities to choose the matrix $\widetilde{B}$, we see that the non–zero elements of the columns 2 and 4 in $\widetilde{B}$ are linear independent. From the above equality we get that $V$ is not invertible.

Hence, we could find two equivalent matrices in the set $\mathcal{N}'$ only if both have the rows 2 and 4 zero or the columns 1 and 3 zero. It is clear that we may reduce the study of the equivalent matrices $\widetilde{A} = \tilde{\varphi}_{1\sigma}(a,b,u), \widetilde{B} = \tilde{\varphi}_{2\sigma}(a,b,u)$, which have the rows 2 and 4 zero. Let $U, V \in \mathcal{M}_{4\times 4}(K)$ be invertible matrices such that $\widetilde{A}U = V\widetilde{B}$. We may transform the reduced matrices $\widetilde{A}, \widetilde{B}$ such that the last two rows are zero. Let

$$\widetilde{A} = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix},\ \widetilde{B} = \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix},\ U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix},\ V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix},$$

be the decomposition of our matrices in $2 \times 2$ blocks. Comparing the elements in the above equality, we obtain contradiction with the fact that $U$ is invertible.

In the same way we check that if $\tilde{\varphi}_{1\sigma}(a, b, u)$ and $\tilde{\varphi}_{1\tau}(n, p, v)$ are different, then they are not equivalent. $\square$

Let $M(\sigma, a, b)$, $M'(\tau, n, p)$ be two rank one MCM-modules corresponding to lines and $N(\sigma, a, b)$, $N'(\tau, n, p)$ be two rank one MCM-modules corresponding to conics (that is $\mathrm{Coker}\big(\varphi_\sigma(a, b)\big)$, $\mathrm{Coker}\big(\psi_\sigma(a, b)\big)$ by [EP]).

**Remark 5.5.** *There exists an indecomposable extension in $Ext^1_R(M(\sigma, a, b), M'(\tau, n, p))$ only if $\sigma = \tau$. In this case, for fixed $M(\sigma, a, b)$ there exists 4 non–orientable MCM–modules, which are extensions $E$ of the form*

$$0 \to M'_i \to E \to M(\sigma, a, b) \to 0 \ .$$

*for some $M'_i$, $i = 1, 2$ of type $M'(\sigma, n, p)$. So we have $4 \times 27$ non–orientable MCM–modules. Similarly, taking now extensions $F$ of the form*

$$0 \to N_i \to F \to N(\sigma, a, b) \to 0, i = 1, 2$$

*we obtain another $4 \times 27$ non–orientable MCM–modules. Thus all are $216 = 8 \times 27$.*

### 5.2 Non–orientable, rank 2, 5–generated MCM modules

As in Section 3, let $u, a, b \in K$, with

$$a^3 = b^3 = -1, \ u^2 + u + 1 = 0,$$

$\sigma = (i \ j \ s)$ be a permutation of the set $\{2, 3, 4\}$ with $i < j$ and set

$$\begin{array}{rclrcl} w_{\sigma 1} & = & x_1 - a x_s, & w_{\sigma 2} & = & x_i - b x_j, \\ v_{\sigma 1} & = & x_1^2 + a x_1 x_s + a^2 x_s^2, & v_{\sigma 2} & = & x_i^2 + b x_i x_j + b^2 x_j^2. \end{array}$$

We have

$$v_{\sigma 1} = v'_{\sigma 1} v''_{\sigma 1}, \ v_{\sigma 2} = v'_{\sigma 2} v''_{\sigma 2}$$

for

$$\begin{array}{rclrcl} v'_{\sigma 1} & = & x_1 - uax_s, & v''_{\sigma 1} & = & x_1 + (1+u)ax_s, \\ v'_{\sigma 2} & = & x_i - ubx_j, & v''_{\sigma 2} & = & x_i + (1+u)bx_j. \end{array}$$

Consider the following ideals:

Set

$$\begin{aligned}
J_{1\sigma}(a,b,u) &= (v_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v'_{\sigma 2}, v''_{\sigma 1}v''_{\sigma 2}),\\
J_{2\sigma}(a,b,u) &= (v_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v'_{\sigma 2}, v''_{\sigma 1}v'_{\sigma 2}),\\
J_{3\sigma}(a,b,u) &= (v_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v'_{\sigma 2}, v''_{\sigma 1}(v'_{\sigma 2}+v''_{\sigma 2})),\\
J_{4\sigma}(a,b,u) &= (v_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v''_{\sigma 2}, v''_{\sigma 1}v'_{\sigma 2}),\\
J_{5\sigma}(a,b,u) &= (v_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v''_{\sigma 2}, v''_{\sigma 1}v''_{\sigma 2}),\\
J_{6\sigma}(a,b,u) &= (v_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v''_{\sigma 2}, v''_{\sigma 1}(v'_{\sigma 2}+v''_{\sigma 2})),\\
J_{7\sigma}(a,b,u) &= (w_{\sigma 1}v'_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v'_{\sigma 2}, w_{\sigma 1}v''_{\sigma 2}),\\
J_{8\sigma}(a,b,u) &= (w_{\sigma 1}v'_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v'_{\sigma 2}, w_{\sigma 1}v'_{\sigma 2}),\\
J_{9\sigma}(a,b,u) &= (w_{\sigma 1}v'_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v'_{\sigma 2}, w_{\sigma 1}(v'_{\sigma 2}+v''_{\sigma 2})),\\
J_{10\sigma}(a,b,u) &= (w_{\sigma 1}v'_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v''_{\sigma 2}, w_{\sigma 1}v'_{\sigma 2}),\\
J_{11\sigma}(a,b,u) &= (w_{\sigma 1}v'_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v''_{\sigma 2}, w_{\sigma 1}v''_{\sigma 2}),\\
J_{12\sigma}(a,b,u) &= (w_{\sigma 1}v'_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v''_{\sigma 2}, w_{\sigma 1}(v'_{\sigma 2}+v''_{\sigma 2})),
\end{aligned}$$

and denote by $\mathcal{J}$ the set of these ideals.

