Technische Universität
Kaiserslautern

SCHRIFTEN ZUR

FUNKTIONALANALYSIS
UND GEOMATHEMATIK

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Multiresolution Analysis by Spherical Up Functions

Bericht 2 – September 2003

FACHBEREICH MATHEMATIK

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by

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Abstract

A new class of locally supported radial basis functions on the (unit) sphere is introduced by forming an infinite number of convolutions of "isotropic finite elements". The resulting up functions show useful properties: They are locally supported and are infinitely often differentiable. The main properties of these kernels are studied in detail. In particular, the development of a multiresolution analysis within the reference space of square–integrable functions over the sphere is given. Altogether, the paper presents a mathematically significant and numerically efficient introduction to multiscale approximation by locally supported radial basis functions on the sphere.

AMS-Classification: 41A30, 41A58, 42C15, 42C40, 44A35, 45E10, 65T60.

Keywords: Up Functions, Multiresolution Analysis, Spherical Wavelets, Decomposition and Reconstruction Schemes.

1 Introduction

For future modelling of functions on the sphere a new component of approximation has to come into play, viz. spherical wavelets, at least when short–scale space phenomena, i.e. fine–scale details are of interest. The power of spherical wavelets is based on a multiresolution analysis which enables a balanced amount of both frequency (more accurately, angular momentum) and space localization. Spherical wavelets are used as mathematical means for breaking up a complicated structure of a function into many simple pieces at different scales and positions. Consequently, the representation of data in terms of wavelets is more concentrated on essential features, i.e. more compact, than the original discrete data representation.

A first (parametric) concept of spherical wavelets was proposed by Dahlke, Dahmen, Schmitt, and Weinreich [6], [34] based on a tensor product basis, in which one component is a spline of exponential (E-) type. The so–called E-splines provide not only $C^{(1)}$–wavelets, but also guarantee the reproduction of trigonometric functions within the wavelet expansion. Potts and Tasche [25] form tensor products of interpolatory trigonometric wavelets and polynomial wavelets in accordance with a usual longitude/lattitude parametrization of the sphere. These wavelets, however, satisfy the $C^{(1)}$–assumption described in [34] only in an approximate sense. Starting with a triangulation of the sphere, Schröder and Sweldens [29] construct spherical Haar–type wavelets on the triangles yielding smoother wavelets by virtue of the so–called lifting scheme. A group theoretical approach to a continuous wavelet transform on the sphere is followed by Antoine and Vandergheynst [1], [2]. The parameter choice of their continuous wavelet transform is the product of $SO(3)$ (for the motion on the sphere) and $\mathbb{R}^+$ (for the dilations).

Our constructions [18], [19], [8], [9], [12], [13] are intrinsically based on the specific properties concerning the geometry of the sphere and the theory of "spherical polynomials", i.e. in the jargon of mathematical (geo–)physics spherical harmonics. Two approaches to spherical wavelets have been established: On the one hand, a continuous wavelet transform (and its discretizations) was obtained by taking particular advantage of the conception of spherical singular integrals. Within this framework the wavelets turn out to be (not–necessarily isotropic) kernel functions generated by summing up certain clusters of spherical harmonic expressions. The wavelets are definable either by increasing space localization of the kernels or by decreasing frequency localization of their corresponding symbols (i.e. Fourier transform). Wavelet modelling is provided by a two–parameter family reflecting the different levels of localization and resolution. On the other hand the authors [16], [8] presented a scale discrete wavelet transform involving band–limited as well as non–band–limited kernel representations by forming the so–called P–scale or M–scale wavelet representations. With the help of approximate or exact (for spherical harmonic or spherical splines) interpolatory formulae all wavelet transforms allow fully discrete approximants via tree algorithms (i.e. pyramid schemes) [27], [8], [7], [11].

Seen under numerical aspects, i.e. from a the point of computational implementation, spherical wavelets with local support are of particular significance. Moreover, for reasons of rotational symmetry and structural simplicity, it is canonical to consider radial basis kernel functions (i.e. kernels which are isotropic in the sense that their values depend only on the distance of the argument to a fixed point). In other words, isotropic wavelets with local support seem to be a proper choice for a variety of applications (e.g. invariant pseudodifferential equations [7], regularization of inverse problems [14], etc). However, the locally supported isotropic kernel functions (in brief, isotropic finite elements) on the sphere used so far ([10], [15], [17], [8], [28], [9], [22], [32]) possess a non–monotone $\mathcal{L}^2(\Omega)$–symbol, which prevents them to be used for building up a multiresolution analysis (in the sense that the space of square–integrable functions on the sphere can be decomposed by a nested sequence of subspaces including a closure property). Furthermore, finite elements show only a limited order of differentiability on the sphere, which may be not appropriate when investigating smooth functions. In conclusion, the problem is to introduce a multiscale approximation with the following ingredients for the wavelets to be constructed: isotropy, smoothness, local support, establishment of a multiresolution analysis for the Hilbert space of square–integrable functions on the sphere.

In this paper we transform the idea of the up function (cf. [26]) to the spherical context. Therefore, we introduce generalizations of the already known locally supported kernels and build an infinite convolution with them. This procedure results into a new class of radial basis functions that are infinitely often differentiable and (when the support of the building functions is chosen properly) have also a local support. Another appealing property of these new kernels is, that they show a scale discrete multiresolution analysis of the space of square integrable functions on the sphere. In other words, the radial basis functions constructed in this way are appropriate means for various fields of constructive approximation, including locally supported, smooth spherical wavelets.

The outline of this paper is as follows: In the preliminaries we give the basic definitions and notations. Then, we introduce locally supported kernels on the sphere by an extension of the previously known functions, including unbounded kernels. It is canonical to take iterations of these kernels, since the Fourier transform then becomes non–negative. After these considerations we are able to introduce the spherical up function by infinite convolutions of locally supported kernels. The basic properties of the up function are described in detail. In particular, we show how a multiresolution analysis can be developed by use of these new functions.

2 Preliminaries

In what follows we list some basic notions and definitions used in this paper.

If $x, y \in \mathbb{R}^3$, we write $x \cdot y$ for the Euclidean inner product and $|x| = \sqrt{x \cdot x}$ for the norm. We let $\Omega = \{\xi \in \mathbb{R}^3 \mid |\xi| = 1\}$ denote the unit sphere in $\mathbb{R}^3$. The standard surface measure on Ω is denoted by $d\omega$. On the space $\mathcal{L}^2(\Omega)$ of square–integrable functions on Ω we introduce the inner product $(F, G) = \int_\Omega F(\eta)G(\eta)d\omega(\eta)$.

The *spherical harmonics* Y_n of order n are defined as the everywhere on the unit sphere Ω twice continuously differentiable eigenfunctions of the Beltrami operator Δ^* corresponding to the eigenvalues $-n(n+1), n = 0, 1, \ldots$. The linear space Harm_n of all spherical harmonics of order n is of dimension $2n+1$. As it is well known [24], there exist $2n+1$ linearly independent spherical harmonics $Y_{n,1}, \ldots, Y_{n,2n+1}$. Throughout the remainder of this paper we assume this system to be orthonormalized in the sense of the $\mathcal{L}^2(\Omega)$-inner product: $(Y_{n,l}, Y_{n,k}) = \delta_{lk}$, where δ_{lk} is the Kronecker symbol. The *Fourier transform* of a function $F \in \mathcal{L}^2(\Omega)$ is denoted as $F^\wedge(n, m) = (F, Y_{n,m})$, $n = 0, 1, \ldots$, $m = 1, \ldots, 2n+1$. Clearly,

$$\lim_{N\to\infty} \left\| F - \sum_{n=0}^{N} \sum_{m=1}^{2n+1} F^\wedge(n, m) Y_{n,m} \right\|_{\mathcal{L}^2(\Omega)} = 0$$

for all $F \in \mathcal{L}^2(\Omega)$.

An outstanding result of the theory of spherical harmonics is the *addition theorem*

$$\sum_{m=1}^{2n+1} Y_{n,m}(\xi)Y_{n,m}(\eta) = \frac{2n+1}{4\pi}P_n(\xi\cdot\eta), \quad \xi,\eta\in\Omega, \tag{1}$$

where P_n is the *Legendre polynomial* of degree n.

For later use we present some properties of the Legendre polynomials. First we mention the recurrence relation

$$P_0(t) = 1, \qquad P_1(t) = 1 \tag{2}$$

$$(n+1)P_{n+1}(t) + nP_{n-1}(t) - (2n+1)tP_n(t) = 0. \tag{3}$$

The Legendre polynomial P_n satisfies the estimate (see for example [23], [8])

$$|P_n^{(k)}(t)| \ \le P_n^{(k)}(1), \quad t\in[-1,1] \tag{4}$$

where $P_n^{(k)}(1) = O(n^{2k})$. In particular, we have

$$P_n'(1) = \frac{n(n+1)}{2}. \tag{5}$$

For later use, we mention the estimate (cf. [21])

$$(1-t^2)^{1/4}|P_n(t)| \le \sqrt{\frac{2}{\pi(n+1/2)}}, \quad t\in[-1,1]. \tag{6}$$

Let G be of class $\mathcal{L}^2[-1,1]$. Suppose that $\eta\in\Omega$ us fixed. The η–zonal function $G(\eta\cdot\ ):\Omega\to\mathbb{R}$ given by $\xi\mapsto G(\eta\cdot\xi)$, $\xi\in\Omega$, is in $\mathcal{L}^2(\Omega)$ and is axisymmetric with respect to the axis η, i.e. the value at the point $\xi\in\Omega$ depends only on the inner product $\xi\cdot\eta$. Since $|\xi-\eta| = \sqrt{2-2\xi\cdot\eta}$, zonal functions can be seen to be the spherical counterpart to radial basis functions in Euclidean spaces (see e.g. [5].) The *Funk–Hecke formula* tells us that for any $Y_n\in\mathrm{Harm}_n$ and $\eta\in\Omega$,

$$\int_\Omega G(\eta\cdot\xi)Y_n(\xi)d\omega(\xi) = G^\wedge(n)Y_n(\eta)\ . \tag{7}$$

where the Legendre transform (i.e. the symbol of G) is given by

$$G^\wedge(n) = 2\pi\int_{-1}^{+1} G(t)P_n(t)\ dt, \tag{8}$$

$n = 0,1,\ldots$. For more details the reader is referred for example to [22] or [8]. Applying the addition theorem it follows that the orthogonal expansion in terms of Legendre polynomials of the η–zonal function $G(\eta\cdot\ )$ is

$$G(\eta\cdot\ ) \sim \sum_{n=0}^{\infty}\frac{2n+1}{4\pi}G^\wedge(n)P_n(\eta\cdot\ ).$$

Equivalently,

$$G(t) \sim \sum_{n=0}^{\infty}\frac{2n+1}{4\pi}G^\wedge(n)P_n(t), \qquad t\in[-1,1] \tag{9}$$

$G^\wedge(n)$ is called the symbol of the *Legendre transform* of the function G. Due to a result in [20], the series (9) converges in uniform sense to G on the interval $[-1,1]$, provided that G is Lipschitz–continuous on $[-1,1]$.

