Technische Universität Kaiserslautern

SCHRIFTEN ZUR

FUNKTIONALANALYSIS UND GEOMATHEMATIK

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Wavelet Deformation Analysis for Spherical Bodies

Bericht 9 – Februar 2004

FACHBEREICH MATHEMATIK

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Abstract

In this paper we introduce a multiscale technique for the analysis of deformation phenomena of the Earth. Classically, the basis functions under use are globally defined and show polynomial character. In consequence, only a global analysis of deformations is possible such that, for example, the water load of an artificial reservoir is hardly to model in that way. Up till now, the alternative to realize a local analysis can only be established by assuming the investigated region to be flat.
In what follows we propose a local analysis based on tools (Navier scaling functions and wavelets) taking the (spherical) surface of the Earth into account. Our approach, in particular, enables us to perform a zooming–in procedure. In fact, the concept of Navier wavelets is formulated in such a way that subregions with larger or smaller data density can accordingly be modelled with a higher or lower resolution of the model, respectively.

AMS classification: 33F05, 42C40, 74B05, 86A20

Keywords: spherical multiscale deformation analysis, Navier scaling functions and wavelets, Dirichlet's and Neumann's boundary–value problem.

1 Introduction

Consider an elastic body subjected to an external force thereby identifying the body with the region $\Sigma_{\rm int}$ it occupies. A deformation of $\Sigma_{\rm int}$ is a smooth homeomorphism z of $\Sigma_{\rm int}$ onto a region $z(\Sigma_{\rm int})$ with $\det(\nabla \otimes z) > 0$. The point $z(x)$ is the place occupied by the material point x in the deformation, while $u(x) = z(x) - x,\ x \in \Sigma_{\rm int}$, is the displacement of x. Starting point of the linear theory is the Jacobian $\mathbf{j} = \nabla \otimes u$. The strain tensor $\mathbf{e}$ is defined as the symmetric part of the Jacobian, so that its antisymmetric part has been neglected: $\mathbf{e} = \frac{1}{2}((\nabla \otimes u) + (\nabla \otimes u)^{\rm T})$. The antisymmetric part is used to define the (infinitesimal) rotation tensor $\mathbf{d}$, as $\mathbf{d} = \frac{1}{2}((\nabla \otimes u) - (\nabla \otimes u)^{\rm T})$,while $d = \frac{1}{2}$ curl u is the (infinitesimal) rotation vector. Thus the obvious relation $\mathbf{j} = \mathbf{e} + \mathbf{d}$ is valid and d is the axial vector of $\mathbf{d}$, i.e. for any $a \in \mathbb{R}^3$ we have $\mathbf{d}a = d \wedge a$. While $\mathbf{d}$ describes a rigid displacement field, $\mathbf{e}$ is responsible for the non–rigid displacements. According to Kirchhoff's Theorem, if two displacement fields u and u' correspond to the same strain field, then $u = u' + w$, where w is a rigid displacement field. One calls trace $(\mathbf{e}) = \nabla \cdot u$ the dilatation. (Note that the infinitesimal volume change is zero if and only if trace $(\mathbf{e}) = 0$ on $\Sigma_{\rm int}$). Dilatations, therefore, are determined by the diagonal elements of $\mathbf{e}$, the remaining matrix elements of $\mathbf{e}$ prescribe torsions. Every displacement field can be decomposed into a pure torsion (i.e. $\nabla \cdot u = 0$) and a pure dilatation (i.e. $\nabla \wedge u = 0$).

An elastic body in a strained configuration performs by definition a tendency to recover its original form: this tendency is materialized by a field of forces on each part of the body by the other parts. This field of internal forces called elastic stress, is due to the interaction of the molecules of the body which have been removed from their relative position of equilibrium and to recover it, following the principle of action and interaction. The potential of the molecules forces is known to have a rather short range so that each molecule has a significant interaction only with the closest molecules in the body. More explicitly, if x is a point of an (oriented) surface element in $\Sigma_{\rm int}$ with unit normal ν, then the stress vector $s_\nu(x) = T_\nu(u)(x)$ is the force per unit area at x exerted by the portion of $\Sigma_{\rm int}$ on the side of the surface element toward $\nu(x)$ points on the portion of $\Sigma_{\rm int}$ on the other side. It is a simple matter to establish that for time-independent behaviour and in the absence of body stress fields there exists a symmetric tensor field $\mathbf{s}$, called the stress tensor field, such that $s_\nu = \mathbf{s}\nu$ for each unit vector ν and $\nabla \cdot (\mathbf{s}a) = 0$ for each fixed $a \in \mathbb{R}^3$ (for more details see e.g. [18]). The same consideration also applies when x is located on the boundary $\Sigma = \partial\Sigma_{\rm int}$ and ν is the outward unit normal to $\Sigma = \partial\Sigma_{\rm int}$ at x. In this case $s_\nu(x) = T_\nu(u)(x)$ is called the surface traction at x.

Hooke's law relates the stress to strain, i.e. linear elasticity of the body implies for each $x \in \Sigma_{\rm int}$ that there exists a linear transformation $\mathbf{C}$ from the space of all tensors into the space of all symmetric tensors such that $\mathbf{s} = \mathbf{C}\mathbf{e}$. $\mathbf{C}$ is called the elastic field. If the material is isotropic, $\mathbf{C}$ is given by $\mathbf{C}\mathbf{e} =$

$2\tilde{\mu}\mathbf{e}+\tilde{\lambda}(\text{trace }\mathbf{e})\mathbf{i}$ (see e.g. [18]), where $\mathbf{i}$ stands for the 3×3 identity matrix. The scalars $\tilde{\lambda}$ and $\tilde{\mu}$ (used in this chapter) are called the Lamé moduli. When the material is homogeneous $\tilde{\lambda}$ and $\tilde{\mu}$ are constants. Typical requirements imposed on $\tilde{\lambda}, \tilde{\mu}$ are $\tilde{\mu} > 0, 3\tilde{\lambda} + 2\tilde{\mu} > 0$ (cf. [23]).

In what follows we are concerned with elastic deformation for homogeneous isotropic material corresponding to a body with spherical boundary. First we introduce some basic settings in spherical nomenclature. Then we recapitulate the basic ingredients of the theory of vector spherical harmonics. It follows the representations of the solutions of Dirichlet's and Neumann's (Navier) boundary–value problem of linear elasticity corresponding to vector spherical harmonics as boundary values. Finally, wavelets are introduced to solve the boundary value problems for the Navier equations in a multiscale framework.

2 Spherical Nomenclature

Let us use $x, y, \ldots$ to represent the elements of Euclidean space $\mathbb{R}^3$. For all $x \in \mathbb{R}^3$, $x = (x_1, x_2, x_3)^{\mathrm{T}}$, different from the origin, we have

$$x = r\xi, \quad r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}, \tag{1}$$

where $\xi = (\xi_1, \xi_2, \xi_3)^{\mathrm{T}}$ is the uniquely determined directional unit vector of $x \in \mathbb{R}^3$. The *unit sphere* in $\mathbb{R}^3$ will be denoted by Ω:

$$\Omega = \left\{ \xi \in \mathbb{R}^3 \middle| |\xi| = 1 \right\} \quad . \tag{2}$$

If the vectors $\varepsilon^1, \varepsilon^2, \varepsilon^3$ form the canonical orthonormal basis in $\mathbb{R}^3$ we may represent the points $x \in \mathbb{R}^3$ in cartesian coordinates $x_i = x \cdot \varepsilon^i$; $i = 1, 2, 3$; by

$$x = \sum_{i=1}^{3} (x \cdot \varepsilon^i)\varepsilon^i \; = \; \sum_{i=1}^{3} x_i \varepsilon^i \quad . \tag{3}$$

Inner, vector, and *dyadic (tensor) product* of two elements $x, y \in \mathbb{R}^3$, respectively, are defined by

$$\begin{aligned} x \cdot y &= x^{\mathrm{T}} y = \sum_{i=1}^{3} x_i y_i, && (4) \\ x \wedge y &= (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)^{\mathrm{T}}, && (5) \\ x \otimes y &= x y^{\mathrm{T}} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 \\ x_3 y_1 & x_3 y_2 & x_3 y_3 \end{pmatrix} \quad . && (6) \end{aligned}$$

We set Ω_{int} for the "inner space" of Ω, while Ω_{ext} denotes the "outer space" of Ω. More explicitly,

$$\Omega_{\text{int}} = \{x \in \mathbb{R}^3 |\ |x| < 1\}, \tag{7}$$

$$\Omega_{\text{ext}} = \{x \in \mathbb{R}^3 |\ |x| > 1\}. \tag{8}$$

It is well-known that the total surface $\|\Omega\|$ of Ω is equal to 4π:

$$\|\Omega\| = \int_\Omega d\omega(\xi) = 4\pi. \tag{9}$$

We may represent the points $\xi \in \Omega$ in *polar coordinates* as follows:

$$\begin{aligned} \xi \quad &= \quad t\varepsilon^3 + \sqrt{1-t^2}(\cos\varphi\varepsilon^1 + \sin\varphi\varepsilon^2), \\ &\quad\ -1 \le t \le 1\ ,\ 0 \le \varphi < 2\pi\ ,\ t = \cos\vartheta\ , \end{aligned} \tag{10}$$

($\vartheta \in [0,\pi]$: latitude, φ: longitude, t: polar distance):

$$\xi = (\sin\vartheta\ \cos\varphi,\ \sin\vartheta\ \sin\varphi,\ \cos\vartheta)^{\mathrm{T}}\ . \tag{11}$$

2.1 Spherical Functions

The set of classes of almost everywhere identical (scalar) functions $F : \Omega \to \mathbb{R}$ which are measurable and for which

$$\|F\|_{\mathrm{L}^p(\Omega)} = \left(\int_\Omega |F(\xi)|^p\ d\omega(\xi)\right)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty, \tag{12}$$

is known as $\mathrm{L}^p(\Omega)$. Clearly, $\mathrm{L}^p(\Omega) \subset \mathrm{L}^q(\Omega)$ for $1 \le q \le p$. A function $F : \Omega \to \mathbb{R}$ possessing k continuous derivatives on the unit sphere Ω is said to be of class $\mathrm{C}^{(k)}$ $(0 \le k \le \infty)$. $\mathrm{C}(\Omega)$ $(= \mathrm{C}^{(0)}(\Omega))$ is the class of continuous scalar valued functions on Ω. $\mathrm{C}(\Omega)$ is a complete normed space endowed with

$$\|F\|_{\mathrm{C}(\Omega)} = \sup_{\xi\in\Omega} |F(\xi)|. \tag{13}$$

In connection with $(\cdot,\cdot)_{\mathrm{L}^2(\Omega)}$, $\mathrm{C}(\Omega)$ is a pre-Hilbert space. For each $F \in \mathrm{C}(\Omega)$ we have the *norm estimate*

$$\|F\|_{\mathrm{L}^2(\Omega)} \le \sqrt{4\pi}\ \|F\|_{\mathrm{C}(\Omega)}. \tag{14}$$

$\mathrm{L}^2(\Omega)$ is a Hilbert space with respect to the inner product $(\cdot,\cdot)_{\mathrm{L}^2(\Omega)}$ defined by

$$(F,G)_{\mathrm{L}^2(\Omega)} = \int_\Omega F(\xi)G(\xi)\ d\omega(\xi); \quad F,G \in \mathrm{L}^2(\Omega).$$

$\mathrm{L}^2(\Omega)$ is the completion of $\mathrm{C}(\Omega)$ with respect to the norm $\|\cdot\|_{\mathrm{L}^2(\Omega)}$, i.e.

$$\mathrm{L}^2(\Omega) = \overline{\mathrm{C}(\Omega)}^{\|\cdot\|_{\mathrm{L}^2(\Omega)}}. \tag{15}$$

$l^2(\Omega)$ denotes the space consisting of all square-integrable vector fields on Ω. In connection with the inner product

$$(f, g)_{l^2(\Omega)} = \int_\Omega f(\xi) \cdot g(\xi) \, d\omega(\xi); \quad f, g \in l^2(\Omega); \tag{16}$$

$l^2(\Omega)$ is a Hilbert space. The space $c^{(p)}(\Omega)$, $0 \leq p \leq \infty$, consists of all p-times continuously differentiable vector fields on Ω. For brevity, we write $c(\Omega) = c^{(0)}(\Omega)$. The space $c(\Omega)$ is complete with respect to the norm

$$\|f\|_{c(\Omega)} = \sup_{\xi \in \Omega} |f(\xi)|, \quad f \in c(\Omega). \tag{17}$$

Furthermore,

$$\overline{c(\Omega)} = l^2(\Omega), \tag{18}$$

where the completion is understood with respect to the $l^2(\Omega)$-topology. In analogy to (14) we have for all $f \in c(\Omega)$ the *norm estimate*

$$\|f\|_{l^2(\Omega)} \leq \sqrt{4\pi} \; \|f\|_{c(\Omega)}. \tag{19}$$

In order to separate vector fields into their tangential and normal parts we introduce the projection operators p_{nor} and p_{tan} by

$$p_{\mathrm{nor}} f(\xi) = (f(\xi) \cdot \xi) \, \xi; \quad \xi \in \Omega, f \in c(\Omega); \tag{20}$$

$$p_{\mathrm{tan}} f(\xi) = f(\xi) - p_{\mathrm{nor}} f(\xi); \quad \xi \in \Omega, f \in c(\Omega). \tag{21}$$

We extend their definition in canonical way to vector fields in $l^2(\Omega)$. Furthermore, we define

$$l^2_{\mathrm{nor}}(\Omega) = \{f \in l^2(\Omega) | f = p_{\mathrm{nor}} f\}, \tag{22}$$

$$l^2_{\mathrm{tan}}(\Omega) = \{f \in l^2(\Omega) | f = p_{\mathrm{tan}} f\}. \tag{23}$$

We say $f \in l^2(\Omega)$ is *normal* if $f = p_{\mathrm{nor}} f$ and *tangential* if $f = p_{\mathrm{tan}} f$. Clearly, we have the orthogonal decomposition

$$l^2(\Omega) = l^2_{\mathrm{nor}}(\Omega) \oplus l^2_{\mathrm{tan}}(\Omega). \tag{24}$$

The spaces $c^{(p)}_{\mathrm{nor}}(\Omega)$ and $c^{(p)}_{\mathrm{tan}}(\Omega), 0 \leq p \leq \infty$, are defined in the same fashion.

The projection of the *identity tensor*

$$\mathbf{i} = \sum_{i=1}^{3} \varepsilon^i \otimes \varepsilon^i \tag{25}$$

onto the tangential components at a point $\xi \in \Omega$ defines the *surface identity tensor field* $\mathbf{i}_{\mathrm{tan}}$ given by

$$\mathbf{i}_{\mathrm{tan}}(\xi) = \mathbf{i} - \xi \otimes \xi, \quad \xi \in \Omega. \tag{26}$$

Moreover, we define the *surface rotation (tensor) field* $\mathbf{j}_{\tan}$ by

$$\mathbf{j}_{\tan}(\xi) = \sum_{i=1}^{3} (\xi \wedge \varepsilon^i) \otimes \varepsilon^i, \qquad \xi \in \Omega. \tag{27}$$

The Helmholtz decomposition theorem (see [1], [2], [3], [11], [13]) is based on two operators, viz. the *surface gradient* $\nabla^*$and the *surface curl gradient* L^*. The surface gradient ∇^* contains the tangential derivatives of the gradient ∇ as follows

$$\nabla = \xi \frac{\partial}{\partial r} + \frac{1}{r} \nabla^*, \tag{28}$$

while the surface curl gradient L^* is defined by

$$\mathrm{L}^*_\xi F(\xi) = \xi \wedge \nabla^*_\xi F(\xi), \qquad \xi \in \Omega, \tag{29}$$

$F \in \mathrm{C}^{(1)}(\Omega)$. According to its definition (29), $\mathrm{L}^* F$ is a tangential vector field perpendicular to $\nabla^* F$, i.e.:

$$\nabla^*_\xi F(\xi) \cdot \mathrm{L}^*_\xi F(\xi) = 0, \quad \xi \in \Omega \ . \tag{30}$$

We are now prepared to formulate the vectorial variant of the *Helmholtz decomposition theorem* (see [12], [13] for the proof).