Set:

$$\rho_{1\sigma}(a,b,u) = \begin{pmatrix} 0 & w_{\sigma 1} & -v'_{\sigma 2} & -v''_{\sigma 2} & 0 \\ v'_{\sigma 1} & w_{\sigma 2} & 0 & 0 & -v''_{\sigma 2}v''_{\sigma 1} \\ -v''_{\sigma 2} & 0 & v''_{\sigma 1} & 0 & 0 \\ 0 & 0 & 0 & v'_{\sigma 1} & v_{\sigma 2} \\ 0 & 0 & 0 & -w_{\sigma 2} & w_{\sigma 1}v''_{\sigma 1} \end{pmatrix},$$

$$\omega_{1\sigma}(a,b,u) = \begin{pmatrix} -w_{\sigma 2}v''_{\sigma 1} & w_{\sigma 1}v''_{\sigma 1} & -w_{\sigma 2}v'_{\sigma 2} & 0 & v''_{\sigma 2}v''_{\sigma 1} \\ v_{\sigma 1} & v_{\sigma 2} & v'_{\sigma 1}v'_{\sigma 2} & v''_{\sigma 2}v''_{\sigma 1} & 0 \\ -w_{\sigma 2}v''_{\sigma 2} & w_{\sigma 1}v''_{\sigma 2} & w_{\sigma 1}v'_{\sigma 1} & 0 & (v''_{\sigma 2})^2 \\ 0 & 0 & 0 & w_{\sigma 1}v''_{\sigma 1} & -v_{\sigma 2} \\ 0 & 0 & 0 & w_{\sigma 2} & v'_{\sigma 1} \end{pmatrix},$$

$$\rho_{2\sigma}(a,b,u) = \begin{pmatrix} 0 & w_{\sigma 1} & -v'_{\sigma 2} & 0 & 0 \\ v'_{\sigma 1} & w_{\sigma 2} & 0 & 0 & -v''_{\sigma 1} v'_{\sigma 2} \\ -v''_{\sigma 2} & 0 & v''_{\sigma 1} & -v''_{\sigma 1} & 0 \\ 0 & 0 & 0 & v'_{\sigma 1} & v_{\sigma 2} \\ 0 & 0 & 0 & -w_{\sigma 2} & w_{\sigma 1} v''_{\sigma 1} \end{pmatrix},$$

$$\omega_{2\sigma}(a,b,u) = \begin{pmatrix} -w_{\sigma 2} v''_{\sigma 1} & w_{\sigma 1} v''_{\sigma 1} & -w_{\sigma 2} v'_{\sigma 2} & 0 & v''_{\sigma 1} v'_{\sigma 2} \\ v_{\sigma 1} & v_{\sigma 2} & v'_{\sigma 1} v'_{\sigma 2} & v'_{\sigma 2} v''_{\sigma 1} & 0 \\ -w_{\sigma 2} v''_{\sigma 2} & w_{\sigma 1} v''_{\sigma 2} & w_{\sigma 1} v'_{\sigma 1} & v''_{\sigma 1} w_{\sigma 1} & 0 \\ 0 & 0 & 0 & w_{\sigma 1} v''_{\sigma 1} & -v_{\sigma 2} \\ 0 & 0 & 0 & w_{\sigma 2} & v'_{\sigma 1} \end{pmatrix},$$

$$\rho_{3\sigma}(a,b,u) = \begin{pmatrix} 0 & w_{\sigma 1} & -v'_{\sigma 2} & -v''_{\sigma 2} & 0 \\ v'_{\sigma 1} & w_{\sigma 2} & 0 & 0 & -v''_{\sigma 1}(v'_{\sigma 2} + v''_{\sigma 2}) \\ -v''_{\sigma 2} & 0 & v''_{\sigma 1} & -v''_{\sigma 1} & 0 \\ 0 & 0 & 0 & v'_{\sigma 1} & v_{\sigma 2} \\ 0 & 0 & 0 & -w_{\sigma 2} & w_{\sigma 1} v''_{\sigma 1} \end{pmatrix},$$

$$\omega_{3\sigma}(a,b,u) = \begin{pmatrix} -w_{\sigma 2} v''_{\sigma 1} & w_{\sigma 1} v''_{\sigma 1} & -w_{\sigma 2} v'_{\sigma 2} & 0 & v''_{\sigma 1}(v'_{\sigma 2} + v''_{\sigma 2}) \\ v_{\sigma 1} & v_{\sigma 2} & v'_{\sigma 1} v'_{\sigma 2} & v''_{\sigma 1}(v'_{\sigma 2} + v''_{\sigma 2}) & 0 \\ -w_{\sigma 2} v''_{\sigma 2} & w_{\sigma 1} v''_{\sigma 2} & w_{\sigma 1} v'_{\sigma 1} & v''_{\sigma 1} w_{\sigma 1} & (v''_{\sigma 2})^2 \\ 0 & 0 & 0 & w_{\sigma 1} v''_{\sigma 1} & -v_{\sigma 2} \\ 0 & 0 & 0 & w_{\sigma 2} & v'_{\sigma 1} \end{pmatrix}.$$

Replacing $v'_{\sigma 2}$ by $v''_{\sigma 2}$ and conversely, we get other three pairs of matrices, $\rho_{i\sigma}(a,b,u),\ \omega_{i\sigma}(a,b,u),\ i = 4,5,6$. Next, replacing $w_{\sigma 1}$ by $v''_{\sigma 1}$ and conversely, we get other three pairs of matrices, $\rho_{i\sigma}(a,b,u),\ \omega_{i\sigma}(a,b,u),\ i = 7,8,9$, and, finally, performing the both changes, we get the pairs of matrices $\rho_{i\sigma}(a,b,u),\ \omega_{i\sigma}(a,b,u),\ i = 10,11,12$.

