Locational planning in the Mathematics Curriculum of High Schools

by

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1. Didactical and methodological principles

Location theory considers the possibilities to use mathematical methods in order to find good locations for facilities in industry and administrations. A good location for a central warehouse of a company has, for instance, been found if the sum of the transport and warehouse costs are minimized and if the warehouse is also used up to its capacity. If, on the other hand, an administration is looking for a location of a new firehouse or a hospital, an important criterion for an optimal location is that the longest distance should not surpass a given bound.

The location problems introduced in this publication belong to a group of so-called planar problems since the given locations are defined by two co-ordinates, i.e. the problem is considered in the Euclidean plane. Moreover, we restrict ourselves to searching for a single optimal new location.

Simple location problems can efficiently be solved by triangle constructions, distance computations in coordinate systems and methods of elementary calculus. In this way the issues considered here can primary been dealt with using methods and contents of mathematics classes for 14 – 18 year old students.

This evaluation with respect to its feasibility in application oriented teaching is also shared by a large group of teachers which have participated in continuing education courses, although those teachers were at the beginning surprised regarding the non conventional approach to teach mathematics.

Including location theory as part of school mathematics is not only reasonable from a mathematical point of view. It is also advantageous with respect to didactical and methodological reasons. The approach is contributing to a higher motivation due to the practice oriented teaching and the ability to repeat and strengthen mathematical ideas which have already been taught. In particular one should emphasize the opportunity students will have to acquire a social and activity driven knowledge which is possible by a methodological organisation of mathematics classes.

Moreover location theory is one of the possibilities to accept the challenge of the mathematics curriculum with respect to relevant applications, multidisciplinarity and mathematical modelling. According to the curriculum in German high schools: „A further task of the mathematics curriculum is to show students the process of mathematizing real world problems. Whenever mathematics is used as means to structure a practical problem, to represent essential aspects of complex conditions in a model and to find solutions, interaction between theory and practice can be experienced. (...) Students should be able to interrelate mathematics and facts outside of mathematics, to work on this problem with mathematical means, to interpret solutions which have been found in this mathematical context and to evaluate them. In this process the limits of a specific discipline and in particular of mathematizing should also be recognized.[1]
For the teaching, examples have been chosen which – starting from the every day

[1] Curriculum of mathematics (Lehrplan Mathematik, Grund- und Leistungsfach Jahrgangsstufen 11 bis 13 der gymnasialen Oberstufe (Mainzer Studienstufe), 1998, Seite 7. Sowie „Problemlösendes Verhalten“ und „Mathematisierung von Sachproblemen“ im Lehrplan Mathematik (Klasse 7-9/10), Hauptschule, Realschule, Gymnasium 1984, Seite 11-13

experience of young people – show how to use mathematics to solve problems in management, society and science/technology. The aim is to look at typical problems of modelling based location planning, which can be solved by mathematical methods. These methods correspond to the usual knowledge of students or which can easily be derived by extending well-known results usually taught in schools.

From consulting activities which have been done in the working group Optimization of the University of Kaiserslautern for industry the following three typical types of problems have been selected for the mathematics classes.

1. *The problem of a central warehouse* (cf. 2.1) represents a type of problem in which the sum of distances of given locations – for instance smaller warehouses – and a new facility – e. g. the central warehouse – has to be minimized. This problem comes in several variations:
   - Where to build the central baggage claim in an airport such that the passengers coming from different gates can minimize their walking distance?
   - Where to locate a playground in a neighbourhood such that the children living in neighbourhoods close by have short distances to this playground.

   This problem can be solved in various ways such that working on this topic can be done for different age groups:

   - If we have three given locations we can use the construction of a specific point (the Fermat Point) using specific lines in a triangle (Simson Lines).
   - By minimizing the sum of distances using elementary calculus.

   Such location problems can be discussed with students as young as 14 or 15. In order to give more exact recommendations for the age group one should consider whether existence proofs and the properties of the Fermat Point should indeed be worked on (see 6.1). In this case one has to use congruence theorems but also the theorem of the inscribed tetragon and its inversion (in proof 1) or the property of a rotation as a congruence mapping (in proof 2). There are however important pedagogical and time management reasons which may recommend to delete the mathematically exact proofs of the Fermat Point. This deletion will not lead to a strong loss with respect to quality of the course. In order to solve the problem using triangle constructions – in a coordinate system or directly in a geographical map – constructions have to be used which allow teachers and students to practice and repeat well-known basic constructions. The result also can be generated using a physical model in experimental form for three existing locations.

   In a course <u>for older students</u> this location problem can also be solved using elementary calculus. The problem offers the possibility to introduce functions with two variables. Here it is possible to show that a solution which the students know from functions with one variable can only be carried over if the distance is chosen „smartly“: only if the squared distance is used, one obtains an objective function which can be separated into two parts such that each of these parts is depending only on one variable. In this situation the optimum can be found in the classical way.

   The mathematical model can moreover be extended in order to obtain a better reflection of reality. If the smaller warehouses are used with different frequencies one uses these frequencies as factors by which the distance between central warehouse and small warehouses is multiplied (weighted distance). In the general case of n given locations with different frequencies and squared distance one can derive in this way the optimal location using a closed form expression for the coordinates (cf. 5.1 and 5.2).

2. In setting up the model the computation of the distance between two points is often of crucial importance. Therefore different metrics have been emphasized in advanced education courses for teachers. In schools one usually introduces the Euclidean distance but it is very meaningful to look also at the square of these distances and others. Using the example of the assembly of printed circuit boards (see 2.2) it is shown that the distances have to be computed in different ways depending on the type of movement of the robot. If the robot can only move parallel to the coordinate axes and if the movement has to be done sequentially, the distance of a point $P(x_1,x_2)$ to the origin is given by the so-called ℓ_1-norm: $\ell_1(OP) = |x_1| + |x_2|$.

   An intuitive interpretation of this kind of distance measurement is given if one considers the computation of distances in a city having streets built like a checker board as for instance in Mannheim (Germany) or Manhattan (USA). Therefore this type of distance is also called Manhattan distance.

   In general, students (and also teachers) are not used to deal with this type of distance computation. Consequently it is an often very surprising experience to look at the set of points having constant distance to one or two fixed given points: Perpendicular bisectors turn into broken lines, circles are appearing as regular closed broken lines.

   This example for teaching mathematics can be further extended. Given the conditions for the coordinates of our points not surpassing a certain distance from a fixed given point one can do a complete case analysis using inequalities. The resulting graphs representing the solution sets can be drawn such that this problem can be used for 15-year-old students as an example for linear equations and inequalities.

3. Looking at the problem of planning a fire house, (cf. 2.3) we want to minimize the maximum distance of the location we are looking for to given locations. The problem can be solved using circumcircles of a triangle defined by three given locations. The same procedure can be used if more than three locations are given. One has to consider all combinations of three locations and then chooses the best of the resulting solutions.

   In this approach one has to distinguish between acute and obtuse triangles and one also has to apply the theorem of Thales for a right-angled triangles. Hence this approach is particularly suited for 14-year-old students. For this age group the theorem of Thales can be nicely introduced within this application. Besides practising triangle constructions and description of these constructions there is also the opportunity to formulate algorithms.

   Courses sketched here, are all designed to take approximately five to eight hours. More time should not be spent. By this approach not much time is taken away from the normal mathematics curriculum. In addition the working on practice oriented problems like locational planning leads to new motivation for this classical curriculum. This limitation in time means for the teachers that they have to emphasize some of the topics described in this paper and to work on this chosen topics in an exemplary way. Correspondingly, there is no need to do the complete series of lectures mentioned in this report. They are meant as motivations and hints how the topics described in sections 3 – 5 can be used in different age groups.

In order to complete the process of modelling[2] in which a real world problem is mapped into a mathematical one and solved in this context, one has to look at the solution found by mathematical means and to evaluate this solution in the real world environment. Ever so often, these solutions do not satisfy the requirements from the practical context as it is shown in the following example. If, for instance, one finds the location of a garbage dump (instead of the location of a central warehouse) one would possibly get a location in the centre of the city, since here the sum of the transport roads is minimized!

The reaction of students to given series of lectures was throughout positive. They are highly motivated and were ready and willing to work for several hours on new concepts which from a mathematical point of view are not easy. Very often, it was possible to get into contact with students who usually are not so fond of mathematics. For all students it was interesting to see which kinds of problems can be solved by mathematics and that mathematics doesn't simply end in itself.

In addition to the fact that the motivation has been enhanced by the closeness of mathematical problems to reality, it was also shown that this methodology of teaching mathematics leads to discussion amongst the students regarding solutions of given problems. Students are encouraged to work in small groups such that they can find out for themselves how to use their initiative, their skills in communication and their social competence in order to tackle problems. There is for instance a double purpose in encouraging each of the small groups to present their solutions to the group of all students. On one hand there is a discussion in a larger group of people on the other hand the students acquire an ability to present results and to process new information in a larger context. The feedback obtained from the larger group is then a positive motivation for further work.

Since the students should develop the decisive ideas by themselves teachers should restrict themselves from too much influence. They should only give additional impulses if important aspects are missing to solve the problem or if mathematical methods are needed which are new to the students. In this way teachers are no longer generators of learning processes but help to initiate and carry through such processes. Conscientiously they have to take themselves back and become more consultants in the framework of mathematical teaching.

Within the context of motivating students and key qualifications, it is also important to point out the role of computers as a part of mathematical modelling. The usage of suitable software gives the students the possibility to experiment with a method preferred by students for developing solutions. Simultaneously, students develop in this way a feeling for routine-like usage of PC's.

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[2] compare Ministerium für Bildung, Wissenschaft und Weiterbildung Rheinland-Pfalz: Lehrplan Mathematik. Problemlösen mit mathematischen Methoden – Modellbildung, S. 18 ff., 1998.

2. The significance of locational planning. Introductory examples.

2.1. The Warehouse Problem.

The warehouse problem deals with a set of given small warehouses which should be assigned to a central warehouse in such a way that the overall distance between the central warehouse and the small warehouses is as small as possible.

In order to solve the problem the existing n warehouses $Ex_m(a_{m1},a_{m2})$, m=1,2,...,n, and the central warehouse $X(x_1,x_2)$ which has to be built are represented as points in the plane. As distance measure we choose the Euclidean distance between the central warehouse X and each of the different smaller warehouses Ex_m

$$d(Ex_m, X) = \sqrt{(a_{m1}-x_1)^2 + (a_{m2}-x_2)^2} = \ell_2(Ex_m, X)$$

where ℓ_2: $\mathrm{IR}^2 \to \mathrm{IR}$ is a metric.

The overall distance is defined by the sum of the distances between the central warehouse and the different smaller warehouses

$$\sum_{m=1}^{n} d\left(Ex_m, X\right) = \sum_{m=1}^{n} \sqrt{\left(a_{m1}-x_1\right)^2 + \left(a_{m2}-x_2\right)^2} = \sum_{m=1}^{n} \ell_2\left(Ex_m, X\right).$$

If one wants to consider in this model also the frequencies of ways between the warehouses one has to weight the distances with these frequencies.

A numerical example[4]

In a company there are three small warehouses which are given on a map by the coordinates E_1 = (1/2), E_2 = (7/3) and E_3 = (4/5). The management wants to set up a new central warehouse in a new location such that the sum of the overall distances (it is the sum of all distances between the small warehouses and the central warehouses) is as small as possible.

Model 1: *A simple distance to central warehouse* $X(x_1,x_2)$

$$\sum_{m=1}^{3} \ell_2(\mathrm{Ex}_m, \mathrm{X}) = \sqrt{(1-\mathrm{x}_1)^2 + (2-\mathrm{x}_2)^2} + \sqrt{(7-\mathrm{x}_1)^2 + (3-\mathrm{x}_2)^2} + \sqrt{(4-\mathrm{x}_1)^2 + (5-\mathrm{x}_2)^2}$$

If one chooses for instance X_1 = (4/2) we obtain by replacing X_1 by 4 and X_2 by 2 the overall distance to the small warehouses as

$$\sqrt{9+0} + \sqrt{9+1} + \sqrt{0+9} = 6 + \sqrt{10} \approx 9{,}2 .$$

If we choose X_2 = (4/5), we obtain

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[3] Curriculum of mathematics (Lehrplan Mathematik, Grund- und Leistungsfach Jahrgangsstufen 11 bis 13 der gymnasialen Oberstufe (Mainzer Studienstufe), 1998, Seite 7. Sowie „Problemlösendes Verhalten“ und „Mathematisierung von Sachproblemen“ im Lehrplan Mathematik (Klasse 7-9/10), Hauptschule, Realschule, Gymnasium 1984, Seite 11-13

[4] Horst W. Hamacher: Mathematische Lösungsverfahren für planare Standortprobleme, Vieweg Verlag Braunschweig/Wiesbaden 1995

$\sqrt{9+9} + \sqrt{9+4} + \sqrt{0+0} = \sqrt{18} + \sqrt{13} \approx 7{,}8.$

If there are no additional restrictions in the problem we would therefore prefer location X_2 = (4/5) to the first location X_1= (4/2).