Theorem 2.1 *Let f be of class $\mathrm{c}^{(1)}(\Omega)$. Then there exist uniquely determined functions $F^{(1)} \in \mathrm{C}^{(1)}(\Omega)$ and $F^{(2)}$, $F^{(3)} \in \mathrm{C}^{(2)}(\Omega)$ satisfying*

$$\int_\Omega F^{(i)}(\xi) \ d\omega(\xi) = 0, \qquad i = 2, 3, \tag{31}$$

such that

$$f(\xi) = \xi F^{(1)}(\xi) + \nabla^*_\xi F^{(2)}(\xi) + \mathrm{L}^*_\xi F^{(3)}(\xi), \quad \xi \in \Omega \ . \tag{32}$$

The functions $F^{(i)}$, $i = 1, 2, 3$, are given by

$$F^{(1)}(\xi) = \xi \cdot f(\xi), \tag{33}$$

$$F^{(2)}(\xi) = -\int_\Omega G(\Delta^*; \xi, \eta) \, \nabla^*_\eta \cdot f(\eta) \ d\omega(\eta), \tag{34}$$

$$F^{(3)}(\xi) = -\int_\Omega G(\Delta^*; \xi, \eta) \, \mathrm{L}^*_\eta \cdot f(\eta) \ d\omega(\eta), \tag{35}$$

$\xi \in \Omega$, where

$$G(\Delta^*; \xi, \eta) = \frac{1}{4} \ln(1 - \xi \cdot \eta) + \frac{1}{4\pi} - \frac{1}{4\pi} \ln 2, \tag{36}$$

$\xi, \eta \in \Omega$, $-1 \le \xi \cdot \eta < 1$, is the Green function with respect to the Beltrami operator Δ^* (cf. [7], [8], [9]).

The presence of the *Beltrami operator* Δ^* in the Helmholtz decomposition theorem is not surprising because of the identities

$$\Delta^* = \mathrm{L}^* \cdot \mathrm{L}^* = \mathrm{curl}^* \mathrm{L}^*, \tag{37}$$
$$\Delta^* = \nabla^* \cdot \nabla^* = \mathrm{div}^* \nabla^*, \tag{38}$$

where $\nabla^* \cdot = \mathrm{div}^*$ and $\mathrm{L}^* \cdot = \mathrm{curl}^*$ respectively, denote the *surface divergence* and the *surface curl* given by

$$\nabla^*_\xi \cdot f(\xi) = \sum_{i=1}^{3} \nabla^*_\xi f_i(\xi) \cdot \varepsilon^i \tag{39}$$

and

$$\mathrm{L}^*_\xi \cdot f(\xi) = \sum_{i=1}^{3} \mathrm{L}^*_\xi f_i(\xi) \cdot \varepsilon^i \quad . \tag{40}$$

Note that the surface curl as defined by (40)

$$\xi \mapsto \mathrm{L}^*_\xi \cdot f(\xi) = \mathrm{curl}^*_\xi f(\xi) = \mathrm{div}^*_\xi (f(\xi) \wedge \xi) = \nabla^*_\xi \cdot (f(\xi) \wedge \xi), \quad \xi \in \Omega, \tag{41}$$

represents a scalar–valued function on the unit sphere Ω in $\mathbb{R}^3$.

For a given function $F \in \mathrm{C}^{(1)}(\Omega)$, we set in accordance with the notation presented in [11] and [13]

$$o^{(1)}_\xi F(\xi) = \xi F(\xi), \tag{42}$$
$$o^{(2)}_\xi F(\xi) = \nabla^*_\xi F(\xi), \tag{43}$$
$$o^{(3)}_\xi F(\xi) = \mathrm{L}^*_\xi F(\xi), \tag{44}$$

$\xi \in \Omega$. The operators $o^{(i)}$, $i = 1, 2, 3$, will be transferred to other classes of functions in canonical manner. The triple

$$\left(o^{(1)}_\xi F(\xi), o^{(2)}_\xi F(\xi), o^{(3)}_\xi F(\xi) \right)^{\mathrm{T}}, \quad \xi \in \Omega, \tag{45}$$

supplies us with a system of three orthogonal vectors in each point ξ on the unit sphere Ω. More explicitly,

$$o^{(i)}_\xi F(\xi) \cdot o^{(j)}_\xi F(\xi) = 0, \qquad \xi \in \Omega, \tag{46}$$

for all $i, j \in \{1, 2, 3\}$, $i \neq j$.

The adjoint operators $O^{(i)}$ to $o^{(i)}$, $i = 1, 2, 3$ satisfy the identites

$$\left(o^{(i)} G, f \right)_{\mathrm{l}^2(\Omega)} = \left(G, O^{(i)} f \right)_{\mathrm{L}^2(\Omega)} \quad . \tag{47}$$

Explicitly written out the operators $O^{(i)}$, $i = 1, 2, 3$, read as follows

$$O^{(1)}_\xi f(\xi) = \xi \cdot f(\xi), \tag{48}$$

$$O^{(2)}_\xi f(\xi) = -\mathrm{div}^*_\xi f(\xi) = -\nabla^*_\xi \cdot f(\xi), \tag{49}$$

$$O^{(3)}_\xi f(\xi) = -\mathrm{curl}^*_\xi f(\xi) = -\mathrm{L}^*_\xi \cdot f(\xi), \tag{50}$$

$\xi \in \Omega$, where f is supposed to be an element of class $\mathrm{c}^{(1)}(\Omega)$. Simple calculations show that for $F \in \mathrm{C}^{(2)}(\Omega)$

$$O^{(i)}_\xi o^{(j)}_\xi F(\xi) = 0, \qquad \xi \in \Omega, \tag{51}$$

for $i \neq j; i, j \in \{1, 2, 3\}$, and

$$O^{(i)}_\xi o^{(i)}_\xi F(\xi) = \begin{cases} F(\xi) & \text{if} \quad i = 1 \\ -\Delta^*_\xi F(\xi) & \text{if} \quad i = 2, 3 \end{cases}, \tag{52}$$

$\xi \in \Omega$.

2.2 Spherical Harmonics

Scalar spherical harmonics are defined as the restrictions of homogeneous polynomials that satisfy the Laplace equation (see [4], [8], [29], [30]). To be specific, suppose that $H_n : \mathbb{R}^3 \to \mathbb{R}$ is a homogeneous polynomial of degree n such that $\Delta_x H_n(x) = 0$ for all $x \in \mathbb{R}^3$, then the restriction $Y_n = H_n|\Omega$ is called a *spherical harmonic of degree* n. The space of all spherical harmonics of degree n is denoted by $\mathrm{Harm}_n(\Omega)$. This space is of dimension $2n + 1$, i.e.: $d(\mathrm{Harm}_n(\Omega)) = 2n + 1$. Spherical harmonics of different degrees are orthogonal in the sense of the $\mathrm{L}^2(\Omega)$–inner product

$$(Y_n, Y_m)_{\mathrm{L}^2(\Omega)} = \int_\Omega Y_n(\xi) Y_m(\xi) \, d\omega(\xi) = 0, \qquad n \neq m. \tag{53}$$

Throughout this work, the capital letter Y followed by double indices, for example, $Y_{n,k}$, denotes a member of an $\mathrm{L}^2(\Omega)$–orthonormal system $\{Y_{n,1}, \ldots, Y_{n,2n+1}\} \subset \mathrm{Harm}_n(\Omega), n \in \mathbb{N}_0$.

The *addition theorem* relates the spherical harmonics of degree n to the (univariate) Legendre polynomial in the following way:

$$\sum_{k=1}^{2n+1} Y_{n,k}(\xi) Y_{n,k}(\eta) = \frac{2n+1}{4\pi} P_n(\xi \cdot \eta); \qquad \xi, \eta \in \Omega; \tag{54}$$

where $\{Y_{n,k}\}_{k=1,\ldots,2n+1}$ is an $\mathrm{L}^2(\Omega)$–orthonormal system in $\mathrm{Harm}_n(\Omega)$ and $P_n : [-1, +1] \to [-1, +1]$ is the Legendre polynomial of degree n. It should be remarked that the addition theorem holds for all $\mathrm{L}^2(\Omega)$–orthonormal systems

$Y_{n,1},\ldots,Y_{n,2n+1}$ (in particular for the usually in the geosciences used system (see e.g. [19], [20]) in terms of associated Legendre functions).

For every $Y_n \in \mathrm{Harm}_n(\Omega)$

$$\frac{2n+1}{4\pi}\int_\Omega P_n(\xi\cdot\eta)Y_n(\eta)\,d\omega(\eta) = Y_n(\xi), \qquad \xi\in\Omega. \tag{55}$$

The connection between the orthogonal invariance of the sphere and the addition theorem is established by the *Funk–Hecke formula*

$$\int_\Omega H(\xi\cdot\eta)Y_n(\eta)\,d\omega(\eta) = H^{\wedge_{\mathrm{L}^2[-1,+1]}}(n)Y_n(\xi), \quad \xi\in\Omega, \tag{56}$$

$H \in \mathrm{L}^1[-1,+1]$, $Y_n \in \mathrm{Harm}_n(\Omega)$, where the 'Legendre transform' of $H \in \mathrm{L}^1[-1,+1]$ is given by

$$H^{\wedge_{\mathrm{L}^2[-1,+1]}}(n) = (H,P_n)_{\mathrm{L}^2[-1,+1]} = 2\pi\int_{-1}^{+1} H(t)P_n(t)\,dt. \tag{57}$$

The *Fourier transform* $F\mapsto (FT)(F)$, $F\in\mathrm{L}^1(\Omega)$, is defined by

$$(FT)(F)(n,k) = F^{\wedge_{\mathrm{L}^2(\Omega)}}(n,k) = \int_\Omega F(\eta)Y_{n,k}(\eta)\,d\omega(\eta)\ . \tag{58}$$

The series

$$\sum_{(n,k)\in\mathcal{N}} F^{\wedge_{\mathrm{L}^2(\Omega)}}(n,k)Y_{n,k} = \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} F^{\wedge_{\mathrm{L}^2(\Omega)}}(n,k)Y_{n,k} \tag{59}$$

with

$$\mathcal{N} = \left\{(n,j)\big| n=0,1,\ldots;j=1,\ldots,2n+1\right\} \tag{60}$$

is called the *Fourier expansion* of F (with *Fourier coefficients* $F^{\wedge_{\mathrm{L}^2(\Omega)}}(n,k)$, $n=0,1,\ldots;k=1,\ldots,2n+1$). For all $F\in\mathrm{L}^2(\Omega)$ we have

$$\lim_{N\to\infty}\left\| F - \sum_{n=0}^{N}\sum_{k=1}^{2n+1} F^{\wedge_{\mathrm{L}^2(\Omega)}}(n,k)Y_{n,k}\right\|_{\mathrm{L}^2(\Omega)} = 0. \tag{61}$$

This property in $\mathrm{L}^2(\Omega)$ is equivalent to the Parseval identity

$$(F,F)_{\mathrm{L}^2(\Omega)} = \sum_{(n,k)\in\mathcal{N}}\left(F^{\wedge_{\mathrm{L}^2(\Omega)}}(n,k)\right)^2 = \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1}(F^{\wedge_{\mathrm{L}^2(\Omega)}}(n,k))^2\ . \tag{62}$$

The recovery of a function $F\in\mathrm{L}^2(\Omega)$ by its Fourier expansion (in the sense of $\|\cdot\|_{\mathrm{L}^2(\Omega)}$) is equivalent to the following conditions (see, for example, [5] and [13]):

(i) (*Closure*). The system $\{Y_{n,k}\}_{\substack{n=0,1,\ldots\\k=1,\ldots,2n+1}}$ is closed in $\mathrm{L}^2(\Omega)$, i.e. for any number $\varepsilon > 0$ and any function $F \in \mathrm{L}^2(\Omega)$ there exists a linear combination $Z_N = \sum_{n=0}^{N} \sum_{k=1}^{2n+1} d_{n,k} Y_{n,k}$ such that

$$\|F - Z_N\|_{\mathrm{L}^2(\Omega)} \leq \varepsilon \ .$$

(ii) (*Completeness*). The system $\{Y_{n,k}\}_{\substack{n=0,1,\ldots\\k=1,\ldots,2n+1}}$ is complete in $\mathrm{L}^2(\Omega)$, i.e.: $F \in \mathrm{L}^2(\Omega)$ with $F^{\wedge_{\mathrm{L}^2(\Omega)}}(n,k) = 0$ for all n,k implies $F = 0$.

(iii) The system $\{Y_{n,k}\}_{\substack{n=0,1,\ldots\\k=1,\ldots,2n+1}}$ is a *Hilbert basis* of $\mathrm{L}^2(\Omega)$, i.e.:

$$\overline{\operatorname{span}_{\substack{n=0,1,\ldots\\k=1,\ldots,2n+1}} \{Y_{n,k}\}}^{\|\cdot\|_{\mathrm{L}^2(\Omega)}} = \mathrm{L}^2(\Omega).$$

The closure (and equivalently the completeness) in $\mathrm{L}^2(\Omega)$ states that spherical harmonics are able to represent square–integrable functions on the sphere within arbitrarily given accuracy in the $\mathrm{L}^2(\Omega)$–topology.

Next we introduce vector spherical harmonics. As $o^{(2)}$ and $o^{(3)}$ (see (43), (44)) are operators such that $o^{(2)} Y_{0,1} = o^{(3)} Y_{0,1} = 0$ it does not make any sense to take these vector fields into account for a basis system in the Hilbert space $\mathrm{l}^2(\Omega)$ of square–integrable vector fields on Ω. In conclusion, the abbreviation

$$0_i = \begin{cases} 0, & \text{if} \quad i = 1 \\ 1, & \text{if} \quad i = 2,3 \end{cases} \tag{63}$$

will enable us to derive a unified setup. Now, the vector fields given by

$$y_n^{(i)}(\xi) = o_\xi^{(i)} Y_n(\xi); \qquad \xi \in \Omega;\ n = 0_i, 0_i + 1, \ldots; Y_n \in \mathrm{Harm}_n(\Omega); \tag{64}$$

are called *vector spherical harmonics of degree* n *and type* i. The field $y_n^{(1)}$ describes a normal field, while $y_n^{(2)}, y_n^{(3)}$ are tangential fields of degree n. Obviously, according to our construction (cf. [12]), we have

$$\begin{aligned} \xi \wedge y_n^{(1)}(\xi) &= 0, \\ \xi \cdot y_n^{(2)}(\xi) &= 0, \\ \xi \cdot y_n^{(3)}(\xi) &= 0, \\ \xi \wedge y_n^{(2)}(\xi) &= y_n^{(3)}(\xi), \\ \nabla_\xi^* \cdot (\xi \wedge y_n^{(2)}(\xi)) &= 0 = \mathrm{L}_\xi^* \cdot y_n^{(2)}(\xi), \\ \nabla_\xi^* \cdot y_n^{(3)}(\xi) &= 0. \end{aligned} \tag{65}$$

Moreover,

$$y_n^{(i)}(\xi) \cdot y_n^{(j)}(\xi) = 0, \qquad i \neq j, \tag{66}$$

provided that $y_n^{(i)}$ and $y_n^{(j)}$ are constructed out of the same scalar spherical harmonic Y_n in (64).