Now let us consider the following ideals:

$$\begin{aligned}
T_{1\sigma}(a,b,u) &= (w_{\sigma 2}v''_{\sigma 1}, v_{\sigma 1}, w_{\sigma 2}v'_{\sigma 2}, x_s v''_{\sigma 1}),\\
T_{2\sigma}(a,b,u) &= (w_{\sigma 2}v''_{\sigma 1}, v_{\sigma 1}, w_{\sigma 2}v'_{\sigma 2}, x_j v''_{\sigma 1}),\\
T_{3\sigma}(a,b,u) &= (w_{\sigma 2}v''_{\sigma 1}, v_{\sigma 1}, w_{\sigma 2}v'_{\sigma 2}, (x_j+x_s) v''_{\sigma 1}),\\
T_{4\sigma}(a,b,u) &= (w_{\sigma 2}v''_{\sigma 1}, v_{\sigma 1}, w_{\sigma 2}v''_{\sigma 2}, x_s v''_{\sigma 1}),\\
T_{5\sigma}(a,b,u) &= (w_{\sigma 2}v''_{\sigma 1}, v_{\sigma 1}, w_{\sigma 2}v''_{\sigma 2}, x_j v''_{\sigma 1}),\\
T_{6\sigma}(a,b,u) &= (w_{\sigma 2}v''_{\sigma 1}, v_{\sigma 1}, w_{\sigma 2}v''_{\sigma 2}, (x_j+x_s) v''_{\sigma 1}),\\
T_{7\sigma}(a,b,u) &= (w_{\sigma 2}w_{\sigma 1}, w_{\sigma 1}v'_{\sigma 1}, w_{\sigma 2}v'_{\sigma 2}, x_s w_{\sigma 1}),\\
T_{8\sigma}(a,b,u) &= (w_{\sigma 2}w_{\sigma 1}, w_{\sigma 1}v'_{\sigma 1}, w_{\sigma 2}v'_{\sigma 2}, x_j w_{\sigma 1}),\\
T_{9\sigma}(a,b,u) &= (w_{\sigma 2}w_{\sigma 1}, w_{\sigma 1}v'_{\sigma 1}, w_{\sigma 2}v'_{\sigma 2}, (x_j+x_s) w_{\sigma 1}),\\
T_{10\sigma}(a,b,u) &= (w_{\sigma 2}w_{\sigma 1}, w_{\sigma 1}v'_{\sigma 1}, w_{\sigma 2}v''_{\sigma 2}, x_s w_{\sigma 1}),\\
T_{11\sigma}(a,b,u) &= (w_{\sigma 2}w_{\sigma 1}, w_{\sigma 1}v'_{\sigma 1}, w_{\sigma 2}v''_{\sigma 2}, x_j w_{\sigma 1}),\\
T_{12\sigma}(a,b,u) &= (w_{\sigma 2}w_{\sigma 1}, w_{\sigma 1}v'_{\sigma 1}, w_{\sigma 2}v''_{\sigma 2}, (x_j+x_s) w_{\sigma 1}),
\end{aligned}$$

We denote by $\mathcal{T}$ the set of these ideals and set:

$$\mu_{1\sigma}(a,b,u) = \begin{pmatrix} 0 & -v'_{\sigma 1} & v'_{\sigma 2} & 0 & -x_s \\ w_{\sigma 1} & w_{\sigma 2} & 0 & x_s & 0 \\ -v''_{\sigma 2} & 0 & v''_{\sigma 1} & 0 & 0 \\ 0 & 0 & 0 & v'_{\sigma 1} & -w_{\sigma 2} \\ 0 & 0 & 0 & v_{\sigma 2} & w_{\sigma 1}v''_{\sigma 1} \end{pmatrix},$$

$$\nu_{1\sigma}(a,b,u) = \begin{pmatrix} w_{\sigma 2}v''_{\sigma 1} & v_{\sigma 1} & w_{\sigma 2}v'_{\sigma 2} & -x_s v''_{\sigma 1} & 0 \\ -v''_{\sigma 1}w_{\sigma 1} & v_{\sigma 2} & -v'_{\sigma 2}w_{\sigma 1} & 0 & -x_s \\ w_{\sigma 2}v''_{\sigma 2} & v'_{\sigma 1}v''_{\sigma 2} & -w_{\sigma 1}v'_{\sigma 1} & -x_s v''_{\sigma 2} & 0 \\ 0 & 0 & 0 & w_{\sigma 1}v''_{\sigma 1} & w_{\sigma 2} \\ 0 & 0 & 0 & -v_{\sigma 2} & v'_{\sigma 1} \end{pmatrix},$$

$$\mu_{2\sigma}(a,b,u) = \begin{pmatrix} 0 & -v'_{\sigma 1} & v'_{\sigma 2} & 0 & -x_j \\ w_{\sigma 1} & w_{\sigma 2} & 0 & x_j & 0 \\ -v''_{\sigma 2} & 0 & v''_{\sigma 1} & 0 & 0 \\ 0 & 0 & 0 & v'_{\sigma 1} & -w_{\sigma 2} \\ 0 & 0 & 0 & v_{\sigma 2} & w_{\sigma 1}v''_{\sigma 1} \end{pmatrix},$$