Zonal functions lead us to spherical convolutions, see e.g. [4] or [8] in the following way: For $G \in \mathcal{L}^2[-1,1]$ and $F \in \mathcal{L}^2(\Omega)$

$$(G * F)(\xi) = \int_\Omega G(\xi \cdot \eta) F(\eta) d\omega(\eta), \qquad \xi \in \Omega. \tag{10}$$

It easily follows that $G * F \in \mathcal{L}^2(\Omega)$. Moreover,

$$(G * F)^\wedge(n,m) = G^\wedge(n) F^\wedge(n,m). \tag{11}$$

Of particular importance is the convolution with a second zonal function. Let $H \in \mathcal{L}^2[-1,1]$. An easy application of the Funk–Hecke formula shows that

$$(G * H)(\xi,\zeta) = \int_\Omega G(\xi \cdot \eta) H(\eta \cdot \zeta) d\omega(\eta), \quad \xi,\zeta \in \Omega, \tag{12}$$

depends only on the inner product of ξ and ζ. Thus $G*H$ is considered as another zonal function and can be seen to be a continuous function defined on the interval $[-1,1]$. It easily follows that for $G,\ H \in \mathcal{L}^2[-1,1]$

$$(G * H)^\wedge(n) = G^\wedge(n) H^\wedge(n), \quad n = 0,1,\ldots\ . \tag{13}$$

The convolution of a function $G \in \mathcal{L}^2[-1,1]$ with itself constitutes the so–called iterated function:

$$G^{(2)} = G * G, \quad ,G^{(k+1)} = G * G^{(k)}, \quad k = 2,3,\ldots \tag{14}$$

Obviously, $(G^{(2)})^\wedge(n) = (G^\wedge(n))^2$ and $G^{(2)} \in C[-1,1]$.

3 Locally Supported Kernels on the Unit Sphere

Starting point of our considerations are the functions

$$B_{h,\lambda}(t) = \begin{cases} 0 & \text{for} \quad -1 \le t \le h \\ (t-h)^\lambda & \text{for} \quad h < t \le 1 \end{cases}$$

which we consider for $t \in [-1,1]$, $h \in (-1,1)$ and $\lambda > -1$. Note that in contrast to earlier investigations of these kernels (see e.g. [28]) we let the parameter λ be real, and allow the functions to be unbounded (for $-1 < \lambda < 0$), but with finite integral. Letting $\eta \in \Omega$ be fixed, we get a radial basis function

$$\Omega \ni \xi \mapsto B_{h,\lambda}(\eta \cdot \xi)$$

which in accordance with our construction has the local support

$$\text{supp} B_{h,\lambda}(\eta\cdot\ ) = \{\xi \in \Omega | h \le \xi \cdot \eta \le 1\}.$$

Next we are interested in the symbol (or Legendre transform) $B_{h,\lambda}^\wedge(n)$ of the series

$$B_{h,\lambda} \sim \sum_{n=0}^\infty \frac{2n+1}{4\pi} B_{h,\lambda}^\wedge(n) P_n.$$

By virtue of the Funk–Hecke formula we find

$$B_{h,\lambda}^{\wedge}(n) = 2\pi \int_h^1 B_{h,\lambda}(t) P_n(t) dt.$$

Based on the recursion relations (3) we are able to derive recursion formulas for $B_{h,\lambda}^{\wedge}(n)$, where we follow the lines predetermined by [22], [31], [10], [8], [28]. A straightforward integration yields

$$B_{h,\lambda}^{\wedge}(0) = 2\pi \frac{(1-h)^{\lambda+1}}{\lambda+1}, \quad B_{h,\lambda}^{\wedge}(1) = \frac{\lambda+1+h}{\lambda+2} B_{h,\lambda}^{\wedge}(0). \tag{15}$$

In particular, by using the recurrence formula (3), we obtain for $n \geq 1$

$$(n+1)B_{h,\lambda}^{\wedge}(n+1) + nB_{h,\lambda}^{\wedge}(n-1) - (2n+1)B_{h,\lambda+1}^{\wedge}(n) - (2n+1)hB_{h,\lambda}^{\wedge}(n) = 0.$$

By partial integration and the recursion formulas for Legendre polynomials we find

$$(2n+1)B_{h,\lambda+1}^{\wedge}(n) = -(\lambda+1)\left[B_{h,\lambda}^{\wedge}(n+1) - B_{h,\lambda}^{\wedge}(n-1)\right]. \tag{16}$$

Combining these results we arrive at the following recursion formula

$$(n+\lambda+2)B_{h,\lambda}^{\wedge}(n+1) = (2n+1)hB_{h,\lambda}^{\wedge}(n) - (n-\lambda-1)B_{h,\lambda}^{\wedge}(n-1). \tag{17}$$

For later use we want to scale the kernel $B_{h,\lambda}$ so that the Legendre transform of order zero is 1. Having in mind (15), we define, $L_{h,\lambda}, \lambda > -1$, $h \in (-1,1)$, $t \in [-1,1]$, by setting

$$L_{h,\lambda}(t) = \frac{1}{B_{h,\lambda}^{\wedge}(0)} B_{h,\lambda}(t) = \begin{cases} 0 & \text{for} \quad -1 \leq t \leq h \\ \frac{\lambda+1}{2\pi(1-h)^{\lambda+1}}(t-h)^{\lambda} & \text{for} \quad h < t \leq 1 \end{cases} \tag{18}$$

The previous results immediately lead us to the following recursion relations.

Lemma 3.1. *For $\lambda > -1$, and $h \in (-1,1)$ we have for $n = 1, 2, \ldots$*

$$\begin{aligned} L_{h,\lambda}^{\wedge}(0) &= 1 \\ (\lambda+1)L_{h,\lambda}^{\wedge}(1) &= (\lambda+1+h)L_{h,\lambda}^{\wedge}(0) \\ (n+\lambda+2)L_{h,\lambda}^{\wedge}(n+1) &= (2n+1)hL_{h,\lambda}^{\wedge}(n) - (n-\lambda-1)L_{h,\lambda}^{\wedge}(n-1) \end{aligned}$$

Lemma 3.1 shows that the symbol $L_{h,\lambda}^{\wedge}(n)$ is a polynomial with respect to the variable h. More explicitly, we find

Lemma 3.2. *For $\lambda > -1$ and $h \in (-1,1)$, we have for all $n = 0, 1, \ldots$*

(i) $L_{h,\lambda}^{\wedge}(0) = 1$, $|L_{h,\lambda}^{\wedge}(n)| < 1$, $n \geq 1$

(ii) $\lim_{h \to 1} L_{h,\lambda}^{\wedge}(n) = 1$

Proof. Part (i) is a consequence of the expression

$$L_{h,\lambda}^{\wedge}(n) = \frac{\lambda+1}{(1-h)^{\lambda+1}} \int_h^1 (t-h)^{\lambda} P_n(t) dt \tag{19}$$

and

$$|P_n(t)| \le P_0(t) = 1, \quad t \in [-1,1],$$
$$|P_n(t)| < 1, \quad t \in (-1,1).$$

The limit in part (ii) for

$$L_{h,\lambda}^{\wedge}(n) = 2\pi \int_{-1}^{1} L_{h,\lambda}(t) P_n(t) dt$$

follows from the facts that $2\pi \int_{-1}^{1} L_{h,\lambda}(t)dt = 1$, $L_{h,\lambda}(t) \ge 0$, $P_n(1) = 1$. □

Next, our purpose is to understand, for which h the value of $L_{h,\lambda}^{\wedge}(n)$ becomes zero. The next theorem tells us that (if h is chosen to be close enough to 1) we can ensure, that the symbols are not zero. In particular, we have as an immediate consequence of Lemma 3.2

Theorem 3.3. *Let $\lambda > -1$ be given. For any $N \in \mathbb{N}$ there exists a real number h_0, $h_0 \in (-1,1)$, so that*

$$L_{h,\lambda}^{\wedge}(n) > 0$$

for all $n \le N$ and $h \ge h_0$.

For $\lambda \in \mathbb{N}$ we are able to establish a closed expression of $L_{h,\lambda}^{\wedge}(n)$ in terms of Gegenbauer (or ultraspherical) polynomials:

Theorem 3.4. *Assume that n is a positive integer, i.e. $n \in \mathbb{N}$.*

(i) $L_{h,\lambda}^{\wedge}(n) \ne 0$, $n = 0, 1, \ldots, \lambda + 2$.

(ii) *For* $n = 0, 1, \ldots$

$$L_{h,\lambda}^{\wedge}(n+\lambda+1) = \frac{(1+h)^{\lambda+1}}{2^{\lambda+1}} \frac{1}{\binom{n+2\lambda+2}{n}} C_n^{\lambda+3/2}(h),$$

where $C_n^{\lambda+3/2}$ is the Gegenbauer (or ultraspherical) polynomial of order n.

Proof. Part (i) is an easy consequence of the recurrence relation of Lemma 3.1, since for $n = 0, 1, \ldots, \lambda + 1$ it follows that $n - \lambda - 1 \le 0$.

To show (ii) we use a result of [28]. There it is proved (with a different notation) that

$$L_{h,\lambda}^{\wedge}(n+\lambda+1) = \frac{(1+h)^{\lambda+1}}{2^{\lambda+1}} \left[\frac{1}{n!} \prod_{j=0}^{n-1} (2\lambda+3+j) \right]^{-1} C_n^{\lambda+3/2}(h),$$

but

$$\frac{1}{n!} \prod_{j=0}^{n-1} (2\lambda+3+j) = \frac{(n+2\lambda+1)!}{n!(2\lambda+2)!} = \binom{n+2\lambda+2}{n}.$$

This gives the desired result. $\square$

The Gegenbauer polynomials are normalized in the sense (cf. [21])

$$C_n^{\lambda+3/2}(1) = \binom{n+2\lambda+2}{n},$$

so that the limit relation from Lemma 3.2 follows (for $\lambda \in \mathbb{N}$) from this result.