The vector fields

$$y_{n,j}^{(i)} = (\mu_n^{(i)})^{-1/2} o_\xi^{(i)} Y_{n,j}(\xi); \quad n = 0_i, 0_i+1, \ldots; \tag{67}$$

form an $\mathrm{l}^2(\Omega)$–orthonormal system (with $\{Y_{n,j}\}_{\substack{n=0,1,\ldots \\ j=1,\ldots,2n+1}}$ being always assumed to be $\mathrm{L}^2(\Omega)$–orthonormal); more explicitly, we have

$$\int_\Omega y_{n,k}^{(i)}(\xi) \cdot y_{m,l}^{(j)}(\xi)\, d\omega(\xi) = \delta_{ij}\delta_{nm}\delta_{kl}. \tag{68}$$

Consider the kernel

$$\frac{2n+1}{4\pi}\mathbf{p}_n^{(i,k)}(\xi,\eta) = \sum_{j=1}^{2n+1} y_{n,j}^{(i)}(\xi) \otimes y_{n,j}^{(k)}(\eta); \qquad \xi,\eta \in \Omega. \tag{69}$$

It can be deduced (see [11]) that, for every vector spherical harmonic $y_n^{(i)}$ of degree n and type i, the reproducing property

$$\frac{2n+1}{4\pi}\int_\Omega \mathbf{p}_n^{(i,i)}(\xi,\eta) y_n^{(i)}(\eta) d\omega(\eta) = y_n^{(i)}(\xi), \qquad \xi \in \Omega, \tag{70}$$

is valid. Let $\mathbf{t}$ be an orthogonal transformation. Then it follows that

$$\mathbf{p}_n^{(i,k)}(\mathbf{t}\xi, \mathbf{t}\eta) = \mathbf{t}\mathbf{p}_n^{(i,k)}(\xi,\eta)\mathbf{t}^{\mathrm{T}} \tag{71}$$

for any pair of unit vectors ξ, η and $i,k = 1,2,3$. Therefore, $\mathbf{p}_n^{(i,k)}(\xi,\eta)$ is invariant under orthogonal transformations. By straightforward calculations and observing the structure of the tensor product introduced by (69) we obtain a *vectorial analogue of the addition theorem* (see for the explicit representation [11], [12], [13]). We only mention the cases $i = k$, $i \in \{1,2,3\}$:

$$\begin{aligned} \sum_{j=1}^{2n+1} y_{n,j}^{(i)}(\xi) \otimes y_{n,j}^{(i)}(\eta) &= \left(\left(\mu_n^{(i)}\right)^{-1/2}\right)^2 \sum_{j=1}^{2n+1} o_\xi^{(i)} Y_{n,j}(\xi) \otimes o_\eta^{(i)} Y_{n,j}(\eta) \\ &= \left(\mu_n^{(i)}\right)^{-1} o_\xi^{(i)} o_\eta^{(i)} \sum_{j=1}^{2n+1} Y_{n,j}(\xi) Y_{n,j}(\eta) \\ &= \left(\mu_n^{(i)}\right)^{-1} \frac{2n+1}{4\pi} o_\xi^{(i)} o_\eta^{(i)} P_n(\xi\cdot\eta), \end{aligned} \tag{72}$$

where

$$\mathbf{p}_n^{(i,i)}(\xi,\eta) = \left(\mu_n^{(i)}\right)^{-1} o_\xi^{(i)} o_\eta^{(i)} P_n(\xi\cdot\eta), \quad i \in \{1,2,3\}, \tag{73}$$

is given by

$$\mathbf{p}_n^{(1,1)}(\xi,\eta) \quad = \quad P_n(\xi\cdot\eta)\xi\otimes\eta, \tag{74}$$

$$\mathbf{p}_n^{(2,2)}(\xi,\eta) = \frac{1}{n(n+1)}\left(P_n''(\xi\cdot\eta)\right)(\eta-(\xi\cdot\eta)\xi)\otimes(\xi-(\xi\cdot\eta)\eta) + P_n'(\xi\cdot\eta)\left(\mathbf{i}_{\tan}(\xi)-(\eta-(\xi\cdot\eta)\xi)\otimes\eta\right), \tag{75}$$

$$\mathbf{p}_n^{(3,3)}(\xi,\eta) = \frac{1}{n(n+1)}\left(P_n''(\xi\cdot\eta)(\xi\wedge\eta)\otimes(\eta\wedge\xi)\right. \tag{76}$$
$$\left. + \; P_n'(\xi\cdot\eta)\left((\xi\cdot\eta)\mathbf{i}_{\tan}(\xi)-(\eta-(\xi\cdot\eta)\xi)\otimes\xi\right)\right),$$

$(\xi,\eta)\in\Omega\times\Omega$.

The cartesian components of vector spherical harmonics of degree n of type 1 and 2 are known to be linear combinations of scalar harmonics of degree $n-1$ and $n+1$, while the cartesian components of a vector spherical harmonic of degree n and type 3 are linear combinations of scalar spherical harmonics of degree n.

For all $\xi,\eta\in\Omega$ and $i,k,j\in\{1,2,3\}$,

$$\left|\mathbf{p}_n^{(i,k)}(\xi,\eta)\varepsilon^j\right| \le 1. \tag{77}$$

In particular,

$$\sum_{j=1}^{2n+1}\left|y_{n,j}^{(i)}(\xi)\right|^2 = \frac{2n+1}{4\pi}, \qquad \xi\in\Omega, \tag{78}$$

so that

$$\sup_{\xi\in\Omega}\left|y_{n,j}^{(i)}(\xi)\right| \le \left(\frac{2n+1}{4\pi}\right)^{1/2}; \quad j=1,\ldots 2n+1. \tag{79}$$

The system of $\mathrm{l}^2(\Omega)$–orthonormal vector spherical harmonics $\{y_{n,j}^{(i)}\}$ is closed and complete in $\mathrm{l}^2(\Omega)$ with respect to $\|\cdot\|_{\mathrm{l}^2(\Omega)}$. Thus, every $f\in\mathrm{l}^2(\Omega)$ can be expanded in terms of vector spherical harmonics as follows:

$$\lim_{N\to\infty}\left\| f-\sum_{i=1}^{3}\sum_{n=0_i}^{N}\sum_{j=1}^{2n+1} f^{\wedge_{\mathrm{l}^2_{(i)}(\Omega)}}(n,j)\,y_{n,j}^{(i)}\right\|_{\mathrm{l}^2(\Omega)} = 0, \tag{80}$$

where the Fourier coefficients $f^{\wedge_{\mathrm{l}^2_{(i)}(\Omega)}}(n,j)$; $i=1,2,3$; are given by

$$f^{\wedge_{\mathrm{l}^2_{(i)}(\Omega)}}(n,j) = \left(f,y_{n,j}^{(i)}\right)_{\mathrm{l}^2(\Omega)} = \int_\Omega f(\xi)\cdot y_{n,j}^{(i)}(\xi)\,d\omega(\xi)\;. \tag{81}$$

Consequently, observing the completeness and the orthogonality of vector spherical harmonics we are able to split the space $\mathrm{c}^{(\infty)}(\Omega)$ orthogonally as follows:

$$\mathrm{c}^{(\infty)}(\Omega) = \mathrm{c}^{(\infty)}_{(1)}(\Omega)\oplus\mathrm{c}^{(\infty)}_{(2)}(\Omega)\oplus\mathrm{c}^{(\infty)}_{(3)}(\Omega), \tag{82}$$

where

$$\begin{aligned} \mathrm{c}^{(\infty)}_{(1)}(\Omega) &= \mathrm{c}^{(\infty)}_{\mathrm{nor}}(\Omega), && (83)\\ \mathrm{c}^{(\infty)}_{(2)}(\Omega) &= \left\{ f \in \mathrm{c}^{(\infty)}(\Omega) \,\middle|\, O^{(1)} f = O^{(3)} f = 0 \right\}, && (84)\\ \mathrm{c}^{(\infty)}_{(3)}(\Omega) &= \left\{ f \in \mathrm{c}^{(\infty)}(\Omega) \,\middle|\, O^{(1)} f = O^{(2)} f = 0 \right\} \ . && (85) \end{aligned}$$

In canonical way we extend these definitions to $\mathrm{c}^{(k)}(\Omega)$, $0 \leq k < \infty$, or $\mathrm{l}^2(\Omega)$. Furthermore, we are able to give a more detailed characterization of the decomposition of $\mathrm{l}^2(\Omega)$ (see [11], [13]) in the form

$$\mathrm{l}^2(\Omega) = \mathrm{l}^2_{\mathrm{nor}}(\Omega) \oplus \mathrm{l}^2_{\mathrm{tan}}(\Omega) \tag{86}$$

with

$$\begin{aligned} \mathrm{l}^2_{\mathrm{nor}}(\Omega) &= \mathrm{l}^2_{(1)}(\Omega), && (87)\\ \mathrm{l}^2_{\mathrm{tan}}(\Omega) &= \mathrm{l}^2_{(2)}(\Omega) \oplus \mathrm{l}^2_{(3)}(\Omega) \ . && (88) \end{aligned}$$

Clearly,

$$\mathrm{l}^2_{(i)}(\Omega) = \overline{\mathrm{span}\left\{ y^{(i)}_{n,j} \right\}_{n=0_i,0_i+1,\ldots;j=1,\ldots,2n+1}}^{\|\cdot\|_{\mathrm{l}^2(\Omega)}} \ . \tag{89}$$

3 Boundary–Value Problems of Elasticity

First we introduce some settings which are standard in elasticity theory (see, for example, [22], [26], [32], [33]).

3.1 The Cauchy–Navier Equation

Following the approach formulated in our introduction we obtain the following results in spherical nomenclature: the fundamental system of field equations for the time-independent behaviour of a linear homogeneous isotropic (spherical) body consists of the *strain–displacement relation*

$$\mathbf{e}(x) = \frac{1}{2}\left((\nabla_x \otimes u(x)) + (\nabla_x \otimes u(x))^{\mathrm{T}} \right), \ x \in \Omega_{\mathrm{int}}, \tag{90}$$

the *stress-strain relation*

$$\mathbf{s}(x) = 2\tilde{\mu}\mathbf{e}(x) + \tilde{\lambda}(\mathrm{trace}\ \ \mathbf{e}(x))\mathbf{i}(x), \quad x \in \Omega_{\mathrm{int}}, \tag{91}$$

and the *equation of equilibrium*

$$\nabla_x \cdot \left((2\tilde{\mu}\mathbf{e}(x) + \tilde{\lambda}(\mathrm{trace}\ \ \mathbf{e}(x))\mathbf{i}(x))\varepsilon^i \right) = 0, \quad x \in \Omega_{\mathrm{int}}, \tag{92}$$

for $i = 1,2,3$. For $i = 1,2,3$ we have

$$\nabla_x \cdot \left(\mathbf{s}(x)\varepsilon^i\right) = \tilde{\mu}\Delta_x \left(u(x) \cdot \varepsilon^i\right) + \left(\tilde{\lambda} + \tilde{\mu}\right) \frac{\partial}{\partial x_i}(\nabla_x \cdot u(x)), \quad x \in \Omega_{\text{int}}.$$

In other words, when we like to treat equilibrium problems of an isotropic homogeneous (spherical) elastic body, the field equations reduce to the *Navier equation* (also called *Cauchy–Navier equation*)

$$\tilde{\mu}\Delta_x u(x) + \left(\tilde{\lambda} + \tilde{\mu}\right) \nabla_x(\nabla_x \cdot u(x)) = 0, \quad x \in \Omega_{\text{int}}. \tag{93}$$

This equation plays in the theory of elasticity the same part as the Laplace equation in the theory of harmonic functions, and it formally reduces to it for $\tilde{\mu} = 1, \tilde{\lambda} = -1$.

The Navier equation admits the equivalent formulation

$$\diamondsuit_x u(x) = \Delta_x u(x) + \tau\nabla_x(\nabla_x \cdot u(x)) = 0, \quad x \in \Omega_{\text{int}}, \tag{94}$$

where

$$\tau = \frac{1}{1-2\delta}, \;\; \delta = \frac{\tilde{\lambda}}{2(\tilde{\lambda}+\tilde{\mu})} \tag{95}$$

(δ is called *Poisson's ratio*). Since

$$\Delta_x u(x) = \nabla_x(\nabla_x \cdot u(x)) - \nabla_x \wedge (\nabla_x \wedge u(x)), \quad x \in \Omega_{\text{int}}, \tag{96}$$

we equivalently have

$$\diamondsuit_x u(x) = \left(\tilde{\lambda} + 2\tilde{\mu}\right) \nabla_x(\nabla_x \cdot u(x)) - \tilde{\mu}\nabla_x \wedge (\nabla_x \wedge u(x)), \quad x \in \Omega_{\text{int}}. \tag{97}$$

Suppose now that u is a (sufficiently often differentiable) vector field satisfying the Navier equation. Then it follows that $\Delta_x(\nabla_x \cdot u(x)) = 0$, $\Delta_x(\nabla_x \wedge u(x)) = 0$, $\Delta_x(\Delta_x u(x)) = 0$, hold for all $x \in \Omega_{\text{int}}$. In other words, the displacement field u is biharmonic, and its divergence and curl are harmonic. This shows a deep relation between linear elasticity and potential theory.

3.2 Well–Posedness

In view of the stress–strain and strain displacement relations the corresponding *surface traction field* on Ω is given by

$$s_\nu = T_\nu(u) = \tilde{\mu}\left(\nabla \otimes u + (\nabla \otimes u)^{\mathrm{T}}\right)\nu + \tilde{\lambda}\,\text{trace}\;\;(\nabla \otimes u)\nu. \tag{98}$$

This gives us

$$T_\nu(u) = 2\tilde{\mu}(\nu \cdot \nabla)u + \tilde{\mu}\left((\nabla \otimes u) - (\nabla \otimes u)^{\mathrm{T}}\right)\nu + \tilde{\lambda}(\nabla \cdot u)\nu. \tag{99}$$

But
$$\left((\nabla \otimes u) - (\nabla \otimes u)^{\mathrm{T}}\right) \nu = \nu \wedge (\nabla \wedge u), \tag{100}$$
thus we have the following formula for the surface traction field (stress vector field) on Ω
$$T_\nu(u) = 2\tilde{\mu}(\nu \cdot \nabla)u + \tilde{\mu}\nu \wedge (\nabla \wedge u) + \tilde{\lambda}(\nabla \cdot u)\nu. \tag{101}$$
A formal mathematical role plays the *pseudostress vector field* $N_\nu(u)$ on Σ defined by
$$\begin{aligned} &N_\nu(u) \\ &= \frac{2\tilde{\mu}\left(\tilde{\lambda}+2\tilde{\mu}\right)}{\tilde{\lambda}+3\tilde{\mu}}(\nu \cdot \nabla)u + \frac{\left(\tilde{\lambda}+\tilde{\mu}\right)\left(\tilde{\lambda}+2\tilde{\mu}\right)}{\tilde{\lambda}+3\tilde{\mu}}(\nabla \cdot u)\nu + \frac{\tilde{\lambda}\left(\tilde{\lambda}+\tilde{\mu}\right)}{\tilde{\lambda}+3\tilde{\mu}}\nu \wedge (\nabla \wedge u) \end{aligned} \tag{102}$$
(cf. [24]). The pseudostress operator N plays in the theory of elasticity the same part as the normal derivative operator in the theory of harmonic functions.

A vector field f possessing k (ρ-Hölder) continuous derivatives is said to be of class $\mathrm{c}^{(k)}(\mathrm{c}^{(k,\rho)})$. The space $\mathrm{c}^{(0,\rho)}(\Omega), 0 \leq \rho < 1$, will be the class of ρ–Hölder continuous vector fields on Ω. $\mathrm{c}^{(0,\rho)}(\Omega)$ is a non-complete normed space with $\|\cdot\|_{\mathrm{c}(\Omega)}$ and a Banach space with
$$\|f\|_{\mathrm{c}^{(0,\rho)}(\Omega)} = \sup_{x\in\Omega}|f(x)| + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\rho}. \tag{103}$$
In $\mathrm{c}^{(0,\rho)}(\Omega)$ we have also the scalar product $(\cdot,\cdot)_{\mathrm{l}^2(\Omega)}$ implying the norm $\|\cdot\|_{\mathrm{l}^2(\Omega)}$.

In what follows $\mathrm{pot}(\Omega_{\mathrm{int}})$ denotes the space of vector fields $u \in \mathrm{c}^{(2)}(\Omega_{\mathrm{int}})$ satisfying Navier's equations in Ω_{int}, while $\mathrm{pot}(\Omega_{\mathrm{ext}})$ denotes the space of all vector fields $u \in \mathrm{c}^{(2)}(\Omega_{\mathrm{ext}})$ satisfying Navier's equation in Ω_{ext} and being regular at infinity (see [24]).

