Ultracoherence and Canonical Transformations

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Abstract

The symplectic group of homogeneous canonical transformations is represented in the bosonic Fock space by the action of the group on the ultracoherent vectors, which are generalizations of the coherent states. The intertwining relations between this representation and the algebra of Weyl operators are derived. They confirm the identification of this representation with Bogoliubov transformations.

1 Introduction

The transformations preserving the commutation relations between conjugate variables in quantum theory are known as canonical transformations and have as their classical counterpart the transformations that preserve the Hamilton equations of motion. An incomplete list of the literature about canonical transformations is [1, 2, 3, 4, 5, 6, 7, 8, 9]. The aim of this paper is to give a self-contained presentation of canonical transformations in quantum mechanics and in quantum field theory using ultracoherent vectors, which are generalizations of the well known coherent states [10]. Such a study is of relevance from the point of view of understanding canonical transformations as well as the fact that it has a number of physical consequences. Squeezing, which has acquired much relevance these days from the point of view of quantum optics and quantum computation [11, 12, 13], belongs to the general class of canonical transformations.

The plan of the paper is as follows. In Section 2 we will briefly discuss the Hilbert and the bosonic Fock spaces. We define and discuss the exponential vectors – related to coherent states – and the more general ultracoherent vectors. In Section 3 we introduce the algebra of Weyl operators and its connection with canonical transformations. That is of special importance for the case of an infinite number of degrees of freedom as in quantum field theory. In Section 4 we investigate the symplectic group on finite and infinite dimensional Hilbert spaces. In the case of infinite dimensions the symplectic transformations can be implemented by unitary operators on the Fock space only if an additional Hilbert-Schmidt condition is satisfied [1, 2, 3, 14]. Under this restriction we construct in Section 5 a unitary ray representation of the symplectic group on the bosonic Fock space by defining the action of this group on exponential and ultracoherent vectors. The intertwining relations of this representation with the algebra of Weyl operators are given in Sect. 5.2. If the dimension of the Hilbert space is infinite the symplectic group has one-parameter subgroups with unbounded generators. A reasonable Lie algebra can

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be defined if the generators are restricted to bounded operators [9]. The representation of a basis of this Lie algebra is presented in Sect. 6 by an study of one-parametzer subgroups of the symplectic group. Some detailed calculations for ultracoherent states and for the multiplication law within the representation of the symplectic group are given in the Appendices A and B.

2 Fock space and ultracoherent vectors

2.1 Hilbert spaces and Fock spaces

In this section we recapitulate some basic notations about Hilbert spaces and Fock spaces of symmetric tensors. Let $\mathcal{H}$ be complex Hilbert space with inner product $(f \mid g)$ and with an antiunitary involution $f \to f^*$, $f^{**} \equiv (f^*)^* = f$. Then the mapping

$$f,\, g \to \langle f \mid g\rangle := (f^* \mid g) \in \mathbb{C} \tag{1}$$

is a symmetric bilinear form $\langle f \mid g\rangle = \langle g \mid f\rangle$.[3] The underlying real Hilbert space of $\mathcal{H}$ is denoted as $\mathcal{H}_{\mathbb{R}}$. This space has the inner product $(f \mid g)_{\mathbb{R}} = \mathrm{Re}\,(f \mid g)$ $= \frac{1}{2}\left((f \mid g) + (f^* \mid g^*)\right)$. As a point set the spaces $\mathcal{H}$ and $\mathcal{H}_{\mathbb{R}}$ coincide. For many calculations it is advantageous to identify $\mathcal{H}_{\mathbb{R}}$ with diagonal subspace $\mathcal{H}_{diag}$ of $\mathcal{H} \times \mathcal{H}^*$. This space is defined as the set of all elements $\begin{pmatrix} f \\ g^* \end{pmatrix} \in \mathcal{H} \times \mathcal{H}^*$ which satisfy $g = f$.

We use the following notations for linear operators: the space of all bounded operators A with operator norm $\|A\|$ is $\mathcal{L}(\mathcal{H})$, the space of all Hilbert-Schmidt operators A with norm $\|A\|_2 = \sqrt{\mathrm{tr}_{\mathcal{H}} A^+ A}$ is $\mathcal{L}_2(\mathcal{H})$, the space of all trace class or nuclear operators A with norm $\|A\|_1 = \mathrm{tr}_{\mathcal{H}} \sqrt{A^+ A}$ is $\mathcal{L}_1(\mathcal{H})$. For operators $A \in \mathcal{L}(\mathcal{H})$ the adjoint operator is denoted by A^+. The complex conjugate operator $\bar{A}$ and the transposed operator A^T are defined by the identities

$$\bar{A} f = (A f^*)^*\,,\ A^T f = \left(A^+ f^*\right)^* \tag{2}$$

for all $f \in \mathcal{H}$. The usual relations $A^+ = \left(\bar{A}\right)^T = \overline{(A^T)}$ remain valid. An operator A with the property $A = A^T$ is called a *symmetric operator*. It satisfies the relation

$$\langle f \mid Ag\rangle = \langle Af \mid g\rangle \tag{3}$$

for all $f, g \in \mathcal{H}$. An operator with the property $A = \bar{A}$ is called a *real operator*.

The algebra of symmetric tensors generated by finite sums and products of elements of $\mathcal{H}$ is denoted by $\mathcal{S}_{alg}(\mathcal{H})$, thereby the symmetric tensor product of two elements $F, G \in \mathcal{S}_{alg}(\mathcal{H})$ is written as $F \circ G$. The vacuum vector is denoted by 1_{vac}. The inner product of the tensors $F = f_1 \circ \ldots \circ f_n$ and $G = g_1 \circ \ldots \circ g_n$ is normalized to the permanent

$$(f_1 \circ \ldots \circ f_n \mid g_1 \circ \ldots \circ g_n) = \mathrm{per}\,(f_\mu \mid g_\nu)\,. \tag{4}$$

[3] An explicit and representative example is $\mathcal{H} = l^2$, the space of square summable sequences with inner product $(f \mid g) = \sum_{\mu=1}^{\infty} \overline{f}_\mu g_\mu$ for $f = (f_1, f_2, \ldots) \in \mathbb{C}^\infty$ and $g = (g_1, g_2, \ldots) \in \mathbb{C}^\infty$. The involution is simply complex conjugation $f^* = \left(\overline{f}_1, \overline{f}_2, \ldots\right)$, and the bilinear form is $\langle f \mid g\rangle = (f^* \mid g) = \sum_{\mu=1}^{\infty} f_\mu g_\mu$.

The completion of $\mathcal{S}_{alg}(\mathcal{H})$ with respect to the norm $\|F\| = \sqrt{(F \mid F)}$ is the Fock space $\mathcal{S}(\mathcal{H})$. The closed subspace of tensors of degree $n \in \{0, 1, ...\}$ is called $\mathcal{S}_n(\mathcal{H})$. The involution $f \to f^*$ on $\mathcal{H}$ can be extended uniquely to an involution on $\mathcal{S}(\mathcal{H})$ with the rule $(F \circ G)^* = G^* \circ F^* = F^* \circ G^*$. The mapping $F, G \in \mathcal{S}(\mathcal{H}) \to \langle F \mid G \rangle := (F^* \mid G) \in \mathbb{C}$ is again a bilinear symmetric form.

Let $F \in \mathcal{S}_{alg}(\mathcal{H})$ and $Y \in \mathcal{S}(\mathcal{H})$, then the interior product of these tensors can be defined.

Definition 1 *The interior product or contraction of a tensor $F \in \mathcal{S}_{alg}(\mathcal{H})$ with a tensor $Y \in \mathcal{S}(\mathcal{H})$ is the unique element $Y \lrcorner F$ of $\mathcal{S}_{alg}(\mathcal{H})$ for which the identity*

$$\langle Y \lrcorner F \mid X \rangle = \langle F \mid X \circ Y \rangle \tag{5}$$

is valid for all $X \in \mathcal{S}(\mathcal{H})$.

In general the linear mapping $F \to Y \lrcorner F$ is not continuous in the norm of $\mathcal{S}(\mathcal{H})$. An extension of the domain of definition beyond $\mathcal{S}_{alg}(\mathcal{H})$ will be given in the subsequent sections for some choices of the tensor Y.

2.2 Exponential vectors and coherent states

For all vectors $f \in \mathcal{H}$ the exponential series $\exp f = 1_{vac} + f + \frac{1}{2} f \circ f + ...$ converges within $\mathcal{S}(\mathcal{H})$ with the simple factorization property $\exp f \circ \exp g = \exp(f + g)$. The mapping $f \to \exp f$ is an entire analytic function[4]. The inner product of two exponential vectors is

$$(\exp f \mid \exp g) = \exp (f \mid g) . \tag{6}$$

Coherent states are the normalized exponential vectors $\exp\left(f - \frac{1}{2}\|f\|^2\right) \in \mathcal{S}(\mathcal{H})$. The linear span of all exponential vectors $\{\exp f \mid f \in \mathcal{H}\}$ will be denoted by $\mathcal{S}_{coh}(\mathcal{H})$.

Lemma 1 *The set $\{\exp f \mid f \in \mathcal{H}\}$ of all exponential vectors is linearly independent and their linear span $\mathcal{S}_{coh}(\mathcal{H})$ is dense in $\mathcal{S}(\mathcal{H})$.*

Proof. A proof is given in [16] § 2.1 and in Sect. 19 of [17]. ■

To determine an operator on $\mathcal{S}(\mathcal{H})$ it is therefore sufficient to know this operator on all exponential vectors.

The involution of an exponential vector is $(\exp f)^* = \exp f^*$, $f \in \mathcal{H}$. Let $A \in \mathcal{L}(\mathcal{H})$ be a bounded operator on $\mathcal{H}$ then $\Gamma(A)$ is the bounded operator on $\mathcal{S}(\mathcal{H})$ which maps the exponential vector $\exp f$ onto

$$\Gamma(A) \exp f = \exp Af. \tag{7}$$

For all $f, g \in \mathcal{H}$ the product $f \circ \exp g = \frac{\partial}{\partial \lambda} \exp(g + \lambda f) \mid_{\lambda=0}$ is an element of $\mathcal{S}(\mathcal{H})$, and the interior product (5) $f \lrcorner \exp g = \langle f \mid g \rangle \exp g$ is defined, see Appendix A.3.

[4] In this article analyticity means the existence of norm convergent power series expansions as used e.g. in [15].

Given a vector $f \in \mathcal{H}$ the creation operator $a^+(f)$ and the corresponding annihilation operator $a(f)$ are uniquely determined by

$$a^+(f) \exp g = f \circ \exp g \tag{8}$$
$$a(f) \exp g = f \lrcorner \exp g = \langle f \mid g \rangle \exp g. \tag{9}$$

These operators are related by

$$\left(a^+(f)\right)^+ = a(f^*), \tag{10}$$

and they satisfy the canonical commutation relations

$$\left[a(f), a^+(g)\right] = \langle f \mid g \rangle \; I, \; \left[a^+(f), a^+(g)\right] = [a(f), a(g)] = 0 \tag{11}$$

for all $f, g \in \mathcal{H}$. These relations are equivalent to

$$\left[a^+(f) - a(f^*), a^+(g) - a(g^*)\right] = -2i\omega(f, g) \, I \tag{12}$$

with

$$\omega(f, g) := \frac{1}{2i} \left(\langle f^* \mid g \rangle - \langle f \mid g^* \rangle\right) = \operatorname{Im}\left(f \mid g\right) \in \mathbb{R}, \tag{13}$$

which is an $\mathbb{R}$-bilinear continuous skew symmetric form on the Hilbert space $\mathcal{H}$ – more precisely – on the underlying real space $\mathcal{H}_{\mathbb{R}}$. The creation and annihilation operators are unbounded operators. A domain of definition, on which these operators and their commutators are meaningful is the linear span of all coherent vectors.

Given a linear operator M on $\mathcal{H}$ then $d\Gamma(M)$ is the unique linear operator on $\mathcal{S}(\mathcal{H})$ which satisfies $d\Gamma(M) \exp f = (Mf) \circ \exp f$ for all $f \in \mathcal{H}$. Defining the operator a^+Ma as $\sum_{\mu\nu} (e_\mu \mid Me_\nu) \, a^+(e_\mu) a(e_\nu)$ where $\{e_\mu\}_{\mu \in \mathbb{N}}$ is a complete orthonormal system (ONS) we obtain $(a^+Ma) \exp f = Mf \circ \exp f$. Hence in a notation which is closer to the physics literature

$$d\Gamma(M) = a^+Ma. \tag{14}$$

2.3 Ultracoherent vectors

As a starting point we introduce an isomorphism between tensors in $\mathcal{S}_2(\mathcal{H})$ and symmetric Hilbert-Schmidt operators. The space of symmetric Hilbert-Schmidt operators on $\mathcal{H}$ is called $\mathcal{L}_{2sym}(\mathcal{H})$.