$$\nu_{2\sigma}(a,b,u) = \begin{pmatrix} w_{\sigma 2}v''_{\sigma 1} & v_{\sigma 1} & w_{\sigma 2}v'_{\sigma 2} & -x_j v''_{\sigma 1} & 0 \\ -v''_{\sigma 1}w_{\sigma 1} & v_{\sigma 2} & -v'_{\sigma 2}w_{\sigma 1} & 0 & -x_j \\ w_{\sigma 2}v''_{\sigma 2} & v'_{\sigma 1}v''_{\sigma 2} & -w_{\sigma 1}v'_{\sigma 1} & -x_j v''_{\sigma 2} & 0 \\ 0 & 0 & 0 & w_{\sigma 1}v''_{\sigma 1} & w_{\sigma 2} \\ 0 & 0 & 0 & -v_{\sigma 2} & v'_{\sigma 1} \end{pmatrix},$$

$$\mu_{3\sigma}(a,b,u) = \begin{pmatrix} 0 & -v'_{\sigma 1} & v'_{\sigma 2} & 0 & -(x_j+x_s) \\ w_{\sigma 1} & w_{\sigma 2} & 0 & x_j+x_s & 0 \\ -v''_{\sigma 2} & 0 & v''_{\sigma 1} & 0 & 0 \\ 0 & 0 & 0 & v'_{\sigma 1} & -w_{\sigma 2} \\ 0 & 0 & 0 & v_{\sigma 2} & w_{\sigma 1}v''_{\sigma 1} \end{pmatrix},$$

$$\nu_{3\sigma}(a,b,u) = \begin{pmatrix} w_{\sigma 2}v''_{\sigma 1} & v_{\sigma 1} & w_{\sigma 2}v'_{\sigma 2} & -(x_j+x_s)v''_{\sigma 1} & 0 \\ -v''_{\sigma 1}w_{\sigma 1} & v_{\sigma 2} & -v'_{\sigma 2}w_{\sigma 1} & 0 & -(x_j+x_s) \\ w_{\sigma 2}v''_{\sigma 2} & v'_{\sigma 1}v''_{\sigma 2} & -w_{\sigma 1}v'_{\sigma 1} & -(x_j+x_s)v''_{\sigma 2} & 0 \\ 0 & 0 & 0 & w_{\sigma 1}v''_{\sigma 1} & w_{\sigma 2} \\ 0 & 0 & 0 & -v_{\sigma 2} & v'_{\sigma 1} \end{pmatrix},$$

Replacing $v''_{\sigma 2}$ by $v'_{\sigma 2}$ and conversely, we get other three pairs of matrices, $\mu_{i\sigma}(a,b,u),\ \nu_{i\sigma}(a,b,u),\ i=4,5,6$. Next, replacing $w_{\sigma 1}$ by $v''_{\sigma 1}$ and conversely, we get other three pairs of matrices, $\mu_{i\sigma}(a,b,u),\ \nu_{i\sigma}(a,b,u),\ i=7,8,9$, and, finally, performing the both changes, we get the pairs of matrices $\mu_{i\sigma}(a,b,u),\ \nu_{i\sigma}(a,b,u),\ i=10,11,12$.

Clearly, the pair of matrices $(\rho_{i\sigma}(a,b,u),\omega_{1\sigma}(a,b,u))$ forms a matrix factorization of $\Omega^2_R(J_{i\sigma}(a,b,u)/(f))$ for $1\le i\le 12$, and the pair $(\mu_{i\sigma}(a,b,u),\nu_{i\sigma}(a,b,u))$ forms a matrix factorization of $\Omega^2_R(T_{i\sigma}(a,b,u)/(f))$ for $1\le i\le 12$.

**Lemma 5.6.** *Let $M$ be a graded non–orientable, rank 2, 5–generated MCM $R$–module, without free direct summands. Then there exists an ideal $J\in\mathcal{J}\cup\mathcal{T}$ such that $f\in J$ and $M\cong\Omega^2\big(J/(f)\big)$ or $M^*\cong\Omega^2\big(J/(f)\big)$, where $M^*$ is the dual of $M$. Conversely, for every $J\in\mathcal{J}\cup\mathcal{T}$, the module $\Omega^2\big(J/(f)\big)$ is a non–orientable, rank two, 5–generated MCM $R$–module without free direct summands.*

*Proof.* The second statement follows easily, as we already have the matrix factorizations above of those ideals. Let $M$ be as above. As in the beginning of Section 4 we see that $M\cong\Omega^2\big(J/(f)\big)$, for $J$ an ideal of $S$ containing $f$, with $\mu(J) = 4$, dim $S/J = 2$, depth $S/J$=1 and $\mu(\Omega^1_S(J)) = 5$. We may also suppose $J = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ with $f \in (\alpha_1, \alpha_2)$, where $\alpha_t$ is necessarily either $w_{\sigma t}$ or $v_{\sigma t}$ for $t = 1, 2$ for some $a, b$ and a certain permutation $\sigma$ as above. Clearly we cannot have, simultaneously, $\alpha_t = w_{\sigma t}$ because then $(\alpha_1, \alpha_2)$ is a prime ideal and one cannot find $\alpha_3, \alpha_4$ zero divisors, as we need. We treat the following cases:

**Case I:** $\alpha_1 = w_{\sigma 1}$

Then we have $\alpha_2 = v_{\sigma 2}$ and $(\alpha_1, \alpha_2)$ is the intersection of the prime ideals $(v'_{\sigma 2}, w_{\sigma 1})$, $(v''_{\sigma 2}, w_{\sigma 1})$. Since $\alpha_3, \alpha_4$ must be zero divisors in $S/(\alpha_3, \alpha_4)$ we have the following possibilities:

(I1) $\alpha_3 = v'_{\sigma 2}\beta$, $\alpha_4 = v'_{\sigma 2}\gamma$, (I2) $\alpha_3 = v''_{\sigma 2}\beta$,$\alpha_4 = v''_{\sigma 2}\gamma$,
(I3) $\alpha_3 = v'_{\sigma 2}\beta$, $\alpha_4 = v''_{\sigma 2}\gamma$, (I4) $\alpha_3 = v''_{\sigma 2}\beta$, $\alpha_4 = v'_{\sigma 2}\gamma$,

for some homogeneous $\beta, \gamma$ from $\mathfrak{m} = (x_1, x_2, x_3, x_4)$. In the first case we see that the relations given by the columns of the following matrix:

$$\begin{pmatrix} v_{\sigma 2} & \alpha_3 & \alpha_4 & 0 & 0 \\ -w_{\sigma 1} & 0 & 0 & \gamma & \beta \\ 0 & -w_{\sigma 1} & 0 & 0 & -v''_{\sigma 2} \\ 0 & 0 & -w_{\sigma 1} & -v''_{\sigma 2} & 0 \end{pmatrix},$$

are elements in $\Omega^1_S(J) \subset S^4$. Clearly these columns are part of the minimal system of generators of $\Omega^1_S(J)$ because $w_{\sigma 1}, v''_{\sigma 2}$ form a regular system in $S$. The subcase (I2) is similar, this contradicts Lemma 5.2.

Suppose now (I3) holds. Then the relations given by the columns of the following matrix

$$\begin{pmatrix} v_{\sigma 2} & v'_{\sigma 2}\beta & v''_{\sigma 2}\beta & 0 & 0 \\ -w_{\sigma 1} & 0 & 0 & \beta & \gamma \\ 0 & -w_{\sigma 1} & 0 & -v''_{\sigma 2} & 0 \\ 0 & 0 & -w_{\sigma 1} & 0 & -v'_{\sigma 2} \end{pmatrix},$$

are part of a minimal set of generators of $\Omega^1_S(J)$ (note that $w_{\sigma 1}, v''_{\sigma 2}, v'_{\sigma 2}$ form a regular system in $S$). Contradiction! Case (I4) is similar.

**Case II:** $\alpha_1 = v_{\sigma 1}$, $\alpha_2 = v_{\sigma 2}$

Since $(\alpha_1, \alpha_2) = (v'_{\sigma 1}, v'_{\sigma 2}) \cap (v'_{\sigma 1}, v''_{\sigma 2}) \cap (v''_{\sigma 1}, v''_{\sigma 2}) \cap (v''_{\sigma 1}, v'_{\sigma 2})$, we see that the zero divisors of $S/(\alpha_1, \alpha_2)$ must be in one of the prime ideals of the above decomposition. Suppose $\alpha_3 \in (v'_{\sigma 1}, v'_{\sigma 2})$. If $\alpha_3 = \beta_1 v'_{\sigma 1} + \beta_2 v'_{\sigma 2}$ then, as in the proof of Case III of Proposition 5.4, we see that there are at least four minimal relations between first three $\alpha$. Then all $\alpha$ have at least five minimal relations. Contradiction! Thus, $\alpha_3$ as well $\alpha_4$ are multiples of one $v'_{\sigma t}, v''_{\sigma t}$. So we have the following possibilities:

| | | | |
|---|---|---|---|
| (II1) | $\alpha_3 = v'_{\sigma 1}\beta,\ \alpha_4 = v'_{\sigma 1}\gamma,$ | (II2) | $\alpha_3 = v''_{\sigma 1}\beta,\ \alpha_4 = v''_{\sigma 1}\gamma,$ |
| (II3) | $\alpha_3 = v'_{\sigma 2}\beta,\ \alpha_4 = v'_{\sigma 2}\gamma,$ | (II4) | $\alpha_3 = v''_{\sigma 2}\beta,\ \alpha_4 = v''_{\sigma 2}\gamma,$ |
| (II5) | $\alpha_3 = v'_{\sigma 1}\beta,\ \alpha_4 = v''_{\sigma 1}\gamma,$ | (II6) | $\alpha_3 = v'_{\sigma 1}\beta,\ \alpha_4 = v''_{\sigma 2}\gamma,$ |
| (II7) | $\alpha_3 = v'_{\sigma 2}\beta, \alpha_4 = v''_{\sigma 1}\gamma,$ | (II8) | $\alpha_3 = v'_{\sigma 2}\beta, \alpha_4 = v''_{\sigma 2}\gamma,$ |
| (II9) | $\alpha_3 = v'_{\sigma 1}\beta, \alpha_4 = v'_{\sigma 2}\gamma,$ | (II10) | $\alpha_3 = v''_{\sigma 1}\beta, \alpha_4 = v''_{\sigma 2}\gamma.$ |

**Subcase:** $\alpha_3 = v'_{\sigma 1}\beta,\ \alpha_4 = v'_{\sigma 1}\gamma,\ (v_{\sigma 2}v''_{\sigma 1}, \gamma) \cong 1,\ (v_{\sigma 2}v''_{\sigma 1}, \beta) \cong 1$

We see that the relations given by the columns of the following matrix

$$\begin{pmatrix} v_{\sigma 2} & \beta & \gamma & 0 & 0 \\ -v_{\sigma 1} & 0 & 0 & \alpha_3 & \alpha_4 \\ 0 & -v''_{\sigma 1} & 0 & -v_{\sigma 2} & 0 \\ 0 & 0 & -v''_{\sigma 1} & 0 & -v_{\sigma 2} \end{pmatrix},$$

are part of a minimal system of generators of $\Omega^1_S(J)$, which must be false. Indeed, it is easy to see that the last four columns are part of a minimal system of generators of $\Omega^1_S(J)$. If the first column belongs to the module generated by the last four, then there exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in S$ such that:

$$\begin{aligned} v_{\sigma 2} &= \lambda_1\beta + \lambda_2\gamma, \\ -v_{\sigma 1} &= \lambda_3 v'_{\sigma 1}\beta + \lambda_4 v''_{\sigma 1}\gamma, \\ 0 &= \lambda_1 v''_{\sigma 1} + \lambda_3 v_{\sigma 2}, \\ 0 &= \lambda_2 v''_{\sigma 1} + \lambda_4 v_{\sigma 2}. \end{aligned}$$

It follows that $v_{\sigma 2} \mid \lambda_1$ and $v_{\sigma 2} \mid \lambda_2$ and so we obtain $1 \in (\beta, \gamma)$. Contradiction! If $(v_{\sigma 2}v''_{\sigma 1}, \beta) \ncong 1$, then we are in the subcase (II5), (II6), .... In the same way we treat (II2), (II3), (II4).