For $k = 0, 1, \ldots$ we set
$$\begin{aligned} \mathrm{pot}^{(k,\rho)}\left(\overline{\Omega_{\mathrm{int}}}\right) &= \mathrm{pot}(\Omega_{\mathrm{int}}) \cap \mathrm{c}^{(k,\rho)}\left(\overline{\Omega_{\mathrm{int}}}\right) \\ \mathrm{pot}^{(k,\rho)}\left(\overline{\Omega_{\mathrm{ext}}}\right) &= \mathrm{pot}(\Omega_{\mathrm{ext}}) \cap \mathrm{c}^{(k,\rho)}\left(\overline{\Omega_{\mathrm{ext}}}\right). \end{aligned} \tag{104}$$
The matrix $\mathbf{g}(x), x \in \mathbb{R}^3$ with $|x| \neq 0$, given by
$$\mathbf{g}(x) = \frac{\tilde{\lambda}+3\tilde{\mu}}{2\tilde{\mu}\left(\tilde{\lambda}+2\tilde{\mu}\right)}\left(\left(\varepsilon^i \cdot \varepsilon^k + \frac{\tilde{\lambda}+\tilde{\mu}}{\tilde{\lambda}+3\tilde{\mu}}\frac{\left(x\cdot\varepsilon^i\right)\left(x\cdot\varepsilon^k\right)}{|x|^2}\right)\frac{1}{|x|}\right)_{i,k=1,2,3} \tag{105}$$
is constituted by the *fundamental solutions* $g_k(x) = \mathbf{g}(x)\varepsilon^k,\ k = 1,2,3,$ *of the operator* $\diamondsuit$. It is not hard to see that for $|x| \neq 0$ and $k = 1, 2, 3$
$$g_k(x) = \frac{\tilde{\lambda}+3\tilde{\mu}}{2\tilde{\mu}\left(\tilde{\lambda}+2\tilde{\mu}\right)}\left(\frac{1}{|x|}\varepsilon^k + \frac{\tilde{\lambda}+\tilde{\mu}}{\tilde{\lambda}+3\tilde{\mu}}\frac{\left(x\cdot\varepsilon^k\right)}{|x|^3}x\right) \tag{106}$$

$$
\begin{aligned}
&= \frac{\tilde{\lambda}+3\tilde{\mu}}{2\tilde{\mu}\left(\tilde{\lambda}+2\tilde{\mu}\right)}\left(\frac{1}{|x|}\varepsilon^k - \frac{\tilde{\lambda}+\tilde{\mu}}{\tilde{\lambda}+3\tilde{\mu}} x\left(\varepsilon^k\cdot\nabla_x\right)\frac{1}{|x|}\right) \\
&= \frac{1}{2\left(\tilde{\lambda}+2\tilde{\mu}\right)}\nabla_x\left(\nabla_x\cdot\left(|x|\varepsilon^k\right)\right) - \frac{1}{2\tilde{\mu}}\nabla_x\wedge\left(\nabla_x\wedge\left(|x|\varepsilon^k\right)\right).
\end{aligned}
$$

Furthermore, we obtain

$$
\nabla_x\cdot g_k(x) = -\frac{1}{\tilde{\lambda}+2\tilde{\mu}}\frac{\varepsilon^k\cdot x}{|x|^3}, \nabla_x\wedge g_k(x) = \frac{1}{\tilde{\mu}}\frac{\varepsilon^k\wedge x}{|x|^3}, \tag{107}
$$

$$
\begin{aligned}
\nabla_x\left(\varepsilon^j\cdot g_k(x)\right) &= -\frac{\tilde{\lambda}+3\tilde{\mu}}{2\tilde{\mu}\left(\tilde{\lambda}+2\tilde{\mu}\right)}\delta_{jk}\frac{x}{|x|^3} \\
&\quad + \frac{\tilde{\lambda}+\tilde{\mu}}{2\tilde{\mu}\left(\tilde{\lambda}+2\tilde{\mu}\right)}\left(\frac{\left(x\cdot\varepsilon^j\right)\varepsilon^k + \left(x\cdot\varepsilon^k\right)\varepsilon^j}{|x|^3} - 3\frac{\left(x\cdot\varepsilon^j\right)\left(x\cdot\varepsilon^k\right)}{|x|^5 x}\right).
\end{aligned}
\tag{108}
$$

Moreover, let

$$
\begin{aligned}
T_\nu(g_k)(x) &= \left(\nu(x)\cdot\nabla_x\frac{1}{|x|}\right)t_k^w(x) + t_k^s(x),\ k=1,2,3, \\
N_\nu(g_k)(x) &= \left(\nu(x)\cdot\nabla_x\frac{1}{|x|}\right)n_k^w(x),\ k=1,2,3,
\end{aligned}
\tag{109}
$$

where $t_k^w(x), t_k^s(x)$, respectively, is the weak and strong singular part of $T_\nu(g_k)(x)$ given by

$$
t_k^w(x) = \frac{\tilde{\mu}}{\tilde{\lambda}+2\tilde{\mu}}\varepsilon^k + \frac{3\left(\tilde{\lambda}+\tilde{\mu}\right)}{\tilde{\lambda}+2\tilde{\mu}}\frac{\left(x\cdot\varepsilon^k\right)x}{|x|^2}, \tag{110}
$$

$$
t_k^s(x) = \frac{\tilde{\mu}}{\tilde{\lambda}+2\tilde{\mu}}\frac{\left(x\cdot\varepsilon^k\right)\nu(x) - \left(\nu(x)\cdot\varepsilon^k\right)x}{|x|^3} \tag{111}
$$

and

$$
n_k^w(x) = \frac{2\tilde{\mu}}{\tilde{\lambda}+3\tilde{\mu}}\varepsilon^k + \frac{3\left(\tilde{\lambda}+\tilde{\mu}\right)}{\tilde{\lambda}+3\tilde{\mu}}\frac{\left(x\cdot\varepsilon^k\right)x}{|x|^2} \tag{112}
$$

where ν is – as usual – the outward-drawn unit normal field.

We set

$$
\mathbf{t}^w(x) = \left(t_i^w(x)\cdot\varepsilon^k\right)_{i,k=1,2,3}, \tag{113}
$$

$$
\mathbf{t}^s(x) = \left(t_i^s(x)\cdot\varepsilon^k\right)_{i,k=1,2,3}, \tag{114}
$$

and

$$
\mathbf{n}^w(x) = \left(n_i^w(x)\cdot\varepsilon^k\right)_{i,k=1,2,3}. \tag{115}
$$

The field u_1 given by

$$u_1(x) = \int_\Omega \mathbf{g}(x-y) f(y)\ d\omega(y) \tag{116}$$

is called *the single–layer potential*, while $u_2^{T_v}, u_2^{N_v}$ given by

$$u_2^{T_\nu}(x) = \int_\Omega \left(\left(\frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} \right) \mathbf{t}^w(x-y) + \mathbf{t}^s(x-y) \right) f(y)\ d\omega(y), \tag{117}$$

and

$$u_2^{N_\nu}(x) = \int_\Omega \left(\left(\frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} \right) \mathbf{n}^w(x-y) \right) f(y)\ d\omega(y), \tag{118}$$

are called T_ν*-double layer potential* and N_ν*–double layer potential*, respectively. The integral in (117) stands for its principal value.

The potential operators in elasticity behave near the boundary much like the ordinary harmonic potentials. Their properties have been thoroughly investigated, for example, by [24]. We omit the formulations of the "limit formulae" and "jump relations". We only report on their role in the classical (inner) boundary–value problems.

Inner Dirichlet Problem (idp): Given the vector function $f \in \mathrm{c}^{(0,0)}(\Omega)$. Find $u \in \mathrm{pot}^{(0,0)}(\overline{\Omega_{\mathrm{int}}})$ with

$$u^-(x) = \lim_{\substack{\tau \to 0 \\ \tau > 0}} u(x - \tau\nu(x)) = f(x), x \in \Omega. \tag{119}$$

Inner Neumann Problem (inp): Given $f \in \mathrm{c}^{(0,\rho)}(\Omega)$, $0 < \rho \leq 1$, satisfying

$$\int_\Omega f(y)\ d\omega(y) = 0, \quad \int_\Omega (y \wedge f(y))\ d\omega(y) = 0.$$

Find $u \in \mathrm{pot}^{(1,\rho)}(\overline{\Omega_{\mathrm{int}}})$,

$$\int_\Omega u(y)\ d\omega(y) = 0, \int_\Omega (y \wedge u(y))\ d\omega(y) = 0$$

with

$$T_\nu(u)^-(x) = \lim_{\substack{\sigma \to 0 \\ \sigma > 0}} T_\nu(u)(x - \sigma\nu(x)) = f(x),\ x \in \Omega. \tag{120}$$

Theoretical aspects about uniqueness, existence, and continuous dependence on the boundary values can be found in many books, for example [6], [18], [21], [23], [24], [25], [31].

We recapitulate the essential results for the inner problems:

(idp) Let d^- denote the space of "boundary values"

$$d^- = \left\{ u^- = u|\Omega \ \middle|\ u \in \ \mathrm{pot}^{(0,0)}\left(\overline{\Omega_{\mathrm{int}}}\right) \right\}. \tag{121}$$

Problem *(idp)* corresponding to $f \in \mathrm{c}^{(0,0)}(\Omega)$ has a unique solution representable by an N_ν-double layer potential. It is known that

$$\mathrm{c}^{(0,0)}(\Omega) = d^-. \tag{122}$$

By completion it follows that

$$\mathrm{l}^2(\Omega) = \overline{d^-}^{\|\cdot\|_{\mathrm{l}^2(\Omega)}}. \tag{123}$$

Furthermore, [24] shows that the problem (*idp*) corresponding to $f \in \mathrm{c}^{(0,\rho)}(\Omega)$, $0 < \rho \leq 1$, is uniquely solvable by a T_ν-double layer potential, and it follows that

$$\mathrm{c}^{(0,\rho)}(\Omega) = \left\{ u^- = u|\Omega \;\middle|\; u \in \;\mathrm{pot}^{(0,\rho)}\left(\overline{\Omega_{\mathrm{int}}}\right) \right\}. \tag{124}$$

(inp) Let

$$n^- = \left\{ T_\nu(u)^- \;\middle|\; u \in \;\mathrm{pot}^{(1,\rho)}\left(\overline{\Omega_{\mathrm{int}}}\right), \int_\Omega u(y)\; d\omega(y) = 0,\; \int_\Omega (y \wedge u(y))\; d\omega(y) = 0 \right\},$$

and

$$\begin{aligned} e_{n^-} &= \left\{ u \;\middle|\; u \in \;\mathrm{pot}^{(1,\rho)}\left(\overline{\Omega_{\mathrm{int}}}\right), T_\nu(u)^- = 0 \right\}, \\ e_{n^-}|\Omega &= \{u^- | u \in e_{n^-}\}. \end{aligned}$$

The necessary and sufficient solvability condition reads as follows

$$n^- \perp e_{n^-}|\Omega,\; \mathrm{c}^{(0,\rho)}(\Omega) = n^- \oplus e_{n^-}|\Omega.$$

In other words, the problem (*inp*) is uniquely solvable for a vector function $f \in \mathrm{c}^{(0,\rho)}(\Omega)$ satisfying

$$\int_\Omega f(x)\; d\omega(x) = 0\;, \quad \int_\Omega (f(x) \wedge x)\; d\omega(x) = 0. \tag{125}$$

The uniquely determined solution may be represented as a single–layer potential (with ρ–Hölder continuous density function).

By completion it follows that

$$\mathrm{l}^2(\Omega) = \overline{n^-}^{\|\cdot\|_{\mathrm{l}^2(\Omega)}} \oplus e_{n^-}|\Omega. \tag{126}$$

As in the case of harmonic functions (see [10]) we are able to formulate l^2-regularity theorems.

Theorem 3.1 *For every $\alpha > 0$ (sufficiently small) there exists a constant $C = C(\alpha;\Omega)$ such that the following statements are valid:*

(i) *For all* $K \subset \Omega_{\rm int}$ *with* $\operatorname{dist}(\overline{K}, \Omega) \geq \alpha > 0$ *and all* $u \in \operatorname{pot}^{(0,0)}(\overline{\Omega_{\rm int}})$

$$\sup_{x \in K} |u(x)| \leq C \, \|u^-\|_{l^2(\Omega)} \ .$$

(ii) *For all* $K \subset \Omega_{\rm int}$ *with* dist $(\overline{K}, \Omega) \geq \alpha > 0$ *and for* $u \in \operatorname{pot}^{(1,\rho)}(\overline{\Omega_{\rm int}})$ *satisfying*

$$\int_\Omega u(y) \; d\omega(y) = 0, \; \int_\Omega (y \wedge u(y)) \; d\omega(y) = 0$$

we have

$$\sup_{x \in K} |u(x)| \leq C \; ||T_\nu(u)^-\|_{l^2(\Omega)}.$$

3.3 Vector Spherical Harmonics as Boundary Values

In this section we are interested in determining elastic potentials corresponding to vector spherical harmonics as boundary values.

Lemma 3.1 *Let* $v^{(i)}_{n,j}, \mathbb{R}^3 \to \mathbb{R}^3, i = 1,2,3,$ *be defined by*

$$\begin{aligned}
v^{(1)}_{n,j}(x) &= H_{n,j}(x)x + \alpha_n \left(x^2 - 1\right) \nabla_x H_{n,j}(x); && (127)\\
&\quad n = 0,1,...; j = 1,...,2n+1;\\
v^{(2)}_{n,j}(x) &= (n(n+1))^{-\frac{1}{2}} \left(\nabla_x H_{n,j}(x) - n v^{(1)}_{n,j}(x)\right); && (128)\\
&\quad n = 1,2,...; j = 1,...,2n+1;\\
v^{(3)}_{n,j}(x) &= (n(n+1))^{-\frac{1}{2}} x \wedge \nabla_x H_{n,j}(x); && (129)\\
&\quad n = 1,2,...; j = 1,...,2n+1;
\end{aligned}$$

where

$$\alpha_n = -\frac{n\tau + 2 + 3\tau}{2(n(\tau+2)+1)}, \tag{130}$$

$$H_{n,j}(x) = |x|^n Y_{n,j}(\xi), \qquad x = |x|\xi, \; \xi \in \Omega. \tag{131}$$

Then $v^{(i)}_{n,j} \in \operatorname{pot}^{(0,0)}(\overline{\Omega_{\rm int}})$ *with* $v^{(i)-}_{n,j} = v^{(i)}_{n,j}|\Omega = y^{(i)}_{n,j}$.

Proof: It is not hard to see that

$$\begin{aligned}
\diamondsuit_x v^{(1)}_{n,j}(x) &= 2\nabla_x H_{n,j}(x) + \tau(3+n)\nabla_x H_{n,j}(x) && (132)\\
&\quad + \alpha_n((6 + 4(n-1))\nabla_x H_{n,j}(x) + 2n\tau\nabla_x H_{n,j}(x))\\
&= 0,\\
\diamondsuit_x v^{(2)}_{n,j}(x) &= (n(n+1))^{-\frac{1}{2}} (\diamondsuit_x \nabla_x H_{n,j}(x)) && (133)\\
&\quad - n(n(n+1))^{-\frac{1}{2}} \left((\diamondsuit_x) v^{(1)}_{n,j}(x)\right)\\
&= 0,
\end{aligned}$$

$$\begin{aligned}\diamondsuit_x v^{(3)}_{n,j}(x) &= (n(n+1))^{-\frac{1}{2}}\diamondsuit_x(x\wedge\nabla_x H_{n,j}(x)) \\ &= -2\nabla_x\wedge\nabla_x H_{n,j}(x) \\ &= 0.\end{aligned} \tag{134}$$

Using polar coordinates $x = r\xi,\ r = |x|,\ \xi\in\Omega$, we obtain after simple calculations

$$v^{(1)}_{n,j}(x) = \sigma^{(1)}_n(r)y^{(1)}_{n,j}(\xi)+\tau^{(1)}_n(r)y^{(2)}_{n,j}(\xi), \tag{135}$$

$$v^{(2)}_{n,j}(x) = \sigma^{(2)}_n(r)y^{(1)}_{n,j}(\xi)+\tau^{(2)}_n(r)y^{(2)}_{n,j}(\xi), \tag{136}$$

$$v^{(3)}_{n,j}(x) = \sigma^{(3)}_n(r)y^{(3)}_{n,j}(\xi), \tag{137}$$

where

$$\sigma^{(1)}_n(r) = r^{n-1}\left(r^2+n\alpha_n\left(r^2-1\right)\right), \tag{138}$$

$$\sigma^{(2)}_n(r) = (n(n+1))^{-\frac{1}{2}}n(1+n\alpha_n)r^{n-1}\left(1-r^2\right), \tag{139}$$

$$\sigma^{(3)}_n(r) = r^n, \tag{140}$$

$$\tau^{(1)}_n(r) = \alpha_n(n(n+1))^{+\frac{1}{2}}r^{n-1}\left(r^2-1\right), \tag{141}$$

$$\tau^{(2)}_n(r) = r^{n-1}\left(1-n\alpha_n\left(r^2-1\right)\right). \tag{142}$$

But this shows us that $v^{(i)-}_{n,j} = v^{(i)}_{n,j}|\Omega = y^{(i)}_{n,j}$, as required. ■

Note that the polynomial solution $v^{(i)}_{n,j},\ i=1,2$ corresponding to $y^{(i)}_{n,j}$ on Ω is not homogeneous.

Remark 3.1 *Observe that, under the assumption* $3\tilde\lambda+2\tilde\mu>0,\ \tilde\mu>0$, *it follows that*

$$\tau = \frac{\tilde\lambda+\tilde\mu}{\tilde\mu} = \frac{1}{3}+\frac{3\tilde\lambda+2\tilde\mu}{3\tilde\mu} > \frac{1}{3}. \tag{143}$$

Therefore it is not difficult to deduce that for all $n\geq 3$

$$\begin{aligned}|\alpha_n| &= \frac{1}{2}\frac{n\tau+3\tau+2}{n\tau+2n+1} \\ &= \frac{1}{2}\frac{1+\frac{3\tau}{n\tau}+\frac{2}{n\tau}}{1+\frac{2n}{n\tau}+\frac{1}{n\tau}} \leq \frac{1}{2}\frac{2+\frac{2}{n\tau}}{1+\frac{1}{n\tau}} \leq 1,\end{aligned} \tag{144}$$

while for all $n\geq 1$

$$|\alpha_n| \leq \frac{1}{2}+\frac{\frac{3}{2n}}{1+\frac{2}{n\tau}} \leq 2. \tag{145}$$

The sequence (α_n), *therefore is uniformly bounded with respect to* τ.