Lemma 2 *Let A be a symmetric Hilbert-Schmidt operator on $\mathcal{H}$, then there exists a unique tensor of second degree, in the sequel denoted by $\Omega(A)$, such that*

$$\langle \Omega(A) \mid f \circ g \rangle = \langle f \mid Ag \rangle = \langle g \mid Af \rangle \tag{15}$$

for all $f, g \in \mathcal{H}$. The mapping $A \in \mathcal{L}_{2sym}(\mathcal{H}) \to \Omega(A) \in \mathcal{S}_2(\mathcal{H})$ is an isomorphism between the spaces $\mathcal{L}_{2sym}(\mathcal{H})$ and $\mathcal{S}_2(\mathcal{H})$.

Proof. As $\langle\Omega(A) \mid f\circ g\rangle = 0$ for all $f, g \in \mathcal{H}$ implies $\Omega(A) = 0$, the tensor $\Omega(A)$ is uniquely determined by (15). For the construction of $\Omega(A)$ we choose a real orthonormal basis $e_\mu = e_\mu^*$ of the Hilbert space $\mathcal{H}$. Then any symmetric Hilbert-Schmidt operator A has matrix elements $A_{\mu\nu} = (e_\mu \mid A\,e_\nu) = \langle e_\mu \mid A\,e_\nu\rangle = A_{\nu\mu}$ which are square summable $\sum_{\mu\nu} |A_{\mu\nu}|^2 < \infty$. The tensor $\Omega(A) := \frac{1}{2}\sum_{\mu\nu} A_{\mu\nu}\, e_\mu \circ e_\nu \in \mathcal{S}_2(\mathcal{H})$ then satisfies the identity (15). The tensor norm of $\Omega(A)$ is calculated as

$$\|\Omega(A)\|^2 = \frac{1}{2}\,\|A\|_2^2\,. \tag{16}$$

On the other hand, given a tensor $F \in \mathcal{S}_2(\mathcal{H})$ we have $F = \frac{1}{2}\sum_{\mu\nu} F_{\mu\nu}\, e_\mu \circ e_\nu$ with coefficients $F_{\mu\nu} \in \mathbb{C}$, which are square summable. Then the operator $f \in \mathcal{H} \to Af = \sum_\mu F_{\mu\nu}\,\langle e_\mu \mid f\rangle\, e_v \in \mathcal{H}$ is obviously a symmetric Hilbert-Schmidt operator with $\Omega(A) = F$. ∎

Definition 2 *The set of all symmetric Hilbert–Schmidt operators $A \in \mathcal{L}_{2sym}(\mathcal{H})$ with the property $\|A\| < 1$, i.e., operator norm strictly less than one, is called the Siegel (unit) disc and it is denoted by $\mathcal{D}_1$.*[5]

The set $\mathcal{D}_1$ is open and convex, and it is stable against transformations $A \to UAU^T$ with a unitary operator U. The last statement follows from $\mathrm{tr}_{\mathcal{H}}\bar{U}A^+U^+UAU^T = \mathrm{tr}_{\mathcal{H}}A^+A$ and $\left\|UAU^T\right\| \leq \|A\|$.

If the norm of $\Omega(A)$ is strictly less than $1/\sqrt{2}$, the exponential series $\exp\Omega(A) = 1_{vac} + \Omega(A) + \frac{1}{2!}\Omega(A)\circ\Omega(A) + ...$ converges within $\mathcal{S}(\mathcal{H})$. The inner product of these tensors is

$$(\exp\Omega(A) \mid \exp\Omega(B)) = \left(\det{}_{\mathcal{H}}\left(I - A^+B\right)\right)^{-\frac{1}{2}} = \left(\det{}_{\mathcal{H}}\left(I - BA^+\right)\right)^{-\frac{1}{2}}. \tag{17}$$

The proof of this identity is given in Appendix A. The identities (16) and (17) imply that $\exp(\Omega(A)) \in \mathcal{S}(\mathcal{H})$ if and only if $A \in \mathcal{D}_1$.

For any operator $Z \in \mathcal{D}_1$ and for any $f \in \mathcal{H}$ we now define the *ultracoherent vector*

$$\mathcal{E}\,(Z, f) = \exp\Omega(Z)\circ\exp f = \exp\left(\Omega(Z) + f\right) \in \mathcal{S}(\mathcal{H}). \tag{18}$$

The result (17) can be extended to the inner product of two ultracoherent vectors

$$\begin{aligned}&(\exp(\Omega(A) + f) \mid \exp(\Omega(B) + g))\\ &= (\det{}_{\mathcal{H}}(I - A^+B))^{-\frac{1}{2}} \exp\left(\tfrac{1}{2}\langle f^* \mid Cf^*\rangle + \langle f^* \mid (I - BA^+)^{-1}\, g\rangle + \tfrac{1}{2}\langle g \mid D\, g\rangle\right)\end{aligned} \tag{19}$$

with

$$\begin{aligned}C &= B(I - A^+B)^{-1} = (I - BA^+)^{-1}B\\ D &= A^+ + A^+CA^+ = A^+(I - BA^+)^{-1} = (I - A^+B)^{-1}A^+.\end{aligned} \tag{20}$$

The proof of this identity is given in Appendix A. Since the exponential vectors are dense in $\mathcal{S}(\mathcal{H})$, the ultracoherent vector is uniquely determined by its inner product with the exponential vectors. This inner product is a special case of (19)

$$(\exp z \mid \exp\left(\Omega(A) + f\right)) = \exp\left(\frac{1}{2}\langle z^* \mid Az^*\rangle + \langle z^* \mid f\rangle\right). \tag{21}$$

[5]For Hilbert spaces with finite dimensions this disc has been introduced by Siegel [18].

Remark 1 *For $F \in \mathcal{S}(\mathcal{H})$ the function*

$$\Phi_F(z^*) = (\exp z \mid F) = \langle \exp z^* \mid F \rangle \tag{22}$$

is entire antianalytic in the variable $z \in \mathcal{H}$, and the tensor F is uniquely determined by this function. We denote the linear space of all functions $\{\Phi_F(z^*) \mid F \in \mathcal{S}(\mathcal{H})\}$ by $\mathcal{B}$. Then $\mathcal{B}$ can be equipped with the Hilbert space topology induced by the topology of $\mathcal{S}(\mathcal{H})$, i.e., the inner product $(\Phi_F \parallel \Phi_G)$ of two functions Φ_F and Φ_G is defined as $(\Phi_F \parallel \Phi_G) := (F \mid G)$. With this structure the space $\mathcal{B}$ becomes a Hilbert space with the reproducing kernel $\exp \langle z^* \mid w \rangle$ *[19]. This representation of the bosonic Fock space is called the Bargmann-Fock or complex wave representation, see e.g. Sect. 1.11 of [14]. The exponential vectors and the ultracoherent vectors have a simple representation in this space:* $\exp f \in \mathcal{S}(\mathcal{H})$ *corresponds to the exponential function* $(\exp z \mid \exp f) = \exp \langle z^* \mid f \rangle$ *and the ultracoherent vectors are given by (21).*
If $F \in \mathcal{S}_{alg}(\mathcal{H})$ or $F \in \mathcal{S}_{coh}(\mathcal{H})$ then $\Phi_F(z^*)$ is a tame function, i.e., it depends only on a finite number of variables $\langle z^* \mid f_j \rangle$, $f_j \in \mathcal{H}$, $j = 1, ..., N$. Let $\upsilon(dz, dz^*)$ be the canonical Gaussian promeasure on the Hilbert space $\mathcal{H}_{\mathbb{R}}$, then tame functions can be integrated and the identity

$$\int_{\mathcal{H}_{\mathbb{R}}} \overline{\Phi_F(z^*)} \Phi_F(z^*) \upsilon(dz, dz^*) = (F \mid G) \tag{23}$$

holds for all $F, G \in \mathcal{S}_{alg}(\mathcal{H}) \cup \mathcal{S}_{coh}(\mathcal{H})$. If $A \in \mathcal{D}_1$ is a finite rank operator, then (21) is a tame function. The integral (23) can be used to calculate the inner product (19) – first for finite rank operators A and B and then by a continuity argument for general $A, B \in \mathcal{D}_1$. A calculation of the integral for finite dimensional Hilbert spaces can be found in Appendix II of [4]. The proof of (19), which we present in Appendix A, does not use this technique.

3 Weyl operators and canonical transformations

3.1 Weyl operators

The *Weyl operators* $W(h)$, $h \in \mathcal{H}$, are defined on the linear span of the exponential vectors by

$$W(h) \exp f = \mathrm{e}^{-(h|f) - \frac{1}{2}\|h\|^2} \exp(f + h) = \mathrm{e}^{-\langle h^*|f\rangle - \frac{1}{2}\langle h^*|h\rangle} \exp(f + h). \tag{24}$$

The matrix element between exponential vectors then follows from (6) as

$$(\exp g \mid W(h) \exp f) = \exp \left((g \mid f) + (g \mid h) - (h \mid f) - \frac{1}{2} \|h\|^2 \right). \tag{25}$$

The operators $W(h)$ are obviously invertible with $W^{-1}(h) = W(-h)$. Since the calculation of $(W(-h) \exp g \mid \exp f)$ yields again the same result (25), we have $(W(h))^+ = W(-h) = W^{-1}(h)$ on the linear span of the exponential vectors, and the Weyl operators can be extended to unitary operators on the Fock space $\mathcal{S}(\mathcal{H})$. Calculating $W(f)W(g) \exp h$ and $W(f + g) \exp h$ we obtain the *Weyl relations*

$$W(f)W(g) = \mathrm{e}^{-i\mathrm{Im}(f|g)} W(f + g). \tag{26}$$

The exponent $\mathrm{Im}\,(f \mid g)$ is the skew symmetric form (13) on the underlying real space $\mathcal{H}_{\mathbb{R}}$. The identity (26) implies that $h \in \mathcal{H}_{\mathbb{R}} \to W(h) \in \mathcal{L}(\mathcal{H})$ is a unitary projective representation of the additive group $\mathcal{H}_{\mathbb{R}}$.

The action of the Weyl operator on the ultracoherent vector can be determined from the following identities

$$\begin{aligned}
&(\exp z \mid W(h) \exp(\Omega(A) + f)) = (W(-h) \exp z \mid \exp(\Omega(A) + f)) \\
&\overset{(24)}{=} \mathrm{e}^{-\frac{1}{2}\|h\|^2 + \frac{1}{2}\langle h^* | Ah^* \rangle - \langle h^* | f \rangle} \mathrm{e}^{\langle f + h - Ah^* | z^* \rangle + \frac{1}{2}\langle z^* | Az^* \rangle} \\
&= \mathrm{e}^{-\frac{1}{2}\|h\|^2 + \frac{1}{2}\langle h^* | Ah^* \rangle - \langle h^* | f \rangle} (\exp z \mid \exp(\Omega(A) + f + h - Ah^*))
\end{aligned}$$

which imply

$$W(h) \exp(\Omega(A) + f) = \mathrm{e}^{-\frac{1}{2}\|h\|^2 + \frac{1}{2}\langle h^* | Ah^* - 2f \rangle} \exp(\Omega(A) + f + h - Ah^*)\,. \tag{27}$$

The Weyl operators have a simple representation in terms of the creation and annihilation operators. Differentiating $W(\lambda h) \exp f$ with respect to $\lambda \in \mathbb{R}$ we obtain

$$\frac{\partial}{\partial \lambda} W(\lambda h) \exp f \mid_{\lambda = 0} = h \circ \exp f - \langle h^* \mid f \rangle \exp f = \left(a^+(h) - a(h^*) \right) \exp f.$$

The last identity follows from (8) and (9). Therefore $W(h)$ has the representation

$$W(h) = \exp\left(a^+(h) - a(h^*) \right) \tag{28}$$

and coincides with the displacement operator of quantum optics. One can easily see that the Weyl relations (26) are equivalent to the canonical commutation relations (11). Since the Weyl operators are unitary, the Weyl relations have the advantage that they can be formulated without discussion of the domains of operators.

Using (24) together with (8) and (9) we obtain the well known relations

$$\begin{aligned}
W(h) a^+(f) W^+(h) &= a^+(f) - (h \mid f) = a^+(f) - \langle f \mid h^* \rangle\,, \\
W(h) a(f) W^+(h) &= a(f) - (f^* \mid h) = a(f) - \langle f \mid h \rangle\,.
\end{aligned} \tag{29}$$

Remark 2 *The Weyl operator is closely connected to the Wigner function [20] which is a quasi-probability distribution that gives the phase-space description of quantum mechanics. The expectation value of the Weyl operator is the Fourier transform of the Wigner function [21].*

3.2 Canonical transformations

Canonical transformations are unitary operators S on $\mathcal{S}(\mathcal{H})$ which preserve the canonical commutation relations (11) or (12). To avoid any discussion about the domain of the operators $Sa^+(f)S^+$ and $Sa(f^*)S^+$ we demand the invariance of the Weyl relations (26)

$$SW(f)S^+SW(g)S^+ = SW(f)W(g)S^+ = \mathrm{e}^{-i\mathrm{Im}(f|g)} SW(f+g)S^+. \tag{30}$$

There are essentially two types of canonical transformations

1. Transformations, which generate a c-number shift for the creation and annihilation operators. These transformations are given by the unitary Weyl operators, as can be seen from the relations (29). The invariance of the Weyl relations, i.e.,

$$W(h)W(f)W(g)W(-h) = \mathrm{e}^{-i\mathrm{Im}(f|g)}W(h)W(f+g)W(-h),$$

   where $S = W(h)$, $h \in \mathcal{H}$, is the canonical transformation, easily follow from the Weyl relations (26) themselves.