**Subcase:** $\alpha_3 = v'_{\sigma 1}\beta,\ \alpha_4 = v''_{\sigma 1}\gamma$

We see that the relations given by the columns of the following matrix

$$\begin{pmatrix} v_{\sigma 2} & \beta & \gamma & 0 & 0 \\ -v_{\sigma 1} & 0 & 0 & \alpha_3 & \alpha_4 \\ 0 & -v''_{\sigma 1} & 0 & -v_{\sigma 2} & 0 \\ 0 & 0 & -v'_{\sigma 1} & 0 & -v_{\sigma 2} \end{pmatrix},$$

are elements in $\Omega^1_S(J)$. The columns two and three, together with the last two columns divided by $(\beta, v_{\sigma 2})$, respectively $(\gamma, v_{\sigma 2})$, are part of a minimal system of generators. Since $\mu\big(\Omega^1_S(J)\big) = 4$, we see that the first column is a linear combination of the others, as above. Thus, there exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in S$ such that:

$$\begin{aligned} v_{\sigma 2} &= \lambda_1\beta + \lambda_2\gamma, \\ -v_{\sigma 1} &= \lambda_3 v'_{\sigma 1}\beta/(\beta, v_{\sigma 2}) + \lambda_4 v''_{\sigma 1}\gamma/(\gamma, v_{\sigma 2}), \\ 0 &= \lambda_1 v''_{\sigma 1} + \lambda_3 v_{\sigma 2}/(\beta, v_{\sigma 2}), \\ 0 &= \lambda_2 v'_{\sigma 1} + \lambda_4 v_{\sigma 2}/(\gamma, v_{\sigma 2}). \end{aligned}$$

It follows that $v_{\sigma 2}/(\beta, v_{\sigma 2})|\lambda_1$ and $v_{\sigma 2}/(\gamma, v_{\sigma 2})|\lambda_2$ and so we obtain $1 \in (\beta, \gamma)$, which is false, as above, if $(\beta, v_{\sigma 2}) \cong 1$, $(\gamma, v_{\sigma 2}) \cong 1$. Clearly $\beta, \gamma$ cannot be multiples of $v_{\sigma 2}$ because otherwise $J$ is only 3–generated. The analysis of the possibilities $(\beta, v_{\sigma 2}) = v'_{\sigma 2}$ and $(\beta, v_{\sigma 2}) = v''_{\sigma 2}$ will lead to the conclusion that $J \in \mathcal{J}$. In this way one can discuss all the above cases. □

**Theorem 5.7.** *Let*

$$\mathcal{E}_1 = \{\text{Coker}\big(\rho_{i\sigma}(a,b,u)\big),\ \text{Coker}\big(\mu_{i\sigma}(a,b,u)\big)) \mid \ \sigma, a, b, u,\ i = \overline{1,6}\},$$

*$\mathcal{E}_2$ be the set of the duals of the modules from the set $\mathcal{E}_1$, and $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$.*

1. *The set $\mathcal{E}$ contains only indecomposable, graded, non–orientable, 5–generated MCM $R$–modules of rank 2.*
2. *Every indecomposable, graded, non–orientable, 5–generated MCM module over $R$ of rank 2 is isomorphic with one module of $\mathcal{E}$.*
3. *There are 648 isomorphism classes of indecomposable, graded, non–orientable MCM modules over $R$ of rank 2, with five generators.*

*Proof.* (1) For the proof of indecomposability we may proceed as in the last part of the proof of Theorem 4.18. For example, let $N$ be the module $\text{Coker}\big(\rho_{1\sigma}(a,b,u)\big)$ and suppose that it decomposes. Then $\rho_{1\sigma}(a,b,u)$ is equivalent with a direct sum of two matrices: $A_1$, of order three and $A_2$, of order two. Let $B_1$, $B_2$ be the submatrices of $\rho_{1\sigma}(a,b,u)$ given by the first three lines and columns, respectively the last two lines and columns. Certainly $A_1, A_2, B_1, B_2$ define some maximal Cohen-Macaulay modules of rank one that we denote, respectively, by $N_1, N_2, T_1, T_2$, and due to the particular form of $\rho_{1\sigma}(a,b,u)$ we have the following exact sequence

$$0 \to T_1 \to N_1 \oplus N_2 = N \to T_2 \to 0.$$

Since $\rho_{1\sigma}(a,b,u)$ is modulo $x_j$ the sum of $B_1, B_2$, $T_i/x_jT_i \cong N_i/x_jN_i$ for $i = 1,2$. Looking at the description of rank one maximal Cohen-Macaulay modules we can see that $A_i$ is equivalent with $B_i$ modulo $x_j$ only when $A_i$ is equivalent with $B_i$. Thus $T_i \cong N_i$ for $i = 1,2$ and so $N \cong T_1 \oplus T_2$. By [Mi], this happens only if the above exact sequence splits, that is impossible.
(2) It is enough to observe that the matrices

$$\rho_{i\sigma}(a,b,u), a, b, u, \sigma, i = \overline{1,12}$$

and

$$\mu_{i\sigma}(a,b,u), a,b,u,\sigma, i=\overline{1,12},$$

are pairwise equivalent. Indeed, one may show that

$$\rho_{7\sigma}(a,b,u) \sim \rho_{4\sigma}(a,b,u),\ \rho_{8\sigma}(a,b,u) \sim \rho_{5\sigma}(a,b,u),\ \rho_{9\sigma}(a,b,u) \sim \rho_{6\sigma}(a,b,u),$$

and

$$\rho_{10\sigma}(a,b,u) \sim \rho_{1\sigma}(a,b,u),\ \rho_{11\sigma}(a,b,u) \sim \rho_{2\sigma}(a,b,u),\ \rho_{12\sigma}(a,b,u) \sim \rho_{3\sigma}(a,b,u).$$