In connection with Theorem 3.1 (i) we easily obtain (see [17])

Theorem 3.2 *Suppose that* $f \in \mathrm{c}^{(0,0)}(\Omega)$. *Then the unique solution* u *of the Dirichlet problem* $u \in \mathrm{pot}^{(0,0)}(\overline{\Omega_{\mathrm{int}}})$, $u^- = f$ *is representable in the form*

$$u(x) = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} f_{n,j}^{(i)} \ v_{n,j}^{(i)}(x)$$

for all $x \in K$ *with* $K \subset \Omega_{\mathrm{int}}$ *and* $\mathrm{dist}(\overline{K}, \Omega) > 0$, *where* $f_{n,j}^{(i)}$ *are the Fourier coefficients of* f *with respect to the system* $\{y_{n,j}^{(i)}\}$

$$f_{n,j}^{(i)} = \left(f, y_{n,j}^{(i)}\right)_{\mathrm{l}^2(\Omega)} = \int_{\Omega} f(\eta) \cdot y_{n,j}^{(i)}(\eta) \ d\omega(\eta).$$

From Lemma 3.1 it is not difficult to determine the *stress vector field* $T_\nu(v_{n,j}^{(i)})(x)$ for any point $x \in \Omega_{\mathrm{int}}$:

$$\begin{aligned}
|x|T_\nu\left(v_{n,j}^{(1)}\right)(x) &= \left(\tilde{\mu}(n+2) + \tilde{\lambda}(n+3) + \alpha_n\left(\tilde{\lambda} + \tilde{\mu}\right)\right) H_{n,j}(x)x \\
&\quad + (\tilde{\mu} + 2\tilde{\mu}n\alpha_n)\, x^2 \nabla_x H_{n,j}(x) - 2\alpha_n \tilde{\mu}(n-1)\nabla_x H_{n,j}(x); \\
&\quad\ n = 0, 1, ...; j = 1, ..., 2n+1; \\
|x|T_\nu\left(v_{n,j}^{(2)}\right)(x) &= (n(n+1))^{-\frac{1}{2}}\left(2\tilde{\mu}(n-1)\nabla_x H_{n,j}(x) - nT_\nu\left(v_{n,j}^{(1)}(x)\right)\right); \\
&\quad\ n = 1, 2, ...; j = 1, ..., 2n+1; \\
|x|T_\nu\left(v_{n,j}^{(3)}\right)(x) &= (n(n+1))^{-\frac{1}{2}}\tilde{\mu}(n-1)x \wedge \nabla_x H_{n,j}(x); \\
&\quad\ n = 1, 2, ...; j = 1, ..., 2n+1.
\end{aligned}$$

This leads us to the following theorem.

Theorem 3.3 *Let* f *be of class* $\mathrm{c}^{(0,0)}(\Omega)$. *Suppose that* u *is the solution of the inner Dirichlet problem* $u \in \mathrm{pot}^{(0,0)}(\overline{\Omega_{\mathrm{int}}}), u^- = f$. *Then*

$$|x|T_\nu(u)(x) = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} \left(f, y_{n,j}^{(i)}\right)_{\mathrm{l}^2(\Omega)} T_\nu\left(v_{n,j}^{(i)}\right)(x)$$

for each $x \in \Omega_{\mathrm{int}}$.

Next we note that the fields $v_{n,j}^{(i)}$ admit a decomposition into curl-free and divergence-free parts. For that purpose we formulate the following lemma (see [17]).

Lemma 3.2 *Under the assumptions of Lemma 3.1*

$$\begin{aligned}
v_{n,j}^{(1)}(x) &= \delta_n \nabla_x\left(x^2 H_{n,j}(x)\right) + \varepsilon_n \nabla_x \wedge \nabla_x \wedge \left(\left(x^2 H_{n,j}(x)\right)x\right), \\
v_{n,j}^{(2)}(x) &= (n(n+1))^{-\frac{1}{2}}\nabla_x\left(H_{n,j}(x) - n\delta_n x^2 H_{n,j}(x)\right) \\
&\quad -(n(n+1))^{-\frac{1}{2}} n\varepsilon_n \nabla_x \wedge \nabla_x \wedge \left(\left(x^2 H_{n,j}(x)\right)x\right), \\
v_{n,j}^{(3)}(x) &= -(n(n+1))^{-\frac{1}{2}}\nabla_x \wedge (H_{n,j}(x)x),
\end{aligned}$$

where

$$\delta_n = \frac{n+3+2n\alpha_n}{2(2n+3)} \ , \ \varepsilon_n = \frac{2n\alpha_n - 1}{2(2n+3)}. \tag{146}$$

Proof: Elementary calculations show us that

$$\nabla_x \left(x^2 H_{n,j}(x) \right) = 2H_{n,j}(x)x + x^2 \nabla_x H_{n,j}(x), \tag{147}$$

and

$$\begin{aligned} & \nabla_x \wedge \nabla_x \wedge \left(\left(x^2 H_{n,j}(x) \right) x \right) \\ = \quad & -\nabla_x \left(x^2 x \wedge \nabla_x H_{n,j}(x) \right) \\ = \quad & -2x \wedge (x \wedge \nabla_x H_{n,j}(x)) + x^2 \nabla_x \wedge (\nabla_x \wedge H_{n,j}(x)x) \\ = \quad & -2nH_{n,j}(x)x + 2x^2 \nabla_x H_{n,j}(x) + x^2 (\nabla_x \nabla_x - \Delta_x)(H_{n,j}(x)x) \\ = \quad & -2nH_{n,j}(x)x + (n+3)x^2 \nabla_x H_{n,j}(x) \ . \end{aligned} \tag{148}$$

But this implies that

$$\begin{aligned} & H_{n,j}(x)x \\ & \quad = (2(2n+3))^{-1} \left((n+3)\nabla_x \left(x^2 H_{n,j}(x) \right) - \nabla_x \wedge \nabla_x \wedge \left(x^2 H_{n,j}(x)x \right) \right), \\ & x^2 \nabla_x H_{n,j}(x) \\ & \quad = (2n+3)^{-1} \left(n\nabla_x \left(x^2 H_{n,j}(x) \right) + \nabla_x \wedge \nabla_x \wedge \left(x^2 H_{n,j}(x)x \right) \right) \ . \end{aligned} \tag{149}$$

Therefore, the vector fields $v^{(i)}_{n,j}, i = 1,2,3,$ can be written as indicated by Lemma 3.2. ■

Lemma 3.2 leads us to the following result.

Theorem 3.4 *For given $f \in \mathrm{c}^{(0,0)}(\Omega)$ the uniquely determined solution u of the Dirichlet problem $u \in \mathrm{pot}^{(0,0)}(\overline{\Omega_{\mathrm{int}}}), u^- = f$ is given by*

$$u(x) = \nabla_x Z_1(x) + \nabla_x \wedge \nabla_x \wedge \left(x^2 Z_2(x)x \right) + \nabla_x \wedge (Z_3(x)x)$$

for all $x \in K$ with $K \subset \Omega_{\mathrm{int}}$ and $\mathrm{dist}(\overline{K}, \Omega) > 0$, where the functions $Z_i, i = 1,2,3,$ can be written down as follows:

$$\begin{aligned} & Z_1(x) \\ & = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left(f^{(1)}_{n,j} \delta_n x^2 H_{n,j}(x) + \frac{f^{(2)}_{n,j}\sigma_n}{\sqrt{n(n+1)}} \left(H_{n,j}(x) - n\delta_n x^2 H_{n,j}(x) \right) \right), \end{aligned}$$

$$Z_2(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left(\left(f^{(1)}_{n,j} - n(n+1))^{-1/2} f^{(2)}_{n,j} \right) \varepsilon_n H_{n,j}(x) \right),$$

$$Z_3(x) = -\sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} (n(n+1))^{-1/2} f^{(3)}_{n,j} H_{n,j}(x),$$

where $\sigma_0 = 0$ and $\sigma_n = 1$ for $n > 0$.

Obviously, the vector fields $u_i, i = 1, 2, 3$, given by

$$u_1(x) = \nabla_x Z_1(x), \tag{150}$$
$$u_2(x) = \nabla_x \wedge \nabla_x \wedge \left(x^2 Z_2(x)x\right), \tag{151}$$
$$u_3(x) = \nabla_x \wedge (Z_3(x)x) \tag{152}$$

satisfy

$$\nabla_x \wedge u_1(x) = 0, \tag{153}$$
$$\nabla_x \cdot u_2(x) = 0, \quad x \cdot \nabla_x \wedge u_2(x) = 0, \tag{154}$$
$$\nabla_x \cdot u_3(x) = 0, \quad x \cdot u_3(x) = 0, \tag{155}$$

for all $x \in K \subset \Omega_{\text{int}}$ with $\text{dist}(\overline{K}, \Omega) > 0$. The vector field u_2 is of poloidal type, while u_3 is of torsional type.

Finally we discuss the Neumann problem of determining polynomial solutions from given surface tractions on the unit sphere (see [17]).

Lemma 3.3 *The vector fields $w_{n,j}^{(i)}, i = 1, 2, 3$, defined by*

$$\begin{aligned}
w_{n,j}^{(1)}(x) &= \zeta_n \left(H_{n,j}(x)x + \alpha_n x^2 \nabla_x H_{n,j}(x) - \frac{1 + 2n\alpha_n}{2(n-1)} \nabla_x H_{n,j}(x) \right); \\
&\qquad n = 0, 2, 3, ...; j = 1, ..., 2n + 1; \\
w_{1,j}^{(1)}(x) &= 3\zeta_1 \left(H_{1,j}(x)x + \alpha_1 x^2 \nabla_x H_{1,j}(x) \right); \\
&\qquad j = 1, 2, 3; \\
w_{n,j}^{(2)}(x) &= (n(n+1))^{-1} (2\tilde{\mu}(n-1))^{-1} \nabla_x H_{n,j}(x) - (n(n+1))^{-\frac{1}{2}} n w_{n,j}^{(1)}(x); \\
&\qquad n = 2, 3, ...; j = 1, ..., 2n + 1; \\
w_{n,j}^{(3)}(x) &= (n(n+1))^{-\frac{1}{2}} (\tilde{\mu}(n-1))^{-1} x \wedge \nabla_x H_{n,j}(x); \\
&\qquad n = 2, 3, ...; j = 1, ..., 2n + 1;
\end{aligned}$$

where

$$\alpha_n = -\frac{n\tau + 2 + 3\tau}{2(n(\tau + 2) + 1)}, \ \zeta_n = \frac{1}{(\tilde{\lambda} + \tilde{\mu})(3 + n + 2n\alpha_n) - \tilde{\mu}},$$
$$H_{n,j}(x) = |x|^n Y_{n,j}(\xi), \ x = |x|\xi, \ \xi \in \Omega,$$

satisfy

$$w_{n,j}^{(i)} \in \text{pot}^{(1,1)}(\overline{\Omega_{\text{int}}})$$

and

$$\begin{aligned}
T_\nu \left(w_{n,j}^{(1)} \right)^- &= y_{n,j}^{(1)}; \ n = 0, 2, 3, ...; \ j = 1, ..., 2n + 1; \\
T_\nu \left(w_{1,j}^{(1)} \right)^- &= 2y_{n,j}^{(1)} - \sqrt{2}\ y_{n,j}^{(2)}; \ j = 1, 2, 3; \\
T_\nu \left(w_{n,j}^{(i)} \right)^- &= y_{n,j}^{(i)}; \ i = 2, 3; \ n = 2, 3, ...; \ j = 1, ..., 2n + 1.
\end{aligned}$$

We conclude our considerations with the following theorem.

Theorem 3.5 *Suppose that* $f \in \mathrm{c}^{(0,\rho)}(\Omega)$ *satisfies the condition*

$$\int_\Omega f(\xi)\ d\omega(\xi) = 0, \quad \int_\Omega (f(\xi) \wedge \xi)\ d\omega(\xi) = 0. \tag{156}$$

Then the series

$$u = f^{(1)}_{0,1} w^{(1)}_{0,1} + \sum_{j=1}^{3} f^{(1)}_{1,j} w^{(1)}_{1,j} + \sum_{i=1}^{3} \sum_{n=2}^{\infty} \sum_{j=1}^{2n+1} f^{(i)}_{n,j} w^{(i)}_{n,j}$$

solves Neumann's problem $u \in \mathrm{pot}^{(1,\rho)}(\overline{\Omega_{\mathrm{int}}}), T_\nu(u)^- = f$ *on every* $K \subset \Omega_{\mathrm{int}}$ *with* $\mathrm{dist}(\overline{K}, \Omega) > 0$.

4 The Scaling Function for the Dirichlet Problem

Definition 4.1 *Let* $\varphi_1 : [0,\infty) \to \mathbb{R}$ *be a generator of a scaling function, i.e. it satisfies the admissibility condition*

$$\sum_{n=0}^{\infty} \frac{(2n+1)^2}{4\pi} \sup_{x \in [n,n+1)} (\varphi_1(x))^2 < +\infty\ , \tag{157}$$

and it has the properties:

- φ_1 *is monotonically decreasing on* $[0,\infty)$,
- φ_1 *is continuous at 0 with value* $\varphi_1(0) = 1$.

Its dilates $\varphi_\rho : [0,\infty) \to \mathbb{R}$ *are defined by* $\varphi_\rho(x) = \varphi_1(\rho x)$. *A Navier scaling function* $\mathbf{\Phi}_\rho : \overline{\Omega_{\mathrm{int}}} \times \Omega \to \mathbb{R}^{3\times 3}$, $\rho \in (0,\infty)$, *is defined by*

$$\mathbf{\Phi}_\rho(x,\xi) = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{k=1}^{2n+1} (\varphi_\rho(n))^2\, v^{(i)}_{n,k}(x) \otimes y^{(i)}_{n,k}(\xi)$$

for almost all $(x,\xi) \in \overline{\Omega_{\mathrm{int}}} \times \Omega$ *in the sense of the space* $\mathbf{l}^2(\overline{\Omega_{\mathrm{int}}} \times \Omega)$ $(= \mathrm{L}^2(\overline{\Omega_{\mathrm{int}}} \times \Omega, \mathbb{R}^{3\times 3}))$

Note that it is easy to verify that dilated generators also satisfy the requirements for a generator of a scaling function stated in Definition 4.1.

Theorem 4.1 *Every Navier scaling function belongs to* $\mathbf{l}^2(\overline{\Omega_{\mathrm{int}}} \times \Omega)$.