In order to develop smoothness properties of the iterations of the kernels $L_{h,\lambda}$, it is of importance to have an estimation for $L_{h,\lambda}^{\wedge}(n)$ for $n \to \infty$ and fixed h.

The following theorem extends an earlier result which was proved in [15] for $\lambda \in \mathbb{N}$ (see also [8]).

Theorem 3.5. *Suppose that* $\lambda > -1$, $h \in (-1,1)$. *Then*

$$L_{h,\lambda}^{\wedge}(n) = O(n^{-3/2-\lambda}), \quad n \to \infty.$$

Proof. We start with the following remarks: We know from [33] that

$$\int_x^1 P_{n-1}(t)dt = C_n^{-1/2}(x) \tag{20}$$

with the Gegenbauer polynomial $C_n^{-1/2}$. For these polynomials, we find in [21] the estimate

$$|C_n^{-1/2}(t)| \leq \frac{C}{n^{3/2}} \tag{21}$$

for all $n \geq 1$ and a positive constant C.

Let $h \in (-1,1)$ be fixed. First we discuss the case $-1 < \lambda \leq 0$. From (19) it is clear that it suffices to verify

$$\int_h^1 (t-h)^\lambda P_n(t)dt = O(n^{-3/2-\lambda}). \tag{22}$$

Given ε such that $0 < \varepsilon < (1-h)/2$. We split the integral (22) into

$$\int_h^1 (t-h)^\lambda P_n(t)dt = \int_h^{h+\varepsilon} (t-h)^\lambda P_n(t)dt + \int_{h+\varepsilon}^1 (t-h)^\lambda P_n(t)dt. \tag{23}$$

From (6) we can deduce that

$$\begin{aligned} \left| \int_h^{h+\varepsilon} (t-h)^\lambda P_n(t)dt \right| &\leq \int_h^{h+\varepsilon} (t-h)^\lambda dt \max_{t\in[h,h+\varepsilon]} |P_n(t)| \\ &\leq \left. \frac{(t-h)^{\lambda+1}}{\lambda+1} \right|_h^{h+\varepsilon} \cdot \frac{C}{\sqrt{n}} \\ &= \frac{\varepsilon^{\lambda+1}}{\lambda+1} \frac{C}{\sqrt{n}} \end{aligned}$$

which holds uniformly with respect to all ε with $0 < \varepsilon < (1-h)/2$. We choose $\varepsilon = 1/n$ and get

$$\int_h^{h+\varepsilon} (t-h)^\lambda P_n(t)dt = O(n^{-3/2-\lambda}). \tag{24}$$

For the second term in (23) we obtain with partial integration and the remark from the beginning of the proof

$$\int_{h+\varepsilon}^{1} (t-h)^{\lambda} P_n(t)dt = (t-h)^{\lambda} C_n^{-1/2}(t)\Big|_{h+\varepsilon}^{1} - \int_{h+\varepsilon}^{1} \lambda (t-h)^{\lambda-1} C_n^{-1/2}(t)dt.$$

Since the maximum of $(t-h)^{\lambda}$ for $t \in [h+\varepsilon, 1]$ is attained for $t = h+\varepsilon$, we can estimate the first summand in view of (21)

$$\left| (t-h)^{\lambda} C_n^{-1/2}(t)\Big|_{h+\varepsilon}^{1} \right| \leq \varepsilon^{\lambda} \frac{C}{n^{3/2}}. \tag{25}$$

For the second summand it follows from (21) that

$$\begin{aligned} \left| \int_{h+\varepsilon}^{1} \lambda (t-h)^{\lambda-1} C_n^{-1/2}(t)dt \right| &\leq \frac{C}{n^{3/2}} \int_{h+\varepsilon}^{1} \lambda (t-h)^{\lambda-1} dt & (26) \\ &\leq \frac{C}{n^{3/2}} \, (t-h)^{\lambda} \Big|_{h+\varepsilon}^{1} & (27) \\ &= \frac{C}{n^{3/2}} \varepsilon^{\lambda}. & (28) \end{aligned}$$

Combining (25) and (28) and taking again $\varepsilon = 1/n$, we obtain

$$\int_{h+\varepsilon}^{1} (t-h)^{\lambda} P_n(t)dt = O(n^{-3/2-\lambda}), \quad n \to \infty.$$

Together with (24), it follows that

$$\int_{h}^{1} (t-h)^{\lambda} P_n(t)dt = O(n^{-3/2-\lambda}), \quad n \to \infty,$$

which, finally, yields the assertion for all λ with $-1 < \lambda \leq 0$.

To prove the result for $\lambda > 0$, we first deduce from (16) and the definition of $L_{h,\lambda}$, that

$$L_{h,\lambda+1}^{\wedge}(n) = -\frac{\lambda+1}{2n+1} \left[L_{h,\lambda}^{\wedge}(n+1) - L_{h,\lambda}^{\wedge}(n-1) \right].$$

Thus, the assertion follows for all λ with $0 < \lambda \leq 1$, and, recursively, for all $\lambda > -1$. □

The last theorem shows that, for $\lambda > -1/2$,

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \left[L_{h,\lambda}^{\wedge}(n) \right]^2 < \infty.$$

In consequence, $L_{h,\lambda}(\eta \cdot\ )$ is of class $\mathcal{L}^2(\Omega)$ for every $\eta \in \Omega$. Clearly, $L_{h,\lambda}(\eta \cdot\ ) \in \mathcal{L}^1(\Omega)$ for every $\eta \in \Omega$ and all $\lambda > -1$.

Finally, we provide illustrations of some kernels: Figure 1 shows the kernel $\vartheta \mapsto L_{h,\lambda}(\cos\vartheta)$, $\vartheta \in [-\pi, \pi]$, for $\lambda = 1$ and different values of h. Figure 2 illustrates the kernels $\vartheta \mapsto L_{h,\lambda}(\cos\vartheta)$, $\vartheta \in [-\pi, \pi]$, for different values of λ.

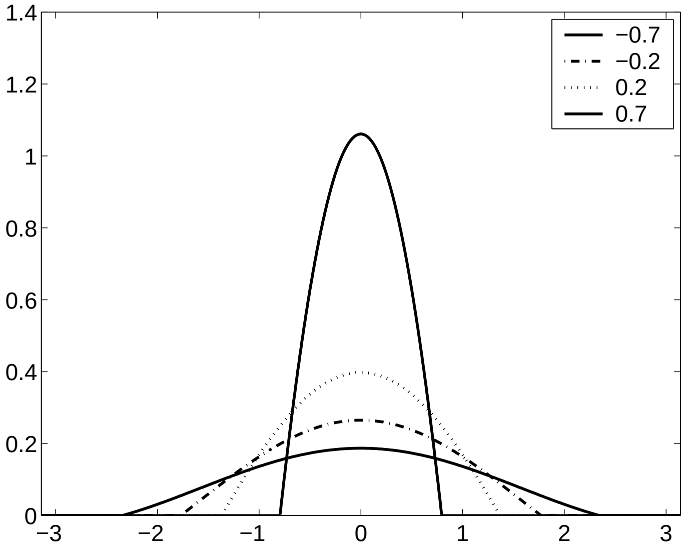

Figure 1: The kernels $\vartheta \mapsto L_{h,\lambda}(\cos\vartheta)$ for $\lambda = 1$, $h = -0.7, -0.2, 0.2, 0.7$.

4 Iterated Locally Supported Kernels

Next, we will apply the earlier described concept of iterated kernels to the locally supported radial basis functions of the last section. It turns out, that the iterated kernels have some appealing properties. To be more concrete, they are still locally supported, their Legendre transform is non–negative, and they show a certain degree of smoothness.

Let $h \in (0,1)$ and $\lambda > -1$. Then it is known (see e.g. [8]) that the iterated kernel

$$L_{h,\lambda}^{(2)} = L_{h,\lambda} * L_{h,\lambda}$$

has the support

$$\mathrm{supp} L_{h,\lambda}^{(2)}(\eta\cdot\ ) = \left\{\xi \in \Omega | 2h^2 - 1 \le \xi\cdot\eta \le 1\right\}. \tag{29}$$

Since the support of the aforementioned radial basis functions will become an important issue when we consider infinite convolutions, the statement (29) should be explained in more detail: The support of $L_{h,\lambda}(t)$ is $[h,1]$, so that the function $\vartheta \mapsto L_{h,\lambda}(\cos\vartheta)$, $\vartheta \in [0,\pi]$, is supported in $[0, \arccos h]$. The support of the iterated kernel $\vartheta \mapsto L_{h,\lambda}^{(2)}(\cos\vartheta)$ is then twice as large, i.e. $[0, 2\arccos h]$, which is obvious when the kernel is considered as a radial basis function over the sphere Ω. Thus, the support of $t \mapsto L_{h,\lambda}^{(2)}(t)$ is $[\cos(2\arccos h), 1] = [2h^2 - 1, 1]$.

For the Legendre series of the iterated kernels we have formally

$$L_{h,\lambda}^{(2)}(t) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \left[L_{h,\lambda}^{\wedge}(n)\right]^2 P_n(t),$$

so that

$$L_{h,\lambda}^{(2)\wedge}(n) = \left[L_{h,\lambda}^{\wedge}(n)\right]^2 \ge 0, \quad n \ge 0.$$

A consequence of Theorem 3.5 is the asymptotic relation

$$L_{h,\lambda}^{(2)\wedge}(n) = O(n^{-3-2\lambda}), \quad n \to \infty. \tag{30}$$

This leads us to

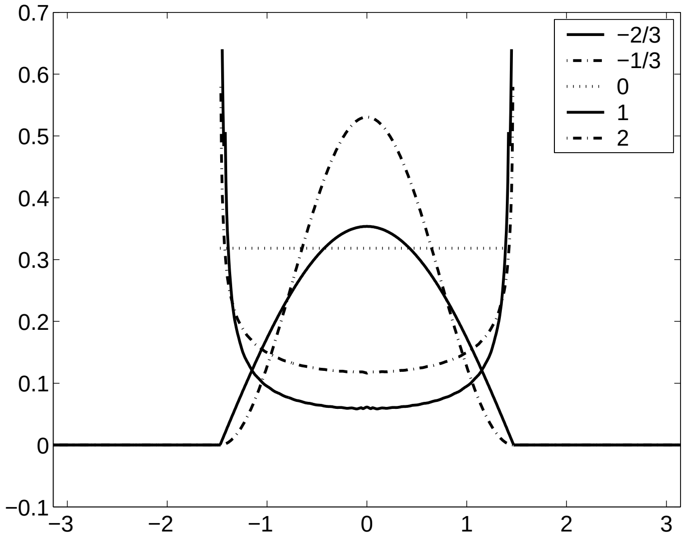

Figure 2: The kernels $\vartheta \mapsto L_{h,\lambda}(\cos\vartheta)$ for $h = 0.1$ and $\lambda = -2/3, -1/3, 0, 1, 2$.