2. The second type of canonical transformations generate linear transformations between the creation and annihilation operators

$$\begin{aligned} Sa^+(f)S^+ &= a^+(Uf) - a(\bar{V}f) \\ Sa(f)S^+ &= -a^+(Vf) + a(\bar{U}f) \end{aligned} \tag{31}$$

   such that

$$\begin{aligned} S\left(a^+(f) - a(f^*)\right)S^+ &= -a(\bar{V}f) + a^+(Uf) - a(\bar{U}f^*) + a^+(Vf^*) \\ &= a^+(Uf + Vf^*) - a\left(\bar{V}f + \bar{U}f^*\right). \end{aligned}$$

   The canonical transformations of this type are usually called *Bogoliubov transformations*. The Weyl form of these transformations is

$$SW(f)S^+ = W(Uf + Vf^*). \tag{32}$$

   Here U and V are bounded linear transformations on $\mathcal{H}$. The Weyl relations (and consequently the canonical commutation relations) are preserved, if the skew symmetric form (13) is invariant against the $\mathbb{R}$-linear mapping

$$f \in \mathcal{H}_\mathbb{R} \to R(U,V)f := Uf + Vf^* \in \mathcal{H}_\mathbb{R}, \tag{33}$$

   i.e.,

$$\omega(Rf, Rg) \equiv \omega(Uf + Vf^*, Ug + Vg^*) = \omega(f,g). \tag{34}$$

   for all $f, g \in \mathcal{H}_\mathbb{R}$.

The transformations (33) which satisfy (34) form the *symplectic group* of the Hilbert space $\mathcal{H}_\mathbb{R}$, and the transformations (32) generate a unitary ray representation of this group on the Fock space $\mathcal{S}(\mathcal{H})$. So far we have assumed that U and V are bounded linear operators on $\mathcal{H}$ (and consequently also on $\mathcal{H}_\mathbb{R}$). For infinite dimensional Hilbert spaces $\mathcal{H}$ – needed for quantum field theory – an additional constraint turns out to be necessary: In order to obtain a unitary ray representation on $\mathcal{S}(\mathcal{H})$ the operator V has to be a Hilbert-Schmidt operator, see [1, 2, 3] or [14] Chap. 4.

4 The symplectic group

4.1 Definition

In this Section we give a more explicit definition of the symplectic transformations. We identify $\mathcal{H}_\mathbb{R}$ with the diagonal subspace $\mathcal{H}_{diag} \subset \mathcal{H} \times \mathcal{H}^*$, see the beginning of Sect. 2.1.

The space $\mathcal{H}\times\mathcal{H}^*$ has elements $\begin{pmatrix} f \\ g^* \end{pmatrix}$ with $f, g \in \mathcal{H}$. On $\mathcal{H}\times\mathcal{H}^*$ we define the operators

$$\Delta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \text{ and } \hat{M} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{35}$$

The matrix array

$$\hat{R} = \begin{pmatrix} U & V \\ X & Y \end{pmatrix} \tag{36}$$

of operators $U, V, X, Y \in \mathcal{L}(\mathcal{H})$ yields a bounded linear operator on $\mathcal{H}\times\mathcal{H}^*$

$$\begin{pmatrix} U & V \\ X & Y \end{pmatrix}\begin{pmatrix} f \\ g^* \end{pmatrix} = \begin{pmatrix} Uf + Vg^* \\ Xf + Yg^* \end{pmatrix}.$$

Definition 3 *The operator (36) is a symplectic transformation, if it satisfies the constraints*

$$\hat{R}\hat{M}\hat{R}^+ = \hat{M} \tag{37}$$

and

$$\Delta\overline{\hat{R}}\Delta = \hat{R}. \tag{38}$$

The set of all these transformations is denoted by $\widehat{Sp}(\mathcal{H})$.

For finite dimensional Hilbert spaces $\mathcal{H} = \mathbb{C}^n$ the constraint (37) implies that $\hat{R}$ is an element of the group $SU(n,n)$. The reality constraint (38) implies that (36) has the form

$$\hat{R} = \hat{R}(U,V) = \begin{pmatrix} U & V \\ \bar{V} & \bar{U} \end{pmatrix} \tag{39}$$

and it guarantees that an operator $\hat{R} \in \widehat{Sp}(\mathcal{H})$ maps the diagonal subspace $\mathcal{H}_{diag}$ into itself. Thereby $\bar{U}$ and $\bar{V}$ are the complex conjugate operators of U and V, respectively, as defined in Sect 2.1. The identity operator $\hat{I} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ is an element of $\widehat{Sp}(\mathcal{H})$. The product of two matrix operators (39)

$$\hat{R}_2\hat{R}_1 = \begin{pmatrix} U_2 & V_2 \\ \bar{V}_2 & \bar{U}_2 \end{pmatrix}\begin{pmatrix} U_1 & V_1 \\ \bar{V}_1 & \bar{U}_1 \end{pmatrix} = \begin{pmatrix} U_2U_1 + V_2\bar{V}_1 & U_2V_1 + V_2\bar{U}_1 \\ \bar{V}_2U_1 + \bar{U}_2\bar{V}_1 & \bar{V}_2V_1 + \bar{U}_2\bar{U}_1 \end{pmatrix} \tag{40}$$

is also an element of $\widehat{Sp}(\mathcal{H})$. From (37) follows the inverse of $\hat{R}$ as

$$\hat{R}^{-1} = \hat{M}\hat{R}^+\hat{M} = \begin{pmatrix} U^+ & -V^T \\ -V^+ & U^T \end{pmatrix}. \tag{41}$$

On the other hand, if a matrix operator (39) satisfies $\hat{R}^{-1} = \hat{M}\hat{R}^+\hat{M}$, then the conditions of Definition 3 apply to $\hat{R}$, and $\hat{R}$ is an element of $\widehat{Sp}(\mathcal{H})$. From $\hat{R}^{-1}\hat{R} = I$ we have $\hat{R}^+\hat{M}\hat{R} = \hat{M}$ such that

$$\hat{R}^+ = \begin{pmatrix} U^+ & V^T \\ V^+ & U^T \end{pmatrix} \in \widehat{Sp}(\mathcal{H}). \tag{42}$$

But then also the operator (41) is an element of $\widehat{Sp}(\mathcal{H})$ and the set $\widehat{Sp}(\mathcal{H})$ is a group with identity $\hat{I}$ and multiplication (40).

The (equivalent) identities $\hat{R}\hat{R}^{-1} = I$ and $\hat{R}^{-1}\hat{R} = I$ (with $\hat{R}^{-1}$ given by (41)) are satisfied if the following (again equivalent) conditions hold

$$UU^{+} - VV^{+} = I,\ UV^{T} = VU^{T}, \tag{43}$$
$$U^{+}U - V^{T}\bar{V} = I,\ U^{T}\bar{V} = V^{+}U. \tag{44}$$

Hence $\|U\| \geq 1$ and U has an inverse. Then the identities

$$U^{-1}V = V^{T}\left(U^{-1}\right)^{T}, \quad \bar{V}U^{-1} = \left(U^{-1}\right)^{T} V^{+} \tag{45}$$

follow. Therefore the operators $U^{-1}V$ and $\bar{V}U^{-1}$ are symmetric. Moreover we obtain from (43) – (45)

$$I - \left(U^{-1}V\right)\left(U^{-1}V\right)^{+} = \left(U^{+}U\right)^{-1}, \tag{46}$$
$$I - \left(\bar{V}U^{-1}\right)^{+}\left(\bar{V}U^{-1}\right) = \left(UU^{+}\right)^{-1}. \tag{47}$$

The operator norms of $U^{-1}V$ and $\bar{V}U^{-1}$ therefore satisfy

$$\left\|U^{-1}V\right\|^{2} = \left\|\bar{V}U^{-1}\right\|^{2} = 1 - \|U\|^{-2} < 1. \tag{48}$$

The group element $\hat{R}(U,V) \in \widehat{S}p(\mathcal{H})$ maps $\mathcal{H}_{diag} = \left\{ \begin{pmatrix} f \\ f^{*} \end{pmatrix},\ f \in \mathcal{H} \right\}$ into itself

$$\hat{R}(U,V)\begin{pmatrix} f \\ f^{*} \end{pmatrix} = \begin{pmatrix} U & V \\ \bar{V} & \bar{U} \end{pmatrix}\begin{pmatrix} f \\ f^{*} \end{pmatrix} = \begin{pmatrix} Uf + Vf^{*} \\ \bar{V}f + \bar{U}f^{*} \end{pmatrix} \in \mathcal{H}_{diag}. \tag{49}$$

The operator $\hat{R}(U,V)$ is uniquely determined by the following $\mathbb{R}$-linear mapping $R(U,V)$ on $\mathcal{H}$ (more precisely on $\mathcal{H}_{\mathbb{R}}$)

$$R(U,V)f = Uf + Vf^{*}. \tag{50}$$

The calculations presented above imply

Lemma 3 *The skew symmetric form (13) is invariant against the $\mathbb{R}$- linear mapping R on $\mathcal{H}_R$ if and only if R has the form (50) where U and V are bounded operators on $\mathbb{H}$, which satisfy the relations (43) and (44).*

In the sequel we often refer to (50) as the symplectic transformation. The product and the inverse follow from (40) and (41) as

$$R(U_2,V_2)R(U_1,V_1) = R(U_2U_1 + V_2\bar{V}_1, U_2V_1 + V_2\bar{U}_1), \tag{51}$$
$$R^{-1}(U,V) = R(U^{+}, -V^{T}). \tag{52}$$

The set of these transformations forms the group of symplectic transformations, which will be denoted by $Sp(\mathcal{H})$. The identity of the group is $R(I,0)$. In order to derive a unitary representation of this group in the Fock space $\mathcal{S}(\mathcal{H})$ an additional constraint is necessary if $\dim\mathcal{H}$ is infinite: The operator V has to be a Hilbert-Schmidt operator [1, 2, 3]. This constraint is stable under the group operations (51) and (52).

Definition 4 *The group $Sp_2(\mathcal{H})$ is the subgroup of all transformations (50) $R(U,V) \in Sp(\mathcal{H})$ with a bounded operator $U \in \mathcal{L}(\mathcal{H})$ and a Hilbert-Schmidt operator $V \in \mathcal{L}_2(\mathcal{H})$.*[6]

For the investigation of the representations it is useful to define the following subgroup of symplectic transformations, which satisfy the additional trace class constraint $U - I \in \mathcal{L}_1(\mathcal{H})$.

Definition 5 *The set of all symplectic transformations, which satisfy $V \in \mathcal{L}_2(\mathcal{H})$ and $U - I \in \mathcal{L}_1(\mathcal{H})$ is denoted by $Sp_1(\mathcal{H})$.*

If $U_1 - I \in \mathcal{L}_1(\mathcal{H})$ and $U_2 - I \in \mathcal{L}_1(\mathcal{H})$ (and $V_{1,2} \in \mathcal{L}_2(\mathcal{H})$ – as assumed anyhow), then $U_2U_1 + V_2\bar{V}_1 - I \in \mathcal{L}_1(\mathcal{H})$ and $U_2V_1 + V_2\bar{U}_1 \in \mathcal{L}_2(\mathcal{H})$. Hence the product (51) satisfies the same restrictions. Moreover, if $R(U,V) \in Sp_1(\mathcal{H})$ then the inverse $R^{-1}(U,V) = R(U^+, -V^T)$ is a symplectic transformation of the same type. The set $Sp_1(\mathcal{H})$ is therefore a subgroup of $Sp_2(\mathcal{H})$.

Remark 3 *If $\dim \mathcal{H} < \infty$ the conditions $V \in \mathcal{L}_2(\mathcal{H})$ and $U - I \in \mathcal{L}_1(\mathcal{H})$ are satisfied by all linear operators and we have $Sp_1(\mathcal{H}) = Sp_2(\mathcal{H}) = Sp(\mathcal{H})$.*

4.2 Special cases and factorization

4.2.1 The group of unitary (isometric) transformations

Let $R(U,V) \in Sp(\mathcal{H})$ be an isometric symplectic transformation $\mathcal{H}_\mathbb{R}$, then $\hat{R}(U,V)$ is a unitary transformation on $\mathcal{H} \times \mathcal{H}^*$. The inequality $\left\| \hat{R}(U,V) \right\| \leq 1$ implies $UU^+ \overset{(43)}{=} I + VV^+ \leq I$ such that $V = 0$ and U is a unitary operator. The isometric symplectic transformations are therefore exactly the operators $R(U,0) \in Sp_2(\mathcal{H})$ with a unitary $U \in \mathcal{L}(\mathcal{H})$. Since the product of such operators is again an operator of this type, the isometric symplectic operators form a group. Since $R(U,0)f = Uf$ for all $f \in \mathcal{H}_\mathbb{R}$ we simply write U for the isometric symplectic transformation.