Proof: With the inner product

$$(\mathbf{f},\mathbf{g})_{\mathbf{l}^2(\overline{\Omega_{\text{int}}}\times\Omega)} = \int_{\overline{\Omega_{\text{int}}}}\int_\Omega \mathbf{f}(x,\xi)\cdot\mathbf{g}(x,\xi)\ d\omega(\xi)\ dx;$$

$\mathbf{f},\mathbf{g}\in\mathbf{l}^2(\overline{\Omega_{\text{int}}}\times\Omega)$; where $\mathbf{f}\cdot\mathbf{g}$ denotes the inner product of second order tensors

$$\mathbf{f}\cdot\mathbf{g} := \sum_{i,j=1}^{3} F_{ij}G_{ij}, \quad \mathbf{f} = (F_{ij})_{i,j=1,2,3}\,,\ \mathbf{g} = (G_{ij})_{i,j=1,2,3}\,,$$

which is sometimes also denoted by $\mathbf{f} : \mathbf{g}$ in literature and called double–dot product, we easily verify that

$$\left(v_{n,k}^{(i)}\otimes y_{n,k}^{(i)}, v_{m,l}^{(j)}\otimes y_{m,l}^{(j)}\right)_{\mathbf{l}^2(\overline{\Omega_{\text{int}}}\times\Omega)} = \delta_{ij}\delta_{mn}\delta_{kl}\left\|v_{n,k}^{(i)}\right\|^2_{\mathrm{l}^2(\overline{\Omega_{\text{int}}})}.$$

Due to (144) we have for $n\geq 3$

$$\begin{aligned}
&\int_{\overline{\Omega_{\text{int}}}}\left|v_{n,j}^{(1)}(x)\right|^2\ dx\\
=\ &\int_0^1 r^2\int_\Omega\left(\left(\sigma_n^{(1)}(r)\right)^2\left|y_{n,j}^{(1)}(\xi)\right|^2+\left(\tau_n^{(1)}(r)\right)^2\left|y_{n,j}^{(2)}(\xi)\right|^2\right)\ d\omega(\xi)\ dr\\
=\ &\int_0^1\left(\sigma_n^{(1)}(r)\right)^2 r^2\ dr+\int_0^1\left(\tau_n^{(1)}(r)\right)^2 r^2\ dr\\
\leq\ &\int_0^1 r^{2n-2}\left(1+n|\alpha_n|\right)^2 r^2\ dr+\int_0^1\alpha_n^2(n(n+1))r^{2n-2}r^2\ dr\\
\leq\ &\frac{(1+n)^2}{2n+1}+\frac{n(n+1)}{2n+1}\leq 2(n+1),\\
&\int_{\overline{\Omega_{\text{int}}}}\left|v_{n,j}^{(2)}(x)\right|^2\ dx = \int_0^1\left(\left(\sigma_n^{(2)}(r)\right)^2+\left(\tau_n^{(2)}(r)\right)^2\right)r^2\ dr\\
\leq\ &\frac{(n(n+1))^2}{n(n+1)}\int_0^1 r^{2n-2}r^2\ dr+(1+n)^2\int_0^1 r^{2n-2}r^2\ dr\\
=\ &\frac{n(n+1)}{2n+1}+\frac{(1+n)^2}{2n+1}\leq 2(n+1),\\
&\int_{\overline{\Omega_{\text{int}}}}\left|v_{n,j}^{(3)}(x)\right|^2\ dx = \int_0^1 r^{2n+2}\ dr = \frac{1}{2n+3}\ .
\end{aligned}$$

The orthogonality yields

$$\|\mathbf{\Phi}_\rho\|^2_{\mathbf{l}^2(\overline{\Omega_{\text{int}}}\times\Omega)} = \sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{k=1}^{2n+1}\left(\varphi_\rho(n)\right)^4\left\|v_{n,k}^{(i)}\right\|^2_{\mathrm{l}^2(\overline{\Omega_{\text{int}}})} < +\infty\ . \tag{158}$$

■

Definition 4.2 *Let $\{\mathbf{\Phi}_\rho\}_{\rho\in(0,\infty)}$ be a Navier scaling function. The convolution $\mathbf{\Phi}_\rho * f$, $f \in \mathrm{l}^2(\Omega)$, is given by*

$$\mathbf{\Phi}_\rho * f = \int_\Omega \mathbf{\Phi}_\rho(\cdot,\xi) f(\xi)\ d\omega(\xi).$$

Moreover, the scale spaces $V_\rho, \rho \in (0,\infty)$, are defined by

$$V_\rho = \left\{ \mathbf{\Phi}_\rho * f \,\middle|\, f \in \mathrm{c}^{(0,0)}(\Omega) \right\}.$$

Theorem 4.2 *Let $\mathbf{\Gamma} \in \mathbf{l}^2(\overline{\Omega_{\mathrm{int}}} \times \Omega)$ be a function which is representable by*

$$\mathbf{\Gamma}(x,\xi) = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{k=1}^{2n+1} \gamma(n)\ v_{n,k}^{(i)}(x) \otimes y_{n,k}^{(i)}(\xi)$$

(for almost every $(x,\xi) \in \overline{\Omega_{\mathrm{int}}} \times \Omega$) in the sense of $\mathbf{l}^2(\overline{\Omega_{\mathrm{int}}} \times \Omega)$, where γ is admissible in the sense of (157). Then the convolution

$$\mathbf{\Gamma} * f = \int_\Omega \mathbf{\Gamma}(\cdot,\xi) f(\xi)\ d\omega(\xi), f \in \mathrm{l}^2(\Omega),$$

is an element of $\mathrm{l}^2(\overline{\Omega_{\mathrm{int}}})$ with the Fourier series

$$\mathbf{\Gamma} * f = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{k=1}^{2n+1} \gamma(n) \left(f, y_{n,k}^{(i)} \right)_{\mathrm{l}^2(\Omega)} v_{n,k}^{(i)}\ .$$

Moreover, if $f \in \mathrm{c}^{(0,0)}(\Omega)$, then this series converges uniformly in every $K \subset \overline{\Omega_{\mathrm{int}}}$ with $\mathrm{dist}(\overline{K},\Omega) > 0$.

Proof: Since $\mathbf{\Gamma}$ is the strong $\mathbf{l}^2(\overline{\Omega_{\mathrm{int}}} \times \Omega)$-limit of the sequence of partial sums

$$\mathbf{S}_N(x,\xi) = \sum_{i=1}^{3} \sum_{n=0_i}^{N} \sum_{k=1}^{2n+1} \gamma(n) v_{n,k}^{(i)}(x) \otimes y_{n,k}^{(i)}(\xi),\ N \in \mathbb{N},$$

and f can be expanded into a Fourier series

$$f = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{k=1}^{2n+1} \left(f, y_{n,k}^{(i)} \right)_{\mathrm{l}^2(\Omega)} y_{n,k}^{(i)}$$

in the sense of $\mathrm{l}^2(\Omega)$, we may conclude that

$$\mathbf{\Gamma} * f \quad = \quad \lim_{N,M\to\infty} \int_\Omega \mathbf{S}_N(\cdot,\xi) \left(\sum_{j=1}^{3} \sum_{m=0_j}^{M} \sum_{l=1}^{2m+1} \left(f, y_{m,l}^{(j)} \right)_{\mathrm{l}^2(\Omega)} y_{m,l}^{(j)}(\xi) \right) d\omega(\xi)$$

$$\begin{aligned} &= \lim_{N,M\to\infty} \sum_{i,j=1}^{3} \sum_{n=0_i}^{N} \sum_{m=0_j}^{M} \sum_{k=1}^{2n+1} \sum_{l=1}^{2m+1} \gamma(n) \left(f, y_{m,l}^{(j)} \right)_{\mathrm{l}^2(\Omega)} v_{n,k}^{(i)} \\ &\quad \times \int_\Omega \left(y_{n,k}^{(i)}(\xi) \right)^{\mathrm{T}} y_{m,l}^{(j)}(\xi) \; d\omega(\xi) \\ &= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{k=1}^{2n+1} \gamma(n) \left(f, y_{n,k}^{(i)} \right)_{\mathrm{l}^2(\Omega)} v_{n,k}^{(i)}, \end{aligned}$$

where the equality is understood in $\mathrm{l}^2(\overline{\Omega_{\mathrm{int}}})$–sense. The convergence in $\mathrm{l}^2(\overline{\Omega_{\mathrm{int}}})$ can be seen from

$$\int_{\overline{\Omega_{\mathrm{int}}}} \left| \int_\Omega \mathbf{\Gamma}(x,\xi) f(\xi) \; d\omega(\xi) \right|^2 \; dx \leq \int_{\overline{\Omega_{\mathrm{int}}}} \int_\Omega \left| \mathbf{\Gamma}(x,\xi) \right|^2 \; d\omega(\xi) \int_\Omega |f(\xi)|^2 \; d\omega(\xi) \; dx,$$

such that

$$\begin{aligned} &\int_{\overline{\Omega_{\mathrm{int}}}} \left| \int_\Omega \left(\mathbf{\Gamma} - \mathbf{S}_N \right)(x,\xi) f(\xi) \; d\omega(\xi) \right|^2 \; dx \\ &\qquad \leq \int_{\overline{\Omega_{\mathrm{int}}}} \int_\Omega \left| \left(\mathbf{\Gamma} - \mathbf{S}_N \right)(x,\xi) \right|^2 \; d\omega(\xi) \; dx \, \|f\|^2_{\mathrm{l}^2(\Omega)} \\ &\qquad = \|\mathbf{\Gamma} - \mathbf{S}_N\|^2_{\mathbf{l}^2(\overline{\Omega_{\mathrm{int}}} \times \Omega)} \|f\|^2_{\mathrm{l}^2(\Omega)} \longrightarrow 0, \quad N \to \infty \end{aligned}$$

and

$$\begin{aligned} &\int_{\overline{\Omega_{\mathrm{int}}}} \left| \int_\Omega \mathbf{S}_N(x,\xi) \left(f(\xi) - \sum_{j=1}^{3} \sum_{m=0_j}^{M} \sum_{l=1}^{2m+1} \left(f, y_{m,l}^{(j)} \right)_{\mathrm{l}^2(\Omega)} y_{m,l}^{(j)}(\xi) \right) d\omega(\xi) \right|^2 \; dx \\ &\qquad \leq \int_{\overline{\Omega_{\mathrm{int}}}} \int_\Omega \left| \mathbf{S}_N(x,\xi) \right|^2 \; d\omega(\xi) \; dx \\ &\qquad \quad \times \int_\Omega \left| \sum_{j=1}^{3} \sum_{m=M+1}^{\infty} \sum_{l=1}^{2m+1} \left(f, y_{m,l}^{(j)} \right)_{\mathrm{l}^2(\Omega)} y_{m,l}^{(j)}(\xi) \right|^2 \; d\omega(\xi) \\ &\qquad \longrightarrow 0, \quad M \to \infty \ . \end{aligned}$$

The uniform convergence in K is a consequence of Theorem 3.1, since (78) and (157) obviously imply that a function g of the form

$$g = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{k=1}^{2n+1} \gamma(n) \left(f, y_{n,k}^{(i)} \right)_{\mathrm{l}^2(\Omega)} y_{n,k}^{(i)},$$

satisfies the estimate

$$\left| \sum_{i=1}^{3} \sum_{n=N}^{\infty} \sum_{k=1}^{2n+1} \gamma(n) \left(f, y_{n,k}^{(i)} \right)_{\mathrm{l}^2(\Omega)} y_{n,k}^{(i)}(\xi) \right|$$

$$
\begin{aligned}
&\leq \left(\sum_{i=1}^{3} \sum_{n=N}^{\infty} \sum_{k=1}^{2n+1} \left(f, y_{n,k}^{(i)} \right)^2_{\mathrm{l}^2(\Omega)} \right)^{1/2} \left(\sum_{i=1}^{3} \sum_{n=N}^{\infty} \sum_{k=1}^{2n+1} (\gamma(n))^2 \left| y_{n,k}^{(i)}(\xi) \right|^2 \right)^{1/2} \\
&= \left(\sum_{i=1}^{3} \sum_{n=N}^{\infty} \sum_{k=1}^{2n+1} \left(f, y_{n,k}^{(i)} \right)^2_{\mathrm{l}^2(\Omega)} \right)^{1/2} \left(3 \sum_{n=N}^{\infty} (\gamma(n))^2 \frac{2n+1}{4\pi} \right)^{1/2} . \qquad (159)
\end{aligned}
$$

It is clear, that (159) tends to 0 as $N \to \infty$, hence, g is an element of $\mathrm{c}^{(0,0)}(\Omega)$, and $\mathbf{\Gamma} * f \in \mathrm{pot}^{(0,0)}(\overline{\Omega_{\mathrm{int}}})$ is the solution of the corresponding Dirichlet problem given by Theorem 3.2. ■

Since φ_ρ^2 is, in particular, admissible, Theorem 4.2 is applicable to the convolution with a scaling function. This, however, immediately implies the multiresolution analysis of the solution of the Dirichlet boundary value problem.

Theorem 4.3 *Let $\{\mathbf{\Phi}_\rho\}_{\rho \in (0,\infty)}$ be a Navier scaling function. Then the unique solution u of the (inner) Dirichlet problem $u \in \mathrm{pot}^{(0,0)}(\overline{\Omega_{\mathrm{int}}})$ unknown, $u^- = f \in \mathrm{c}^{(0,0)}(\Omega)$ given, allows a multiscale approximation:*

$$\lim_{\substack{\rho \to 0 \\ \rho > 0}} \|\mathbf{\Phi}_\rho * f - u\|_{\mathrm{c}(K)} = 0 \qquad (160)$$

for all $K \subset \Omega_{\mathrm{int}}$ with $\mathrm{dist}(\overline{K}, \Omega) \geq \gamma > 0$. Moreover, if $\frac{\varphi_\rho}{\varphi_{\rho'}}$, $\rho > \rho'$, is also admissible in the sense that

$$\sum_{\substack{n=0 \\ \varphi_{\rho'}(n) \neq 0}}^{\infty} \frac{(2n+1)^2}{4\pi} \left(\frac{\varphi_\rho(n)}{\varphi_{\rho'}(n)} \right)^2 < +\infty \qquad (161)$$

then the scale spaces form a multiresolution analysis

$$
\begin{aligned}
&V_\rho \subset V_{\rho'} \subset \mathrm{pot}^{(0,0)}(\overline{\Omega_{\mathrm{int}}}), \\
&\overline{\bigcup_{\rho \in \mathbb{R}^+} V_\rho}^{\|\cdot\|_{\mathrm{c}(K)}} \supset \mathrm{pot}^{(0,0)}(\overline{\Omega_{\mathrm{int}}})|K \ .
\end{aligned}
$$

Proof: Due to Theorem 3.2 and Theorem 4.2 we have

$$
\begin{aligned}
u(x) &= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} \left(f, y_{n,j}^{(i)} \right)_{\mathrm{l}^2(\Omega)} v_{n,j}^{(i)}(x), \\
\left(\mathbf{\Phi}_\rho * f \right)(x) &= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} (\varphi_\rho(n))^2 \left(f, y_{n,j}^{(i)} \right)_{\mathrm{l}^2(\Omega)} v_{n,j}^{(i)}(x)
\end{aligned}
$$

for all $x \in K$ with $K \subset \Omega_{\mathrm{int}}$ and $\mathrm{dist}(\overline{K}, \Omega) \geq \gamma > 0$. From (144) we obtain for $x \in K$, $r = |x|$, $n \geq 3$

$$\left| \sigma_n^{(1)}(r) \right| \leq (1-\gamma)^{n-1}(1+n),$$

$$\begin{aligned}
\left|\sigma_n^{(2)}(r)\right| &\leq (n(n+1))^{-1/2}\, n(n+1)(1-\gamma)^{n-1} = (n(n+1))^{1/2}\,(1-\gamma)^{n-1}, \\
\left|\sigma_n^{(3)}(r)\right| &\leq (1-\gamma)^n, \\
\left|\tau_n^{(1)}(r)\right| &\leq (n(n+1))^{1/2}\,(1-\gamma)^{n-1}, \\
\left|\tau_n^{(2)}(r)\right| &\leq (1-\gamma)^{n-1}(1+n),
\end{aligned}$$

and, consequently,

$$\left|v_{n,j}^{(i)}(x)\right| \leq (1-\gamma)^{n-1}(1+n)2\sqrt{\frac{2n+1}{4\pi}}; \quad i=1,2,3.$$

Hence, we may conclude that

$$\begin{aligned}
&\left|\left(\mathbf{\Phi}_\rho * f\right)(x) - u(x)\right| \\
&= \left|\sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{j=1}^{2n+1}\left((\varphi_\rho(n))^2-1\right)\left(f, y_{n,j}^{(i)}\right)_{\mathrm{l}^2(\Omega)} v_{n,j}^{(i)}(x)\right| \\
&\leq \left(\sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{j=1}^{2n+1}\left((\varphi_\rho(n))^2-1\right)^2\left|v_{n,j}^{(i)}(x)\right|^2\right)^{1/2} \|f\|_{\mathrm{l}^2(\Omega)} \ . \qquad (162)
\end{aligned}$$

For $x \in K$ the series in (162) is uniformly convergent with respect to $\rho \in \mathbb{R}^+$, since $0 \leq 1-(\varphi_\rho(n))^2 \leq 1$ and

$$\sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{j=1}^{2n+1}\left\|v_{n,j}^{(i)}\right\|_{\mathrm{c}(K)}^2 < +\infty \ .$$

Thus, we are allowed to write

$$\begin{aligned}
0 &\leq \lim_{\substack{\rho\to 0\\ \rho>0}} \left\|\mathbf{\Phi}_\rho * f - u\right\|_{\mathrm{c}(K)} \\
&\leq \lim_{\substack{\rho\to 0\\ \rho>0}} \left(\sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{j=1}^{2n+1}\left((\varphi_\rho(n))^2-1\right)^2\left\|v_{n,j}^{(i)}\right\|_{\mathrm{c}(K)}^2\right)^{1/2}\|f\|_{\mathrm{l}^2(\Omega)}, \\
&= \left(\sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{j=1}^{2n+1}\left(\lim_{\substack{\rho\to 0\\ \rho>0}}(\varphi_\rho(n))^2-1\right)^2\left\|v_{n,j}^{(i)}\right\|_{\mathrm{c}(K)}^2\right)^{1/2}\|f\|_{\mathrm{l}^2(\Omega)} \\
&= 0 \ .
\end{aligned}$$

Moreover, note that $\mathbf{\Phi}_\rho * f$ can be regarded as solution of the Dirichlet problem $\mathbf{\Phi}_\rho * f \in \mathrm{pot}^{(0,0)}(\overline{\Omega_{\mathrm{int}}})$, $(\mathbf{\Phi}_\rho * f)^- = g_\rho \in \mathrm{c}^{(0,0)}(\Omega)$, where

$$g_\rho = \sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{j=1}^{2n+1}(\varphi_\rho(n))^2\left(f, y_{n,j}^{(i)}\right)_{\mathrm{l}^2(\Omega)} y_{n,j}^{(i)},$$

as we already derived in the proof of Theorem 4.2. Thus, since $0 \leq \varphi_\rho(n) \leq \varphi_{\rho'}(n)$ for all $n \in \mathbb{N}_0$, we may conclude that $u_\rho \in V_\rho$, $u_\rho^- = g_\rho \in \mathrm{c}^{(0,0)}(\Omega)$ implies the existence of a function

$$h_{\rho,\rho'} = \sum_{i=1}^{3} \sum_{\substack{n=0_i \\ \varphi_{\rho'}(n)\neq 0}}^{\infty} \sum_{j=1}^{2n+1} \left(\frac{\varphi_\rho(n)}{\varphi_{\rho'}(n)}\right)^2 \left(f, y_{n,j}^{(i)}\right)_{\mathrm{l}^2(\Omega)} y_{n,j}^{(i)}$$

in $\mathrm{c}^{(0,0)}(\Omega)$ in analogy to (159) due to the requirement that $\frac{\varphi_\rho}{\varphi_{\rho'}}$ is also admissible. Obviously, $\mathbf{\Phi}_{\rho'} * h_{\rho,\rho'} = \mathbf{\Phi}_\rho * f$, such that $\mathbf{\Phi}_\rho * f \in V_{\rho'}$.