Lemma 4.1. *Suppose that $h \in (-1, 1)$. Then the following statements hold true:*

(i) If $\lambda > -1$ then $L^{(2)}_{h,\lambda}(\eta\cdot\ ) \in \mathcal{L}^2(\Omega)$.

(ii) If $\lambda > -1/2$ then $L^{(2)}_{h,\lambda}(\eta\cdot\ ) \in \mathcal{C}(\Omega)$.

(iii) If $\lambda > k/2 - 1/2$ then $L^{(2)}_{h,\lambda}(\eta\cdot\ ) \in \mathcal{C}^{(k)}(\Omega)$, $k \in \mathbb{N}$.

Proof. Part (i) can easily deduced from (30). The second and third statement are consequences of the the Sobolev Lemma (see [8]). □

Note that for λ with $-1 < \lambda \leq -1/2$, the continuity of the kernels cannot be assured. However, if the parameters $h_1, h_2 \in (-1, 1)$ define two kernels, then it was already pointed out in our prelimiaries that the convolution $L^{(2)}_{h_1,\lambda} * L^{(2)}_{h_2,\lambda}$ is a continuous kernel. To be more specific, let $\xi,\ \eta \in \Omega$, and let $K : [-1, 1] \to \mathbb{R}$ be an $\mathcal{L}^2$–kernel, i.e. $K(\eta\cdot\ ) \in \mathcal{L}^2(\Omega)$. Then we obtain by aid of the Cauchy–Schwarz inequality

$$\begin{aligned} |K^{(2)}(\xi\cdot\eta)| &= |K * K(\xi\cdot\eta)| = \left|\int_\Omega K(\xi\cdot\zeta)K(\zeta\cdot\eta)d\omega(\zeta)\right| \\ &\leq \int_\Omega |K(\xi\cdot\eta)|^2 d\omega(\zeta)) \\ &= \left|\int_\Omega K(\xi\cdot\zeta)K(\zeta\cdot\xi)d\omega(\zeta)\right| \\ &= K^{(2)}(\xi\cdot\xi) = K^{(2)}(1). \end{aligned}$$

Summarizing our results we therefore obtain

Theorem 4.2. *If $K : [-1, 1] \to \mathbb{R}$ is an $\mathcal{L}^2$–kernel, i.e. $K(\eta\cdot\ ) \in \mathcal{L}^2(\Omega)$, then the iterated kernel*

$K^{(2)} : [-1,1] \to \mathbb{R}$ *is continuous and satisfies the estimate*

$$|K^{(2)}(t)| \leq K^{(2)}(1) = \|K(\eta\cdot\ )\|^2_{\mathcal{L}^2(\Omega)}$$

This theorem justifies the use of unbounded kernels for approximation purposes, since the iterates of $\mathcal{L}^2$–kernels attain their maximum at $t = 1$.

In view of Lemma 4.1 we have

Theorem 4.3. *If* $\lambda > -1/2$, *then*

$$0 \leq L^{(2)}_{h,\lambda}(t) \leq L^{(2)}_{h,\lambda}(1) = \frac{(\lambda+1)^2}{2\pi(2\lambda+1)(1-h)}.$$

If $-1 < \lambda \leq -1/2$, then $L_{h,\lambda}$ is not an $\mathcal{L}^2$–kernel, but for any $-1 < h_1, h_2 < 1$ the kernel $L_{h_1,\lambda} * L_{h_2,\lambda}$ is of class $\mathcal{L}^2[-1,1]$. Thus,

$$0 \leq L^{(2)}_{h_1,\lambda} * L^{(2)}_{h_2,\lambda}(t) \leq L^{(2)}_{h_1,\lambda} * L^{(2)}_{h_2,\lambda}(1), \tag{31}$$

which makes these kernels to useful structures for approximation problems an the sphere.

In Figure 3, we show the iterated kernels $\vartheta \mapsto L^{(2)}_{h,\lambda}(\cos\vartheta)$. Note that, according to (31), $L^{(2)}_{h,\lambda}$ cannot be guaranteed to be continuous for $\lambda = -2/3$. However, $L^{(2)}_{h,\lambda} * L^{(2)}_{h,\lambda}$ is continuous for all $\lambda > -1$.

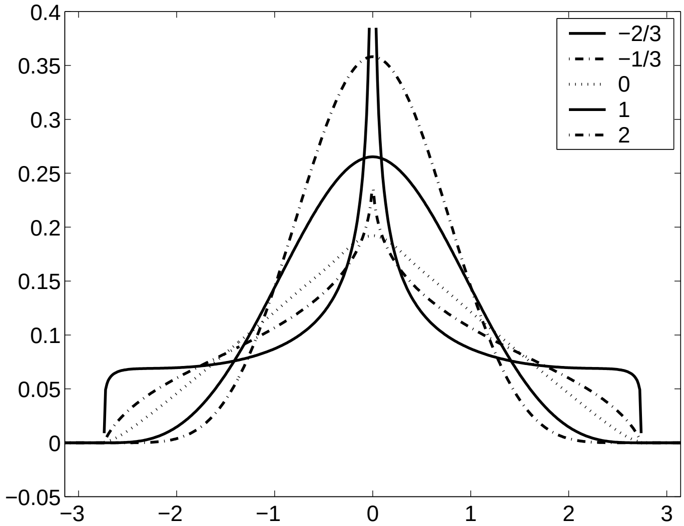

Figure 3: The kernels $\vartheta \mapsto L^{(2)}_{h,\lambda}(\cos\vartheta)$ for $h = 0.2$ and $\lambda = -2/3, -1/3, 0, 1, 2$.

5 The Spherical Up Function

Now, we deal with a spherical counterpart of the so–called up function which is, for one dimensional problems, described e.g. in [26]. The main idea is to build an infinite convolution of locally supported functions, where the support of each of the building blocks is chosen carefully to ensure that the resulting convolution is additionally locally supported. Even more, the infinite convolution turns out to be infinitely often differentiable. The reason is that the symbol of the up function decays for increasing n faster than any rational function (in n).

Definition 5.1 *Suppose that $h \in (-1,1)$, and $\lambda > -1$. We let $\varphi_0 = \arccos h$ and introduce*

$$\varphi_i = 2^{-i}\varphi_0, \ , \ h_i = \cos\frac{\varphi_i}{2}, \quad i = 1, 2, \dots . \tag{32}$$

Then $Up_{h,\lambda}$ defined by

$$Up_{h,\lambda} = L^{(2)}_{h_1,\lambda} * L^{(2)}_{h_2,\lambda} * \dots = \mathop{\Huge *}_{i=1}^{\infty} \ L^{(2)}_{h_i,\lambda} \tag{33}$$

is called up function (more precisely: (h,λ)–up function).

Each $\vartheta \mapsto L_{h_i,\lambda}(\cos\vartheta)$ possesses the support $[0, \varphi_i/2]$, so that $\vartheta \mapsto L^{(2)}_{h_i,\lambda}(\cos\vartheta)$ has the support $[0,\varphi_i]$. Thus, the function $\vartheta \mapsto Up_{h,\lambda}(\cos\vartheta)$ has the support $[0, \sum_{i=1}^{\infty}\varphi_i] = [0,\varphi_0]$, so that $\operatorname{supp} Up_{h,\lambda}(t) = [h,1]$ (what justifies our way of writing).

We know that, for each i, we have

$$0 \le L^{(2)\wedge}_{h_i,\lambda}(n) \le L^{(2)\wedge}_{h_i,\lambda}(0) = 1, \quad n = 1, 2, \dots$$

so that the infinite convolution (33) is well–defined, and we have

$$Up^{\wedge}_{h,\lambda}(n) = \prod_{i=1}^{\infty} L^{(2)\wedge}_{h_i,\lambda}(n). \tag{34}$$

In particular,

$$0 \le Up^{\wedge}_{h,\lambda}(n) \le Up^{\wedge}_{h,\lambda}(0) = 1, \quad n = 1, 2, \dots \tag{35}$$

From Lemma 3.2 it follows that

$$\lim_{h\to 1} Up^{\wedge}_{h,\lambda}(n) = 1, \quad n = 1, 2, \dots$$

Furthermore, as a consequence of Theorem 4.2 we have for every $k \in \mathbb{N}$

$$Up^{\wedge}_{h,\lambda}(n) = O(n^{-k}), \quad n \to \infty. \tag{36}$$

Hence we are able to deduce from the Sobolev Lemma that $Up_{h,\lambda}(\eta\cdot\ ) \in \mathcal{C}^{(\infty)}(\Omega)$ for every $\eta \in \Omega$.

We summarize the properties of the (h,λ)– spherical up function.

Theorem 5.2. *Let for $h \in (-1,1)$ and $\lambda > -1$ the (h,λ)–up function $Up_{h,\lambda} : [-1,1] \to \mathbb{R}$ be defined as in (33). Then the following statements are valid:*

(i) $Up_{h,\lambda}$ is locally supported with $\operatorname{supp} Up_{h,\lambda} = [h,1]$.

(ii) For every $\eta \in \Omega$: $Up_{h,\lambda}(\eta\cdot\ )$ is of class $\mathcal{C}^{(\infty)}(\Omega)$.