Model 2: *Weighted distances*

If we know by our data analysis that we would have to go from the new central warehouses five times and three times more often to the small warehouses E_1 and E_2 than to E_3, respectively. A more realistic model for the distance is given by multiplying the simple distance between E_1 and X with the factor $w_1 = 5$ and between E_2 and X with $w_2 = 3$. Therefore we get the weighted distance as

$$\sum_{m=1}^{3} w_m \ell_2(Ex_m, X) =$$

$$5 \cdot \sqrt{(1-x_1)^2 + (2-x_2)^2} + 3 \cdot \sqrt{(7-x_1)^2 + (3-x_2)^2} + \sqrt{(4-x_1)^2 + (5-x_2)^2}$$

Checking the weighted distance of the two candidate locations considered above we obtain for

X_1 (4/2) $\quad 5 \cdot \sqrt{9+0} + 3 \cdot \sqrt{9+1} + 1 \cdot \sqrt{0+9} = 18 + 3 \cdot \sqrt{10} \approx 27{,}5$

X_2 (4/5) $\quad 5 \cdot \sqrt{9+9} + 3 \cdot \sqrt{9+1} + 1 \cdot \sqrt{0+1} = 15 \cdot \sqrt{2} + 3 \cdot \sqrt{13} \approx 32{,}0$

In this example X_1 (4/2) would be a better location than the second one for the central warehouse.

Using these simple examples one gets a feeling that the question how to find the best possible location has to be reflected more keenly.

Model 3: More definitions of distances

It is necessary to improve the way distances are measured in the models. Euclidean distance only makes sence if it is possible to move directly between the central and the small warehouses. This applies to helicopters, cars in the dessert or possibly for boats in waters where they can move freely, but not for all problems. Therefore it should be possible to find out the distance $d(Ex_m,X)$ between any possible location X (x_1/x_2) and corresponding small warehouses Ex_m from a network or street network. (For instance distance tables as in road maps).

While setting up the model it has thus to be checked which mathematical definitions have to be used in order to model the distances.

2.2. Production of Printed Circuit Boards

Another example which emphasizes the importance of choosing a good distance measure (a metric) is the production of printed circuit boards using robots:

In assembly of printed circuit boards M parts belonging to N different classes (transistors, storage elements etc.) are sampled on specific points Ex_m of the printed circuit board. Here we assume $N \leq M$. It is required to set up locations for the bins in which the different types of parts are stored. While trying to find a good location for $X_1,...,X_N$ for these bins one has to observe the following facts.

1. There should be some safety margin between those bin locations and the printed circuit board.
2. The time of assembling the printed circuit board should be minimized.

For the model we assume that the sequence in which the parts are inserted is fixed. (This is by the way another interesting mathematical problem which can be modelled.) For our further considerations it is important to know how the robot arm is moving since this is determining the distances of the routes.

a) Two linear engines which allow a simultaneous movement in two coordinate directions.

   The next insertion can only be started if both of the engines stop. In this situation the following "distance" between an insertion point Ex_m (a_{m1},a_{m2}) and the location X (x_1,x_2) for a bin holds: $d(Ex_m, X) = \max\{|a_{m1} - x_1|, |a_{m2} - x_2|\}$
   One also uses the denotation $d(Ex_m, X) = \ell_\infty(Ex_m, X)$ in this case.

b) If the robot arm has only one driving engine the movement along the coordinate axes have to be done sequentially and the „distance" between insertion point and location of the bin is given by $d(Ex_m, X)= |a_{m1} - x_1| + |a_{m2} - x_2| = \ell_1(Ex_m, X)$.

Numerical example for the production of printed circuit boards

The situation is considered in which two bins (N = 2) containing round and angular parts for insertion points on the printed circuit boards denoted Ex_1, ..., Ex_4 (M = 4) should be used. The safety distance between printed circuit board and bin is represented by a rectangle around the printed circuit board (cf. Fig. 1a).

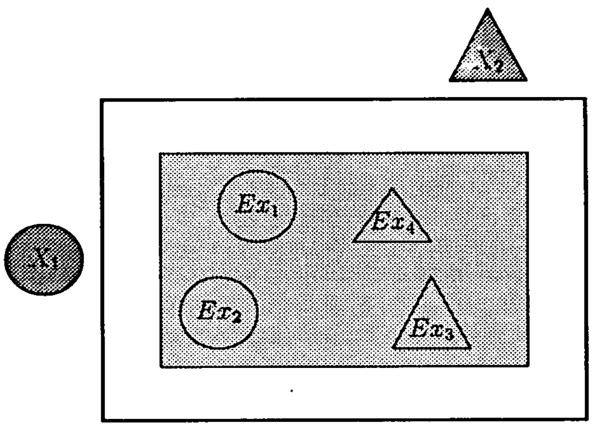


Figure 1a:
Printed circuit board in a rectangle with fixed insertion points for four parts belonging to two type classes[5].

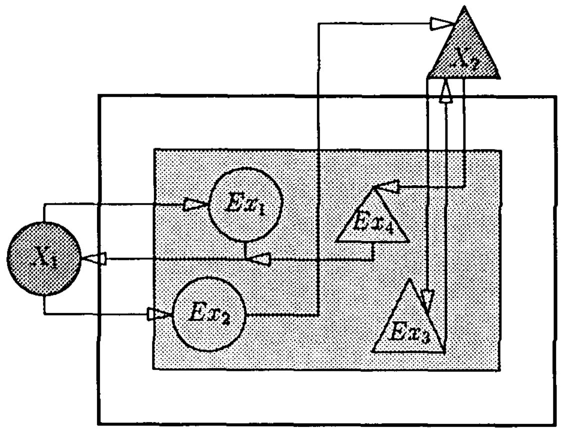
X_1 is a possible location for bins with round parts and
X_2 is a possible location for bins with angular parts.

From each type two parts have to be inserted.

[5] From: Horst W. Hamacher: Mathematische Lösungsverfahren für planare Standortproblem, Vieweg Braunschweig/Wiesbaden, 1995

The robot arm is working on the board following the sequence $Ex_1, ..., Ex_4$. The working sequence is abbreviated by the notation (1,2,3,4).

Figure 1b: A robot tour based on the insertion sequence (1, 2, 3, 4)



The tour of the robot arm following the insertion sequence (1, 2, 3, 4) with X_1 as starting and ending point has an overall length which is depending on X_1 and X_2:

$$f(X_1,X_2) = d(X_1, Ex_1) + d(Ex_1, X_1) + d(X_1, Ex_2) + d(Ex_2, X_2) + d(X_2, Ex_3,) + d(Ex_3, X_2) + + d(X_2, Ex_4,) + d(Ex_4, X_1) \quad (1)$$

$$= 2\, d(Ex_1, X_1) + d(Ex_2, X_1) + d(Ex_2, X_2) + 2\, d(Ex_3, X_2) + d(Ex_4, X_2) + d(Ex_4, X_1) \quad (2)$$

$$= 2\, d(Ex_1, X_1) + d(Ex_2, X_1) + 0 \cdot d(Ex_3, X_1) + d(Ex_4, X_1) + 0 \cdot d(Ex_1, X_2) + d(Ex_2, X_2) + + 2\, d(Ex_3, X_2) + d(Ex_4, X_2) \quad (3)$$

$$= f_1(X_1) + f_2(X_2) \text{ with} \quad (4)$$

$$f_1(X_1) = 2\, d(Ex_1, X_1) + d(Ex_2, X_1) + 0 \cdot d(Ex_3, X_1) + d(Ex_4, X_1) \text{ and} \quad (5)$$

$$f_2(X_2) = 0 \cdot d(Ex_1, X_2) + d(Ex_2, X_2) + 2\, d(Ex_3, X_2) + d(Ex_4, X_2) \quad (6)$$

In part (1) the tour is written in that way starting from X_1, inserting at Ex_1, going back to X_1, inserting at Ex_2. The round parts are now inserted. The robot arm now is going from inserting location Ex_2 to bin X_2, inserting an angular part at Ex_3, back to the bin X_2, and inserting a last part at Ex_4. From here the robot arm is going „home“ to bin X_1.

Remark:

d is a distance measure. In this example it is a metric in a mathematical sense. One of the characteristics of a metric is the symmetry (i.e. the direct way from A to B has the same length than from B to A). This is used coming from (1) to (2): $d(X_1, Ex_1) = d(Ex_1, X_1)$

In (2) equal terms are collected together, in (3) the **terms** are sorted depending either on X_1 or X_2. (4), (5) and (6) are defining **functions** f_1 and f_2 depending either on X_1 or on X_2.

$2 \cdot d(Ex_1, X_1)$ has the meaning that a round part taken from the bin located at X_1 is inserted in EX_1 and the robot arms than moves back to the X_1.

$0 \cdot d(Ex_3, X_1)$ means that there is no round part connecting Ex_3 and X_1.

Since the robot arm moves back to bin at Ex_1 after inserting Ex_4 there is the part one times d (Ex_4, X_1).

In the same way the part of the robot tour is computed which has to do with insertion of angular parts which are stored in a bin located at X_2.

Here the term $d(Ex_2, X_2)$ reflects the situation that after inserting the round parts the robot arm has to move to the bin with angular parts to start from there inserting angular parts. If one calculates the minimum of the functions $f_1(X_1)$ and $f_2(X_2)$ we get locations in which the bins storing the round and angular parts respectively should be located in order to minimize the overall length f of the tour of the robot arm. The bins can however not be placed without the restriction since we have to deal with a security distance, Hence we have to choose locations from a given part of the plane excluding the rectangle.

2.3. Planning of a firehouse

In order to keep a certain maximum reaction time the planning of a firehouse includes the important question where to put it. If one wants for instance to set up a company internal firehouse dealing with three production sites at Ex_1, Ex_2, Ex_3 one may want to have a reaction time of at most 10 minutes. Thus the location X of the new firehouse has to fulfil the following constraints:

$\ell_2(Ex_m, X) \leq s$ für m = 1, 2, 3. Here s is the distance which can be covered within 10 minutes.

It would even be better if the fire brigade can arrive the location of fire as quickly as possible. That is one wants to find an r which satisfies $\ell_2 (Ex_m, X) \leq r$ such that r is as small as possible (m = 1, 2, 3). Again ℓ_2 is a Euclidean distance.
It may be helpful to start the search for a procedure to find X by interpreting a possible result. If the location X of the firehouse would be known we know that $\ell_2 (Ex_m, X) \leq r$ for all m. Hence the production sites Ex_1, Ex_2, Ex_3 are contained within a circle centred at X with radius r or on its boundary. Consequently we are looking for the centre of a circle which covers the given points Ex_1, Ex_2, Ex_3 with a radius as small as possible.

3. Geometrical solutions for selected location problems

Fortunately, several of the examples mentioned above can be solved by using simple constructions which are known from geometric classes of age group 13 – 15.

3.1 Solution of the central warehouse problem by finding the Fermat point.

Recall: Given three existing locations Ex_m, m =1,2,3, (existing warehouses), which are not lying on a line, we want to find a new location for a central warehouse in such a way that the (weighted) sum of the distances is minimized. The geometrical procedure for the case of equal weights consists of the following steps (see fig. 2):

Step 1: Draw the triangle A,B,C (A = Ex_1, B = Ex_2, C = Ex_3).

Step 2: Construct on every side of the triangle an equilateral triangle. By this procedure the points A*, B*, C* are generated.

Step 3: Connect each of the newly constructed points A*, B*, C* in an equilateral triangle with the related corner point of the triangle A, B, C, which does not belong to this equilateral triangle (A with A*, etc.). (Simson-lines[6]).

Step 4: The Simson-line are intersected in one point, the optimal location, called the Fermat-point F.

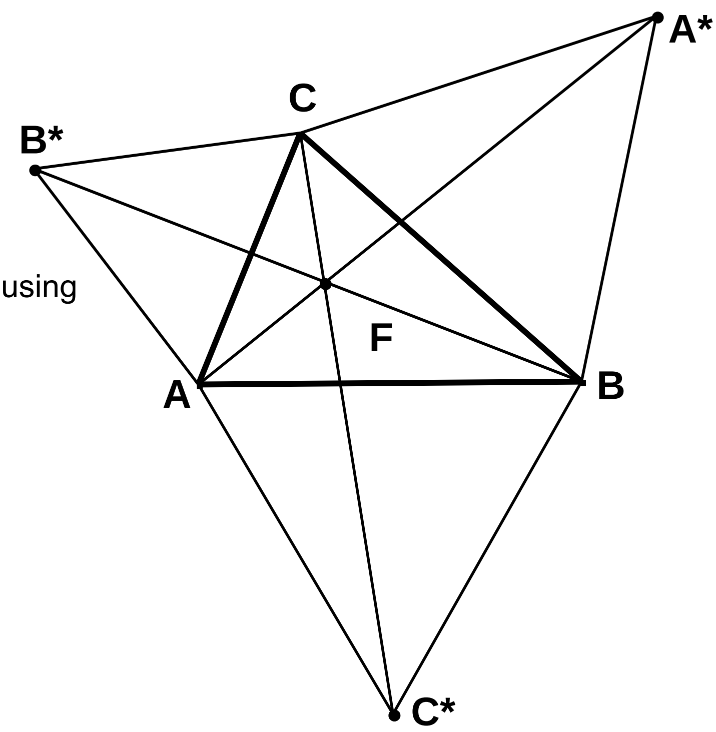


Figure 2:
Construction of the Fermat-point using Simson-lines

[6] Named by Robert Simson, an English mathematician 1687 - 1768

Using geometrical procedures the central warehousing problem can also be solved for weighted distances and if four existing locations given. In all other cases we have to use iterative procedures.