4.2.2 The subset of positive operators in $Sp_2(\mathcal{H})$

Assume that $\hat{S} \in \widehat{S}p_2(\mathcal{H})$ is a positive operator $\hat{S} = \begin{pmatrix} U & V \\ \bar{V} & \bar{U} \end{pmatrix} \geq 0$, then $U \geq I$ and $V^T = V$ follow. Since $V \in \mathcal{L}_2(\mathcal{H})$, the identity (43) implies that $U - I = \sqrt{I + VV^+} - I$ is a nuclear operator. Hence $\hat{S} - I$ is a Hilbert-Schmidt operator. Since $\hat{S}$ is invertible there exists a (unique) self-adjoint Hilbert-Schmidt operator $\hat{\Xi}$ such that $\hat{S} = \exp \hat{\Xi}$. From $\hat{S}^{-1} = \exp\left(-\hat{\Xi}\right) = \hat{M}\left(\exp \hat{\Xi}\right)\hat{M} = \exp \hat{M}\hat{\Xi}\hat{M}$ then follows $\hat{\Xi} = -\hat{M}\hat{\Xi}\hat{M}$; and the identity $\Delta\overline{\hat{S}}\Delta = \hat{S}$ implies the additional constraint $\Xi = \Xi^T$. The operator $\hat{\Xi}$ has therefore the form $\hat{\Xi} = \begin{pmatrix} 0 & \Xi \\ \Xi^+ & 0 \end{pmatrix}$ with a symmetric Hilbert-Schmidt operator Ξ. The

[6] In [2] and [9] the elements of $Sp_2(\mathcal{H})$ are called restricted symplectic transformations, in [3] proper canonical transformations.

exponential series $\exp \hat{\Xi}$ converges to a self-adjoint positive operator on $\mathcal{H} \times \mathcal{H}^*$

$$\begin{aligned} \hat{S}(\Xi) &= \exp \hat{\Xi} = \hat{R}(U,V) \text{ with} \\ U &= \cosh \sqrt{\Xi\bar{\Xi}} \geq I \quad \text{and} \quad V = \Xi \frac{\sinh \sqrt{\bar{\Xi}\Xi}}{\sqrt{\bar{\Xi}\Xi}} = \frac{\sinh \sqrt{\Xi\bar{\Xi}}}{\sqrt{\Xi\bar{\Xi}}} \Xi = V^T. \end{aligned} \tag{53}$$

The corresponding operator $R(U,V)$ will be denoted by $S(\Xi)$. Since $\hat{S}$ is a positive operator on $\mathcal{H}_{diag} \subset \mathcal{H} \times \mathcal{H}^*$, the operator $S(\Xi)$ is a positive mapping on $\mathcal{H}_{\mathbb{R}}$.

For real positive Hilbert-Schmidt operators $\Xi = \bar{\Xi} = \Phi$ the operator $S(\Xi)$ simplifies to

$$S(\Phi) = R(\cosh \Phi, \sinh \Phi). \tag{54}$$

4.2.3 Factorization

Any symplectic transformation R can be represented as a product of two transformations, which belong to the classes considered above.

Theorem 4 *Any $R \in Sp(\mathcal{H})\,(Sp_2(\mathcal{H}))$ is the product $R = US_1 = S_2U$ of a unitary transformation U and positive transformations $S_{1,2} \in Sp(\mathcal{H})\,(Sp_2(\mathcal{H}))$.*

Proof. Factorization theorems have been derived in [2] and in [17]. For completeness we give a proof which essentially follows the article [6], where the finite dimensional case has been investigated. Any $\hat{R} \in \widehat{Sp}(\mathcal{H})$ has the unique polar decompositions $\hat{R} = \hat{U}\hat{S}_1 = \hat{S}_2\hat{U}$ with a unitary operator $\hat{U}$ and the positive operators $\hat{S}_1 = \sqrt{\hat{R}^+\hat{R}}$ and $\hat{S}_2 = \sqrt{\hat{R}\hat{R}^+}$, see e.g. [22]. The first identity (41) is equivalent to $\hat{R}^{-1T} = \hat{M}\overline{\hat{R}}\hat{M}$. Inserting $\hat{R} = \hat{U}\hat{S}_1$ we obtain $\hat{U}^{-1T}\hat{S}_1^{-1T} = \hat{M}\overline{\hat{U}\hat{S}_1}\hat{M} = \hat{M}\overline{\hat{U}}\hat{M}\hat{M}\overline{\hat{S}_1}\hat{M}$. This identity is again a (unique) polar decomposition, hence

$$\hat{U}^{-1T} = \hat{M}\overline{\hat{U}}\hat{M} \text{ and } \hat{S}_1^{-1T} = \hat{M}\overline{\hat{S}_1}\hat{M}. \tag{55}$$

The same type of argument can be used for the identity (38). Hence

$$\Delta\overline{\hat{U}}\Delta = \hat{U} \text{ and } \Delta\overline{\hat{S}_1}\Delta = \hat{S}_1. \tag{56}$$

As a consequence of (55) and (56) we obtain that $\hat{U}$ and $\hat{S}_1$ are symplectic transformations. But then the operator $\hat{S}_2 = \hat{R}\hat{U}^{-1}$ is also a symplectic transformation, and we have derived

$$\hat{U} \in \widehat{Sp}(\mathcal{H}) \text{ and } \hat{S}_{1,2} \in \widehat{Sp}(\mathcal{H}). \tag{57}$$

If we start from symplectic transformations of the restricted group $R(U,V) \in Sp_2(\mathcal{H})$, then the operators R^+R and RR^+ are positive elements of $Sp_2(\mathcal{H})$. For such operators we know that $R^+R - I$ and $RR^+ - I$ are Hilbert-Schmidt operators, see Sect. 4.2.2. But then also $S_1 - I = \sqrt{R^+R} - I$ and $S_2 - I = \sqrt{RR^+} - I$ are Hilbert-Schmidt operators, and $S_{1,2} \in Sp_2(\mathcal{H})$ follows. ∎

The result of Theorem 4 can be extended using

Lemma 5 *Let A be a symmetric Hilbert-Schmidt operator, then there exists a unitary operator U such that $UAU^T = B$ is a real positive Hilbert-Schmidt operator.*

Proof. For the proof see Lemma 11 and the Remark 8. ■

Let $S(\Xi)$ be a positive symplectic transformation with a symmetric Hilbert-Schmidt operator $\Xi \in \mathcal{L}_{2sym}(\mathcal{H})$. The operator Ξ can be factorized into $\Xi = U_1 \Phi U_1^T$ with a real positive Hilbert-Schmidt operator Φ and a unitary operator U_1. The operator $\hat{\Xi}$ therefore factorizes into

$$\hat{\Xi} = \begin{pmatrix} 0 & \Xi \\ \Xi^+ & 0 \end{pmatrix} = \begin{pmatrix} U_1 & 0 \\ 0 & \bar{U}_1 \end{pmatrix} \begin{pmatrix} 0 & \Phi \\ \Phi & 0 \end{pmatrix} \begin{pmatrix} U_1^+ & 0 \\ 0 & U_1^T \end{pmatrix},$$

and $S(\Xi)$ has the product representation $S(\Xi) = R(U_1, 0) S(\Phi) R(U_1^+, 0)$ where $S(\Phi)$ has the form (54). If this product is inserted into the representation of Theorem 4 we obtain

Corollary 6 *Any $R \in Sp_2(\mathcal{H})$ is a product*

$$R = U_1 S(\Phi) U_2 \tag{58}$$

where Φ is a positive real Hilbert-Schmidt operator and U_1 and U_2 are unitary operators.

Remark 4 *For finite dimensional Hilbert spaces Lemma 5 and the Corollary have been derived in Appendix II of [4] and in [23].*

4.2.4 One parameter subgroups of $Sp_2(\mathcal{H})$

In finite dimensions $\dim \mathcal{H} < \infty$ the symplectic group is a semisimple Lie group, see e.g. [8]. But if $\dim \mathcal{H}$ is infinite the situation is more subtle.

Let $K(t)$, $t \in \mathbb{R}$, be a one parameter group of unitary operators on $\mathcal{H}$. This group has a unique (bounded or unbounded) skew-self-adjoint generator Ψ such that $K(t) = K_\Psi(t) = \exp(\Psi t)$. The operators $K_\Psi(t)$ or – in the equivalent representation by operators on $\mathcal{H} \times \mathcal{H}^*$ – $\hat{K}_\Psi(t) = \exp \begin{pmatrix} \Psi t & 0 \\ 0 & \bar{\Psi} t \end{pmatrix} = \begin{pmatrix} \exp \Psi t & 0 \\ 0 & \exp \bar{\Psi} t \end{pmatrix}$ form a one parameter subgroup of unitary operators within $\hat{S}p_2(\mathcal{H})$. As a consequence of the spectral representation any unitary element of $\hat{S}p(\mathcal{H})$ is an element of such a subgroup. Given an operator (53) $\hat{S}(\Xi) \in \hat{S}p_2(\mathcal{H})$ this operator is an element of the one-parameter group $\hat{S}(\Xi t)$, $t \in \mathbb{R}$, of positive operators. Therefore the two classes of symplectic transformations introduced in Sect. 4.2 are covered by one-parameter subgroups.

If we restrict the generators Ψ of the unitary transformations to bounded operators, the generators of these two classes of subgroups form the basis of a Lie algebra, which is usually denoted by $sp_2(\mathcal{H})$, see [9]. In the notation of operators on $\mathcal{H} \times \mathcal{H}^*$ this Lie algebra is given by the operators[7]

$$\begin{pmatrix} \Psi & \Xi \\ \bar{\Xi} & \bar{\Psi} \end{pmatrix}, \ \Psi = -\Psi^+ \in \mathcal{L}(\mathcal{H}), \ \Xi = \Xi^T \in \mathcal{L}_2(\mathcal{H}). \tag{59}$$

[7] For the Lie product see [9]. The finite dimensional case can be found in [6] and [8].

Given a group $K_\Psi(t)$, $\Psi = -\Psi^+ \in \mathcal{L}(\mathcal{H})$, of unitary transformations then for any $R(U,V) \in Sp_2(\mathcal{H})$ the operators $R(t) = R(U,V)K_\Psi(t)R^{-1}(U,V)$ $= R(U,V)K_\Psi(t)R(U^+, -V^T)$ form again a one parameter subgroup. The generator of this group

$$\hat{R}(U,V)\begin{pmatrix} \Psi & 0 \\ 0 & \bar{\Psi} \end{pmatrix}\hat{R}(U^+,-V^T) = \begin{pmatrix} U\Psi U^+ - V\bar{\Psi}V^+ & -U\Psi V^T + V\bar{\Psi}U^T \\ \bar{V}\Psi U^+ - \bar{U}\bar{\Psi}V^+ & \bar{U}\bar{\Psi}U^T - \bar{V}\Psi V^T \end{pmatrix} \tag{60}$$

is an element of $sp_2(\mathcal{H})$ with a skew-self-adjoint $U\Psi U^+ - V\bar{\Psi}V^+ \in \mathcal{L}(\mathcal{H})$ and a symmetric $-U\Psi V^T + V\bar{\Psi}U^T \in \mathcal{L}_2(\mathcal{H})$. But for the investigation of the dynamics of quantum fields isometric groups $R_\Psi(t)$ with an unbounded $\Psi = -iM$, where M is a real positive and invertible operator, are of special interest. Then $K(t) = R(U,V)K_\Psi(t)R^{-1}(U,V)$ is again a one-parameter group within $Sp_2(\mathcal{H})$ with a well defined generator. But the identification of the generator with (60) is only formal, since the separate entries of the matrix (60) might not be defined for unbounded M.

4.3 Transformations of the Siegel disc

The Siegel disc $\mathcal{D}_1$ has already been introduced in Definition 2. A symmetric Hilbert-Schmidt operator Z is an element of $\mathcal{D}_1$ if

$$I - ZZ^+ > 0 \quad \text{(all eigenvalues strictly positive).} \tag{61}$$

Lemma 7 *For all $R \in Sp_2(\mathcal{H})$ the transformation*

$$Z \to \widetilde{Z} = \zeta(R;Z) := (UZ+V)\left(\bar{U}+\bar{V}Z\right)^{-1} = \left(U^+ + ZV^+\right)^{-1}\left(V^T + ZU^T\right) \tag{62}$$

is an automorphism of the set $\mathcal{D}_1$. Thereby the group $Sp_2(\mathcal{H})$ acts transitively on $\mathcal{D}_1$.