(iii) $Up_{h,\lambda} : [-1,1] \to \mathbb{R}$ *admits the uniformly convergent orthogonal expansion in terms of Legendre polynomials*

$$Up_{h,\lambda} = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} Up_{h,\lambda}^{\wedge}(n) P_n \tag{37}$$

where $Up_{h,\lambda}^{\wedge}(0) = 1$ *and*

$$0 \le Up_{h,\lambda}^{\wedge}(n) = \prod_{i=1}^{\infty} \left(L_{h_i,\lambda}^{\wedge}(n)\right)^2 \le 1, \ n = 0, 1, 2, \ldots. \tag{38}$$

(iv) *For* $n = 1, 2, \ldots$

$$\lim_{h \to 1} Up_{h,\lambda}^{\wedge}(n) = 1. \tag{39}$$

(v) *For all* $t \in [-1,1]$

$$0 \le Up_{h,\lambda}(t) \le Up_{h,\lambda}(1) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} Up_{h,\lambda}^{\wedge}(n). \tag{40}$$

(vi) *For any* $k \in \mathbb{N}$,

$$Up_{h,\lambda}^{\wedge}(n) = O(n^{-k}), \quad n \to \infty. \tag{41}$$

Proof. In the light of the previous considerations, we only have to prove the statement (v): Since $Up_{h,\lambda}$ is built by a convolution out of positive functions $L_{h_i,\lambda}$, it is clear that $Up_{h,\lambda}(t) \ge 0$ for $t \in [-1,1]$. Since $Up_{h,\lambda}^{\wedge}(n) \ge 0$ for all $n = 0, 1, \ldots$, and $|P_n(t)| \le P_n(1) = 1$ for all $t \in [-1,1]$, the series expansion (37) lead us to the inequality

$$0 \le Up_{h,\lambda}(t) \le Up_{h,\lambda}(1) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} Up_{h,\lambda}^{\wedge}(n),$$

which completes the proof. □

6 Finite Truncations of the Up Function

From a numerical point of view, it is of advantage that the infinite convolution (33) can be replaced by a finite one with arbitrary accuracy. This result will be developed in the following. Assume that $h \in (-1,1)$ and $\lambda > -1$. We consider, for $N \in \mathbb{N}$, the splitting

$$Up_{h,\lambda} = Up_{h,\lambda}^{1,\ldots,N} * Up_{h,\lambda}^{N+1,\ldots,\infty} = \mathop{\huge\ast}_{i=1}^{N} L_{h_i,\lambda}^{(2)} * \mathop{\huge\ast}_{i=N+1}^{\infty} L_{h_i,\lambda}^{(2)}$$

Our aim is to estimate the expression $|Up_{h,\lambda}^{1,\ldots,N}(t) - Up_{h,\lambda}(t)|, t \in [-1,1]$. We start with a series of lemmata:

Lemma 6.1. *Suppose that* $K \in \mathcal{L}^1[-1,1]$, $K \ge 0$, $K^{\wedge}(0) = 1$, *and assume that* K *is locally supported:* $\mathrm{supp} K = [h,1]$. *Then, for every* $F \in \mathcal{C}(\Omega)$,

$$\|K * F - F\|_{\mathcal{C}(\Omega)}^2 \le \max_{\xi \cdot \eta \ge h} |F(\eta) - F(\xi)|.$$

Proof. For $\xi \in \Omega$ we have

$$\begin{aligned}
|K * F(\xi) - F(\xi)| &= \left| \int_\Omega K(\xi \cdot \eta) F(\eta) d\omega(\eta) - F(\xi) \right| \\
&= \left| \int_\Omega K(\xi \cdot \eta) [F(\eta) - F(\xi)] d\omega(\eta) \right| \\
&\leq \int_\Omega K(\xi \cdot \eta) d\omega(\eta) \max_{\xi \cdot \eta \geq h} |F(\eta) - F(\xi)| \\
&= \max_{\xi \cdot \eta \geq h} |F(\eta) - F(\xi)|.
\end{aligned}$$

□

Lemma 6.2. *Assume that* $K \in \mathcal{L}^1[-1,1]$, $K \geq 0$, $K^\wedge(0) = 1$, $\mathrm{supp} K = [h,1]$. *Let* $H \in \mathcal{C}^{(1)}[-1,1]$. *Then for every* $t \in [-1,1]$

$$|K * H(t) - H(t)| \leq \sqrt{2}\sqrt{1-h^2} \max_{\tau \in [-1,1]} |H'(\tau)|.$$

Proof. We deduce from the last lemma, that for every $\xi, \eta \in \Omega$

$$|K * H(\xi \cdot \eta) - H(\xi \cdot \eta)| \leq \max_{\eta \cdot \zeta \geq h} |H(\xi \cdot \eta) - H(\xi \cdot \zeta)|.$$

For $\eta \cdot \zeta \geq h$ we have

$$\begin{aligned}
|\xi \cdot \eta - \xi \cdot \zeta| = |\xi \cdot (\eta - \zeta)| &\leq \sqrt{(\eta - \zeta)^2} \\
&= \sqrt{2 - 2\eta \cdot \zeta} \\
&\leq \sqrt{2}\sqrt{1-h^2}\ .
\end{aligned}$$

Hence, the result stated in Lemma 6.2 easily follows from the mean value theorem. □

Lemma 6.3. *Let* $K : [-1,1] \to \mathbb{R}$ *be of class* $\mathcal{H}^2$, *i.e.*

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} |K^\wedge(n)|^2 \left(n + \frac{1}{2}\right)^4 < \infty.$$

Assume further, that $K^\wedge(n) \geq 0$ *for all* $n \in \mathbb{N}$. *Then* K *is continuously differentiable and*

$$|K'(t)| \leq K'(1) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \frac{n(n+1)}{2} K^\wedge(n).$$

Proof. It follows from the Sobolev Lemma that K is continuously differentiable. Furthermore, we obtain the uniformly convergent series

$$|K'(t)| = \left| \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} K^\wedge(n) P_n'(t) \right|$$

$$\begin{aligned}
&\leq \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} K^\wedge(n) P_n'(1) \\
&= \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \frac{n(n+1)}{2} K^\wedge(n) \\
&= C'(1),
\end{aligned}$$

□

Using these preliminaries we are able to prove

Theorem 6.4. *Suppose that $h \in (-1,1)$, $\lambda > -1$. For a given $\varepsilon > 0$, choose $N \in \mathbb{N}$, $N \geq 5$, so that*

$$\sin \frac{\arccos h}{2^{N+1}} \leq \frac{\varepsilon}{\sqrt{2} \frac{d}{dt} Up_{h,\lambda}^{1,\ldots,N}(1)}. \tag{42}$$

Then, for all $t \in [-1,1]$,

$$|Up_{h,\lambda}^{1,\ldots,N}(t) - Up_{h,\lambda}(t)| < \varepsilon. \tag{43}$$

Proof. The assertion is a consequence of the previous lemmata using the following facts:

- $\operatorname{supp} Up_{h,\lambda}^{N+1,\ldots,\infty} = [h_N, 1]$ with

  $$h_N = \cos \frac{\varphi_N}{2} = \cos \frac{2^{-N}}{2} \varphi_0 = \cos \frac{\varphi_0}{2^{N+1}},$$

  so that

  $$\begin{aligned}
  \sqrt{1-h_N^2} &= \sqrt{1-(\cos 2^{-(N+1)}\varphi_0)^2} \\
  &= \sin \frac{\varphi_0}{2^{N+1}} \\
  &= \sin \frac{\arccos h}{2^{N+1}}.
  \end{aligned}$$

- If $N \geq 5$, we deduce from (30) that $Up_{h,\lambda}^{1,\ldots,N}$ is of class $\mathcal{H}^2$ for $\lambda > -1$. Hence, we conclude from Lemma 6.3 that $Up_{h,\lambda}^{1,\ldots,N}$ is continuously differentiable with

  $$\left| \frac{d}{dt} Up_{h,\lambda}^{1,\ldots,N}(t) \right| \leq \frac{d}{dt} Up_{h,\lambda}^{1,\ldots,N}(1)$$

  so that Lemma 6.2 can be applied.

□

For a larger λ (i.e. the kernels $L_{h,\lambda}^{(2)}$ are in a higher smoothness class) $\frac{d}{dt} Up_{h,\lambda}^{1,\ldots,N}(1)$ can be substituted by the same expression with a $M < N$, as long as the kernel $Up_{h,\lambda}^{1,\ldots,M}$ fulfills the assumptions of Lemma 6.3. This is true because if $|L_{h,\lambda}^\wedge(n)| \leq 1$, so that

$$\frac{d}{dt} Up_{h,\lambda}^{1,\ldots,M}(1) \geq \frac{d}{dt} Up_{h,\lambda}^{1,\ldots,N}(1),$$

for $M < N$, as long as the terms are well–defined.

We close this section by giving a graphical illustration of some (h, λ)–up functions. Figure 4 shows the kernels $\vartheta \mapsto Up_{h,\lambda}(\cos\vartheta)$ for different values of λ.

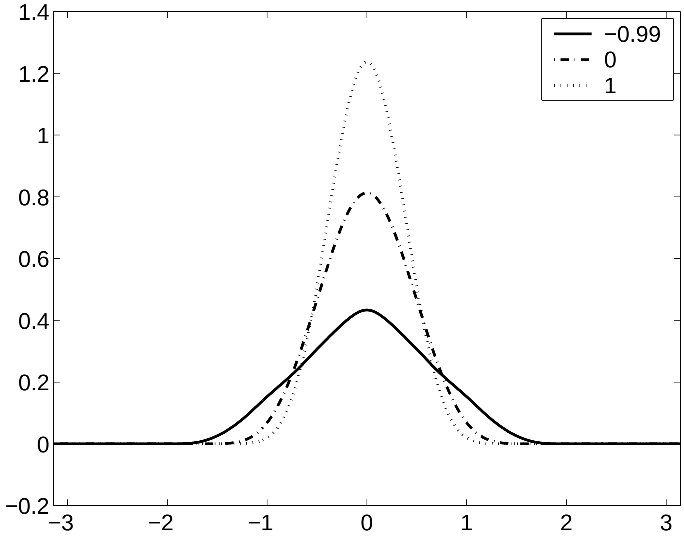

Figure 4: The kernels $\vartheta \mapsto Up_{h,\lambda}(\cos\vartheta)$ for $h = -0.5$ and $\lambda = -0.99, 0, 1$.