Finally, for $u \in \mathrm{pot}^{(0,0)}(\overline{\Omega_{\mathrm{int}}})$ there exists $f \in \mathrm{c}^{(0,0)}(\Omega)$ such that $u^- = f$ and $\mathbf{\Phi}_\rho * f \to u$ uniformly on every $K \subset \Omega_{\mathrm{int}}$ with $\mathrm{dist}(\overline{K}, \Omega) \geq \gamma > 0$. Since $\mathbf{\Phi}_\rho * f \in V_\rho$ for all $\rho \in \mathbb{R}^+$ we have

$$u \in \overline{\bigcup_{\rho>0} V_\rho}^{\|\cdot\|_{\mathrm{c}(K)}}$$

■

Note that the additional requirement that $\frac{\varphi_\rho}{\varphi_{\rho'}}$, $\rho > \rho'$, is admissible is satisfied by all bandlimited generators and a series of non–bandlimited generators such as the Abel-Poisson generator, where

$$\sum_{n=0}^{\infty} \frac{(2n+1)^k}{4\pi} \left(\frac{e^{-R\rho n}}{e^{-R\rho' n}}\right)^2 = \sum_{n=0}^{\infty} \frac{(2n+1)^k}{4\pi} e^{-2R(\rho-\rho')n} < \infty,\ k \in \mathbb{N},$$

and the Gauß–Weierstraß generator, where

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(2n+1)^k}{4\pi} \left(\frac{e^{-R\rho n(\rho n+1)}}{e^{-R\rho' n(\rho' n+1)}}\right)^2 \\ = &\sum_{n=0}^{\infty} \frac{(2n+1)^k}{4\pi} e^{-2R\left(n^2\left(\rho^2-(\rho')^2\right)+n\left(\rho-\rho'\right)\right)} < +\infty,\ k \in \mathbb{N}\ . \end{aligned}$$

5 The Scaling Functions for the Neumann Problem

In analogy to the Dirichlet boundary value problem we can also construct an approximate identity for the solution of the inner Neumann problem by defining the corresponding scaling function ${}^{\mathrm{N}}\mathbf{\Phi}_\rho \in \mathbf{l}^2(\overline{\Omega_{\mathrm{int}}} \times \Omega)$ as

$${}^{\mathrm{N}}\mathbf{\Phi}_\rho(x,\xi) = w_{0,1}^{(1)}(x) \otimes y_{0,1}^{(1)}(\xi) + \sum_{k=1}^{3} (\varphi_\rho(1))^2\, w_{1,k}^{(1)}(x) \otimes y_{1,k}^{(1)}(\xi)$$

$$+ \quad \sum_{i=1}^{3} \sum_{n=2}^{\infty} \sum_{k=1}^{2n+1} (\varphi_\rho(n))^2 \, w_{n,k}^{(i)}(x) \otimes y_{n,k}^{(i)}(\xi) \tag{163}$$

(for almost all $(x,\xi) \in \overline{\Omega_{\mathrm{int}}} \times \Omega$).

Note that (cf. Lemma 3.3)

$$\frac{3}{n} + \frac{2}{n\tau} - \frac{2}{\tau} - \frac{1}{n\tau} = \frac{3\tau + 2 - 2n - 1}{n\tau} = \frac{3\tau - 2n + 1}{n\tau} < 0$$

for $n > \frac{3\tau+1}{2}$, such that

$$0 > \alpha_n = -\frac{1}{2} \frac{1 + \frac{3}{n} + \frac{2}{n\tau}}{1 + \frac{2n}{n\tau} + \frac{1}{n\tau}} > -\frac{1}{2}$$

and, consequently,

$$3 + n + 2n\alpha_n > 3 + n - n = 3$$

for $n \geq \frac{3\tau+1}{2}$. Hence, we have

$$\begin{aligned} |\zeta_n| &\leq \frac{1}{(\tilde{\lambda} + \tilde{\mu})3 - \tilde{\mu}} = \frac{1}{3\tilde{\lambda} + 2\tilde{\mu}}, \\ |\alpha_n| &\leq 1, n \geq 3, \end{aligned}$$

and with $(x = |x|\xi)$

$$\begin{aligned} w_{n,j}^{(1)}(x) &= \zeta_n \Bigg(|x|^{n+1} Y_{n,j}(\xi)\xi + \alpha_n |x|^2 n |x|^{n-1} Y_{n,j}(\xi)\xi + \alpha_n |x|^{n+1} \nabla_\xi^* Y_{n,j}(\xi) \\ &\quad - \frac{1 + 2n\alpha_n}{2(n-1)} \left(n|x|^{n-1} Y_{n,j}(\xi)\xi + |x|^{n-1} \nabla_\xi^* Y_{n,j}(\xi) \right) \Bigg) \\ &= \zeta_n \left(|x|^{n-1} \left(|x|^2 (1 + \alpha_n n) - n \frac{1 + 2n\alpha_n}{2(n-1)} \right) y_{n,j}^{(1)}(\xi) \right. \\ &\quad \left. + |x|^{n-1} \left(\alpha_n |x|^2 - \frac{1 + 2n\alpha_n}{2(n-1)} \right) y_{n,j}^{(2)}(\xi) (n(n+1))^{1/2} \right), \\ w_{n,j}^{(2)}(x) &= (n(n+1))^{-1} (2\tilde{\mu}(n-1))^{-1} \left(n|x|^{n-1} Y_{n,j}(\xi)\xi + |x|^{n-1} \nabla_\xi^* Y_{n,j}(\xi) \right) \\ &\quad - (n(n+1))^{-1/2} \, n w_{n,j}^{(1)}(x), \\ w_{n,j}^{(3)}(x) &= (n(n+1))^{-1/2} (\tilde{\mu}(n-1))^{-1} \, \xi \wedge \nabla_\xi^* Y_{n,j}(\xi) |x|^n \\ &= (\tilde{\mu}(n-1))^{-1} |x|^n y_{n,j}^{(3)}(\xi) \end{aligned}$$

for $n = 2, 3, \ldots; \, j = 1, \ldots, 2n+1$.

Hence, we have for $n \geq \max(3, \frac{3\tau+1}{2})$

$$\int_{\Omega_{\mathrm{int}}} \left| w_{n,j}^{(1)}(x) \right|^2 \, dx \quad \leq \quad \zeta_n^2 \left(\int_0^1 r^{2n} \, dr \left((1+n) + n \frac{1+2n}{2(n-1)} \right)^2 \right.$$

$$
\begin{aligned}
& + \int_0^1 r^{2n}\, dr \left(1 + \frac{1+2n}{2(n-1)}\right)^2 n(n+1)\Bigg) \\
\leq \;& \frac{1}{(3\tilde{\lambda}+2\tilde{\mu})^2}\left(\frac{(2+3n)^2}{2n+1} + 9\frac{n(n+1)}{2n+1}\right) \\
\leq \;& \frac{1}{(3\tilde{\lambda}+2\tilde{\mu})^2}\left(2(3n+2)+9n\right),
\end{aligned}
$$

$$
\begin{aligned}
\left\| w_{n,j}^{(2)} \right\|_{\mathbf{l}^2(\overline{\Omega_{\mathrm{int}}})} \leq\;& \left\| w_{n,j}^{(2)} + (n(n+1))^{-1/2} n w_{n,j}^{(1)} \right\|_{\mathbf{l}^2(\overline{\Omega_{\mathrm{int}}})} \\
&+ \frac{n}{(n(n+1))^{1/2}} \left\| w_{n,j}^{(1)} \right\|_{\mathbf{l}^2(\overline{\Omega_{\mathrm{int}}})} \\
\leq\;& (n(n+1))^{-1} (2\tilde{\mu}(n-1))^{-1} \left(\left(n^2 + n(n+1)\right) \int_0^1 r^{2n}\, dr \right)^{1/2} \\
&+ \left\| w_{n,j}^{(1)} \right\|_{\mathbf{l}^2(\overline{\Omega_{\mathrm{int}}})} \\
\leq\;& \frac{(n^2+n(n+1))^{1/2}}{(2n+1)^{1/2}(n(n+1))(2\tilde{\mu}(n-1))} + \frac{(15n+4)^{1/2}}{3\tilde{\lambda}+2\tilde{\mu}} \\
=\;& O\left(n^{1/2}\right), \quad n \to \infty, \\
\left\| w_{n,j}^{(3)} \right\|_{\mathbf{l}^2(\overline{\Omega_{\mathrm{int}}})} =\;& (\tilde{\mu}(n-1))^{-1} \frac{1}{2n+3} n(n+1) = O(1), \quad n \to \infty.
\end{aligned}
$$

This guarantees that ${}^{\mathrm{N}}\mathbf{\Phi}_J \in \mathbf{l}^2(\overline{\Omega_{\mathrm{int}}} \times \Omega)$ in the Neumann case in analogy to (158). Moreover, we get for sufficiently large n and $y \in K$, where $K \subset \Omega_{\mathrm{int}}$ with $\operatorname{dist}(\overline{K}, \Omega) > \gamma > 0$

$$
\begin{aligned}
\left| w_{n,j}^{(1)}(y) \right| \leq\;& |\zeta_n| \Bigg((1-\gamma)^{n-1} \left(n+1+n\frac{1+2n}{2n-2}\right) \sqrt{\frac{2n+1}{4\pi}} \\
&+ (1-\gamma)^{n-1}\left(1 + \frac{1+2n}{2n-2}\right) \sqrt{\frac{2n+1}{4\pi}} (n(n+1))^{1/2} \Bigg) \\
=\;& (1-\gamma)^{n-1} O\left(n^{3/2}\right), \quad n \to \infty, \\
\left| w_{n,j}^{(2)}(y) \right| \leq\;& (n(n+1))^{-1} \frac{(1-\gamma)^{n-1}}{2\tilde{\mu}(n-1)} \left(n + (n(n+1))^{1/2}\right) \sqrt{\frac{2n+1}{4\pi}} \\
&+ (1-\gamma)^{n-1} O\left(n^{3/2}\right) \\
=\;& (1-\gamma)^{n-1} O\left(n^{3/2}\right),\ n \to \infty, \\
\left| w_{n,j}^{(3)}(y) \right| \leq\;& (\tilde{\mu}(n-1))^{-1} (1-\gamma)^n \sqrt{\frac{2n+1}{4\pi}}\ .
\end{aligned}
$$

Thus, we get in analogy to Theorem 4.3 the following theorem.

Theorem 5.1 *Let $\{ {}^{\mathrm{N}}\mathbf{\Phi}_\rho \}_{\rho \in (0,\infty)}$ be a Navier scaling function for the Neumann problem as introduced in (163). Then the solution u of the (inner) Neumann problem $u \in \mathrm{pot}^{(1,\alpha)}(\overline{\Omega_{\mathrm{int}}})$, $T_\nu(u)^- = f \in \mathrm{c}^{(0,\alpha)}(\Omega)$,*

$$\int_\Omega f(\xi)\ d\omega(\xi) = 0, \qquad \int_\Omega (f(\xi) \wedge \xi)\ d\omega(\xi) = 0$$

allows a multiscale approximation:

$$\lim_{\substack{\rho \to 0 \\ \rho > 0}} \left\| {}^{\mathrm{N}}\mathbf{\Phi}_\rho * f - u \right\|_{\mathrm{c}(K)} = 0$$

for all $K \subset \Omega_{\mathrm{int}}$ with $\mathrm{dist}(\overline{K}, \Omega) \geq \gamma > 0$.

Moreover, the scale spaces, defined by

$$V_\rho = \left\{ {}^{\mathrm{N}}\mathbf{\Phi}_\rho * h \,\middle|\, h \in \mathrm{c}^{(0,\alpha)}(\Omega) \right\},$$

form a multiresolution analysis, if $\frac{\varphi_\rho}{\varphi_{\rho'}}$, $\rho > \rho'$, is admissible in the sense of (161):

$$\begin{aligned} & V_\rho \subset V_{\rho'} \subset \mathrm{pot}^{(1,\alpha)}\left(\overline{\Omega_{\mathrm{int}}}\right), \\ & \overline{\bigcup_{\rho \in \mathbb{R}^+} V_\rho}^{\|\cdot\|_{\mathrm{c}(K)}} \supset \mathrm{pot}^{(1,\alpha)}\left(\overline{\Omega_{\mathrm{int}}}\right)\Big| K \ . \end{aligned}$$

Proof: It remains to prove that $g \in \mathrm{c}^{(0)}(\Omega)$ with

$$g = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} (p(n))^2 \left(f, y_{n,j}^{(i)} \right)_{\mathrm{l}^2(\Omega)} y_{n,j}^{(i)},$$

where p is admissible, is α–Hölder continuous. Note that the conditions (156) are only needed for the uniqueness of the solution and not for its existence. Obviously, we have

$$\left(g, y_{n,j}^{(i)} \right)_{\mathrm{l}^2(\Omega)} = (p(n))^2 \left(f, y_{n,j}^{(i)} \right)_{\mathrm{l}^2(\Omega)},$$

such that

$$\sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} (p(n))^{-4} \left(g, y_{n,j}^{(i)} \right)_{\mathrm{l}^2(\Omega)}^2 < +\infty,$$

since $f \in \mathrm{l}^2(\Omega)$. Consulting [14] we come to the conclusion that g is an element of a vectorial Sobolev space $h(\{A_n\}; \Omega)$ with $A_n = (p(n))^{-2}$, where the summability condition

$$\sum_{n=0}^{\infty} \frac{(n+\frac{1}{2})^3}{A_n^2} = \sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^3 (p(n))^4 \leq \left(\sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{3/2} (p(n))^2\right)^2 < +\infty$$

implies that $h(\{A_n\};\Omega) \subset \mathrm{c}^{(1)}(\Omega)$. Hence, g is α–Hölder continuous. With this property in mind we are able to transfer the proofs of the Theorems 4.2 and 4.3. ■

Note that for $f \in \mathrm{l}^2(\Omega)$ we already get that the convolution $(\Phi_\rho * f)^-$ yields a Hölder–continuous function.

6 Scale Continuous Wavelets

The development of wavelets that generate band pass filters filling the gaps between low pass filters constructed via scaling functions is here motivated by the approaches in, for example, [12], [15], [16], [27], and [28]. We will, therefore, concentrate on a brief, concise treatment of this subject. We obtain the following reconstruction formulae.

Definition 6.1 *Let $\varphi_1 : [0,\infty) \to \mathbb{R}$ be a piecewise continuously differentiable generator of a scaling function and ψ_1 be a generator of the mother wavelet, i.e. ψ_1 satisfies the admissibility condition (157) and the differential equation*

$$\psi_1(t) = \left(-t\frac{d}{dt}(\varphi_1(t))^2\right)^{1/2} = (-2t\varphi_1(t)\varphi_1'(t))^{1/2}. \tag{164}$$

Its dilates ψ_ρ are defined in analogy to the dilates of a generator of a scaling function. Then $\tilde{\mathbf{\Psi}}_\rho \in \mathbf{l}^2(\overline{\Omega_{\mathrm{int}}} \times \Omega)$ and $\mathbf{\Psi}_\rho \in \mathbf{l}^2(\Omega\times\Omega)$, $\rho \in (0,\infty)$, are defined by

$$\tilde{\mathbf{\Psi}}_\rho(x,\xi) = \sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{k=1}^{2n+1} \psi_\rho(n) v_{n,k}^{(i)}(x) \otimes y_{n,k}^{(i)}(\xi),$$

$$\mathbf{\Psi}_\rho(\xi,\eta) = \sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{k=1}^{2n+1} \psi_\rho(n) y_{n,k}^{(i)}(\xi) \otimes y_{n,k}^{(i)}(\eta),$$

for almost all $(x,\xi) \in \overline{\Omega_{\mathrm{int}}} \times \Omega$, $(\xi,\eta) \in \Omega^2$. $\{\mathbf{\Psi}_\rho\}_{\rho\in(0,\infty)}$ and $\{\tilde{\mathbf{\Psi}}_\rho\}_{\rho\in(0,\infty)}$ are called (scale continuous) decomposition and reconstruction Navier wavelets, respectively.