Proof. For $R \in Sp_2(\mathcal{H})$ we have $V \in \mathcal{L}_2(\mathcal{H})$ and $\widetilde{Z}$ is a Hilbert-Schmidt operator. From (48) $\|U^{-1}V\|^2 = \left\|\bar{V}U^{-1}\right\|^2 = 1 - \|U\|^{-2} < 1$ and $|Z| < 1$ we know that $\|U^{-1}VZ\| < 1$, therefore the operator $U + VZ = U\left(I + U^{-1}VZ\right)$ is invertible. Hence

$$\begin{aligned}
&I - \widetilde{Z}\widetilde{Z}^+ = I - \left(U^+ + ZV^+\right)^{-1}\left(ZU^T + V^T\right)\left(\bar{U}Z^+ + \bar{V}\right)\left(U + VZ^+\right)^{-1} \\
&= \left(U^+ + ZV^+\right)^{-1}\left\{\left(U^+ + ZV^+\right)\left(U + VZ^+\right) - \left(ZU^T + V^T\right)\left(\bar{U}Z^+ + \bar{V}\right)\right\}\left(U + VZ^+\right)^{-1} \\
&= \left(U^+ + ZV^+\right)^{-1}\left\{I - ZZ^+\right\}\left(U + VZ^+\right)^{-1} > 0,
\end{aligned}$$

since $I - ZZ^+ > 0$.
The proof of the transitivity follows as in the finite dimensional case, see [18]. Let $Z \in \mathcal{D}_1$ then $I - ZZ^+ > 0$ and we can determine a $U \in \mathcal{L}(\mathcal{H})$ such that $U\left(I - ZZ^+\right)U^+ = I$. A special choice is

$$U = \left(I - ZZ^+\right)^{-\frac{1}{2}} \geq I. \tag{63}$$

The pair U and

$$V = UZ \in \mathcal{L}_2(\mathcal{H}) \tag{64}$$

satisfies the identities (43) and we easily derive $\zeta(R;0) = U^{+-1}V^T = Z$. ∎

The proof allows a stronger statement. The operator (63) is positive and has the property $U - I = \sqrt{I + VV^+} - I \in \mathcal{L}_1(\mathcal{H})$. Therefore $R = R(U,V)$ is an element of the restricted group $Sp_1(\mathcal{H})$.

Corollary 8 *The orbit* $\{Z = \zeta(R;0) \mid R \in Sp_1(\mathcal{H})\}$ *covers the disc* $\mathcal{D}_1$.

The mapping ζ satisfies the rules

$$\begin{array}{l} \zeta(id;Z) = Z \\ \zeta(R_2;\zeta(R_1;Z)) = \zeta(R_2R_1;Z). \end{array} \tag{65}$$

Hence $R \to \zeta(R;\,.\,)$ is a (nonlinear) representation of the group $Sp(\mathcal{H})$.

If $R = R(U,0)$ with a unitary operator U then

$$\zeta(R;Z) = UZU^T. \tag{66}$$

5 Unitary representations of the symplectic group

A unitary ray representation of $Sp_2(\mathcal{H})$ in $\mathcal{S}(\mathcal{H})$ has the following properties:

$$\begin{array}{c} R \in Sp_2(\mathcal{H}) \longmapsto T\,(R) \ \text{unitary operator on}\ \mathcal{S}(\mathcal{H}) \\ T(id) = I,\, T^{-1}\,(R) = T^{+}\,(R) = T\,(R^{-1}) \\ T(R_2)T(R_1) = \chi(R_2,R_1)T(R_2R_1) \\ \text{with}\ \chi(R_2,R_1) \in \mathbb{C},\ |\chi(R_2,R_1)| = 1. \end{array} \tag{67}$$

5.1 Representation of the group $Sp_2(\mathcal{H})$

5.1.1 Ansatz for coherent states

Let $R = R(U,V)$ be a symplectic transformation of the group $Sp_2(\mathcal{H})$ – i.e., $U \in \mathcal{L}(\mathcal{H})$ and $V \in \mathcal{L}_2(\mathcal{H})$ – then $|U| := \sqrt{UU^+} = \sqrt{I + VV^+} \geq I$ has the property $|U| - I \in \mathcal{L}_1(\mathcal{H})$ and the determinants $\det|U| \geq 1$ and $\det|U|^{-1} = (\det|U|)^{-1}$ are well defined.

The representation $T(R)$ of the group $Sp_2(\mathcal{H})$ is now defined on the set of exponential vectors by

$$T(R)\exp f := (\det|U|)^{-\frac{1}{2}} \exp\left(\Omega\left(U^{+-1}V^T\right) + U^{+-1}f - \frac{1}{2}\left\langle f \mid V^+U^{+-1}f\right\rangle\right). \tag{68}$$

Since V is a Hilbert-Schmidt operator, the relations (45) and (48) imply that the mapping $U^{+-1}V^T = V\bar{U}^{-1}$ is an element of the Siegel unit disc. Hence the exponentials are tensors within the Fock space $\mathcal{S}(\mathcal{H})$. In the special case of a unitary transformation $R(U,0)$, $U = U^{+-1}$ unitary, the ansatz (68) has the simple form $T(R)\exp f = \exp(Uf)$ such that, see (7),

$$T(R(U,0)) = \Gamma(U). \tag{69}$$

The transformation $T(R)$ defined in (68) can then be extended by linearity onto the linear span $\mathcal{S}_{coh}(\mathcal{H})$ of all exponential vectors. Thereby the identity of the group $R(I_{\mathcal{H}},0)$ is mapped onto the unit operator on $\mathcal{S}_{coh}(\mathcal{H})$. In the following part of this Section it is shown that $R \in Sp_2(\mathcal{H}) \to T(R)$ is actually a unitary ray representation of the group $Sp_2(\mathcal{H})$ on the Fock space $\mathcal{S}(\mathcal{H})$.

Lemma 9 *The transformation (68) has a unique extension to an isometric linear mapping on* $\mathcal{S}(\mathcal{H})$.

Proof. For the proof of this statement we calculate the inner product

$$(T(R)\exp f \mid T(R)\exp g) = \det |U|^{-1} \exp\left(-\tfrac{1}{2}\overline{\langle f \mid V^{+}U^{+-1}f\rangle} - \tfrac{1}{2}\langle g \mid V^{+}U^{+-1}g\rangle\right) \\ \times \left(\exp\left(\Omega\left(U^{+-1}V^{T}\right) + U^{+-1}f\right) \mid \exp\left(\Omega\left(U^{+-1}V^{T}\right) + U^{+-1}g\right)\right).$$

The inner product

$$\left(\exp\left(\Omega\left(U^{+-1}V^{T}\right) + U^{+-1}f\right) \mid \exp\left(\Omega\left(U^{+-1}V^{T}\right) + U^{+-1}g\right)\right) = \det\left(I - U^{+-1}V^{T}\bar{V}U^{-1}\right)^{-\frac{1}{2}} \\ \times \exp\left(\tfrac{1}{2}\langle U^{+-1}g \mid D\, U^{+-1}g\rangle + \tfrac{1}{2}\overline{\langle U^{+-1}f \mid D\, U^{+-1}f\rangle}\right) \\ \times \exp\left\langle U^{T-1}f^{*} \mid (I - U^{+-1}V^{T}\bar{V}U^{-1})^{-1}\, U^{+-1}g\right\rangle,$$

follows from (19). Thereby D is given by $D = \bar{V}U^{-1}(I - U^{+-1}V^{T}\bar{V}U^{-1})^{-1}$. Since $I - U^{+-1}V^{T}\bar{V}U^{-1} \overset{(47)}{=} (UU^{+})^{-1}$ we have $D = \bar{V}U^{+}$ and $\det\left(I - U^{+-1}V^{T}\bar{V}U^{-1}\right) = (\det|U|)^{-2}$. We finally obtain

$$(T(R)\exp f \mid T(R)\exp g) = (\exp f \mid \exp g) \tag{70}$$

for all $f, g \in \mathcal{H}$. By linearity the operator $T(R)$ can be extended to $\mathcal{S}_{coh}(\mathcal{H})$ and (70) implies $\|T(R)F\| = \|F\|$ for all $F \in \mathcal{S}_{coh}(\mathcal{H})$. Since $\mathcal{S}_{coh}(\mathcal{H})$ is dense in $\mathcal{S}(\mathcal{H})$ the Lemma is proven. ■

The calculation of $(\exp g \mid T^{+}(R)\exp f) = (T(R)\exp g \mid \exp f)$ using (21) yields

$$T^{+}(R)\exp f = (\det|U|)^{-\frac{1}{2}} \exp\left(-\Omega\left(U^{-1}V\right) + U^{-1}f + \frac{1}{2}\langle f \mid \bar{V}U^{-1}f\rangle\right). \tag{71}$$

Inserting (41) into (68) we obtain

$$T^{+}(R) = T(R^{-1}) \tag{72}$$

first on $\mathcal{S}_{coh}(\mathcal{H})$ and by continuity on $\mathcal{S}(\mathcal{H})$. Since $T(R^{-1})$ is isometric the operator $T^{+}(R)$ is also an isometric mapping. We have therefore derived

Theorem 10 *The transformation (68) has a unique extension to a unitary mapping on* $\mathcal{S}(\mathcal{H})$.

If $R = R(U,V)$ is an element of the restricted group $Sp_1(\mathcal{H}) \subset Sp_2(\mathcal{H})$ – i.e., $U - I$ is a nuclear operator – then $\det U$ is well defined and $\det U = \tau \det|U|$ with a phase factor $\tau \in \mathbb{C}$, $|\tau| = 1$. In this case it is convenient to include the phase factor in the definition of the representation

$$T_1(R)\exp f := \left(\det U^{+}\right)^{-\frac{1}{2}} \exp\left(\Omega\left(U^{+-1}V^{T}\right) + U^{+-1}f - \frac{1}{2}\langle f \mid V^{+}U^{+-1}f\rangle\right). \tag{73}$$

The operators (68) and (73) are related by

$$T_1(R) = \sqrt{\frac{\det|U|}{\det U^{+}}}\,T(R). \tag{74}$$

With an appropriate choice of the square root the identity (72), i.e., $T_1^{+}(R) = T_1(R^{-1})$, is also valid for this representation.

In the following Section we shall see that $R \in Sp_1(\mathcal{H}) \to T_1(R)$ is a unitary ray representation with a multiplier χ which takes the values ± 1. This representation is the infinite dimensional extension of the representation investigated in [4] using the Bargmann-Fock representation.

5.1.2 Extension to ultracoherent vectors

With help of the relation (19) we can derive a closed formula for $T(R)$ operating on ultracoherent vectors

$$\begin{aligned} &(\exp z \mid T(R) \exp(\Omega(Z)+f)) = (T^{+}(R)\exp z \mid \exp(\Omega(Z)+f)) \\ &\overset{(71)}{=} (\det|U|)^{-\frac{1}{2}} \left(\exp\left(-\Omega(U^{-1}V)+U^{-1}z\right) \mid \exp(\Omega(Z)+f)\right) \exp \tfrac{1}{2}\overline{\left\langle z \mid \bar{V}U^{-1}z\right\rangle} \\ &\overset{(19)}{=} (\det|U|)^{-\frac{1}{2}} \left(\det(I+ZV^{+}U^{+-1})\right)^{-\frac{1}{2}} \exp\left(\tfrac{1}{2}\left\langle z^{*} \mid V\bar{U}^{-1}z^{*}\right\rangle + \tfrac{1}{2}\left\langle \bar{U}^{-1}z^{*} \mid C\bar{U}^{-1}z^{*}\right\rangle\right) \\ &\qquad \times \exp\left(\left\langle \bar{U}^{-1}z^{*} \mid (I+ZV^{+}U^{+-1})^{-1}f\right\rangle + \tfrac{1}{2}\left\langle f \mid Df\right\rangle\right) \end{aligned} \tag{75}$$

with the operators

$$\begin{aligned} C &= (I+ZV^{+}U^{+-1})^{-1}Z, \\ D &= -V^{+}(U^{+}+ZV^{+})^{-1}. \end{aligned} \tag{76}$$

Since $U^{+-1}C\bar{U}^{-1}+V\bar{U}^{-1} = (U^{+}+ZV^{+})^{-1}(V^{T}+ZU^{T}) = \zeta(R;Z)$, we finally obtain

$$\begin{aligned} &T(R)\exp(\Omega(Z)+f) = (\det|U|)^{-\frac{1}{2}} \left(\det(I+ZV^{+}U^{+-1})\right)^{-\frac{1}{2}} \\ &\times \exp\left(\Omega(\zeta(R;Z)) + (U^{+}+ZV^{+})^{-1}f - \tfrac{1}{2}\left\langle f \mid V^{+}(U^{+}+ZV^{+})^{-1}f\right\rangle\right). \end{aligned} \tag{77}$$

For the representation (74) $T_1(R)$ of the group $Sp_1(\mathcal{H})$ the determinants can be combined to just one factor, and we get

$$\begin{aligned} &T(R)\exp(\Omega(Z)+f) = \left(\det(U^{+}+ZV^{+})\right)^{-\frac{1}{2}} \\ &\times \exp\left(\Omega(\zeta(R;Z)) + (U^{+}+ZV^{+})^{-1}f - \tfrac{1}{2}\left\langle f \mid V^{+}(U^{+}+ZV^{+})^{-1}f\right\rangle\right). \end{aligned} \tag{78}$$

5.1.3 The multiplication law

In the next step we prove

$$T(R_2)T(R_1) = \chi(R_2,R_1)T(R_3) \quad \text{if } R_2R_1 = R_3 \tag{79}$$

with a multiplier $\chi(R_2,R_1) \in \mathbb{C}$, $|\chi(R_2,R_1)| = 1$.