Remark:

The construction of the Fermat – point with equilateral triangles is a special case of a so-called *Torricelli configuration*[7]. On each side of an arbitrary given triangle ΔABC directly similar triangles ΔABC*, ΔBCA*, Δ ACB* are constructed (see fig. 3).

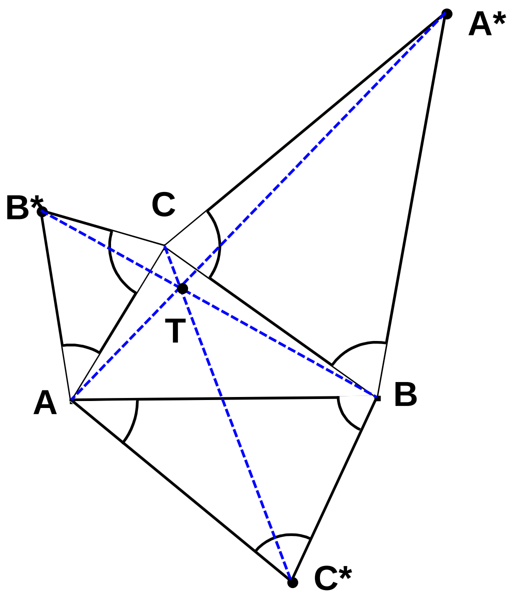
For the constellation in fig. 3 the following facts hold:

The „diagonals“ $\overline{AA*}$, $\overline{BB*}$, and $\overline{CC*}$ in this drawing are intersecting to a point, the *Torricelli-point* T. In this intersection point one can find again the angles α‘, β‘, γ‘ of the additional triangles and the length of the „diagonals“ $\overline{AA*}$, $\overline{BB*}$, $\overline{CC*}$ have the same relation as the altitude of these triangles.

In the special case discussed above with equilateral triangles (α‘ = β‘ = γ‘ = 60°), the diagonals $\overline{AA*}$, $\overline{BB*}$, $\overline{CC*}$ have the same length and include angles of 60°.

Figure 3: *Torricelli-configuration*

with α' = 40°,
β' = 65°,
γ' = 75°



For this special case we observe:

If the triangle ΔABC does not contain an angle larger than 120° the Simpson lines intersect inside the triangle. The sum of the distances of a point P to the corner points A, B, C is reaching its minimum if and only if P is the intersection point of the diagonals T. The sum of these distances then corresponds to the sum of diagonals. This extreme value problem has been formulated by P. de Fermat[10] and that is the reason why T is also called the Fermat point F (see fig. 2). In the English speaking literature the problem is associated with Robert Simson who taught at about 1750 at the University of Glasgow[11]

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[7] Evangelista Torricelli (1608-1647)

[8] Pierre de Fermat (1601 - 1665) posed this problem in his „Discours de Maxima et Minima“ 1643/44.

[9] See Homepage „History of mathematics archive“ of St. Andrews University, Scotland. http://www-history.mcs.st-and.ac.uk/history/Mathematicians (under Torricelli, Simson, Fermat)

[10] Pierre de Fermat (1601 - 1665) posed this problem in his „Discours de Maxima et Minima“ 1643/44.

[11] See Homepage „History of mathematics archive“ of St. Andrews University, Scotland. http://www-history.mcs.st-and.ac.uk/history/Mathematicians (under Torricelli, Simson, Fermat)

In the case that there is an angle of 120° the Fermat point coincides with the vertex of an angle. If there is an angle which is larger than 120° the intersection point of the Simson lines lies outside of the triangle and is not the point minimizing the sum of the distances. This can also be seen using dynamic geometry software if one moves one of the corner points of the triangle.

3.2 Planning of a firehouse

The location problem can be described as follows: With respect to existing companies a firehouse is to be built such that the maximum distance between one of the companies and the firehouse is as small as possible.
We use a planar model with Euclidean distance.
If we have found a location X for the firehouse such that a distance to all companies Ex_i is less then s and everyone can be reached in less than a given time, for instance ten minutes, then the circle centred at X contains all given locations[12].

3.2.1 Problem for 2 locations Ex_1, Ex_2

The location X is just the centre of the line segment $\overline{Ex_1 Ex_2}$

3.2.2 Problem for 3 locations Ex_1, Ex_2, Ex_3

a) Ex_1, Ex_2, Ex_3 build an acute triangle. In this case the circumcircle of the triangle $\Delta Ex_1 Ex_2 Ex_3$ is the smallest circle containing the given locations and the centre of this circle is the location X for the firehouse.

b) If triangle $\Delta Ex_1 Ex_2 Ex_3$ is obtuse then there is a better solution than the centre of the circumcircle. The centre of the longest side of the triangle is the optimal location of the firehouse.

3.2.3 Procedure for more than three locations

In this case we are looking for the smallest circle (i.e. the circle with the smallest radius), covering all given locations Ex_m. Such a circle contains at least two locations on its boundary.

Solution procedure:

Step 1: Draw all circumcircles of triples of locations (Ex_i, Ex_j, Ex_k) with $i \neq j$, $i \neq k$, $j \neq k$ for all i, j, k = 1, ..., M

Step 2: Draw all circles centred at the centres of the line segments $\overline{Ex_i Ex_j}$, which contain Ex_i and Ex_j on their boundaries.

Step 3: Delete all circles which do not cover all locations.

Step 4: Choose among the remaining circles one with smallest radius.

Numerical Example with 4 Locations:

Ex_1 =(0|3), Ex_2 =(5|8), Ex_3 =(9|0), Ex_4 =(9|3)

Sub problems

[12] With circle the interior of a circle joined with the border is meant.

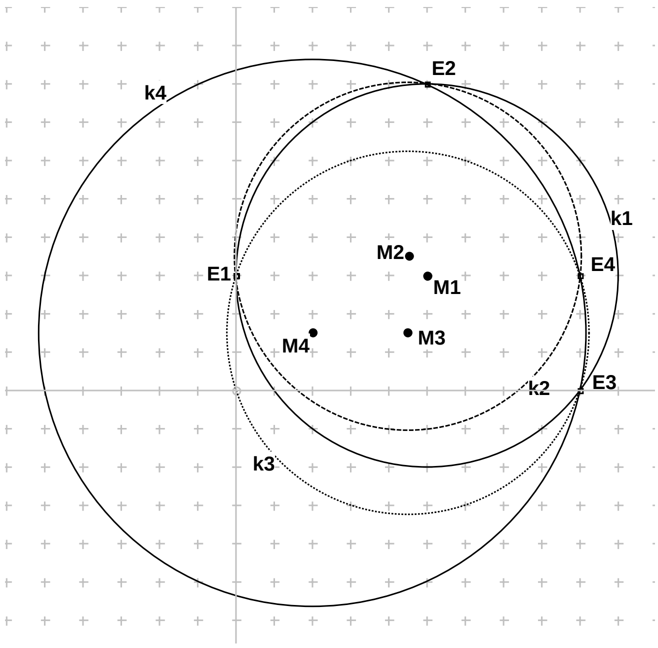
1. Draw the points in a coordinate system!
2. How many pairs and how many triples do we have?
3. Draw all circumcircles given by three of the given locations!
4. Are there feasible circles which are only defined by 2 locations?
5. Which of the circles is the best?
6. What happens if Ex_4 =(10/3) or Ex_4 =(11/8) are considered instead?

The solution (see fig. 4) has been determined using triples:
Δ E_1 E_2 E_3: circumcircle K_1 has the radius $r_1 \approx$ 4,6 L.E.
Δ E_1 E_2 E_4: circumcircle K_2 has the radius $r_2 \approx$ 5 L.E.
Δ E_1 E_3 E_4: circumcircle K_3 has the radius $r_3 \approx$ 4,7 L.E.
Δ E_2 E_3 E_4: circumcircle K_4 has the radius $r_4 \approx$ 7 L.E.
(we are dealing with an obtuse triangle)

$K_i = \{P| \left|\overline{PM_i}\right| \leq r_i\}$
K_1 contains all 4 points
K_2 does not contain E_3
K_3 does not contain E_2
K_4 does contain all 4 points, but $r_4 > r_1$

Hence the centre of the circle K_1 is the optimal location for the firehouse.
If we change E_4 to E_4 =(10/3) K_1 remains the optimal circle since all existing locations are lying on its boundary.

Figure 4:
Search for an optimal location of the firehouse with respects to given 4 companies (numerical example). Shown are four circumcircles for four triangles built by each triple of the four points E_1, E_2, E_3, E_4. M_1, M_2, M_3, M_4 are the centres of that circumcircles.



For E_4=(11/8) however triangle Δ E_1 E_2 E_4 is obtuse and K_3 becomes the “optimal” circle and consequently the centre of K_3 the optimal location for the fire house.
The questions remains unanswered whether an optimal circle is always unique.

3.3 Restrictive Location Problems

In many problems there is an additional restriction for the choice of locations.

Examples for Restrictions:

1. printed circuit board problem:

Along the boundary of the printed circuit board we have to maintain a certain security distance in which we can not position any bins. (Similar requirements have to be satisfied in conveyor belt production.)

2. Firehouse

A nature reserve cannot be used for the location of a firehouse.

In the mathematical model we will represent the restriction by excluding the interior of a polygon R. On the boundary ∂R however we can locate facilities.

Solution approach:

Step 1: Solve the unrestrictive problem (i.e. determine all optimal locations without taking the restriction to the region R into consideration).

Step 2: If we have found in step 1 a location which is feasible, the procedure terminates.

Step 3: Otherwise augment the objective value until we have found a location X* with the objective value which is feasible.

In the firehouse problem we obtain an optimal location by extending the radii of the circles about the location of the companies until we generate an intersecting region which is contained inside the forbidden region but touches the boundary.

Example: Find the optimal location in a model with restrictions (firehouse, figure 5)!

In this example we are looking for a location of a firehouse serving the companies located at $Ex_1,..,Ex_4$ where the region inside the pentagon P_1,P_2,P_3,P_4,P_5 is a nonfeasible region. We are looking for the centre of such a circle which covers the location of the companies (here 4). For such centre points the intersection points of perpendicular bisectors of $\overline{Ex_iEx_j}$ with the boundary of the infeasible area (here denoted by A_i and B_j) and the projections of the location of the companies (corner points) Ex_k on the boundary of the nonfeasible region are possible candidates (here L_1, L_2 and L_4, the other projections are not existing on the pentagon.)

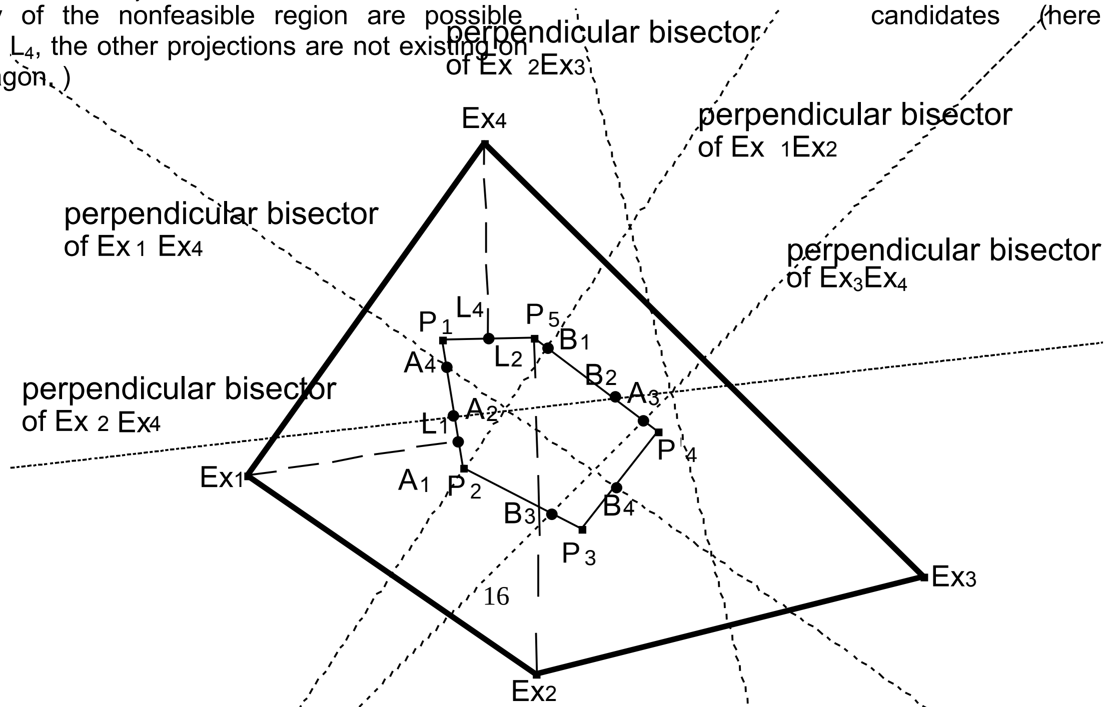


Figure 5: Determining the optimal location in the model with restrictions.

For all circles centred at Ex_m having a radius larger than the minimum radius found in step 1 we conclude that their intersection is a circular lune. The „corner“ of this circular lune is built by the intersection of two circles and its boundaries consist of circular arcs (see fig. 6).