Theorem 6.1 *Let $\{\mathbf{\Phi}_\rho\}_{\rho\in(0,\infty)}$ be a Navier scaling function and let $\{\tilde{\mathbf{\Psi}}_\rho\}_{\rho\in(0,\infty)}$ and $\{\mathbf{\Psi}_\rho\}_{\rho\in(0,\infty)}$ be the corresponding reconstruction and decomposition wavelets, respectively.*

If $u \in \mathrm{pot}^{(0,0)}(\overline{\Omega_{\mathrm{int}}})$ is given corresponding to the Dirichlet type data $u^- = f \in \mathrm{c}^{(0,0)}(\Omega)$, then

$$
\begin{aligned}
\mathbf{\Phi}_{R_1} * f &= \mathbf{\Phi}_{R_2} * f + \int_{R_1}^{R_2} \int_\Omega \tilde{\mathbf{\Psi}}_\rho(\cdot,\xi) \int_\Omega \mathbf{\Psi}_\rho(\xi,\eta) f(\eta)\ d\omega(\eta)\ d\omega(\xi) \frac{1}{\rho}\ d\rho, \\
\mathbf{\Phi}_{R_1} * f &= \int_{R_1}^{\infty} \int_\Omega \tilde{\mathbf{\Psi}}_\rho(\cdot,\xi) \int_\Omega \mathbf{\Psi}_\rho(\xi,\eta) f(\eta)\ d\omega(\eta)\ d\omega(\xi) \frac{1}{\rho}\ d\rho,
\end{aligned}
$$

in the sense of $\mathrm{l}^2(\overline{\Omega_{\mathrm{int}}})$ and

$$
u = \int_0^{\infty} \int_\Omega \tilde{\mathbf{\Psi}}_\rho(\cdot,\xi) \int_\Omega \mathbf{\Psi}_\rho(\xi,\eta) f(\eta)\ d\omega(\eta)\ d\omega(\xi) \frac{1}{\rho}\ d\rho,
$$

in the sense of $\mathrm{c}^{(0)}(K)$, $K \subset \Omega_{\mathrm{int}}$, $\mathrm{dist}(\overline{K},\Omega) \geq \gamma > 0$ for all $R_1, R_2 \in (0,\infty)$ with $R_1 < R_2$.

Proof: Let I be an interval of the form $[R_1, R_2]$, $[R_1,\infty)$ or $(0,\infty)$. Then we obtain by interchanging the order of integration

$$
\begin{aligned}
&\int_I \int_\Omega \tilde{\mathbf{\Psi}}_\rho(\cdot,\xi) \int_\Omega \mathbf{\Psi}_\rho(\xi,\eta) f(\eta)\ d\omega(\eta)\ d\omega(\xi) \frac{1}{\rho}\ d\rho \\
&= \int_I \int_\Omega f(\eta) \int_\Omega \tilde{\mathbf{\Psi}}_\rho(\cdot,\xi) \mathbf{\Psi}_\rho(\xi,\eta)\ d\omega(\xi)\ d\omega(\eta) \frac{1}{\rho}\ d\rho \\
&= \int_I \int_\Omega f(\eta) \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{k=1}^{2n+1} (\psi_\rho(n))^2\, v_{n,k}^{(i)} \otimes y_{n,k}^{(i)}(\eta)\ d\omega(\eta) \frac{1}{\rho}\ d\rho \\
&= \int_\Omega f(\eta) \int_I \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{k=1}^{2n+1} (\psi_\rho(n))^2\, v_{n,k}^{(i)} \otimes y_{n,k}^{(i)}(\eta) \frac{1}{\rho}\ d\rho\ d\omega(\eta)\ .
\end{aligned}
$$

Using the addition theorem for vector spherical harmonics (Equation (69)) and Inequality (77) we may modify the series in the following way (see also Lemma 3.1 and note that $\psi_\rho(0) = 0$)

$$
\begin{aligned}
&\sum_{i=1}^{3} \sum_{n=1}^{\infty} \sum_{k=1}^{2n+1} (\psi_\rho(n))^2 v_{n,k}^{(i)}(x) \otimes y_{n,k}^{(i)}(\eta) \\
&= \sum_{n=1}^{\infty} \frac{2n+1}{4\pi} \left(\sigma_n^{(1)}(|x|) \mathbf{p}_n^{(1,1)}\left(\frac{x}{|x|},\eta\right) + \tau_n^{(1)}(|x|) \mathbf{p}_n^{(2,1)}\left(\frac{x}{|x|},\eta\right) \right. \\
&\qquad + \sigma_n^{(2)}(|x|) \mathbf{p}_n^{(1,2)}\left(\frac{x}{|x|},\eta\right) + \tau_n^{(2)}(|x|) \mathbf{p}_n^{(2,2)}\left(\frac{x}{|x|},\eta\right) \\
&\qquad \left. + \ \sigma_n^{(3)}(|x|) \mathbf{p}_n^{(3,3)}\left(\frac{x}{|x|},\eta\right) \right) (\psi_\rho(n))^2 .
\end{aligned}
$$

The absolute value of each component of this tensor is less or equal

$$\sum_{n=1}^{\infty} \frac{2n+1}{4\pi}|x|^{n-1}\Big((1+2n)+2(n(n+1))^{+\frac{1}{2}}+(n(n+1))^{-\frac{1}{2}}n(1+2n)$$
$$+1+2n+|x|\Big)(\psi_\rho(n))^2 \ .$$

For fixed $x \in \Omega_{\rm int}$ this series is integrable with respect to ρ on $[R,\infty)$, $R>0$ according to the Beppo Levi Theorem. Thus, we obtain for $M \in \mathbb{R}^+ (M > R)$:

$$\begin{aligned}
&\int_R^M \sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{k=1}^{2n+1} (\psi_\rho(n))^2\, v_{n,k}^{(i)}(x)\otimes y_{n,k}^{(i)}(\eta)\frac{1}{\rho}\ d\rho \\
=\ & \sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{k=1}^{2n+1} \int_R^M (\psi_\rho(n))^2\,\frac{1}{\rho}\ d\rho\ v_{n,k}^{(i)}(x)\otimes y_{n,k}^{(i)}(\eta) \\
=\ & \sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{k=1}^{2n+1} \left((\varphi_R(n))^2-(\varphi_M(n))^2\right) v_{n,k}^{(i)}(x)\otimes y_{n,k}^{(i)}(\eta) \\
=\ & \mathbf{\Phi}_R(x,\eta)-\mathbf{\Phi}_M(x,\eta). \qquad (165)
\end{aligned}$$

Consequently, we obtain in the sense of $\mathrm{l}^2(\overline{\Omega_{\rm int}})$ and $\mathrm{c}(K)$

$$\int_{R_1}^{R_2}\int_\Omega \tilde{\mathbf{\Psi}}_\rho(\cdot,\xi)\int_\Omega \mathbf{\Psi}_\rho(\xi,\eta)f(\eta)\ d\omega(\eta)\ d\omega(\xi)\frac{1}{\rho}\ d\rho$$
$$= \mathbf{\Phi}_{R_1}*f-\mathbf{\Phi}_{R_2}*f \ .$$

Since the series in (165) is uniformly convergent with respect to M (cf. the admissibility condition) we also get

$$\int_{R_1}^{\infty}\int_\Omega \tilde{\mathbf{\Psi}}_\rho(\cdot,\xi)\int_\Omega \mathbf{\Psi}(\xi,\eta)f(\eta)\ d\omega(\eta)\ d\omega(\xi)\frac{1}{\rho}\ d\rho = \mathbf{\Phi}_{R_1}*f$$

such that Theorem 4.3 finally yields

$$\int_0^{\infty}\int_\Omega \tilde{\mathbf{\Psi}}_\rho(\cdot,\xi)\int_\Omega \mathbf{\Psi}(\xi,\eta)f(\eta)\ d\omega(\eta)\ d\omega(\xi)\frac{1}{\rho}\ d\rho = u$$

in the sense of $\mathrm{c}(K)$. ■

The analogous reconstruction wavelets for the Neumann problem may be defined by

$${}^{\rm N}\tilde{\mathbf{\Psi}}_\rho(x,\xi)=\sum_{k=1}^{3}\psi_\rho(1)w_{1,k}^{(1)}(x)\otimes y_{1,k}^{(1)}(\xi)+\sum_{i=1}^{3}\sum_{n=2}^{\infty}\sum_{k=1}^{2n+1}\psi_\rho(n)w_{n,k}^{(i)}(x)\otimes y_{n,k}^{(i)}(\xi)$$

for almost every $(x, \xi) \in \overline{\Omega_{\mathrm{int}}} \times \Omega$.

The decomposition wavelets are identical in case of the Dirichlet and the Neumann problem. The derivation of the reconstruction formulae in the Neumann case is completely analogous to the Dirichlet case.

Hence, if $f \in \mathrm{c}^{(0,\alpha)}(\Omega)$ satisfies the condition

$$\int_\Omega f(\xi)\ d\omega(\xi) = 0, \quad \int_\Omega (f(\xi) \wedge \xi)\ d\omega(\xi) = 0,$$

then a solution $u \in \mathrm{pot}^{(1,\alpha)}(\overline{\Omega_{\mathrm{int}}})$ of Neumann's problem $T_\nu(u)^- = f$ is given by

$$u = \int_0^\infty \int_\Omega {}^{\mathrm{N}}\tilde{\mathbf{\Psi}}_\rho(\cdot, \xi) \int_\Omega \mathbf{\Psi}_\rho(\xi, \eta) f(\eta)\ d\omega(\eta)\ d\omega(\xi) \frac{1}{\rho} d\rho$$

is the sense of $\mathrm{c}(K)$ for all $K \subset \Omega_{\mathrm{int}}$ with $\mathrm{dist}(\overline{K}, \Omega) \geq \gamma > 0$.

7 Scale Discrete Wavelets

Definition 7.1 *Let $\varphi_0^{\mathrm{D}} := \varphi_1 : [0,\infty) \to \mathbb{R}$ be a generator of a scaling function and $\psi_0^{\mathrm{D}} : [0,\infty) \to \mathbb{R}$ be a corresponding generator of a mother wavelet, i.e. it satisfies the admissibility condition (157) and, in addition, the difference equation*

$$\left(\psi_0^{\mathrm{D}}(t)\right)^2 = \left(\varphi_0^{\mathrm{D}}(t/2)\right)^2 - \left(\varphi_0^{\mathrm{D}}(t)\right)^2, \qquad t \in [0,\infty)\ . \tag{166}$$

The dilation is given by $\psi_j^{\mathrm{D}}(x) = \psi_0^{\mathrm{D}}(2^{-j}x)$; $x \in [0,\infty), j \in \mathbb{Z}$. Then the (scale discrete) decomposition and reconstruction Navier wavelets $\{\mathbf{\Psi}_j^{\mathrm{D}}\}_{j\in\mathbb{Z}} \subset \mathbf{l}^2(\Omega \times \Omega)$ and $\{\tilde{\mathbf{\Psi}}_j^{\mathrm{D}}\}_{j\in\mathbb{Z}} \subset \mathbf{l}^2(\overline{\Omega_{\mathrm{int}}} \times \Omega)$ are given by

$$\mathbf{\Psi}_j^{\mathrm{D}}(\xi, \eta) = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{k=1}^{2n+1} \psi_j^{\mathrm{D}}(n) y_{n,k}^{(i)}(\xi) \otimes y_{n,k}^{(i)}(\eta)$$

and

$$\tilde{\mathbf{\Psi}}_j^{\mathrm{D}}(x, \xi) = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{k=1}^{2n+1} \psi_j^{\mathrm{D}}(n) v_{n,k}^{(i)}(x) \otimes y_{n,k}^{(i)}(\xi)$$

(for almost all $(\xi, \eta) \in \Omega^2$, $(x, \xi) \in \overline{\Omega_{\mathrm{int}}} \times \Omega$), respectively.

For the convolutions

$$\begin{aligned} \mathbf{\Psi}_j^{\mathrm{D}} * f &= \int_\Omega \mathbf{\Psi}_j^{\mathrm{D}}(\cdot, \xi) f(\xi)\ d\omega(\xi), \\ \tilde{\mathbf{\Psi}}_j^{\mathrm{D}} * g &= \int_\Omega \tilde{\mathbf{\Psi}}_j^{\mathrm{D}}(\cdot, \xi) g(\xi)\ d\omega(\xi) \end{aligned}$$

it is easy to verify in analogy to the previous considerations that

$$\begin{aligned}\tilde{\mathbf{\Psi}}_j^{\mathrm{D}} * \left(\mathbf{\Psi}_j^{\mathrm{D}} * f\right) &= \sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{k=1}^{2n+1} \left(\psi_j^{\mathrm{D}}(n)\right)^2 \left(f, y_{n,k}^{(i)}\right)_{\mathrm{l}^2(\Omega)} v_{n,k}^{(i)} \\ &= \mathbf{\Phi}_{2^{-(j+1)}} * f - \mathbf{\Phi}_{2^{-j}} * f\end{aligned}$$

in the sense of $\mathrm{l}^2(\overline{\Omega_{\mathrm{int}}})$ and $\mathrm{c}^{(0)}(K)$ for $K \subset \Omega_{\mathrm{int}}$ with $\mathrm{dist}(\overline{K}, \Omega) \geq \gamma > 0$. Then, we are able to formulate the following result.

Theorem 7.1 *Let $\{\mathbf{\Phi}_\rho\}_{\rho\in(0,\infty)}$ be a given Navier scaling function and $\{\tilde{\mathbf{\Psi}}_j^{\mathrm{D}}\}_{j\in\mathbb{Z}}$, $\{\mathbf{\Psi}_j^{\mathrm{D}}\}_{j\in\mathbb{Z}}$ be the corresponding scale discrete wavelets.*

If $u \in \mathrm{pot}^{(0,0)}(\overline{\Omega_{\mathrm{int}}})$ with $u^- = f \in \mathrm{c}^{(0,0)}(\Omega)$, then

$$\begin{aligned}\mathbf{\Phi}_{2^{-J_2}} * f &= \mathbf{\Phi}_{2^{-J_1}} * f + \sum_{j=J_1}^{J_2-1} \tilde{\mathbf{\Psi}}_j^{\mathrm{D}} * \left(\mathbf{\Psi}_j^{\mathrm{D}} * f\right), \\ u &= \mathbf{\Phi}_{2^{-J_1}} * f + \sum_{j=J_1}^{\infty} \tilde{\mathbf{\Psi}}_j^{\mathrm{D}} * \left(\mathbf{\Psi}_j^{\mathrm{D}} * f\right)\end{aligned}$$

in the sense of $\mathrm{c}(K)$*,* $K \subset \Omega_{\mathrm{int}}$*,* $\mathrm{dist}\ (\overline{K}, \Omega) \geq \gamma > 0$*, for all* $J_1, J_2 \in \mathbb{Z}$ *with* $J_1 < J_2$*.*

In the scale discrete situation the decomposition wavelets for the Neumann problem again coincide with the decomposition wavelets for the Dirichlet problem, where the reconstruction wavelets ${}^{\mathrm{N}}\tilde{\mathbf{\Psi}}_j^{\mathrm{D}} \in \mathbf{l}^2(\overline{\Omega_{\mathrm{int}}} \times \Omega)$, $j \in \mathbb{Z}$, are defined by

$${}^{\mathrm{N}}\tilde{\mathbf{\Psi}}_j^{\mathrm{D}}(x,\xi) = \sum_{k=1}^{3} \psi_j^{\mathrm{D}}(1) w_{1,k}^{(1)}(x) \otimes y_{1,k}^{(1)}(\xi) + \sum_{i=1}^{3}\sum_{n=2}^{\infty}\sum_{k=1}^{2n+1} \psi_j^{\mathrm{D}}(n) w_{n,k}^{(i)}(x) \otimes y_{n,k}^{(i)}(\xi)$$

for almost every $(x,\xi) \in \overline{\Omega_{\mathrm{int}}} \times \Omega$.

The corresponding scale step property is analogous to Theorem 7.1.

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