Let

$$\begin{aligned} Z_1 &= \zeta(R_1;Z) = (U_1^{+}+ZV_1^{+})^{-1}(V_1^{T}+ZU_1^{T}) \\ Z_2 &= \zeta(R_2;Z_1) = \zeta(R_3;Z) \end{aligned} \tag{80}$$

see (65), then we obtain from (77)

$$\begin{aligned} &T(R_1)\exp(\Omega(Z)+f) = (\det|U_1|)^{-\frac{1}{2}} \left(\det(I+ZV_1^{+}U_1^{+-1})\right)^{-\frac{1}{2}} \\ &\times \exp\left(\Omega(Z_1) + (U_1^{+}+ZV_1^{+})^{-1}f - \tfrac{1}{2}\left\langle f \mid V_1^{+}(U_1^{+}+ZV_1^{+})^{-1}f\right\rangle\right) \end{aligned}$$

and

$$\begin{aligned} &T(R_2)T(R_1)\exp(\Omega(Z)+f) \\ &= (\det|U_2|)^{-\frac{1}{2}} \left(\det(I+Z_1V_2^{+}U_2^{+-1})\right)^{-\frac{1}{2}} (\det|U_1|)^{-\frac{1}{2}} \left(\det(I+ZV_1^{+}U_1^{+-1})\right)^{-\frac{1}{2}} \\ &\times \exp\left(\Omega(Z_2) + (U_2^{+}+Z_1V_2^{+})^{-1}(U_1^{+}+ZV_1^{+})^{-1}f\right) \\ &\times \exp\left(-\tfrac{1}{2}\left\langle (U_1^{+}+ZV_1^{+})^{-1}f \mid V_2^{+}(U_2^{+}+Z_1V_2^{+})^{-1}(U_1^{+}+ZV_1^{+})^{-1}f\right\rangle\right) \\ &\times \exp\left(-\tfrac{1}{2}\left\langle f \mid V_1^{+}(U_1^{+}+ZV_1^{+})^{-1}f\right\rangle\right). \end{aligned} \tag{81}$$

The tensor of second degree in the exponent immediately follows as $Z_2 = \zeta(R_2; Z_1) = \zeta(R_3; Z)$. The operator product $(U_2^+ + Z_1 V_2^+)^{-1}(U_1^+ + ZV_1^+)^{-1}$ is calculated with

$$\begin{aligned} U_2^+ + Z_1 V_2^+ &= (U_1^+ + ZV_1^+)^{-1} \left((U_1^+ + ZV_1^+)U_2^+ + (V_1^T + ZU_1^T)V_2^+ \right) \\ &= (U_1^+ + ZV_1^+)^{-1} \left(U_1^+ U_2^+ + ZV_1^+ U_2^+ + V_1^T V_2^+ + ZU_1^T V_2^+ \right) = (U_1^+ + ZV_1^+)^{-1}(U_3^+ + ZV_3^+) \end{aligned}$$

as

$$(U_2^+ + Z_1 V_2^+)^{-1}(U_1^+ + ZV_1^+)^{-1} = (U_3^+ + ZV_3^+)^{-1}. \tag{82}$$

Hence

$$\begin{aligned} T(R_2)T(R_1) \exp(\Omega(Z) + f) &= \chi \, T(R_3) \exp(\Omega(Z) + f) \\ &\quad \times \exp\left(-\frac{1}{2} \left(\alpha_{12} - \left\langle f \mid V_3^+ (U_3^+ + ZV_3^+)^{-1} f \right\rangle \right) \right) \end{aligned} \tag{83}$$

where

$$\chi = \sqrt{\frac{\det|U_3|}{\det|U_1| \det|U_2|}} \sqrt{\frac{\det(I + ZV_3^+ U_3^{+-1})}{\det(I + ZV_1^+ U_1^{+-1}) \det(I + Z_1 V_2^+ U_2^{+-1})}} \tag{84}$$

and the exponent α_{12} is

$$\alpha_{12} = \left\langle (U_1^+ + ZV_1^+)^{-1} f \mid V_2^+ (U_2^+ + Z_1 V_2^+)^{-1} (U_1^+ + ZV_1^+)^{-1} f \right\rangle + \left\langle f \mid V_1^+ (U_1^+ + ZV_1^+)^{-1} f \right\rangle . \tag{85}$$

Now we choose $Z = 0$. Then (84) simplifies to

$$\chi(R_2, R_1) = \sqrt{\frac{\det|U_3|}{\det|U_1| \det|U_2| \det\left(U_1^{+-1} U_3^+ U_2^{+-1} \right)}}, \tag{86}$$

which depends only on the group elements.

In the next step we evaluate (85) for $Z = 0$

$$\begin{aligned} \alpha_{12} &= \left\langle U_1^{+-1} f \mid V_2^+ (U_2^+ + U_1^{+-1} V_1^T V_2^+)^{-1} U_1^{+-1} f \right\rangle + \left\langle f \mid V_1^+ U_1^{+-1} f \right\rangle \\ &= \left\langle f \mid \left(\bar{U}_1^{-1} V_2^+ (U_2^+ + U_1^{+-1} V_1^T V_2^+)^{-1} U_1^{+-1} + V_1^+ U_1^{+-1} \right) f \right\rangle . \end{aligned}$$

Since $(U_2^+ + U_1^{+-1} V_1^T V_2)^{-1} = \left(U_1^+ U_2^+ + V_1^T V_2 \right)^{-1} U_1^+$ we have

$$\begin{aligned} & \bar{U}_1^{-1} V_2^+ (U_2^+ + U_1^{+-1} V_1^T V_2^+)^{-1} U_1^{+-1} + V_1^+ U_1^{+-1} \\ &= \left(V_1^+ U_1^{+-1} \left(U_1^+ U_2^+ + V_1^T V_2^+ \right) + \bar{U}_1^{-1} V_2^+ \right) \left(U_1^+ U_2^+ + V_1^T V_2^+ \right)^{-1} \\ &\overset{(40)(45)}{=} \left(V_1^+ U_2^+ + \bar{U}_1^{-1} (\bar{V}_1 V_1^T + I) V_2^+ \right) U_3^{+-1} \\ &\overset{(43)}{=} \left(V_1^+ U_2^+ + \bar{U}_1^{-1} \bar{U}_1 U_1^T V_2^+ \right) U_3^{+-1} = \left(V_1^+ U_2^+ + U_1^T V_2^+ \right) U_3^{+-1} \overset{(40)}{=} V_3^+ U_3^{+-1}, \end{aligned}$$

and

$$\alpha_{12} = \left\langle f \mid V_3^+ U_3^{+-1} f \right\rangle \tag{87}$$

follows.

The identity (83) together with (86) and (87) imply

$$T(R_2)T(R_1) \exp f = \chi(R_1, R_2) \, T(R_3) \exp f \tag{88}$$

for all $f \in \mathcal{H}$. But then (79) is true as an operator identity. Since we already know that the operators $T(R)$ are unitary, the modulus of χ is $|\chi(R_1, R_2)| = 1$.

To obtain this result it is not necessary to evaluate (84) and (85) for $Z \neq 0$. But for completeness we shall present the calculation also for general $Z \in \mathcal{D}_1$ in Appendix B.

Remark 5 *The identity (72) implies* $\chi(R,R^{-1}) = \chi(R^{-1},R) = 1$.

Remark 6 *Since* $\det|KUK^{-1}| = \det|U|$ *the definition (68) implies that*

$$T(KRK^{-1}) = \Gamma(K)T(R)\Gamma(K^{+}) \tag{89}$$

is true for all unitary transformations K *and all* $R \in Sp_2(\mathcal{H})$ *without phase factor.*

Remark 7 *If the symplectic transformations are restricted to the group* $Sp_1(\mathcal{H})$*, we can define the representation* $T_1(R)$*, see (73). Then the multiplier for this representation* $\chi_1(R_2,R_1) = \sqrt{\frac{\det\left(|U_1||U_2|U_3^{+}\right)}{\det\left(U_1^{+}U_2^{+}|U_3|\right)}}\chi(R_2,R_1)$ *has only the values* ± 1 *related to the two branches of the square root* $\sqrt{\det U}$.

5.2 Weyl operators and Bogoliubov transformations

In this Section we show that the operators $T(R)$ are Bogoliubov transformations as defined in Sect.3.2, i.e., we have to derive, see (32),

$$T\left(R\right)W(f)T^{-1}\left(R\right) = W(Uf+Vf^{*}) \equiv W\left(R(U,V)f\right). \tag{90}$$

For $f,g \in \mathcal{H}$ and $R(U,V) \in Sp_2(\mathcal{H})$ we calculate

$$\begin{aligned} &T(R)W(f)\exp g = \exp\left(-\langle f^{*} \mid g\rangle - \tfrac{1}{2}\|f\|^{2}\right)T(R)\exp(f+g) \\ &= (\det|U|)^{-\frac{1}{2}}\exp\left(\Omega\left(U^{+-1}V^{T}\right)+U^{+-1}(f+g)\right) \\ &\times\exp\left(-\langle f^{*}\mid g\rangle - \tfrac{1}{2}\|f\|^{2} - \tfrac{1}{2}\langle f+g \mid V^{+}U^{+-1}(f+g)\rangle\right) \end{aligned} \tag{91}$$

and

$$\begin{aligned} &W(Rf)T(R)\exp g \overset{(68)}{=} \det|U|^{-\frac{1}{2}}\exp\left(-\tfrac{1}{2}\langle g \mid V\bar{U}^{-1}g\rangle\right) \\ &\quad\times W(Uf+Vf^{*})\exp\left(\Omega\left(V\bar{U}^{-1}\right)+U^{+-1}g\right) \\ &\overset{(27)}{=} (\det|U|)^{-\frac{1}{2}}\exp\left(-\tfrac{1}{2}\langle g\mid V^{+}U^{+-1}g\rangle - \left\langle \bar{U}f^{*}+\bar{V}f \mid U^{+-1}g\right\rangle\right) \\ &\quad\times\exp\left(+\tfrac{1}{2}\left\langle \bar{U}f^{*}+\bar{V}f \mid V\bar{U}^{-1}(\bar{U}f^{*}+\bar{V}f) - (Uf+Vf^{*})\right\rangle\right) \\ &\quad\times\exp\left(\Omega\left(U^{+-1}V^{T}\right)+Uf+Vf^{*}+U^{+-1}g - V\bar{U}^{-1}(\bar{U}f^{*}+\bar{V}f)\right)\right) \\ &= (\det|U|)^{-\frac{1}{2}}\exp\left(\Omega(U^{+-1}V^{T})+U^{+-1}(f+g)\right) \\ &\times\exp(-\langle f^{*}\mid g\rangle - \tfrac{1}{2}\|f\|^{2} - \tfrac{1}{2}\langle f+g\mid V^{+}U^{+-1}(f+g)\rangle). \end{aligned} \tag{92}$$

The last identity follows from

$$\begin{aligned} V\bar{U}^{-1}(\bar{U}f^{*}+\bar{V}f) - (Uf+Vf^{*}) &= V\bar{U}^{-1}\bar{V}f - Uf \\ \overset{(45)}{=} U(U^{-1}VV^{+}U^{+-1} - I)f &\overset{(46)}{=} -U^{+-1}f. \end{aligned}$$

Hence we have derived the identity $T(R)W(f) = W(Rf)T(R)$ on $\mathcal{S}_{coh}(\mathcal{H})$. But then (90) is true on $\mathcal{S}(\mathcal{H})$.

6 Generators of one-parameter subgroups

Let $K(t)$, $t \in \mathbb{R}$, be a one parameter subgroup of unitary operators on the space $\mathcal{H}$. Then this subgroup of symplectic transformations is represented on the Fock space by $\Gamma(K(t))$, see (69). If M is the self-adjoint generator of the group $K(t) = \exp(-iMt)$ then $i\frac{d}{dt}\left(\Gamma(K(t))\exp f\right) = i\frac{d}{dt}\exp\left(K(t)f\right) = (Mf) \circ \exp\left(K(t)f\right)$ determines the generator of the group $\Gamma(K(t))$ as $d\Gamma(M) = a^+Ma$. This case corresponds exactly to a free quantum field, where M is the positive – and usually unbounded – operator of the one-particle energy and $d\Gamma(M)$ is the Hamiltonian on the Fock space. The operator M is defined on a dense domain $\mathcal{D}(M)$ within $\mathcal{H}$. The operator $d\Gamma(M)$ is essentially self-adjoint on the linear span of $\{\exp f \mid f \in \mathcal{D}(M)\}$.