Here the following cases can occur:

a) The „corner“ of a circular lune is touching the boundary of the forbidden region.
b) The boundary of the polygon describing the restrictions is becoming a tangent to one of the circle segments.

From this we deduce a rule for finding the location:

a) Consider all intersection points between perpendicular bisectors of $\overline{Ex_iEx_j}$, $i \neq j$, $i, j = 1, ..., M$ and the boundary ∂R of the restrictions.
b) Consider all those points on the boundary ∂R in which the line segment which is part of the restricting polygon is a tangent to the circle centred at Ex_m, $m = 1, ..., M$. (This means that all projection points of Ex_m, $m = 1, ..., M$ which exists on the boundary ∂R are considered.)

The best candidate from a) and b) is an optimal location for the problem.

Numerical example

$Ex_1=(5|5)$, $Ex_2=(15|7)$, $Ex_3=(7|15)$ are given locations. The restriction is described by a polygon with the corner points $P_1=(6|5)$, $P_2=(8|14)$, $P_3=(10|14)$, $P_4=(14|7)$ (see. figure 6).

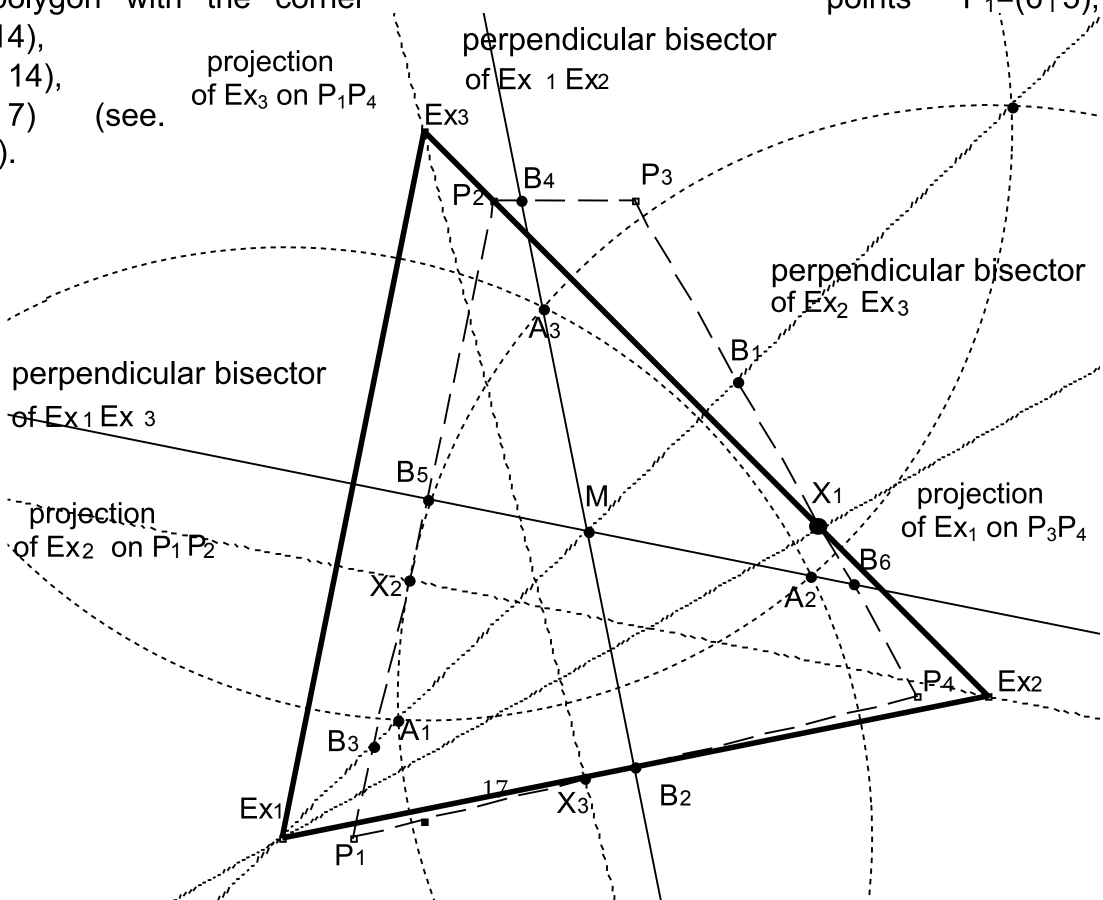


Figure 6:
Visualisation of the algorithm to find an optimal location subject to restrictions: X_i are the projections of Ex_i to the boundary, B_i are the intersection points of perpendicular bisectors of the sides of the triangle with the boundary of the forbidden region.

We find the location following our solution approach:

1. Without restriction: The centre M of the circumcircle of $\Delta\, Ex_1\, Ex_2\, Ex_3$ is determined.
2. The centre point is located in the infeasible region R. Search for a feasible location.

3a) Consider the intersection points between the perpendicular bisectors of the sides of the triangle and the boundary ∂R! From all of these B_i B_5 is the best candidate as intersection point between $\overline{P_1P_2}$ and the perpendicular bisector on $\overline{Ex_1Ex_2}$.
3b) Draw the projection points of Ex_1, Ex_2, Ex_3 on ∂R to obtain X_1, X_2, X_3!
Taking X_2 we get a circle with the smallest radius since the arc triangle A_1, A_2, A_3 is touching the polygon at X_2.
3c) Comparing the "best points" from 3a) and 3b) the circle centred at B_5 has a radius which is slightly larger than the circle centred at X_2.

The solution is $X_2 = (6{,}8 \,|\, 8{,}7)$ with $r \approx 9{,}8$.
(To get the mathematical strength of the applied algorithm cf. Hamacher (1995), chapter 4.4.)

4. Angular circles and broken perpendicular bisectors as modelling tools

If one moves in a coordinate system along the axis (example robot arm) or in a rectangular street network, the Euclidean distance is not suitable measuring distances.

A better metric for this model is the so called ℓ_1-norm, which is defined as follows: $\ell_1 : \mathbb{R}^2 \to \mathbb{R}; (x_1 | x_2) \to |x_1| + |x_2|$.
The ℓ_1-norm is often called rectangular distance or Manhattan distance. The distance is expressing the distance from the point X $(x_1|x_2)$ to the origin. In details this means:

I. quadrant $x_1 \geq 0 \wedge x_2 \geq 0 \rightarrow \ell_1(x_1, x_2) = x_1 + x_2$
II. quadrant $x_1 < 0 \wedge x_2 \geq 0 \rightarrow \ell_1(x_1, x_2) = -x_1 + x_2$
III. quadrant $x_1 < 0 \wedge x_2 < 0 \rightarrow \ell_1(x_1, x_2) = -x_1 - x_2$
IV. quadrant $x_1 > 0 \wedge x_2 < 0 \rightarrow \ell_1(x_1, x_2) = x_1 - x_2$

The notion of the geometric objects circle (centred at a given centre with a fixed radius and perpendicular bisector (with respect to a given line segment) which often used in Euclidean geometry have in this metric a somewhat unusual interpretation but the basic definition is the same.

For instance the unit circle $K(0,1) := \{(x_1, x_2) \in \mathbb{R}^2 | \ \ell_1(x_1, x_2) = 1\}$ has the following representation in a coordinate system (see fig. 7).

Using a case analysis we obtain
for the I. quadrant: $x_1 + x_2 = 1$
for the II. quadrant: $-x_1 + x_2 = 1$
for the III. quadrant: $-x_1 - x_2 = 1$
for the IV. quadrant: $x_1 - x_2 = 1$

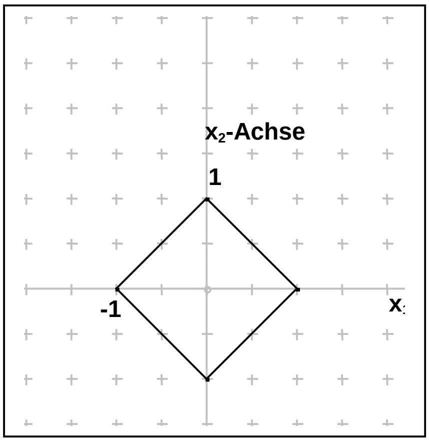


Figure 7:
Unit circle in the metric corresponding to the ℓ_1-norm.

Example:
In Manhattan there are two post offices located at Ex_1 and Ex_2. The post system would like to find out which of the households is the closest to which of the post offices. I.e. it wants to partition Manhattan into two parts. Each household of each part is associated to one and only one post office.
In order to find the separating line between the two parts we have to find the set IB_1 of all points which with respect to the ℓ_1-norm have the same distance from $Ex_1(a_1,a_2)$ and from $Ex_2(b_1,b_2)$ with the coordinates (a_1,a_2), (b_1,b_2) of Ex_1, EX_2 (finding the perpendicular bisector):

$IB_1 = \{X \in IR^2 | \ \ell_1(X - Ex_1) = \ell_1(X - Ex_2)\}$
$= \{(x_1, x_2) \in IR^2 | \ |x_1 - a_1| + |x_2 - a_2| = |x_1 - b_1| + |x_2 - b_2|\}$

The length of the paths in such a metric is depending on the fact whether the paths are parallel to the coordinate axis or not. Hence the graphs of the perpendicular bisectors distinguish from each other whether the location of the two post offices within a street map is on a parallel to the coordinate axis.

Numerical example a): $Ex_1(0|0)$, $Ex_2(4|0)$, i.e. one of the coordinates are equal.

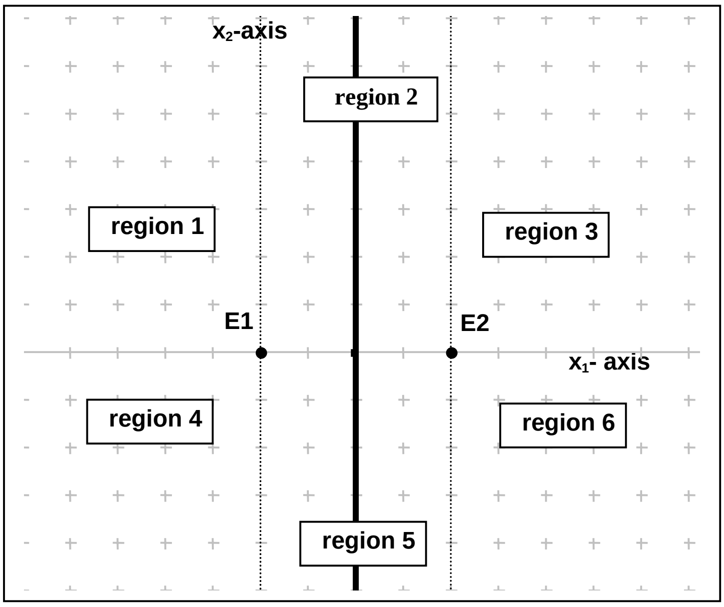


Figure 8:
Determination of the perpendicular bisector in numerical example a)

With the condition $|x_1 - 0| + |x_2 - 0| = |x_1 - 4| + |x_2 - 0|$, $(x_1, x_2) \in IR^2$ the perpendicular bisector is defined. A case distinction leads to the definition of 6 regions in IR^2 :

Region 2: $0 \le x_1 \le 4 \wedge x_2 \ge 0$

Condition for the perpendicular bisectors:

$$(x_1 - 0) + (x_2 - 0) = -(x_1 - 4) + (x_2 - 0) \wedge x_2 \ge 0$$
$$\Leftrightarrow \quad x_1 + x_2 = -x_1 + x_2 + 4$$
$$\Leftrightarrow \quad x_1 = 2 \wedge x_2 \ge 0 \text{ arbitrarily}$$

$IL_2 = \{(x_1, x_2) \in IR^2 \mid x_1 = 2 \wedge x_2 \ge 0\}$

The graph is a parallel to the x_2-axis with a distance 2 in the 1st quadrant.

Region 5: $0 \le x_1 \le 4 \wedge x_2 < 0$

Condition: $(x_1 - 0) - (x_2 - 0) = -(x_1 - 4) - (x_2 - 0) \wedge x_2 < 0$

$\Leftrightarrow x_1 - x_2 = -x_1 + 4 - x_2$

$\Leftrightarrow x_1 = 2 \wedge x_2 < 0$

$IL_5 = \{(x_1, x_2) \in IR^2 \mid x_1 = 2 \wedge x_2 < 0\}$

The graph is a parallel to the x_2 -axis with a distance 2 in the 4th quadrant.

Region 1: $0 < x_1 \wedge x_2 \ge 0$

Condition: $-(x_1 - 0) + (x_2 - 0) = -(x_1 - 4) + (x_2 + 0)$, $x_2 \ge 0$

$\Leftrightarrow \quad -x_1 + x_2 = -x_1 + 4 + x_2$

$\Leftrightarrow \quad 0 = 4 \wedge x_2 \ge 0$

$IL_1 = \{ \}$

For regions 3, 4, 6 there is as in region 1 a new pair of numbers for the solution hence there are no points in the plane which satisfy the conditions. Thus we get a perpendicular bisector as in the case of the Euclidean distance.

<u>Numerical example b):</u> $Ex_1\ (0|0)$, $Ex_2\ (4|2)$, both coordinates are different.