One may find, in each case, a pair of some permutations matrices $U_i, V_i$ such that

$$U_i\rho_{i\sigma}(a,b,u) = \rho_{(i-3)\sigma}(au,b,u)V_i,\ i=7,8,9,$$

and

$$U_i\rho_{i\sigma}(a,b,u) = \rho_{(i-9)\sigma}(au,b,u)V_i,\ i=10,11,12.$$

In a similar way one may group in pairs the matrices $\mu_{i\sigma}(a,b,u)$.
(3) It is a laborious task to prove the that the modules of the list $\mathcal{E}$ are pairwise non–isomorphic. One can use the following procedure in SINGULAR:

```
LIB"matrix.lib";
option(redSB);
proc isomorph5(matrix X, matrix Y)
{
matrix U[5][5]=u(1..25);
matrix V[5][5]=v(1..25);
matrix C=U*X-Y*V;
ideal I=flatten(C);
ideal J=det(U)-1,det(V)-1;
for (int j=1;j<=size(I);j++)
{
J=J+transpose(coef(I[j],x(1)*x(2)*x(3)*x(4)))[2];
}
ideal K=std(J);
return(K);
}
```

□

**Corollary 5.8.** *Let*

$$\mathcal{F}_1 = \{\mathrm{Coker}\big(\omega_{i\sigma}(a,b,u)\big),\ \mathrm{Coker}\big(\nu_{i\sigma}(a,b,u)\big) \mid \sigma, a, b, u,\ i = \overline{1,6}\},$$

*$\mathcal{F}_2$ be the set of the duals of the modules from the set $\mathcal{F}_1$, and $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$.*

1. *The set $\mathcal{F}$ contains only indecomposable, graded, non–orientable, 5–generated MCM $R$–modules of rank 3.*
2. *Every indecomposable, graded, non–orientable, 5–generated MCM module over $R$ of rank 3 is isomorphic with one module of $\mathcal{F}$.*
3. *There are 648 isomorphism classes of indecomposable, graded, non–orientable MCM modules over $R$ of rank 3, with 5 generators.*

*Proof.* The map $M \mapsto \Omega^1_R(M)$ is a bijection between the 5–generated, indecomposable, graded, MCM $R$–modules of rank 2 and the 5–generated, indecomposable, graded, MCM $R$–modules of rank 3. □

**Remark 5.9.** *For each 2-gen MCM module $M$ (line or conic) there exist two non-isomorphic 3-gen MCM modules $P_1, P_2$ and 3 non-isomorphic extensions for each:*

$$0 \to P_i \to E_{ij} \to M \to 0,$$

$i = 1, 2,\ j = 1, 2, 3$. *So there are* $6 \times 54$ *MCM of type* $E_{ij}$. *Taking the duals we get another* $6 \times 54$ *MCM. Thus all are* $648 = 12 \times 54$.

**Lemma 5.10.** *There exist no graded, indecomposable, non–orientable, rank 2, 6–generated MCM modules.*

*Proof.* Suppose there exist such MCM module $M$. Then $M \cong \Omega^2_R\big(J/(f)\big)$ for a certain 5–generated ideal $J = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ of $S$ as hinted at in the first part of Section 4. Then any four elements from the $\alpha_t$ must generate an ideal $J''$ in $\mathcal{J} \cup \mathcal{T}$ because, otherwise, $\mu(\Omega^1_S\big(J''/(f)\big) > 4$ and so, obviously $\mu(\Omega^1_S(J/(f)) > 5$. So we may suppose $\alpha_t = v_{\sigma t}$ for $t = 1, 2$ and after some permutations $\alpha_3 = v'_{\sigma 1} v''_{\sigma 2}$. Set $J' = (\alpha_1, \alpha_2, \alpha_3)$. If $(J', \alpha_4) \in \mathcal{J}$. and $(J', \alpha_5) \in \mathcal{J}$ then there are 4 minimal relations of $(J', \alpha_4)$ and 4 minimal relations of $(J', \alpha_5)$ over $S$, among them at least 6 minimal relations of $J$ over $S$ which contradicts Lemma 5.2. In the same way we treat the other cases. □

**Corollary 5.11.** *There exist no indecomposable, graded, non–orientable, rank 4, 6–generated MCM modules.*

# References


[AC] E. Arrondo, L. Costa, Vector bundles on Fano 3-folds without intermediate cohomology, Comm. Algebra 28, no. 8, pp. 3899–3911, 2000.

[AF] E. Arrondo, D. Faenzzi, Vector bundles with no intermediate cohomology on Fano threefolds of type $V_{22}$, Pacific J. Math. , to appear.

[At] M. F. Atiyah, Vector bundles over an elliptic curve, Proc. London Math. Soc. **7**(3), pp. 415–452 (1957).

[AR] M. Auslander, I. Reiten, Almost split sequences for $\mathbb{Z}$–graded rings, Singularities, representation of algebras, and vector bundles,(Lambrtecht, 1985), Lecture Notes in Math. , vol. 1273, Springer, Berlin, pp. 232–243 (1987).

[Ba1] C.Baciu, Maximal Cohen-Macaulay modules over a non–isolated hypersurface singularity, Preprint Nr. 337, Technische Universität Kaiserslautern, 2003.

[Ba2] C.Baciu, Maximal Cohen-Macaulay modules and stable vector bundles, Proceedings of the NATO Advanced Research Workshop on Computational Commutative and Non–Commutative Algebraic Geometry, Chisinau, Moldova, 6–11 June 2004.