We now consider a one parameter subgroup of positive symplectic transformations $S_\Phi(t) := S(\Phi t)$, $t \in \mathbb{R}$, which follows from (54) after the substitution $\Phi \to \Phi t$ with a real symmetric $\Phi \in \mathcal{L}_2(\mathcal{H})$. This group is represented by the corresponding transformation (68)

$$T(S_\Phi(t))\exp f = (\det\cosh(\Phi t))^{-\frac{1}{2}}\exp\left(\Omega(\tanh(\Phi t) + (\cosh(\Phi t))^{-1} f - \frac{1}{2}\langle f \mid \tanh(\Phi t) f\rangle\right).$$

The generator $D_\Phi = i\frac{d}{dt}T(S_\Phi(t))\mid_{t=0}$ follows from

$$\frac{d}{dt}T(S_\Phi(t))\exp f\mid_{t=0} = \Omega(\Phi)\circ\exp f - \frac{1}{2}\langle f \mid \Phi f\rangle \exp f \overset{(111)}{=} \Omega(\Phi)\circ\exp f - \Omega(\Phi) \lrcorner \exp f.$$

as $D_\Phi F = \Omega(i\Phi)\circ F + \Omega(-i\Phi) \lrcorner F, \quad F \in \mathcal{S}_{coh}(\mathcal{H})$. In the notation introduced at the end of Appendix A.3 we have

$$D_\Phi = \frac{i}{2}a^+\Phi a^+ - \frac{i}{2}a\Phi a.$$

The more general case of a subgroup of positive operators $S_\Xi(t) := S(\Xi t)$ where Ξ is symmetric but not real, can be calculated using Lemma 5. We have $S(\Xi) = U_1 S(\Phi) U_1^+$ with a unitary operator U_1 and a real symmetric $\Phi \in \mathcal{L}_2(\mathcal{H})$. Then $T(S_\Phi(t)) = \Gamma(U_1)T(S_\Phi(t))\Gamma(U_1^+)$, see (89), leads to the generator

$$D_\Xi = i\frac{d}{dt}T(S_\Xi(t))\mid_{t=0} = \frac{i}{2}a^+\Xi a^+ - \frac{i}{2}a\overline{\Xi}a$$

of the positive symplectic transformations.

If the generators of the unitary subgroups are restricted to bounded operators, we obtain the Lie algebra of Sect. 4.2.4. The operators $d\Gamma(M)$ with self-adjoint $M \in \mathcal{L}(\mathcal{H})$ and D_Ξ with symmetric $\Xi \in \mathcal{L}_2(\mathcal{H})$ span a representation of this Lie algebra by unbounded operators. The ultracoherent vectors form a domain of analytic vectors for this representation.

A Calculations for ultracoherent states

A.1 Diagonalization of symmetric operators

In the sequel we use the following representation of operators in $\mathcal{L}_{2sym}(\mathcal{H})$:

Lemma 11 *Let A be a symmetric Hilbert-Schmidt operator on the Hilbert space $\mathcal{H}$. Then it admits a representation*

$$Af = \sum_{\mu \in \mathbf{M}} \alpha_\mu f_\mu \langle f_\mu \mid f \rangle = \sum_{\mu \in \mathbf{M}} \alpha_\mu f_\mu \left(f_\mu^* \mid f \right) , \tag{93}$$

where $\{f_\mu\}$ is a set of orthonormal vectors in $\mathcal{H}$ and the α_μ are complex numbers such that the series $\sum_{\mu \in \mathbf{M}} |\alpha_\mu|^2$ converges. The index set $\mathbf{M} \subset \mathbb{N}$ is finite or countable.

Proof. The proof can be obtained from the general representation of Hilbert-Schmidt operators, given e.g. in Sect. 2.2 of [24]:

$$Af = \sum_{\mu \in \mathbf{M}} \lambda_\mu f_\mu \left(e_\mu \mid f \right) = \sum_{\mu \in \mathbf{M}} \lambda_\mu f_\mu \left\langle e_\mu^* \mid f \right\rangle , \tag{94}$$

where $\{e_\mu\}$ and $\{f_\mu\}$ are orthonormal sets in $\mathcal{H}$ and the λ_μ are positive numbers such that the series $\sum_{\mu \in \mathbf{M}} \lambda_\mu^2$ converges. The numbers λ_μ may be ordered $\lambda_1 \geq \lambda_2 \geq ... > 0$. They are the square roots of the strictly positive eigenvalues of A^+A (and AA^+): $A^+Ae_\mu = \lambda_\mu^2 e_\mu$, $AA^+ f_\mu = \lambda_\mu^2 f_\mu$. The multiplicity of these eigenvalues is finite. The symmetry of A implies $A^+Af_\mu^* = \lambda_\mu^2 f_\mu^*$. If there is no degeneracy, $\lambda_1 > \lambda_2 > \lambda_3 > ... > 0$, then $e_\mu^* = \theta_\mu f_\mu$ with phase factors $\theta_\mu \in \mathbb{C}$, $|\theta_\mu| = 1$, follows, and the representation (93) is derived. If degeneracies exist (the multiplicity is always finite), a slightly more involved argument implies again the representation (93). See the proof given for Lemma 1 in Appendix II of [4], where a corresponding result has been derived for finite dimensional Hilbert spaces. ∎

The ONS $\{f_\mu\}$ in Lemma 11 need not to be a complete system. If it is not complete, we extend it to a complete system. The corresponding numbers α_μ with $\mu \notin \mathbf{M}$ are defined as $\alpha_\mu = 0$. The vectors f_μ in Lemma 11 need not to be real. But starting from any real complete ONS $\left\{ e_\mu = e_\mu^* \right\}$ there exists a unitary operator U such that $f_\mu = U^{-1} e_\mu$, $\mu = 1, 2, ...$ Then the symmetric operator $UAU^T = D$ has the representation

$$Df = \sum_{\mu=1}^{\infty} \alpha_\mu e_\mu \langle e_\mu \mid f \rangle = \sum_{\mu=1}^{\infty} \alpha_\mu e_\mu \left(e_\mu \mid f \right) \tag{95}$$

in the real ON basis $\{e_\mu\}$. The complex eigenvalues α_μ of D are the numbers of (93) for $\mu \in \mathcal{M}$ and zero for $\mu \notin \mathcal{M}$.

Remark 8 *The phases of the numbers α_μ in (93) can be absorbed in the vectors f_μ. Therefore Lemma 11 is also true with positive numbers α_μ. With such a choice of the ON system $\{f_\mu\}$ the eigenvalues of $UAU^T = D$ are also positive, and Lemma 5 follows.*

A.2 Gaussian functionals

Let $F \in \mathcal{S}_m(\mathcal{H})$ and $F \in \mathcal{S}_n(\mathcal{H})$ be tensors of degree m and n, then the normalization (4) implies the estimate $\|F \circ G\|^2 \leq \frac{(m+n)!}{m!n!} \|F\|^2 \|G\|^2$. For $H \in \mathcal{S}_2(\mathcal{H})$ we obtain $\|H^{\circ 2n}\| \leq \frac{(2n)!}{2^n} \|H\|^{2n}$, and the exponential series $\exp H = 1 + H + \frac{1}{2!} H^{\circ 2} + \frac{1}{3!} H^{\circ 3} + ...$ converges within

$\mathcal{S}(\mathcal{H})$ with the norm estimate $\|\exp H\|^2 \leq \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right)^2 \frac{(2n)!}{2^n} \|H\|^{2n} < \infty$ if $\|H\|^2 < 1/2$. The mapping $H \in \mathcal{S}_2(\mathcal{H}) \to \exp H \in \mathcal{S}(\mathcal{H})$ is therefore an analytic function within the open ball $\left\{H \mid \|H\| < 1/\sqrt{2}\right\}$. As a consequence of Lemma 11 we therefore know that $\exp \Omega(A) \in \mathcal{S}(\mathcal{H})$ if $A \in \mathcal{D}_1 \subset \mathcal{L}_{2sym}(\mathcal{H})$ and $A \in \mathcal{D}_1 \to \exp \Omega(A) \in \mathcal{S}(\mathcal{H})$ is analytic.

The identity (15) implies $\langle (\Omega(A))^* \mid f \circ g\rangle = \overline{\langle \Omega(A) \mid g^* \circ f^*\rangle} = \overline{\langle g^* \mid Af^*\rangle} = \overline{(g \mid Af^*)} = \langle f \mid A^+ g\rangle$ and we obtain $(\Omega(A))^* = \Omega(A^+) = \Omega(\bar{A})$. The exponential series of a vector $f \in \mathcal{H}$ or a tensor $\Omega(A)$ then satisfy

$$\begin{gathered}
\langle \exp f \mid \exp g\rangle = \exp \langle f \mid g\rangle \\
\exp(f+g) = \exp f \circ \exp g \\
\exp\left(\Omega(A) + f\right) = \exp \Omega(A) \circ \exp f \\
(\exp f)^* = \exp f^* \\
(\exp \Omega(A))^* = \exp\left(\Omega(A)\right)^* = \exp \Omega(\bar{A}).
\end{gathered} \tag{96}$$

The mapping

$$F \in \mathcal{S}(\mathcal{H}) \to \langle \exp \Omega(A) \mid F\rangle \in \mathbb{C} \tag{97}$$

is a linear continuous functional on the Fock space $\mathcal{S}(\mathcal{H})$. The restriction of this functional to coherent states yields by direct evaluation of the power series

$$\langle \exp \Omega(A) \mid \exp z\rangle = \exp \frac{1}{2} \langle \Omega(A) \mid z \circ z\rangle = \exp \frac{1}{2} \langle z \mid Az\rangle\,, \tag{98}$$

and the functional (97) is uniquely determined by these values. Moreover, we see that $F \in \mathcal{S}_{coh}(\mathcal{H}) \to \langle \exp \Omega(A) \mid F\rangle$ is a linear functional for all $A \in \mathcal{L}_{2sym}(\mathcal{H})$ without restriction of the operator norm.

For $B \in \mathcal{L}(\mathcal{H})$ the operator $\Gamma(B)$ is defined by $\Gamma(B)\exp f = \exp Bf$ as bounded operator on $\mathcal{S}(\mathcal{H})$. Then the identity $\langle \Gamma(B) \exp \Omega(A) \mid \exp z\rangle = \left\langle \exp \Omega(A) \mid \Gamma(B^T) \exp z\right\rangle$ $= \left\langle \exp \Omega(A) \mid \exp B^T z\right\rangle = \exp \frac{1}{2} \left\langle z \mid BAB^T z\right\rangle$ implies

$$\Gamma(B) \exp \Omega(A) = \exp \Omega(BAB^T). \tag{99}$$

In the main part of this subsection we calculate the function

$$\Phi(\bar{A}, B, f) := (\exp \Omega(A) \mid \exp \Omega(B) \circ \exp f) = \left\langle \exp \Omega(\bar{A}) \mid \exp \Omega(B) \circ \exp f\right\rangle, \tag{100}$$

which is antianalytic in $A \in \mathcal{D}_1$ (analytic in $\bar{A} \in \mathcal{D}_1$) and analytic in $B \in \mathcal{D}_1$. This function is uniquely determined by its values on the diagonal $B = A$. To evaluate $\Phi(\bar{A}, A, f)$ we choose a unitary operator U such that $UAU^T = D \in \mathcal{D}_1$ is diagonal in the real basis $\{e_\mu\}_{\mu \in \mathbb{N}}$. Then we obtain from (99)

$$\Phi(\bar{A}, A, f) = (\Gamma(U) \exp \Omega(A) \mid \Gamma(U) \exp\left(\Omega(A) + f\right)) = (\exp \Omega(D) \mid \exp\left(\Omega(D) + g\right)) \tag{101}$$

with $g = Uf$. Since D has the representation (99) (with the additional constraint $\sum_\mu |\alpha_\mu|^2 < 1/2$) we have $\Omega(D) = \frac{1}{2} \sum_\mu \alpha_\mu e_\mu \circ e_\mu$. That yields the product representations $\exp \Omega(D) = \prod_\mu \exp\left(\frac{1}{2} \alpha_\mu e_\mu \circ e_\mu\right)$ and

$$\begin{aligned}
\Phi(\bar{A}, A, f) &= \prod_\mu \varphi(\overline{\alpha_\mu}, \alpha_\mu, \gamma_\mu) \quad \text{with} \qquad (102) \\
\varphi(\overline{\alpha_\mu}, \alpha_\mu, \gamma_\mu) &= \sum_{k,m,n=0}^{\infty} \frac{1}{k!m!(2n)!} \left(\left(\frac{\alpha_\mu}{2} e_\mu \circ e_\mu\right)^k \mid \left(\frac{\alpha_\mu}{2} e_\mu \circ e_\mu\right)^m \circ (\gamma_\mu e_\mu)^{2n} \right)
\end{aligned}$$

where $\gamma_\mu = (e_\mu \mid g) = (e_\mu \mid Uf)$. The inner product vanishes unless $k = m + n$. The remaining sum

$$\varphi(\overline{\alpha}, \alpha, \gamma) = \sum_{m,n=0}^{\infty} \frac{2^{-2m-n}\overline{\alpha}^{m+n}\alpha^m\gamma^{2n}}{(m+n)!m!(2n)!}(2m+2n)!$$

can be evaluated using the identity

$$\sum_{m=0}^{\infty} \frac{(2m+2n)!}{(m+n)!m!} z^m = \frac{(2n)!}{n!} \sum_k \left(\frac{2n+1}{2}\right)_k \frac{(4z)^k}{k!} = \frac{(2n)!}{n!}(1-4z)^{-\frac{2n+1}{2}}$$

such that

$$\varphi(\overline{\alpha}, \alpha, \gamma) = \left(1 - |\alpha|^2\right)^{-\frac{1}{2}} \sum_n \frac{1}{n!} 2^{-n}(1-|\alpha|^2)^{-n}\overline{\alpha}^n\gamma^{2n} = \left(1 - |\alpha|^2\right)^{-\frac{1}{2}} \exp\left[\frac{1}{2}(1-|\alpha|^2)^{-1}\overline{\alpha}\gamma^2\right].$$