Using the $\ell_1(X, Ex_1) = \ell_1(X, Ex_2)$ leading to $|x_1 - 0| + |x_2 - 0| = |x_1 - 4| + |x_2 - 2|$ we have to distinguish now 9 regions:

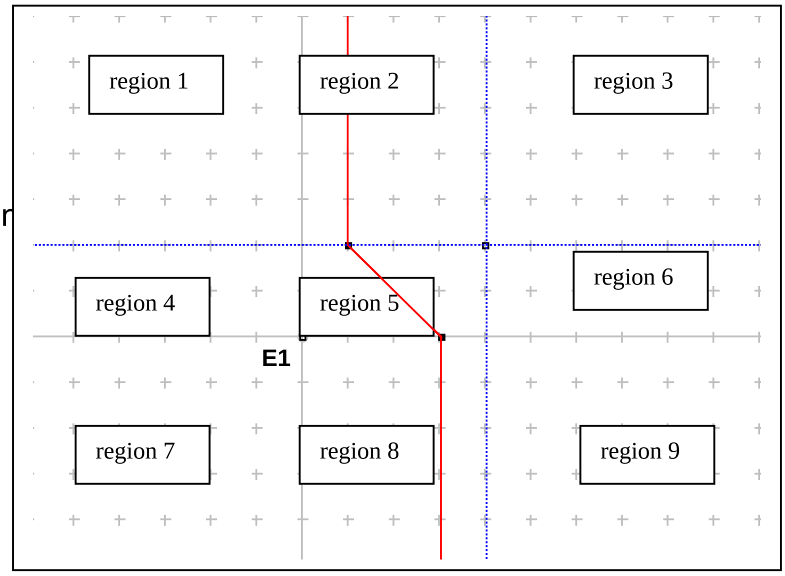


Figure 9:
Perpendicular bisector, separatin
numerical example b)

<u>Region 1:</u> $x_1 < 0 \wedge x_2 > 2$

There is no solution that means the perpendicular bisector does not cross this region.

<u>Region 2:</u> $0 \le x_1 \le 4 \wedge x_2 > 2$

Condition: $(x_1 - 0) + (x_2 - 0) = -(x_1 - 4) + (x_2 - 2)$

$\Leftrightarrow\ x_1 + x_2 = -x_1 + x_2 + 2$

$\Leftrightarrow\ 2x_1 = 2$

$\Leftrightarrow\ x_1 = 1 \wedge x_2 > 2$ arbitrarily

$\mathbb{L}_2 = \{(x_1, x_2) \in \mathbb{R}^2 |\ x_1 = 1 \wedge x_2 > 2\}$.

The graph is a parallel to the x_2-axis with distance 1 for points with ordinates greater than 2.

<u>Region 3:</u> $x_1 > 4 \wedge x_2 > 2$ $\Rightarrow$ no solution

<u>Region 4:</u> $x_1 < 0 \wedge 0 \le x_2 \le 2$ $\Rightarrow$ no solution

<u>Region 5:</u> $0 \le x_1 \le 4 \wedge 0 \le x_2 \le 2$

Condition: $(x_1 - 0) + (x_2 - 0) = -(x_1 - 4) - (x_2 - 2)$

$\Leftrightarrow\ x_1 + x_2 = -x_1 + 4 - x_2 + 2$

$\Leftrightarrow\ 2x_1 + 2x_2 = 6$

$\Leftrightarrow\ x_2 = -x_1 + 3$

$\mathbb{L}_5 = \{(x_1, x_2) \in \mathbb{R}^2 |\ x_2 = -x_1 + 3 \wedge 0 \le x_1 \le 4 \wedge 0 \le x_2 \le 2\}$

The graph is a line segment with slope - 1 passing (0|3)

Region 6: $x_1 > 4 \wedge 0 \leq x_2 \leq 2 \quad \Rightarrow$ no solution

Region 7: $x_1 < 0 \wedge x_2 < 0 \quad \Rightarrow$ no solution

Region 8: $0 \leq x_1 \leq 4 \wedge x_2 < 0$

Condition: $(x_1 - 0) - (x_2 - 0) = - (x_1 - 4) - (x_2 - 2)$

$\Leftrightarrow \quad x_1 - x_2 = - x_1 - x_2 + 6$

$\Leftrightarrow \quad 2\, x_1 = 6$

$\Leftrightarrow \quad x_1 = 3 \wedge x_2 < 0$

$\mathbb{L}_8 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 3 \wedge x_2 < 0\}.$

The graph is a parallel to the x_2-axis with distance 3.

Region 9: $x_1 > 4 \wedge x_2 < 0 \quad \Rightarrow$ no solution

Numerical example c): Ex_1 (0/0), Ex_2 (4/4), both coordinates are different.

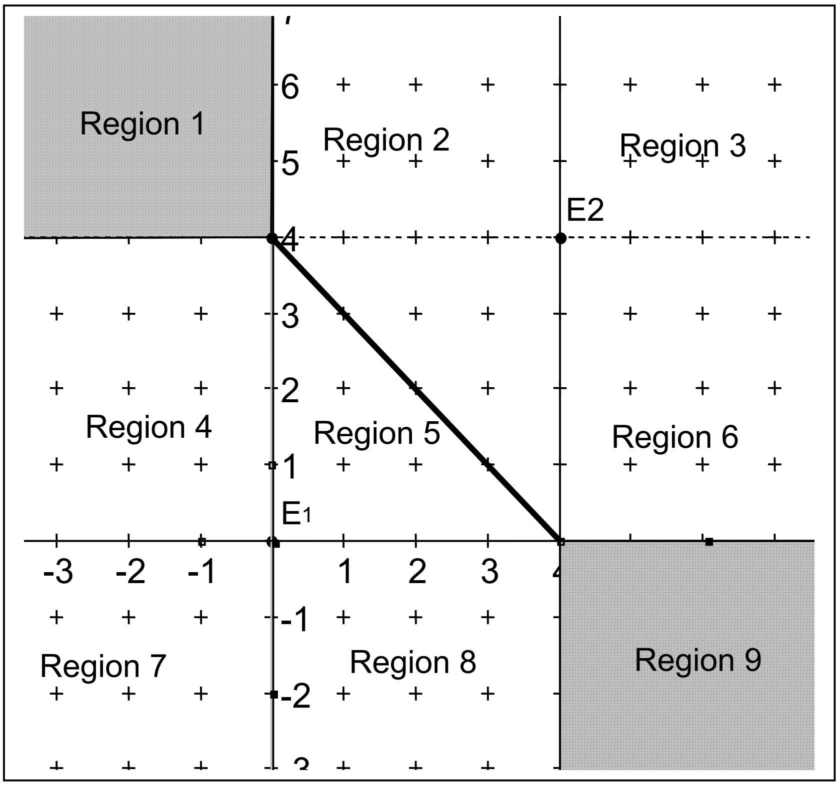


Figure 10: Perpendicular bisector for numerical example c). In the regions 1 and 9 all points are contained in the set of solutions.

There are solutions in Region 1, Region 5, Region 9

Region 1: $x_1 < 0 \wedge x_2 > 4$

Condition: $- (x_1 - 0) + (x_2 - 0) = - (x_1 - 4) + (x_2 - 4)$

$\Leftrightarrow \quad - x_1 + x_2 = - x_1 + 4 + x_2 - 4$

$\Leftrightarrow \quad 0 = 0 \wedge x_2 > 4$

$\mathbb{L}_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 < 0 \wedge x_2 > 4 \}$

The whole region is part of the solution set!

Region 5: $0 \leq x_1 \leq 4 \wedge 0 \leq x_2 \leq 4$

Condition: $(x_1 - 0) + (x_2 - 0) = -(x_1 - 4) - (x_2 - 4)$

$\Leftrightarrow x_1 + x_2 = -x_1 + 4 - x_2 + 4$

$\Leftrightarrow 2x_1 + 2x_2 = 8$

$\Leftrightarrow x_2 = -x_1 + 4$

$\mathbb{L}_5 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = -x_1 + 4, 0 \leq x_1 \leq 4 \wedge 0 \leq x_2 \leq 4\}$

The graph is a line segment in region 5 with slope – 1 and the x_2-crossing at 4

Region 9: $x_1 > 4 \wedge x_2 < 0$

Condition: $(x_1 - 0) - (x_2 - 0) = (x_1 - 4) - (x_2 - 4)$

$\Leftrightarrow x_1 - x_2 = x_1 - 4 - x_2 + 4$

$\Leftrightarrow 0 = 0$

$\mathbb{L}_9 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 4 \wedge x_2 < 0\}$

The whole region is part of the solution set.

For the remaining regions there are no pairs x_1, x_2 satisfying the conditions.

Generalization of the results:

Case 1: $a_{12} = a_{22} \wedge a_{11} \neq a_{21}$ The perpendicular bisector consists of a vertical line.

Case 2: $a_{11} = a_{21} \wedge a_{12} \neq a_{22}$ The perpendicular bisector consists of a horizontal line.

Case 3: $a_{11} \neq a_{21} \wedge a_{12} \neq a_{22}$ with

Case 3a) $|a_{11} - a_{21}| > |a_{12} - a_{22}|$

The perpendicular bisector consists of two vertical half lines connected by a diagonal line segment

Case 3b) $|a_{11} - a_{21}| < |a_{12} - a_{22}|$

The perpendicular bisector consists of two horizontal half lines connected by a diagonal line segment

Case 3c) $|a_{11} - a_{21}| = |a_{12} - a_{22}|$

The perpendicular bisector consists of two quadrants and a diagonal connecting line segment

Case 4: (degenerate case) $a_{11} = a_{21} \wedge a_{12} = a_{22}$

The perpendicular bisector is the whole plain.

5. Solving location problems using discussion of differentiable functions

5.1 The problem of a central warehouse with respect to ℓ_2^2 - distances

Numerical example: Small warehouses are located in Ex_1 (1 | 1), Ex_2 (1 | 4), Ex_3 (2 | 1), Ex_4 (4 | 1), Ex_5 (4 | 4).

The number of trips for a week to the small warehouses Ex_i is w_i. Hence let us consider in the following computations the weights $w_1 = 2$, $w_2 = 1$, $w_3 = 1$, $w_4 = 2$, $w_5 = 4$. Without any further information we assume that the distance is an Euclidean one (ℓ_2 – metric, see section 2.1). The solution of our problem consists in minimizing the sum of weighted distances:

$$d(Ex_m, X) = \sqrt{(a_{m1} - x_1)^2 + (a_{m2} - x_2)^2} = \ell_2(Ex_m, X)$$

$$S(X) = \sum_{m=1}^{M} w_m \sqrt{(a_{m1} - x_1)^2 + (a_{m2} - x_2)^2}$$

First we look at the sum of the quadratic distance and minimize these.[13] We thus consider:

$$f(X) = \sum_{m=1}^{M} w_m \left((a_{m1} - x_1)^2 + (a_{m2} - x_2)^2 \right)$$

For our numerical example this yields:

$$f(X) = 2 \cdot \left((1-x_1)^2 + (1-x_2)^2\right) + 1 \cdot \left((1-x_1)^2 + (4-x_2)^2\right)$$
$$+ 1 \cdot \left((2-x_1)^2 + (1-x_2)^2\right) + 2 \cdot \left((4-x_1)^2 + (1-x_2)^2\right)$$
$$+ 4 \cdot \left((4-x_1)^2 + (4-x_2)^2\right)$$

$$f(X) = 3 \cdot (1-x_1)^2 + 1 \cdot (2-x_1)^2 + 6 \cdot (4-x_1)^2 + 5 \cdot (1-x_2)^2 + 5 \cdot (4-x_2)^2$$

$$f(X) = f_1(x_1) + f_2(x_2)$$

The sum can be partitioned into the sum of two functions depending on a single variable x_1 and x_2 respectively.

In order to minimize f (X) we have to minimize $f_1(x_1)$ and $f_2(x_2)$. It is thus sufficient to get x_i such that $f'_i(x_i) = 0 \wedge f_i''(x_i) > 0$, i =1,2.

For the numerical example yields:

$$f_1(x_1) = 10 x_1^2 - 58 x_1 + 103 \Rightarrow f_1'(x_1) = 20 x_1 - 58 \quad \wedge \quad f_1''(x_1) = 20 > 0$$

$$f_2(x_2) = 10 x_2^2 - 50 x_2 + 85 \Rightarrow f_2'(x_2) = 20 x_2 - 50 \quad \wedge \quad f_2''(x_2) = 20 > 0$$

$$f_1'(x_1) = 0 \Rightarrow x_1 = 2{,}9 \quad ; \quad f_2'(x_2) = 0 \Rightarrow x_2 = 2{,}5$$

---

13 The associated quadratic function does not yield the same solutions since the squaring of the distance puts a weight on points which are further away. On the other hand $[S(X)]^2$ is on $\mathbb{R}$ differentiable whereas S (X) is not differentiable in (a_{m1}, a_{m2}).

Hence we have found: The central warehouse is located at X* (2,9│2,5) using ℓ_2^2 - distances.

If we want to solve the problem using parameters instead of specific numbers we get an explicit formula allowing us to compute the optimal location. Before we get this formula we want to go through another specific example:

Given:

Ex_1 (1,5│18), Ex_2 (18,6│21), Ex_3 (22,5│15), Ex_4 (19,2│3), Ex_5 (0│0)

$w_1 = 4$, $w_2 = 8$, $w_3 = 6$, $w_4 = 4$, $w_5 = 5$

Compute the optimal location with respect to ℓ_2^2 – distance!