7 Multiresolution Analysis Using the Up Function

Next we come to the characterization of a multiresolution analysis within the space $\mathcal{L}^2(\Omega)$ involving the spherical up function. We start with

Theorem 7.1. *Suppose that $\lambda > -1$. For all $F \in \mathcal{L}^2(\Omega)$,*

$$\lim_{h\to 1} \left\| Up_{h,\lambda} * F - F \right\|_{\mathcal{L}^2(\Omega)} = 0. \tag{44}$$

Proof. From the completeness of the spherical harmonics in $\mathcal{L}^2(\Omega)$ we know, that convergence in norm is equivalent to the convergence of the Fourier transform. Therefore, the limit relation (44) is equivalent to

$$\lim_{h\to 1} \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} |Up_{h,\lambda}^{\wedge}(n) F^{\wedge}(n,m) - F^{\wedge}(n,m)|^2 = 0.$$

Given $\varepsilon > 0$. For $F \in \mathcal{L}^2(\Omega)$, there exist a number $N_\varepsilon \in \mathbb{N}$ so that

$$\sum_{n=N_\varepsilon}^{\infty} \sum_{m=1}^{2n+1} |F^{\wedge}(n,m)|^2 < \frac{\varepsilon}{4}.$$

Observing that $\lim_{h\to 1} Up_{h,\lambda}^{\wedge}(n) = 1$, it follows that there exist an h_ε so that for all $h > h_\varepsilon$

$$\sum_{n=0}^{N_\varepsilon - 1} (2n+1) |Up_{h,\lambda}^{\wedge}(n) - 1|^2 < \frac{\varepsilon}{2\,\|F\|_{\mathcal{L}^2(\Omega)}^2}$$

Thus, we are led to the following estimate

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} |\,Up^{\wedge}_{h,\lambda}(n)F^{\wedge}(n,m) - F^{\wedge}(n,m)|^2 \\
&= \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} |\,Up^{\wedge}_{h,\lambda}(n) - 1|^2 |F^{\wedge}(n,m)|^2 \\
&= \left(\sum_{n=0}^{N_\varepsilon-1} + \sum_{n=N_\varepsilon}^{\infty}\right)\sum_{m=1}^{2n+1} |\,Up^{\wedge}_{h,\lambda}(n) - 1|^2 |F^{\wedge}(n,m)|^2 \\
&\le \|F\|^2_{\mathcal{L}^2(\Omega)} \sum_{n=0}^{N_\varepsilon-1}\sum_{m=1}^{2n+1} |\,Up^{\wedge}_{h,\lambda}(n) - 1|^2 + \sum_{n=N_\varepsilon}^{\infty}\sum_{m=1}^{2n+1} |F^{\wedge}(n,m)|^2 \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

where we have used the fact that $|\,Up^{\wedge}_{h,\lambda}(n) - 1| \le 1$. (Note that $0 \le Up^{\wedge}_{h,\lambda}(n) \le 1$). □

Furthermore, from $|\,Up^{\wedge}_{h,\lambda}(n)| \le 1$ it follows that

$$\left\|\,Up_{h,\lambda} * F\right\|_{\mathcal{L}^2(\Omega)} \le \|F\|_{\mathcal{L}^2(\Omega)} \tag{45}$$

which motivates the terminology of a multiresolution analysis $\mathcal{L}^2(\Omega)$. However, when $h_1 \ge h_2$, we are *not* able to guarantee that

$$\left\|\,Up_{h_2,\lambda} * F\right\|_{\mathcal{L}^2(\Omega)} \le \left\|\,Up_{h_1,\lambda} * F\right\|_{\mathcal{L}^2(\Omega)} \tag{46}$$

holds for all $F \in \mathcal{L}^2(\Omega)$. This is due to the fact that the Legendre transform $Up^{\wedge}_{h,\lambda}(n)$ is not monotone with respect to h. As an counterexample for (46) take values h_1 and h_2 so that $Up^{\wedge}_{h_1,\lambda}(n) = 0$ for an $n \in \mathbb{N}$, but $Up^{\wedge}_{h_2,\lambda}(n) \ne 0$. Then it can be easily seen that (46) is *not* true for a spherical harmonic of order n, $F = Y_n \in \mathrm{Harm}_n$.

We can overcome this calamity by restricting ourselves to discrete values of h, i.e. we are looking for a scale discrete multiresolution analysis of $\mathcal{L}^2(\Omega)$.

We assume from now on, that $h \in (-1,1)$ and $\lambda > -1$ are fixed. For this h, the numbers h_i, $i = 1, 2, \dots$ are defined as in (32). The scaling function $\Phi^j_{h,\lambda} : [-1,1] \to \mathbb{R}$ is introduced by

$$\Phi^j_{h,\lambda} = Up^{j,\dots,\infty}_{h,\lambda} = \mathop{*}_{i=j}^{\infty} L^{(2)}_{h_j,\lambda}, \quad j = 1, 2, \dots. \tag{47}$$

By construction, $\mathrm{supp}\Phi^j_{h,\lambda} = [h_{j-1}, 1]$, and we have the refinement equation

$$\Phi^{j+1}_{h,\lambda} * L^{(2)}_{h_j,\lambda} = \Phi^j_{h,\lambda}, \quad j \ge 1. \tag{48}$$

Using the previous results we, therefore, obtain for every $F \in \mathcal{L}^2(\Omega)$

$$\lim_{j\to\infty} \left\|\Phi^j_{h,\lambda} * F - F\right\|_{\mathcal{L}^2(\Omega)} = 0.$$

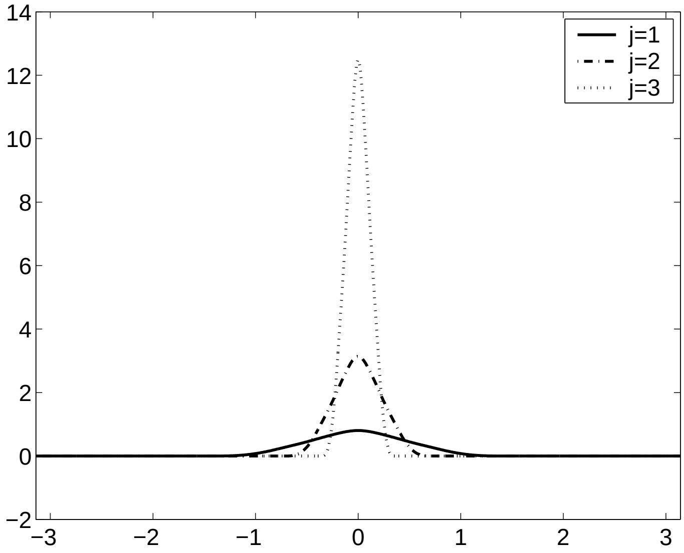

Figure 5: The scaling functions $\vartheta \mapsto \Phi^j_{h,\lambda}(\cos\vartheta)$ for $j = 1, 2, 3$, $\lambda = -0.9$, and $h = -1$.

Moreover, for every $F \in \mathcal{L}^2(\Omega)$

$$\left\|\Phi^j_{h,\lambda} * F\right\|_{\mathcal{L}^2(\Omega)} \le \left\|\Phi^{j+1}_{h,\lambda} * F\right\|_{\mathcal{L}^2(\Omega)} \le \|F\|_{\mathcal{L}^2(\Omega)} .$$

These facts give rise to interpret the convolution with $\Phi^j_{h,\lambda}$ as low–pass filter. Obviously, we define for $j = 1, 2, \dots$ the projection operators $P_j : \mathcal{L}^2(\Omega) \to \mathcal{C}^{(\infty)}(\Omega) \subset \mathcal{L}^2(\Omega)$ by

$$P_j(F) = \Phi^j_{h,\lambda} * F = \int_\Omega \Phi^j_{h,\lambda}(\eta\cdot\ )F(\eta)d\omega(\eta) \tag{49}$$

and introduce the scale spaces

$$V_j = \{P_j(F)|F \in \mathcal{L}^2(\Omega)\}. \tag{50}$$

Thus, we finally arrive at the following result.

Theorem 7.2. *Suppose that $h \in (-1, 1)$, $\lambda > -1$. The scale spaces*

$$V_j = \{\Phi^j_{h,\lambda} * F|F \in \mathcal{L}^2(\Omega)\}.$$

define a multiresolution of $\mathcal{L}^2(\Omega)$ in the following sense:

(i) $V_j \subset \mathcal{L}^2(\Omega)$ *is a linear subspace with* $V_j \subset \mathcal{C}^{(\infty)}(\Omega)$

(ii) $V_1 \subset V_2 \subset V_3 \subset \dots$

(iii) $\bigcap_{j=1}^{\infty} V_j = V_1$

(iv) $\overline{\bigcup_{j=1}^{\infty} V_j} = \mathcal{L}^2(\Omega)$

Remark 7.3. *By setting $h = -1$ (which was excluded for the previous analysis), we find for $\lambda = 0$*

$$L_{h,\lambda}(t) = \frac{1}{4\pi} = \frac{1}{4\pi} P_0(t),\ t \in [-1,1],$$

hence,

$$L^{(2)}_{h,\lambda}(t) = \frac{1}{4\pi} P_0(t),\ t \in [-1,1].$$

In consequence, the property (iii) of the previous theorem specializes to

$$\bigcap_{j=1}^{\infty} V_j = \mathrm{Harm}_0.$$

Based on this multiresolution analysis of $\mathcal{L}^2(\Omega)$, spherical wavelets which are locally supported can be constructed in a similar way as described in [8].

8 Locally Supported Wavelets

In what follows, we assume that h and λ are fixed, and that the corresponding h_i are given as in (32). In doing so, we obtain with $\Phi^j = \Phi^j_{h,\lambda}$ the family $\left\{\Phi^j \mid j = 1,2,\ldots\right\}$ which we interpret as scale discrete scaling function. This scaling function allows us to introduce scale discrete locally supported wavelets on the sphere. We represent an $\mathcal{L}^2(\Omega)$–function F by a two parameter family $(j;\eta)$, $j \in \mathbb{N}$, $\eta \in \Omega$, breaking up the function F into "pieces" at different locations and different levels of resolution. The refinement equation corresponding to the scaling function $\{\Phi_j | j = 1,2,\ldots\}$ reads as follows:

$$\Psi^j = \Phi^{j+1} - \Phi^j, \quad j = 1,2,\ldots, \tag{51}$$

Clearly, Ψ^j is a locally supported infinitely often differentiable function with support $\mathrm{supp}\Psi^j = [h_j, 1]$. We use Ψ^j to introduce the spherical wavelet at level j and point $\eta \in \Omega$ by $\Psi_{j;\eta}(\xi) = \Psi^j(\eta \cdot \xi), (\xi,\eta) \in \Omega \times \Omega$.

For the scaling function we analogously write $\Phi_{j;\eta}(\xi) = \Phi^j(\eta \cdot \xi)$. From the definition of the wavelets it is obvious that $(\xi,\eta) \mapsto \Psi_{j;\eta}(\xi) = \Psi^j(\xi \cdot \eta)$ is a radial basis function on the sphere. It is easily seen that

$$\Psi^{j\wedge}(n) = \Phi^{j+1\wedge}(n) - \Phi^{j\wedge}(n) = \Phi^{j+1\wedge}(n)\left[1 - L^{(2)\wedge}_{h_j,\lambda}(n)\right].$$

In particular,

$$\Psi^{j\wedge}(0) = 0 \tag{52}$$

which is nothing else than the zero–mean property known from Euclidean wavelet theory.