As $D = UAU^T$ and $\bar{A} = U^T\bar{D}U$ the function (102) has the form

$$\begin{aligned}
\Phi(\bar{A}, A, f) &= \det\left(1 - \bar{A}A\right)^{-\frac{1}{2}} \exp\frac{1}{2}\left\langle Uf \mid (I - \bar{D}D)^{-1}\bar{D}Uf\right\rangle \\
&= \det\left(1 - \bar{A}A\right)^{-\frac{1}{2}} \exp\frac{1}{2}\left\langle f \mid (I - \bar{A}A)^{-1}\bar{A}f\right\rangle.
\end{aligned}$$

The function (100) is therefore given by

$$\left\langle \exp\Omega(\bar{A}) \mid \exp\Omega(B) \circ \exp f\right\rangle = \det\left(1 - \bar{A}B\right)^{-\frac{1}{2}} \exp\frac{1}{2}\left\langle f \mid (I - \bar{A}B)^{-1}\bar{A}f\right\rangle. \quad (103)$$

From (100) and (103) we finally obtain for all $F \in \mathcal{S}(\mathcal{H})$

$$\left\langle \exp\Omega(\bar{A}) \mid \exp\Omega(B) \circ F\right\rangle = \det\left(1 - \bar{A}B\right)^{-\frac{1}{2}} \left\langle \exp\Omega(C) \mid F\right\rangle \quad (104)$$

where C is the symmetric operator $C = (I - \bar{A}B)^{-1}\bar{A}$.

A.3 Interior products

Let $F \in \mathcal{S}(\mathcal{H})$ and $G \in \mathcal{S}(\mathcal{H})$ be two tensors for which the symmetric tensor product $F \circ G$ is defined, then the following factorization rule holds

$$\langle F \circ G \mid \exp f\rangle = \langle F \mid \exp f\rangle \langle G \mid \exp f\rangle. \quad (105)$$

For the proof it is sufficient to verify this identity for $F = \exp f$ and $G = \exp g$ using (96). The interior product of two tensors has been defined in Definition 1. From $\langle \exp h \circ \exp g) \mid \exp f\rangle = \langle \exp(h+g) \mid \exp f\rangle = \exp\langle h \mid f\rangle \exp\langle g \mid f\rangle$ we conclude

$$\exp h \lrcorner \exp f = e^{\langle h|f\rangle} \exp f. \quad (106)$$

The identity (106) implies
$\exp h \lrcorner (\exp f \circ \exp g) = \exp\langle h \mid f + g\rangle \exp(f+g) = \left(e^{\langle h|f\rangle} \exp f\right) \circ \left(e^{\langle h|g\rangle} \exp g\right)$
$= (\exp h \lrcorner \exp f) \circ (\exp h \lrcorner \exp g)$ and the general rule

$$\exp h \lrcorner (F \circ G) = (\exp h \lrcorner F) \circ (\exp h \lrcorner G) \quad (107)$$

follows for all $F \in \mathcal{S}(\mathcal{H})$ and $G \in \mathcal{S}(\mathcal{H})$ for which the symmetric tensor product $F \circ G$ is defined.

The interior products $\exp g \,\lrcorner\, \exp \Omega(A)$ follows from (96) and

$$\langle \exp \Omega(A) \mid \exp(f+g) \rangle \overset{(98)}{=} \exp \frac{1}{2} \langle f+g \mid A(f+g) \rangle = \mathrm{e}^{\frac{1}{2}\langle g|Ag\rangle} \langle \exp\left(\Omega(A) + Ag\right) \mid \exp f \rangle$$

as

$$\exp g \,\lrcorner\, \exp \Omega(A) = \mathrm{e}^{\frac{1}{2}\langle g|Ag\rangle} \exp\left(\Omega(A) + Ag\right). \tag{108}$$

From (96), (98) and (105) we obtain $\langle \exp\left(\Omega(A) + f\right) \mid \exp g \rangle = \mathrm{e}^{\frac{1}{2}\langle g|Ag\rangle + \langle f|g\rangle}$, which yields

$$\exp \Omega(A) \,\lrcorner\, \exp g = \mathrm{e}^{\frac{1}{2}\langle g|Ag\rangle} \exp g. \tag{109}$$

The identities (106) - (108) imply

$$\begin{aligned} \exp f \,\lrcorner\, \exp\left(\Omega(B) + g\right) &= \left(\exp f \,\lrcorner\, \exp \Omega(B)\right) \circ \left(\exp f \,\lrcorner\, \exp g\right) \\ &= \mathrm{e}^{\frac{1}{2}\langle f|Bf\rangle + \langle f|g\rangle} \exp\left(\Omega(B) + Bf + g\right). \end{aligned} \tag{110}$$

The interior product

$$\Omega(A) \,\lrcorner\, \exp f = \frac{1}{2} \langle f \mid Af \rangle \exp f \tag{111}$$

can be calculated for $A \in \mathcal{L}_{2sym}$ by direct evaluation of $\langle g \circ \exp h \mid \exp f \rangle$ and $\langle \Omega(A) \circ \exp h \mid \exp f \rangle$.

Following [3] the linear operators $F \in \mathcal{S}_{coh}(\mathcal{H}) \to \Omega(A) \circ F \in \mathcal{S}(\mathcal{H})$ and $F \in \mathcal{S}_{coh}(\mathcal{H}) \to \Omega(A) \,\lrcorner\, F \in \mathcal{S}(\mathcal{H})$ will be denoted by

$$\begin{aligned} a^+ A a^+ \, F &:= 2\,\Omega(A) \circ F \\ a A a \, F &:= 2\,\Omega(A) \,\lrcorner\, F. \end{aligned} \tag{112}$$

From $\left(\Omega(A) \circ F \mid G\right) = \left\langle \Omega(\bar{A}) \circ F^* \mid G \right\rangle = \left(F \mid \Omega(\bar{A}) \,\lrcorner\, G\right)$ follows that

$$a\bar{A}a = \left(a^+ A a^+\right)^+ \tag{113}$$

is the adjoint operator of $a^+ A a^+$.

A.4 The inner product

The bilinear form of two ultracoherent vectors follows from (103) and the contraction (110) as

$$\begin{aligned} \langle \exp\left(\Omega(A) + f\right) \mid \exp\left(\Omega(B) + g\right) \rangle &= \langle \exp \Omega(A) \mid \exp f \,\lrcorner\, \exp\left(\Omega(B) + g\right) \rangle \\ &= \mathrm{e}^{\frac{1}{2}\langle f|Bf\rangle + \langle f|g\rangle} \langle \exp \Omega(A) \mid \exp\left(\Omega(B) + Bf + g\right) \rangle \\ &= \det(I - AB)^{-\frac{1}{2}} \exp\left(\tfrac{1}{2} \langle f \mid Cf \rangle + \langle f \mid (I - BA)^{-1} f \rangle + \tfrac{1}{2} \langle g \mid Dg \rangle\right) \end{aligned} \tag{114}$$

with the operators

$$\begin{aligned} C &= (I - BA)^{-1} B, \\ D &= (I - AB)^{-1} A. \end{aligned} \tag{115}$$

Using the lower lines of (96) we see that (114) is equivalent to the sesquilinear inner product (19).

B Calculations for the product rule

In this Appendix we evaluate (84) and (85) for general $Z \in \mathcal{D}_1$.

In the first step we prove that the multiplier (84) does not depend on Z. The relation (82) implies the operator identity $(I + ZV_1^+ U_1^{+-1})U_1^+(I + Z_1 V_2^+ U_2^{+-1})U_2^+ = (I + ZV_3^+ U_3^{+-1})U_3^+$ or

$$(I + ZV_1^+ U_1^{+-1})U_1^+(I + Z_1 V_2^+ U_2^{+-1})U_1^{+-1} = (I + ZV_3^+ U_3^{+-1})U_3^+ U_2^{+-1} U_1^{+-1}. \tag{116}$$

The operators $X = ZV_1^+ U_1^{+-1},\ Z_1 V_2^+ U_2^{+-1},\ ZV_3^+ U_3^{+-1}$ are nuclear, therefore the determinants $\det(I+X)$ are well defined. The operators U_k, $k = 1, 2, 3$, are bounded operators with bounded inverse. We also know that $\det(I + X) = \det(I + U^{-1}XU)$ if $X \in \mathcal{L}_1(\mathcal{H})$ and $U \in \mathcal{L}(\mathcal{H})$ with $U^{-1} \in \mathcal{L}(\mathcal{H})$. The identity (116) therefore implies

$$\det(I+ZV_1^+ U_1^{+-1}) \det(I+Z_1 V_2^+ U_2^{+-1}) = \det(I+ZV_3^+ U_3^{+-1}) \det\left(U_3^+ U_2^{+-1} U_1^{+-1}\right). \tag{117}$$

The operator $U_3^+ U_2^{+-1} U_1^{+-1} = \left[(U_2 U_1)^{-1} U_3\right]^+$ has the form $I + N$ with N nuclear, as can be seen from $U_3 = U_2 U_1 + V_2 \bar{V}_1$. Hence the determinant $\det\left(U_3^+ U_2^{+-1} U_1^{+-1}\right)$ is a well defined function of the group elements R_1 and R_2. Inserting (117) into (84) the dependence on Z is cancelled, and the multiplier χ is the function (86) of the group elements alone.

Now we evaluate (85) for general $Z \in \mathcal{D}_1$. Using the symmetry $Z^T = Z$ and the identity

$$(U_1^+ + ZV_1^+)^{-1} = (U_2^+ + Z_1 V_2^+)(U_3^+ + ZV_3^+)^{-1}, \tag{118}$$

which follows from (82), we obtain

$$\alpha_{12} = \left\langle f \mid \left[(\bar{U}_1 + \bar{V}_1 Z)^{-1} V_2^+ (U_3^+ + ZV_3^+)^{-1} + V_1^+ (U_1^+ + ZV_1^+)^{-1}\right] f \right\rangle. \tag{119}$$

Now

$$\begin{aligned} &(\bar{U}_1 + \bar{V}_1 Z)^{-1} V_2^+ (U_3^+ + ZV_3^+)^{-1} + V_1^+ (U_1^+ + ZV_1^+)^{-1} \\ &= \left[(\bar{U}_1 + \bar{V}_1 Z)^{-1} V_2^+ + V_1^+ (U_2^+ + Z_1 V_2^+)\right] (U_3^+ + ZV_3^+)^{-1}. \end{aligned} \tag{120}$$

We take the term $(\bar{U}_1 + \bar{V}_1 Z)^{-1} V_2^+ + V_1^+ U_2^+ + V_1^+ Z_1 V_2^+$ in the right hand side of the previous equation and define the operator

$$(\bar{U}_1 + \bar{V}_1 Z)^{-1} V_2^+ + V_1^+ Z_1 V_2^+ = X.$$

This implies that $(\bar{U}_1 + \bar{V}_1 Z)^{-1} + V_1^+ Z_1 = XV_2^{+-1}$, from which we get $I + V_1^+(U_1 Z + V_1) = XV_2^{+-1}(\bar{U}_1 + \bar{V}_1 Z)$. Now using (44) this becomes $U_1^T(\bar{U}_1 + \bar{V}_1 Z) = XV_2^{+-1}(\bar{U}_1 + \bar{V}_1 Z)$ and we obtain $X = U_1^T V_2^+$. Thus, Eq. (120) becomes $V_3^+(U_3^+ + ZV_3^+)^{-1}$. Using the product law (40) we finally obtain from (119)

$$\alpha_{12} = \left\langle f \mid V_3^+ (U_3^+ + ZV_3^+)^{-1} f \right\rangle.$$

Hence (79) follows on the space $\mathcal{S}_{coh}(\mathcal{H})$.

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