Solution: $f(x) = 4 \cdot (1{,}5 - x_1)^2 + 8\ (18{,}6 - x_1)^2 + 6 \cdot (22{,}5 - x_1)^2 + 4 \cdot (19{,}2 - x_1)^2$
$+ 5 \cdot (0 - x_1)^2 + 4 \cdot (18 - x_2)^2 + 8 \cdot (21 - x_2)^2 + 6 \cdot (15 - x_2)^2$
$+ 4 \cdot (3 - x_2)^2 + 5 \cdot (0 - x_2)^2$

with $f_1'(x_1) = 0 \Leftrightarrow 8 \cdot (1{,}5 - x_1) \cdot (-1) + 16 \cdot (18{,}6 - x_1) \cdot (-1) + 12 \cdot (22{,}5 - x_1) \cdot (-1)$
$+ 8 \cdot (19{,}2 - x_1) \cdot (-1) + 10 \cdot (0 - x_1) \cdot (-1) = 0$

and $f_2'(x_2) = 0 \Leftrightarrow 8 \cdot (18 - x_2) \cdot (-1) + 16 \cdot (21 - x_2) \cdot (-1) + 12 \cdot (15 - x_2) \cdot (-1)$
$+ 8 \cdot (3 - x_2) \cdot (-1) + 10 \cdot (0 - x_2) \cdot (-1)$

Hence we get the solution X* (13,58 │ 12,67), $f_i''(x_i)=54 > 0$, i=1,2.

With these two examples we realize how to find the **general solution:**

$$f(x) = \sum_{m=1}^{M} w_m\left((a_{m1} - x_1)^2 + (a_{m2} - x_2)^2\right)$$

$$= \sum_{m=1}^{M} w_m (a_{m1} - x_1)^2 + \sum_{m=1}^{M} (a_{m2} - x_2)^2 = f(x_1) + f(x_2)$$

$$f_1'(x_1) = \sum_{m=1}^{M} w_m \cdot 2 \cdot (a_{m1} - x_1) \cdot (-1) = 2 \cdot \sum_{m=1}^{M} w_m x_1 - 2 \cdot \sum_{m=1}^{M} w_m a_{m1}$$

$f_1''(x_1) = 2 \cdot \sum_{m=1}^{M} w_m > 0$, since all weights are positive.

$\Rightarrow$ $f_1(x_1)$ has a minimum for positive weights if the x_1-value is computed as by

$$f'_1(x_1) = 0: \quad x_1 = \frac{\sum_{m=1}^{M} w_m a_{m1}}{\sum_{m=1}^{M} w_m}.$$

Analogously we get from $f_2'(x_2) = 0$ the zero with $x_2 = \frac{\sum_{m=1}^{M} w_m a_{m2}}{\sum_{m=1}^{M} w_m}$

If we insert into this formula the specific value of the exercise we see:

$$x_1 = \frac{w_1 \cdot a_{11} + w_2 \cdot a_{21} + w_3 \cdot a_{31} + w_4 \cdot a_{41} + w_5 \cdot a_{51}}{w_1 + w_2 + w_3 + w_4 + w_5}$$

$$= \frac{4 \cdot 1{,}5 + 8 \cdot 18{,}6 + 6 \cdot 22{,}5 + 4 \cdot 19{,}2 + 5 \cdot 0}{4 + 8 + 6 + 4 + 5} = \frac{366{,}6}{27} = 13{,}5\overline{7}$$

$$x_2 = \frac{w_1 \cdot a_{12} + w_2 \cdot a_{22} + w_3 \cdot a_{32} + w_4 \cdot a_{42} + w_5 \cdot a_{52}}{w_1 + w_2 + w_3 + w_4 + w_5}$$

$$= \frac{4 \cdot 18 + 8 \cdot 21 + 6 \cdot 15 + 4 \cdot 3 + 5 \cdot 0}{4 + 8 + 6 + 4 + 5} = \frac{342}{27} = 12{,}\overline{6}$$

Up to now we have considered the square of the Euclidean distance.
What happens if we use „normal“ Euclidean distance?

We are looking for the minimum of

$$f(x) = \sum_{m=1}^{M} w_m \cdot \sqrt{(a_{m1} - x_1)^2 + (a_{m2} - x_2)^2} .$$

Here we encounter the following difficulties.

1. f(x) cannot be written as a sum of two functions each of which is only depending on a single variable.
2. One has to consider partial derivatives depending on x_1 and. x_2 respectively:
   $$\frac{\partial f}{\partial x_i} = \sum_{m=1}^{M} w_m \frac{(a_{mi} - x_i) \cdot (-1)}{\sqrt{(a_{m1} - x_1)^2 + (a_{m2} - x_2)^2}}, \quad i = 1,2$$
3. The partial derivatives do not exist in the given locations Ex_m, m = 1,.., M.
4. There exists tests which allow us to find out whether one of the existing locations is optimal. If this is not the case one has to use an iterative procedure to determine the zeros of the related partial derivatives.

Further details can be found in Hamacher[14]:.

[14] aus: Horst W. Hamacher: Mathematische Lösungsverfahren für planare Standortprobleme, Vieweg Verlag Braunschweig/Wiesbaden 1995

5.2 Examples in which f (x) is only piecewise differentiable: Central warehouse problem with ℓ_1-distance

In many cases neither the ℓ_2- nor the ℓ_2^2-metric is a suitable tool to model the distances.

For instance the distance in a street network like Manhattan or Mannheim or the movement of a robot arm or the distance between points in a high rise warehouse ask for the rectangular distance (see section 4).

Since a circle in this metric is angular, has corners and is built by line segments the same is true for the set of points which have the same distance from a given point: They are lying on an angular circle centred at a fixed point (a_{m1}, a_{m2}) and satisfy the condition

$$|a_{m1} - x_1| + |a_{m2} - x_2| = r$$

For the numerical example $a_{m1} = 4$, $a_{m2} = 2$ and $r = 3$ the circle consists of

$\mathbb{L} = \{(x_1, x_2) \mid |4 - x_1| + |2 - x_2| = 3\}$ given by the line segments

$x_2 = -x_1 + 3$ for $x_1 \leq 4 \wedge x_2 \leq 2$ and $x_2 = x_1 - 5$ for $x_1 > 4 \wedge x_2 \leq 2$

$x_2 = -x_1 + 9$ for $x_1 > 4 \wedge x_2 > 2$ and $x_2 = x_1 + 1$ for $x_1 \leq 4 \wedge x_2 > 2$

Consequently the sum of distances in the direction of coordinate axis to any point on the circle is 3 length units.

If we use the ℓ_1 - norm for our location problem (central warehouse from chapter 4) we have to calculate the minimum of the following function f:

$$f(X) = \sum_{m=1}^{M} w_m (|a_{m1} - x_1| + |a_{m2} - x_2|)$$

$$= \sum_{m=1}^{M} w_m |a_{m1} - x_1| + \sum_{m=1}^{M} w_m |a_{m2} - x_2| = f_1 (x_1) + f_2 (x_2)$$

Again the function depending on two variables can be written as the sum of two functions each of which depending only on a single variable which we can investigate separately.

As a **numerical example** for locations and weights we choose again the same values as in section 5.1.

Ex_1 (1 | 1), Ex_2 (1 | 4), Ex_3 (2 | 1), Ex_4 (4 | 1), Ex_5 (4 | 4) with the weights $w_1 = 2$, $w_2 = 1$, $w_3 = 1$, $w_4 = 2$ and $w_5 = 4$

Then we get:

$$f_1 (x_1) = 2 \cdot | 1 - x_1 | + 1 \cdot | 1 - x_1 | + 1 \cdot | 2 - x_1 | + 2 \cdot | 4 - x_1 | + 4 \cdot | 4 - x_1 |$$

$$= 3 \cdot | 1 - x_1 | + | 2 - x_1 | + 6 \cdot | 4 - x_1 |$$

Due to the absolute value we have to do a case analysis resulting in two piecewise linear functions $f_1(x_1)$, $f_2(x_2)$ changing their slope in a corner. Hence these functions are continuous but not differentiable in all points.

Case analysis for $f_1(x_1)$:

1. Case $x_1 \leq 1$: $f_1(x_1) = 3 \cdot (1 - x_1) + (2 - x_1) + 6 \cdot (4 - x_1) = 29 - 10\,x_1$

2. Case $1 \leq x_1 \leq 2$: $f_1(x_1) = 3 \cdot (x_1 - 1) + (2 - x_1) + 6 \cdot (4 - x_1) = 23 - 4\,x_1$

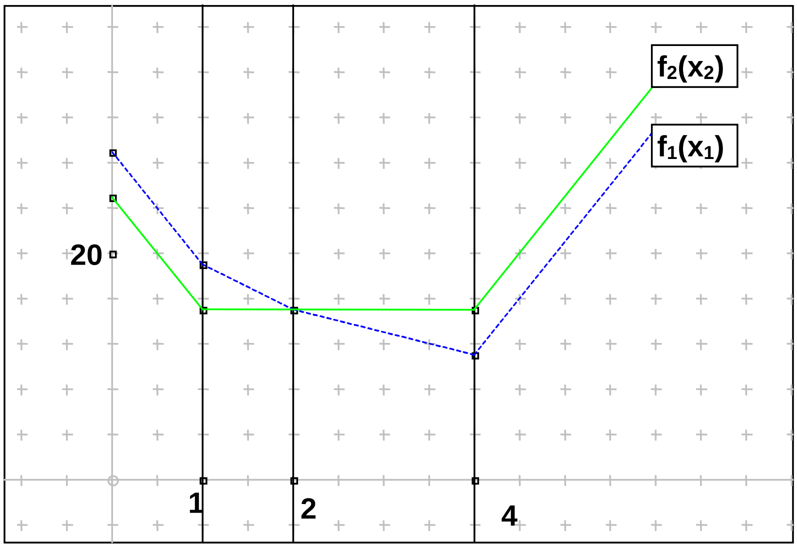
3. Case $2 \leq x_1 \leq 4$: $f_1(x_1) = 3 \cdot (x_1 - 1) + (x_1 - 2) + 6 \cdot (4 - x_1) = 19 - 2\,x_1$

4. Case $x_1 \geq 4$: $f_1(x_1) = 3 \cdot (x_1 - 1) + (x_1 - 2) + 6 \cdot (x_1 - 4) = -29 + 10\,x_1$

If we draw $f_1(x_1)$ we observe the following (see figure 11):

1. $f_1(x_1)$ is piecewise linear. The graph is on each of the intervals a line segment.
2. f_1 has corners in which the function is continuous but not differentiable.
3. The minimum lies in a corner in which the slope changes sign.
4. The x_1 – coordinate of the optimal location is $x_1 = 4$.

Figure 11:
Piecewise differentiable functions $f_1(x_1)$ and $f_2(x_2)$



Case analysis for $f_2(x_2) = 2 \cdot |1 - x_2| + 1 \cdot |4 - x_2| + 1 \cdot |1 - x_2| + 2 \cdot |1 - x_2| + 4 \cdot |4 - x_2|$

$$= 5 \cdot |1 - x_2| + 5 \cdot |4 - x_2|$$

1. Case $x_2 \leq 1$ $f_2(x_2) = 5 \cdot (1 - x_2) + 5 \cdot (4 - x_2) = 25 - 10\,x_2$

2. Case $1 \leq x_2 \leq 4$ $f_2(x_2) = -5 \cdot (1 - x_2) + 5 \cdot (4 - x_2) = 15$

3. Case $4 < x_2$ $f_2(x_2) = -5 \cdot (1 - x_2) - 5 \cdot (4 - x_2) = -25 + 10\,x_2$

Using the drawing of $f_2(x_2)$ we can read of the following results (figure 11):

1. $f_2(x_2)$ is piecewise linear since the graph is on each of the intervals a line segment.

2. f_2 has corners in which the function is continuous but not differentiable.
3. At least one minimum lies in one of those corners.
4. The x_2-coordinate of the optimal location lies in the interval [1; 4], i.e. all points of the line $\overline{AB}$ with A (4 | 1) , B (4 | 4) are minima of $f_2(x_2)$.

The minima are the only points in which we can draw a horizontal line „below the function“ which intersects the function. (This corresponds to a tangent parallel to the x_1-axis in the minimum of a twice differentiable function.) Here the criterion for a minimum is the change of the slope of the function from a negative to a positive value.

The **general question** is as follows:

How do we find the minimum of a function $g(x)=\sum_{m=1}^{M} w_m |a_m - x|$, that means how do we find the corner in which is a change of sign in the slope of the piecewise linear function?

We assume, that in g(x) equal terms have already been collected such that we have $a_1 < a_2 <.. < a_n$ $(n \leq M)$. Now we consider the function on the interval $[a_q; a_{q+1}]$. For m > q+1 we get $|a_m - x| = (a_m - x)$, for $m \leq q$ we assume $|a_m - x| = x - a_m$, hence g(x) can be written as the sum of the two terms:

$$g(x)=\sum_{m=1}^{q} w_m (x - a_m) + \sum_{m=q+1}^{M} w_m (a_m - x) = (\sum_{m=1}^{q} w_m - \sum_{m=q+1}^{M} w_m)\, x + (\sum_{m=q+1}^{M} w_m a_m - \sum_{m=1}^{q} w_m a_m)$$

$=c_q x + b_q$; g (x) is hence a linear function on $[a_q; a_{q+1}]$.