Given a function $F \in \mathcal{L}^2(\Omega)$, we define its wavelet transform by

$$\mathrm{WT}(F)(j;\eta) = (\Psi_{j;\eta}, F), \quad j = 1,2,\ldots, \eta \in \Omega, \tag{53}$$

which allows to break up F into "pieces" at different locations and different scales. This statement is made rigorous in the following theorem, which is a reconstruction formula for linear wavelets:

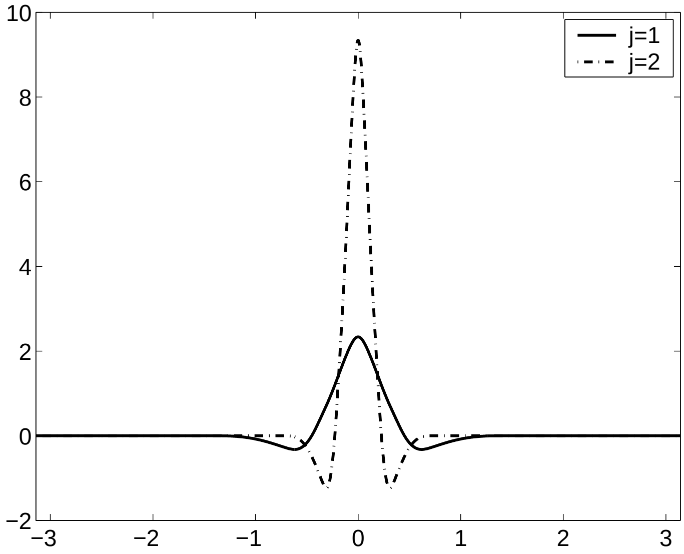

Figure 6: The wavelets $\vartheta \mapsto \Psi^j(\cos\vartheta)$ for $j = 1, 2$, $\lambda = -0.9$, and $h = -1$.

Theorem 8.1. *Let $F \in \mathcal{L}^2(\Omega)$. Then we have in $\mathcal{L}^2(\Omega)$–sense*

$$F(\eta) = (\Phi_{1;\eta}, F) + \sum_{j=1}^{\infty} \mathrm{WT}(F)(j;\eta). \tag{54}$$

Proof. The statement is a reformulation of Theorem 7.1. $\square$

In (49) and (50) the projection operators P_j and the corresponding scale spaces V_j are introduced by

$$P_j(F) = (\Phi_{j;\cdot}, F), \quad V_j = \left\{ P_j(F) \mid F \in \mathcal{L}^2(\Omega) \right\}.$$

Analogously we let the operator R_j and the detail spaces to be given by

$$R_j(F) = (\Psi_{j;\cdot}, F), \quad W_j = \left\{ R_j(F) \mid F \in \mathcal{L}^2(\Omega) \right\}. \tag{55}$$

It follows from the zero–mean property (52) that $F^\wedge(0, 0) = 0$ for all $F \in W_j$. Thus the wavelet transform can be seen as a band–pass filter. By construction, we have

$$V_{J+1} = V_J + W_J = V_1 + \sum_{j=1}^{J} W_j \tag{56}$$

It is worth mentioning that the decomposition (56) is neither direct nor orthogonal.

The described wavelet analysis, which may be seen as a linear wavelet theory, can be extended to a bilinear reconstruction scheme. We introduce a second family of wavelets $\tilde{\Psi}_{j;\eta}$ (dual wavelets) and understand the reconstruction process by a convolution of the wavelet transform against the dual wavelets.

The dual wavelets are given by

$$\tilde{\Psi}^j = \Phi^{j+1} + \Phi^j, \quad j = 1, 2, \ldots\ . \tag{57}$$

As usual,

$$\tilde{\Psi}_{j;\eta}(\xi) = \tilde{\Psi}^j(\eta\cdot\xi), \ \ (\xi,\eta)\in\Omega\times\Omega. \tag{58}$$

The dual wavelet $\tilde{\Psi}^j$ has the local support supp$\tilde{\Psi}^j = [h_j, 1]$, and its Legendre transform reads as follows:

$$\tilde{\Psi}^{j\wedge}(n) = \Phi^{j+1\wedge}(n) + \Phi^{j\wedge}(n) = \Phi^{j+1\wedge}(n)\left[1 + L^{(2)\wedge}_{h_j,\lambda}(n)\right].$$

A reconstruction scheme involving the dual wavelets can be formulated as follows:

Theorem 8.2. *Let $F\in\mathcal{L}^2(\Omega)$. Then it holds in $\mathcal{L}^2(\Omega)$–sense*

$$F(.) = \int_\Omega (\Phi_{1;\eta}, F)\Phi_{1;\eta}(.)d\omega(\eta) + \sum_{j=1}^{\infty}\int_\Omega \mathrm{WT}(F)(j;\eta)\tilde{\Psi}_{j;\eta}(.)d\omega(\eta). \tag{59}$$

Proof. From the completeness of the spherical harmonics in $\mathcal{L}^2(\Omega)$ we are able to deduce, that convergence in $\mathcal{L}^2(\Omega)$ is equivalent to convergence of the Fourier transform. For the first summand in (59) we get

$$\left(\int_\Omega (\Phi_{1;\eta}, F)\Phi_{1;\eta}(.)d\omega(\eta)\right)^\wedge (n,m) = F^\wedge(n,m)\left[\Phi^{1\wedge}(n)\right]^2.$$

Furthermore, we have

$$\begin{aligned}\left(\int_\Omega \mathrm{WT}(F)(j;\eta)\tilde{\Psi}_{j;\eta}(.)d\omega(\eta)\right)^\wedge (n,m) &= F^\wedge(n,m)\Psi^{j\wedge}(n)\tilde{\Psi}^{j\wedge}(n)\\ &= F^\wedge(n,m)\left[\Phi^{j+1\wedge}(n) - \Phi^{j\wedge}(n)\right]\left[\Phi^{j+1\wedge}(n) + \Phi^{j\wedge}(n)\right]\\ &= F^\wedge(n,m)\left[(\Phi^{j+1\wedge}(n))^2 - (\Phi^{j\wedge}(n))^2\right].\end{aligned}$$

In conclusion,

$$\left(\int_\Omega (\Phi_{1;\eta}, F)\Phi_{1;\eta}(.)d\omega(\eta) + \sum_{j=1}^{J}\int_\Omega \mathrm{WT}(F)(j;\eta)\tilde{\Psi}_{j;\eta}(.)d\omega(\eta)\right)^\wedge (n,m) = F^\wedge(n,m)\left(\Phi^{J+1\wedge}(n)\right)^2.$$

Observing the fact that

$$\lim_{J\to\infty}\left(\Phi^{J\wedge}(n)\right)^2 = 1,$$

the proof of Theorem 8.2 follows in a similar way as the proof of Theorem 7.1. □

9 Decomposition and Reconstruction Schemes Involving the Up Functions

For numerical purposes it is important to know, how the wavelet decomposition and reconstruction can be organized in an efficient way, so that information is transported from level to level, which characterizes the essence of a tree algorithm or a pyramid scheme. The decompositions are based an the refinement equation (48)

$$\Phi^{j+1} * L^{(2)}_{h_j,\lambda} = \Phi^j, \quad j = 1,2,\ldots. \tag{60}$$

It is remarkable that we can find a similar relation for the wavelets in the form

$$\Psi^{j+1} * K_j = \Psi^j, \quad j = 1, 2, \ldots, \tag{61}$$

which enables a second variant of the pyramid scheme for the wavelet decomposition.

In the following, we present a series of schemes for the decomposition and reconstruction of a function $F \in \mathcal{L}^2(\Omega)$. We assume, that we start from a finest level $J \in \mathbb{N}$.

The first variant for the wavelet decomposition of a signal $F \in \mathcal{L}^2(\Omega)$ looks as follows:

Wavelet Decomposition (Variant 1)

$$\begin{tikzcd}
F \arrow[r] & (\Phi_{J+1;.}, F) \arrow[r] \arrow[dr] & (\Phi_{J;.}, F) \arrow[r] \arrow[d] \arrow[dr] & \cdots \arrow[r] \arrow[dr] & (\Phi_{1;.}, F) \arrow[d] \\
 & & \mathrm{WT}(F)(J;.) & \cdots & \mathrm{WT}(F)(0;.)
\end{tikzcd}$$

The scheme works because we have from (60) that

$$(\Phi_{j;.}, F) = \Phi^j * F = \Phi^{j+1} * L^{(2)}_{h_j,\lambda} * F = L^{(2)}_{h_j,\lambda} * (\Phi_{j+1;.}, F), \; j = 1, 2, \ldots,$$

and since we can deduce from (51) that

$$\mathrm{WT}(F)(j;.) = (\Phi_{j+1;.}, F)_{\mathcal{L}^2(\Omega)} - (\Phi_{j;.}, F)_{\mathcal{L}^2(\Omega)}.$$

The reconstruction in the linear case (Theorem 8.1) can be organized as follows:

Wavelet Reconstruction (Linear Case)

$$\begin{tikzcd}
\mathrm{WT}(F)(1;.) \arrow[dr] & & \mathrm{WT}(F)(2;.) \arrow[dr] & & \mathrm{WT}(F)(3;.) \arrow[dr] & & \\
(\Phi_{1;.}, F) \arrow[r] & + \arrow[r] & P_2(F) \arrow[r] & + \arrow[r] & P_3(F) \arrow[r] & + \arrow[r] & \cdots
\end{tikzcd}$$

In order to formulate the reconstruction for the bilinear case, we introduce the following variants of the projection operators P_j and R_j:

$$\begin{aligned}
P_j^2(F) &= \int_\Omega (\Phi_{j;\eta}, F)\Phi_{j;\eta}(.)d\omega(\eta) = \Phi^j * \Phi^j * F, \\
R_j^2(F) &= \int_\Omega \mathrm{WT}(F)(j,\eta)\tilde{\Psi}_{j;\eta}(.)d\omega(\eta) = \tilde{\Psi}^j * \Psi^j * F.
\end{aligned}$$

Consequently we obtain the following scheme from Theorem 8.2:

Wavelet Reconstruction (Bilinear Case)

$$\begin{tikzcd}
\mathrm{WT}(F)(1;.) \arrow[d] & & \mathrm{WT}(F)(2;.) \arrow[d] & & \mathrm{WT}(F)(3;.) \arrow[d] & & \\
R_1^2(F) \arrow[dr] & & R_2^2(F) \arrow[dr] & & R_3^3(F) \arrow[dr] & & \\
P_1^2(F) \arrow[r] & + \arrow[r] & P_2^2(F) \arrow[r] & + \arrow[r] & P_3^2(F) \arrow[r] & + \arrow[r] & \cdots
\end{tikzcd}$$