[BEPP] C. Baciu, V. Ene, G. Pfister, D. Popescu, Rank 2 Cohen–Macaulay modules over singularities of type $x_1^3 + x_2^3 + x_3^3 + x_4^3$, to appear in Journal of Algebra

[Bo1] N. Bourbaki, Algebre, Hermann, Paris, 1970-1980, Chapter IX.

[Bo2] N. Bourbaki, Commutative Algebra, Addison–Wesley, Reading, MA, 1972, Chapter VII.

[B] I. Bourban, Abgeleitete Kategorien und Matrixprobleme, Ph. D. , University of Kaiserslautern (2003).

[Br] W. Bruns, “Jede” endliche freie Auflösung ist freie Auflösung eines von drei Elementen erzeugten Ideals, J. of Algebra 39, pp. 429–439 (1976).

[BH] W. Bruns, J. Herzog, Cohen–Macaulay Rings, Cambridge University Press, Cambridge, 1993.

[BGS] R. –O. Buchweitz, G. –M. Greuel, F. –O. ,Schreyer, Cohen-Macaulay modules on hypersurface singularities II, Invent. Math. , no. 1, pp. 165–182, 1987.

[CM1] L. Chiantini, C. Madonna, ACM bundles on a general quintic threefold, Matematiche(Catania) 55, no. 2, pp. 239–258, 2002, Dedicated to Silvio Greco on the occasion of his 60th birthday (Catania, 2001).

[CM2] L. Chiantini, C. Madonna, ACM bundles on a general quintic threefold, Preprint, http:\\arxiv.org\abs\math.AG\0409029, 2004.

[CM3] L. Chiantini, C. Madonna, A splitting criterion for rank two bundles on a general sextic threefold, Internat. J. Math. 15,no. 4, pp. 341–359, 2004.

[DG] Yu. Drozd, G. –M. Greuel, Tame and wild projective curves and classification of vector bundles, Journal of Algebra, 246, pp. 1–54 (2001).

[Ei1] D. Eisenbud, Homological Algebra with an application to group representations. Trans. Amer. Math. Soc. **260**(1980), pp. 35–64.

[Ei2] D. Eisenbud, On the resiliency of determinantal ideals, Advanced Stud. in Pure Math. 11, Commutative Algebra and Combinatorics, pp. 29-38 (1987).

[EH] D. Eisenbud, J. Herzog, The classification of homogeneous Cohen–Macaulay rings of finite representation type, Math. Ann. 280, no. 2, pp. 347–352, 1988.

[EP] V. Ene, D. Popescu, Rank one Maximal Cohen-Macaulay modules over singularities of type $Y_1^3 + Y_2^3 + Y_3^3 + Y_4^3$, in: Commutative Algebra, Singularities and Computer Algebra, J. Herzog and V. Vuletescu (eds.), Kluwer Academic Publishers, (2003), pp. 141–157.

[F1] D. Faenzi, Bundles over the Fano threefold $V_5$, Comm. Algebra, to appear.

[F2] D. Faenzi, Rank 2 arithmetically Cohen–Macaulay bundles on a nonsingular cubic surface, Available at `http://www.math.unifi.it/~faenzi/`2005.

[GK] G.-M. Greuel, H. Knörrer, Einfache Kurvensingukaritäten und torsionfreie Moduln, Math. Ann. 270, no. 3, pp. 417–425, 1985.

[GPS] G.-M. Greuel, G. Pfister and H. Schönemann, SINGULAR 2.0. A Computer Algebra System for Polynomial Computations. Centre for Computer Algebra, University of Kaiserslautern, (2001), \http://www.singular.uni-kl.de

[Ha] R.Hartshorne, Algebraic Geometry, Springer Verlag, New York (1977).

[HK] J. Herzog, M. Kuhl, Maximal Cohen- Macaulay modules over Gorenstein rings and Bourbaki sequences, in: Commutative Algebra and Combinatorics, Adv. Stud. Pure Math., Vol. 11, 1987, pp. 65–92.

[IK] A. Iarrobino, V. Kanev, Power Sums, Gorenstein Algebras, and Determinantal Loci, Springer Lect. Notes in Math. 1721, Berlin, 1999.

[TJP] T.de Jong, G.Pfister, Local analytic geometry, Advanced Lectures in Mathematics (2000).

[JP] T. Jozefiak, P. Pragacz, Ideals generated by Pfaffians, J. of Algebra, **61**(1979), pp. 189–198.

[KL] S. L. Kleiman, J. Landolfi, Geometry and deformation of Special Schubert varieties, Proceedings of the 5th Nordic Summer–School in Mathematics, Oslo, August 5–25, pp. 97–124 (1970).

[LPP] R. Laza, G. Pfister, D. Popescu, Maximal Cohen–Macaulay modules over the cone of an elliptic curve, J. of Algebra, **253**(2002), pp. 209–236.

[Ma1] C. Madonna, Rank-two vector bundles on a general quartic hypersurface in $\mathbb{P}^4$, Rev. Mat. Complut.13, n0. 2, pp. 287–301, 2000.

[Ma2] C. Madonna, ACM vector bundles on prime Fano threefolds and complete intersection Calabi–Yau threefolds, Rev. Roumaine Math. Pures Appl. 47, no. 2, pp. 211–222, 2003.

[Mi] J. Migliore, Introduction to Liaison Theory and Deficiency Modules, Progress in Math., 165, Birkhäuser, Boston, 1998.

[S1] Ø. Solberg, A graded ring of finite Cohen–Macaulay type, Comm. Algebra 16, no. 10, pp. 2121–2124, 1988.

[S2] Ø. Solberg, Hypersurface singularities of finite Cohen–Macaulay type, Proc. London Math. Soc. (3), 58, no. 2, pp. 258–280, 1989.

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