We have thus to find a_q in which the slope of g(x) is changing sign.

The slope "left of a_q" (i.e. for $a_{q-1}<x< a_q$) is denoted with $g^-(a_q)=\sum_{m=1}^{q-1} w_m - \sum_{m=q}^{M} w_m$,

the slope "right of a_q" (i.e. for $a_q<x< a_{q+1}$) is denoted with $g^+(a_q)=\sum_{m=1}^{q} w_m - \sum_{m=q+1}^{M} w_m$.

Hence we are looking for a_q for which $g^-(a_q) \leq 0 \wedge g^+(a_q) \geq 0$ holds.

If we apply these ideas to the example above

$f_1(x_1) = 3\cdot| 1 - x_1 | + | 2 - x_1 | + 6\cdot| 4 - x_1 |$ we obtain:

Slope left of a_1 : $g^-(1) = \sum_{m=1}^{0} w_m - \sum_{m=1}^{3} w_m = -10$,

right of a_1 : $g^+(1) = \sum_{m=1}^{1} w_m - \sum_{m=2}^{3} w_m = 3 - 7 = -4$.

Slope left of a_2: $g^-(2) = g^+(1) = -4$,

right of a_2: $g^+(2) = \sum_{m=1}^{2} w_m - \sum_{m=3}^{3} w_m = 4 - 6 = -2$.

Slope left of a_3: $g^-(4) = g^+(2) = -2$, right of a_3: $g^+(4)=\sum_{m=1}^{3} w_m - \sum_{m=4}^{3} w_m = 10$,

hence the change of sign in the slope is identified at $a_3 = 4$ (see figure 11).

By changing the upper summation index we add each time twice the weight w_q by going from a_q to a_{q+1} since it appears in the additive term and disappears in the sum which needs to be subtracted. (See example where $w_1+w_2+w_3 = 10$ and hence $g^-(1)= -10$, $-10 + 2 \cdot 3 = -4 = g^-(2)$ and so on.)
In other words one looks for the smallest q for which $g^+(q) \geq 0$ holds, i.e. where a change of sign occurs.

$$\sum_{m=1}^{q} w_m \geq \sum_{m=q+1}^{M} w_m \qquad | + \sum_{m=1}^{q} w_m$$

$$2 \cdot \sum_{m=1}^{q} w_m \geq \sum_{m=1}^{M} w_m \qquad | :2 \quad \Rightarrow \sum_{m=1}^{q} w_m \geq \frac{1}{2} \cdot \sum_{m=1}^{M} w_m \quad (*)$$

If the inequality (*) is satisfied with a strict inequality then a_q is the unique minimizer. (See solution of $f_1(x_1)$: the median is 5 and $\sum_{m=1}^{3} w_m = 10$ hence a_3 is the unique minimizer of the function.)
If the inequality (*) is satisfied however with equality sign, then all values between a_q and a_{q+1} minimize the function. (See $f_2(x_2)$: the median value is 5 and $\sum_{m=1}^{1} w_m = 5$.)
In summary the following four cases can occur:

Case 1: The solution for x_1 and x_2 is unique, then the optimal location of the location problem is given by point $P(x_1,x_2)$.
Case 2: The solution for x_1 is unique and for x_2 is not unique, then the solution set is represented by a line segment which is parallel to the x_2-axis.
Case 3: The solution for x_1 is not unique and for x_2 is unique, then the solution set is a line segment which is parallel to the x_1-axis.
Case 4: The solution is not unique neither for x_1 nor for x_2. Then the graph of the optimal solution set is not given by a finite rectangle in $\mathbb{R}^2$.

6. Appendix - Proofs relating to the Fermat point

In case there is enough time for the teaching unit „locational planning“ or it is taught to a class of the students which are very talented in mathematics we would like to present two possibilities to formally proof the Fermat point.

6.1 Proof 1 related to the Fermat point

In this proof we will show the following properties of the Fermat point:

- the points looked for can be found by triangles sitting on the sides of the given triangle and the diagonals intersect in one point.
- only this point minimizes the sum of the distance to the corner points.

The proof uses the theorem of the inscribed tetragon and is only valid for triangles with angles less than 120^o.

Assumption:

1. $\triangle$ABC is an arbitrary triangle with angles less than 120°.
2. On each of the triangle sides we have constructed equilateral triangles $\triangle$BCG, $\triangle$ACD, $\triangle$ABE.
3. We have drawn the lines $\overline{CE}$, $\overline{AG}$ und $\overline{BD}$ (cf. fig.12a).

Claim:

1. $\overline{CE}$, $\overline{AG}$ and $\overline{BD}$ intersect in a common intersection point Q.
2. $|\overline{AQ}|$ + $|\overline{BQ}|$ + $|\overline{CQ}|$ < $|\overline{AR}|$ + $|\overline{BR}|$ + $|\overline{CR}|$ yields where R is any point inside the triangle ABC with R≠Q. (In this case Q is a point with the minimum sum of distances to the corner points of the triangle, hence the Fermat point.)

Proof of Claim 1:

The proof is relatively difficult and constructs first the Fermat point in a different way namely as intersection point of the circumcircles of the equilateral triangles (part I), and then shows that this point is lying on the „diagonal lines“ (part II).

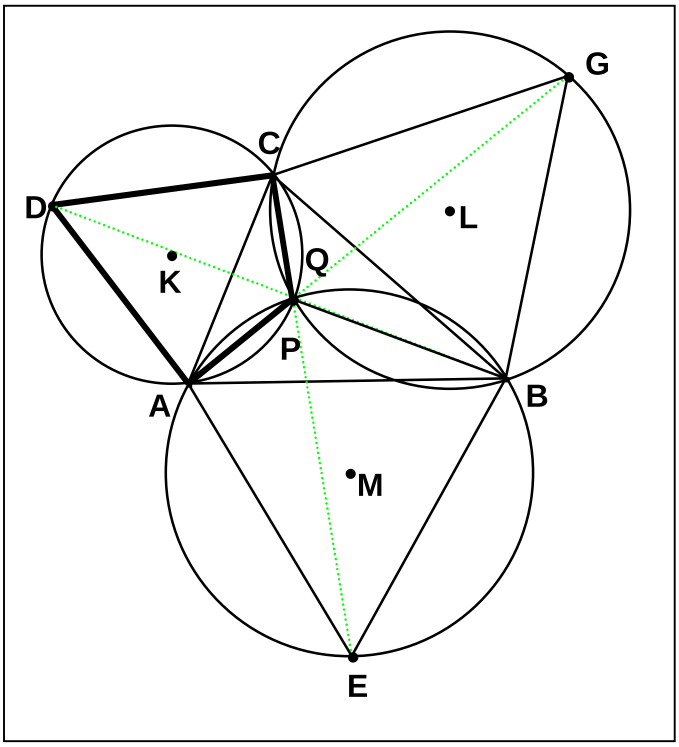
If one wants to use this proof in mathematics classes parts should have been discussed before.

Part 1: To each of the equilateral triangles we construct a circumcircle with its corresponding centre (fig. 12a):

The circumcentre of triangle Δ ACD is K, of Δ BCG is L and of Δ ABE is M.

The circumcircles of Δ ADC and Δ ABE intersect in point A and an additional point P.

Fig 12a:
Situation of the proof of claim 1.



Lets consider now the quadrangle DAPC. It is an inscribed tetragon of the circle with centre K. Angle ∢ ADC is 60°. In an inscribed tetragon the sum of two opposite angles is 180°. Therefore ∢ CPA = 120° yields. APBE is an inscribed tetragon, too, and, in the same way, ∢ BEA = 60° (by assumption) and therefore ∢ APB = 120° yields.

From ∢ APB + ∢ CPA + ∢ BPC = 360° one concludes ∢ BPC = 120°.

The contraposition of the theorem of the inscribed tetragon holds (only that quadrangle has a circumcircle, by which the sum of two opposite angles is 180°). Therefore with the assumption ∢ CGB = 60° and the fact ∢ BPC = 120° it follows, that BQCG is an inscribed tetragon with centre L. C and Q are the intersection points of the circumcircles of Δ ADC and Δ CGB. Q coincides with P. (i.e. the third circumcircle intersects with the other two circles in one point, the point P.)

<u>Result part I</u>: The circumcircles of the equilateral triangles sitting on the sides of the given ΔABC intersect in one point.

What is missing up to now? The proof, that this point is the intersection point of the line segments $\overline{AG}$, $\overline{BD}$, $\overline{CE}$, i.e. it yields: $P \in \overline{AG}$, $P \in \overline{BD}$ and $P \in \overline{CE}$.

<u>Part II:</u> If one connects P with the points A, B, C, D, E and G (cf. Fig. 12b), so the following angles are equal: ∢ DPA = ∢ CPD = ∢ GPC = ∢ BPG = ∢ EPB = ∢ APE = 60° (Proof look see below). It follows, that the points A, P, G as well as D, P, B and C, P, E are lying on a straight line.

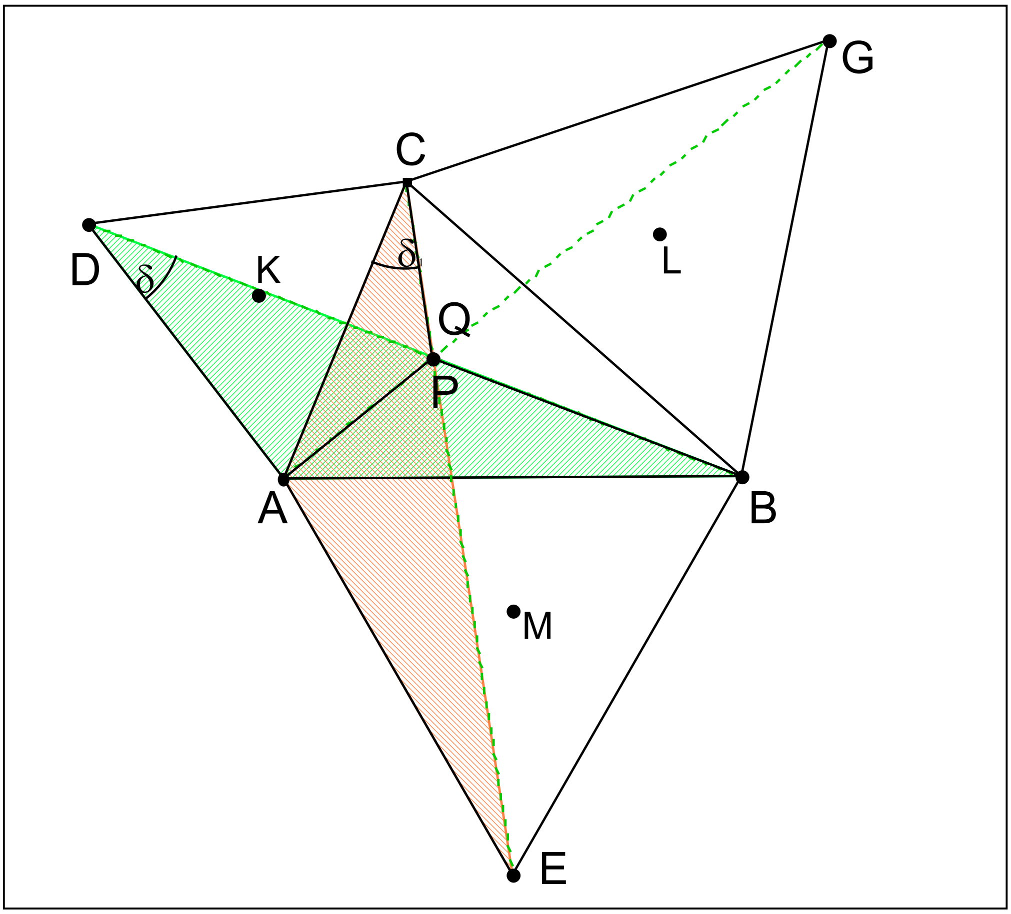


Fig. 12b : Sketch 1st Proof, Part II

<u>Proof:</u> ∢ DPA =∢ CPD = ∢ GPC = ∢ BPG = ∢ EPB = ∢ APE = 60°

By construction of the equilateral triangles sitting on the original triangle sides (Fig. 12b) the congruence identity of the following triangles is shown with the help of the congruence theorem sas (side-angle-side): $\Delta CAE \cong \Delta DAB$ („is congruent").

1. Equal side length: $|\overline{AC}| = |\overline{AD}|$, because Δ ACD is by assumption equilateral.

2. Equal side length: $|\overline{AB}| = |\overline{AE}|$, because Δ ABE is by assumption equilateral.

3. The angles ∢ EAC and ∢ BAD are equal, for:

   ∢ EAC =∢ BAC + ∢ EAB=∢ BAC + 60°=∢ BAC + ∢ CAD=∢ BAD.

   The triangles are congruent and so the equality of ∢ ADB = ∢ ACE = δ yields. One can write ∢ CPD as ∢ CPD = 180°- (60°+δ) - (60°-δ)= 60°.