Next, we want to develop a reformulation of the decomposition scheme. The already given first variant transports information from scale space to scale space. Our aim now is to construct a second version where information is transported from detail space to detail space. For this purpose, we need a scale relation for the wavelets of the from $\Psi^{j+1} * K_j = \Psi^j, \;\; j \geq 1$. Remembering the definition (51) we are able to rewrite (61) as follows:

$$(\Phi^{j+2} - \Phi^{j+1}) * K_j = \Phi^{j+1} - \Phi^j.$$

Using (48) we arrive at

$$(\Phi^{j+2} - \Phi^{j+2} * L^{(2)}_{h_{j+1},\lambda}) * K_j = \Phi^{j+2} * L^{(2)}_{h_{j+1},\lambda} - \Phi^{j+2} * L^{(2)}_{h_{j+1},\lambda} * L^{(2)}_{h_j,\lambda}, \tag{62}$$

which is satisfied, when the kernel K_j fulfills

$$K_j - L^{(2)}_{h_{j+1},\lambda} * K_j = L^{(2)}_{h_{j+1},\lambda} - L^{(2)}_{h_{j+1},\lambda} * L^{(2)}_{h_j,\lambda}. \tag{63}$$

This equation can be solved in spectral language using the Legendre transform. More explicitly,

$$K_j^\wedge(n) = \frac{1 - L^{(2)\wedge}_{h_j,\lambda}(n)}{1 - L^{(2)\wedge}_{h_{j+1},\lambda}(n)} L^{(2)\wedge}_{h_{j+1},\lambda}(n), \quad n = 1, 2, \ldots \tag{64}$$

for $n \geq 1$. Note that $K_j^\wedge(0)$ is *not* specified by this relation. But this is clear, because of the fact that $\Psi^{j+1\wedge}(0) = \Psi^{j\wedge}(0) = 0$ (zero mean property). For $n = 1$ equation (64) is well–defined, since we know that $|L^\wedge_{h,\lambda}(n)| < 1$ for all $n \geq 1$.

Summarizing our results we obtain

Theorem 9.1. *For all $j = 1, 2, \ldots$ there exists a radial basis function K_j satisfyig the scale relation*

$$\Psi^{j+1} * K_j = \Psi^j.$$

K_j is given by

$$K_j^\wedge(n) = \frac{1 - L^{(2)\wedge}_{h_j,\lambda}(n)}{1 - L^{(2)\wedge}_{h_{j+1},\lambda}(n)} L^{(2)\wedge}_{h_{j+1},\lambda}(n), \quad n \geq 1,$$

where $K_j^\wedge(0)$ is arbitrary.

Since $\lim_{n\to\infty} L^\wedge_{h_j,\lambda}(n) = \lim_{n\to\infty} L^\wedge_{h_{j+1},\lambda}(n) = 0$, $K_j^\wedge(n) \sim L^{(2)\wedge}_{h_{j+1},\lambda}(n)$, $n \to \infty$, so that K_j is in the same Sobolev space as $L^{(2)}_{h_{j+1},\lambda}$. Furthermore, for $n \geq 1$, we have $K_j^\wedge(n) = 0$ if and only if $L^{(2)\wedge}_{h_{j+1},\lambda}(n) = 0$.

From $\Psi^{j+1} * K = \Psi^j$, it follows that $\mathrm{WT}(F)(j;.) = K_j * \mathrm{WT}(F)(j+1;.)$ so that we finally end up with the following modification of the decomposition scheme:

Wavelet Decomposition (Variant 2)

$$\begin{tikzcd}
F \arrow[r] \arrow[dr] & \mathrm{WT}(F)(J;.) \arrow[r] & \mathrm{WT}(F)(J-1;.) \arrow[r] & \cdots \arrow[r] & \mathrm{WT}(F)(1;.) \\
& (\Phi_{1;.}, F) & & &
\end{tikzcd}$$

For the numerical implementation of the convolutions with K_j, $K_j^\wedge(0)$ should be chosen, so that the kernel K_j is strongly localized. This can be achieved, when $K(-1) = 0$, so that the Legendre coefficient of order zero should be chosen in accordance with

$$\frac{1}{4\pi} K_j^\wedge(0) = -\sum_{n=1}^{\infty} \frac{2n+1}{4\pi} (-1)^n K_j^\wedge(n).$$

Acknowledgements: Financial support by "Stiftung Rheinland-Pfalz für Innovation" is gratefully acknowledged by the first author.

References

[1] J.-P. Antoine, L. Demanet, L. Jaques, P. Vandergheynst (2002) *Wavelets on the Sphere: Implementations and Approximations,* Appl. Comput. Harmon Anal. **13**:177-200.

[2] J.-P. Antoine, P. Vandergheynst (1999) *Wavelets on the 2–sphere: A Group–theoretic Approach,* Appl. Comp. Harmon. Anal., **7**:1-30.

[3] M. Bayer (1996) Spherical Wavelets and Their Application to Linear Inverse Satellite Problems, Diploma Thesis, University of Kaiserslautern, Geomathematics Group.

[4] S. Bochner (1950) *Quasi-analytic Functions, Laplace Operator, Positive Kernels,* Ann. of Math., **51**:68-91.

[5] M. Buhmann (2000) *Radial Basis Functions: the State of the Art and New Results,* Acta Numerica, **9**:1-37.

[6] S. Dahlke, W. Dahmen, E. Schmitt, I. Weinreich (1995) *Multiresolution Analysis and Wavelets on S^2 and S^3,* Numer. Funct. Anal. Optimiz., **16**(1&2):19-41.

[7] W. Freeden (1999) Multiscale Modelling of Spaceborne Geodata. B.G. Teubner, Stuttgart, Leipzig.

[8] W. Freeden, T. Gervens, M. Schreiner (1998) Constructive Approximation on the Sphere (With Applications to Geomathematics). Oxford Science Publications, Clarendon.

[9] W. Freeden, K. Hesse (2002) *On the Multiscale Solution of Satellite Problems by Use of Locally Supported Kernel Functions Corresponding to Equidistributed Data on Spherical Orbits*, Stud. Sci. Math. Hung., **39**:37-74.

[10] W. Freeden, J.C. Mason (1990) *Uniform Piecewise Approximation on the Sphere*, in: *Algorithms for Approximation II* (J.C. Mason, M.G. Cox, eds.), Chapman & Hall, 320-333.

[11] W. Freeden, T. Maier (2002) *On Multiscale Denoising of Spherical Functions: Basic Theory and Numerical Aspects,* Electron. Trans. Numer. Anal., **14**: 40-62.

[12] W. Freeden, C. Mayer (2003) *Wavelets Generated by Layer Potentials,* Appl. Comput. Harmon. Anal., **14**:195-237.

[13] W. Freeden, C. Mayer, M. Schreiner (2003) *Tree Algorithms in Wavelet Approximation by Helmholtz Potential Operators,* Numer. Func. Anal. Optim. (accepted for publication).

[14] W. Freeden, F. Schneider (1998) *Regularization Wavelets and Multiresolution,* Inverse Problems, **14**:225-243.

[15] W. Freeden, M. Schreiner (1995) *Non-orthogonal Expansions on the Sphere,* Math. Meth. in the Appl. Sci., **18**:83-120.

[16] W. Freeden, M. Schreiner, (1997) *Orthogonal and Non-orthogonal Multiresolution Analysis, Scale Discrete and Exact Fully Discrete Wavelet Transform on the Sphere*, Constr. Approx., **14**:493-515.

[17] W. Freeden, M. Schreiner, R. Franke (1997) *A Survey on Spherical Spline Approximation,* Surv. Math. Ind., **7**: 29-85.

[18] W. Freeden, U. Windheuser (1996) *Spherical Wavelet Transform and Its Discretization,* Adv. Comput. Math., **5**:51-94.

[19] W. Freeden, U. Windheuser (1997) *Combined Spherical Harmonic and Wavelet Expansion — A Future Concept in Earth's Gravitational Determination*, Appl. Comput. Harm. Anal., **4**:1-37.

[20] T. Gronwall (1914) *On the Degree of Convergence of Laplace Series,* Trans. Am. Math. Soc., **15**:1-30.

[21] W. Magnus, F. Oberhettinger, R.P. Soni (1966) Formulas and Theorems for the Special Functions of Mathematical Physics. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 52, Springer.

[22] P.A. Meissl (1971) A Study of Covariance Functions Related to the Earth's Disturbing Potential. The Ohio State University, Department of Geodetic Science and Surveying, Columbus, OSU Report No. 152.

[23] C. Müller (1952) *Über die ganzen Lösungen der Wellengleichung (nach einem Vortrag von G. Herglotz),* Math. Ann., **124**:235-264.

[24] C. Müller (1966) Spherical Harmonics. Lecture Notes in Mathematics, **17**, Springer, Berlin, Heidelberg.

[25] D. Potts, M. Tasche (1995) *Interpolatory Wavelets on the Sphere,* in: C.K. Chui, L.L. Schumaker, eds. Approximation Theory VIII, 335-342, World Scientific, Singapore.

[26] V.A. Rvachev (1990) *Compactly Supported Solutions of Functional-Differential Equations and Their Applications,* Russian Math. Surveys, **45**, No. 1:87-120.

[27] M. Schreiner (1996) *A Pyramid Scheme for Spherical Wavelets*, AGTM Report, No. 170, University of Kaiserslautern, Geomathematics Group.

[28] M. Schreiner (1997) *Locally Supported Kernels for Spherical Spline Interpolation*, J. Approx. Theory, **89**:172-194.

[29] P. Schröder, W. Sweldens, (1995) *Spherical Wavelets: Efficiently Representing Functions on the Sphere,* in: Computer Graphics Proceedings (SIGGRAPH95), 161-175.

[30] I.H. Sloan, R.S. Womersly (2000) *Constructive Polynomial Approximation on the Sphere,* J. Approx. Theory, **103**:91-118.

[31] L. Sjöberg (1980) *A Recurrence Relation for the β_n-Function,* Bull. Géod., **54**:69-72.

[32] S.L. Svensson (1984) *Finite Elements on the Sphere,* J. Approx. Theory, **40**:246-260.

[33] G. Szegö (1959) Orthogonal Polynomials. Amer. Math. Soc. Colloq. Publ., Vol. 23, American Mathematical Society, Providence, RI.

[34] I. Weinreich (2001) *A Construction of $C^{(1)}$–wavelets on the Two–dimensional Sphere,* Appl. Comput. Harmon. Anal., **10**:1-26.

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