With other triangles one can repeat the proof for the remaining angles. On the other hand with the help of the above results related to the inscribed tetragon (cf. fig. 12a) one can conclude with ∢ CPA = 120° that ∢ DPA = 60° yields.

As vertical angle there holds $\sphericalangle$ EPB = $\sphericalangle$ BPG = 60°, with $\sphericalangle$ APB = 120° it follows $\sphericalangle$ APE = $\sphericalangle$ APB - $\sphericalangle$ EPB = 120° - 60° = 60°, and with $\sphericalangle$ BPC = 120° there follows $\sphericalangle$ GPC = $\sphericalangle$ BPC - $\sphericalangle$ BPG = 120° - 60° = 60°.

Now claim 1 is proved: $\overline{AG}$, $\overline{CE}$ and $\overline{BD}$ intersect in one point. This point is the intersection point of the circumcircles of the additionally constructed equilateral triangles sitting on the sides of the given triangle ABC or the intersection point of the connection lines $\overline{AG}$, $\overline{BD}$, $\overline{CE}$. Additionally we got the result that these connection lines intersect under a 120° angle: $\sphericalangle$ APB= $\sphericalangle$ BPC= $\sphericalangle$ CPA =120°.

Proof of claim 2:

At first we prove the lemma holding for equilateral triangles:

The sum of the distances of an arbitrary point inside an equilateral triangle to the sides of that triangle is constant (Fig. 13).

This interesting and perplexed circumstances can be proved in other parts of classes.

Proof of the lemma:

Construction:

a) Choose a point out of the equilateral triangle ABC and draw the perpendiculars to all sides from that point. The feet of the perpendiculars are called R, Q, S and it yields: $\overline{PR} \perp \overline{AB}$, $\overline{PQ} \perp \overline{BC}$ and $\overline{PS} \perp \overline{AC}$.

b) Construct the midperpendicular (i.e. the height in the equilateral triangle) on $\overline{AB}$ with foot D; it holds $\overline{AB} \perp \overline{CD}$.

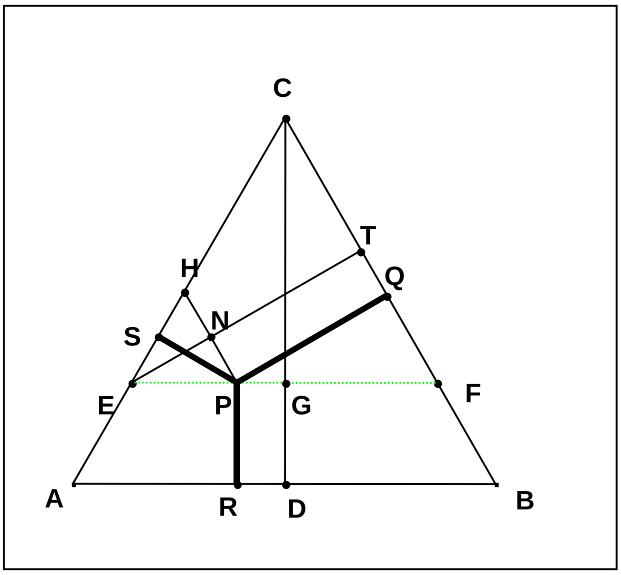
c) Draw the parallel to $\overline{AB}$ through P, which intersects $\overline{CD}$ in G, $\overline{AC}$ in E and $\overline{BC}$ in F! Than by construction it holds $|\overline{PR}| = |\overline{GD}|$.

d) The foot of the perpendicular to $\overline{BC}$ through E say T. The resulting triangle CEF is equilateral (figure of the intercept theorems with Δ ABC or statement with corresponding angle with $\overline{AB} \parallel \overline{EF}$) and its height is $\overline{CG}$, $\overline{ET}$ respectively. Because in a equilateral triangle all heights have the same length, it holds $|\overline{CG}| = |\overline{ET}|$.

e) Draw a parallel to $\overline{CB}$ through P, which intersects $\overline{AC}$ in H and $\overline{ET}$ in N: $|\overline{NT}| = |\overline{PQ}|$

f) Because the triangle EHP is equilateral (reasoning like d), the heights in this triangle have also the same length: $|\overline{SP}| = |\overline{EN}|$

Figure 13:
Sketch to prove the lemma for equilateral triangles:
The sum of distances of an arbitrary point inside the triangle from the sides is constant



Thus holds: $\left|\overline{PS}\right| + \left|\overline{PQ}\right| = \left|\overline{EN}\right| + \left|\overline{NT}\right|$ (following e, f)

$= \left|\overline{ET}\right| = \left|\overline{CG}\right|$ (following d).

With $\left|\overline{PR}\right| = \left|\overline{GD}\right|$ (it follows from c): The distance of the point from the side is:

$\left|\overline{PS}\right| + \left|\overline{PQ}\right| + \left|\overline{PR}\right| = \left|\overline{CG}\right| + \left|\overline{GD}\right| = \left|\overline{CD}\right|$. Consequently it is for all equilateral triangles ABC a constant, the length of the height in this equilateral triangle.

To perform <u>claim 2</u> we have to use an equilateral triangle to apply the lemma. On the other hand we don't need the distances to the sides but to the sum of distances to the points of the given triangle. This is combined in the following construction:

In claim 1 we got the result that the connection lines from the Fermat point to the points of the given triangle include an angle with 120°. Therefore in the following construction M is the Fermat point in the triangle ABC (Fig. 14):

a) In triangle ABC (no angle larger than 120°) let M be the point inside the triangle with $\sphericalangle$ CMA = $\sphericalangle$ AMB = $\sphericalangle$ BMC = 120°

b) Draw in A, B, C respectively the perpendiculars to $\overline{AM}$, $\overline{BM}$, $\overline{CM}$ intersecting in P, Q, R.

c) Δ PQR is equilateral due to the following facts:

1.) $\sphericalangle$ CQB is 60°, because in quadrangle CMBQ $\sphericalangle$ MCQ and $\sphericalangle$ MBQ are rectangles by construction and $\sphericalangle$ CMB is 120° by initial situation.

2.) $\sphericalangle$ ARC = 60°, because in quadrangle CRAM $\sphericalangle$ MAR = 90° = $\sphericalangle$ RCM holds by construction and $\sphericalangle$ CMA = 120° holds by assumption.

3.) With sum of angles in a triangle equals 180° for $\sphericalangle$ BPA yields 60°.

d) Let D be another point in triangle ABC, $|\overline{DA}|$, $|\overline{DB}|$ and $|\overline{DC}|$ be its distances to the corners A, B, C.

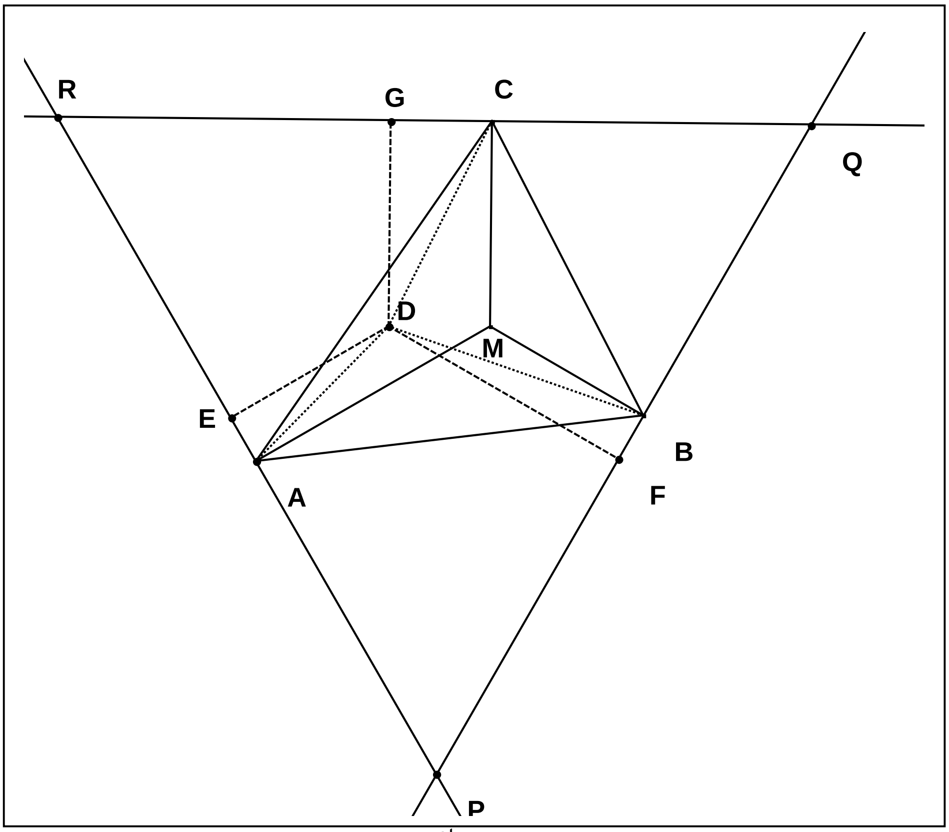


Figure 14: Situation 1st proof, part 2

Now we have to prove: The sum of the distances from M to the corners is less than the distance from D to the corners:

$$|\overline{MA}| + |\overline{MB}| + |\overline{MC}| < |\overline{DA}| + |\overline{DB}| + |\overline{DC}|$$

Proof:

(*) $|\overline{MA}| + |\overline{MB}| + |\overline{MC}| = |\overline{DE}| + |\overline{DF}| + |\overline{DG}|$ (with lemma and $\overline{DE}$, $\overline{DF}$, $\overline{DG}$ are the perpendicular to the sides of the triangle PQR.

If we substitute in our equation (*) on the right side the perpendiculars $\overline{DE}$, $\overline{DF}$, $\overline{DG}$ with the direct connections from D to the points A, B, C, with

$$|\overline{DE}| + |\overline{DF}| + |\overline{DG}| < |\overline{DA}| + |\overline{DB}| + |\overline{DC}|$$

it holds $|\overline{MA}| + |\overline{MB}| + |\overline{MC}| < |\overline{DA}| + |\overline{DB}| + |\overline{DC}|$ qed.

The construction of the Fermat point results into a point inside the triangle if and only if there is no angle 120° or larger. In the case $\sphericalangle$ ACB = 120° the Fermat point is equal with point C. If there is an angle larger than 120° the intersection point of that construction lies outside the triangle and the vertex of this angle has the minimal sum of distances from the given triangle points A, B, C.

6.2 Second proof to the Fermat point

Given a triangle ABC with no angle larger than 120°. Wanted is that point P, for which the sum of distances to the corners is minimal.[15]

In the following we assume that all angles are less than 120°. The proof is shorter compared to the first one; it has the basic idea that the straight line is the shortest distance between a start an end point compared to all possible broken lines between these two points. It uses congruent triangles arisen by rotation.[16] By rotation the distances from P to the corners of the triangles are built into a broken line. The sum of distances from P to the corners of the triangle is minimal, if and only if the resulting broken line is a straight line.

Assumption:

1. Let P be an arbitrary point inside the triangle ABC.
2. D is outside of Δ ABC and Δ ADC is equilateral, i.e. it holds $|\overline{AC}| = |\overline{AD}| = |\overline{DC}|$ and $\measuredangle$ CAD = $\measuredangle$ DCA = $\measuredangle$ADC = 60°.

Claim:

If P is the point with minimal sum of distances then P lies on $\overline{BD}$ and it holds $|\overline{BD}| = |\overline{PA}| + |\overline{PB}| + |\overline{PC}|$.

Proof:

1. Choose P inside the triangle and rotate ΔAPC about A by 60° (Fig. 15), while C is mapped onto D. Let the image point from P be P' and $|\overline{CP}| = |\overline{DP'}|$.
2. ΔAPP' is equilateral, because $\measuredangle$PAP' = 60° and $|\overline{AP}| = |\overline{AP'}|$ because of the rotation $\Rightarrow |\overline{AP}| = |\overline{PP'}|$.
3. For the sum of the line segments it yields:

$$
\begin{aligned}
|\overline{AP}| + |\overline{BP}| + |\overline{CP}| &= |\overline{AP'}| + |\overline{BP}| + |\overline{DP'}| \text{ (from 1. and 2.)} \\
&= |\overline{PP'}| + |\overline{BP}| + |\overline{DP'}| \text{ (from 2.)} \\
&= |\overline{BP}| + |\overline{PP'}| + |\overline{P'D}| \geq |\overline{BD}|
\end{aligned}
$$

(equality holds if and only if P and P' are on $\overline{BD}$.)

The sum $|\overline{PP'}| + |\overline{BP}| + |\overline{DP'}|$ is minimal, if B, P, P' and D lie on a straight line and it yields $|\overline{AP}| + |\overline{BP}| + |\overline{CP}| = |\overline{BD}|$, which just is the shortest connection.

[15] The idea of this proof has its origin from the mathematical historian Joseph Ehrenfried Hofmann in 1929

[16] One can find this proof in an older German school book Lambacher-Schweizer: Geometrie 1, 1983 , Klett-Verlag, Stuttgart, ISBN 3-12-730500-1, pages 104-105. An application is given in that book: Three cities plan to build a new common water tower.

Conversely yields: If P and P' are on $\overline{BD}$ then the angles formed between the straight lines $\overline{PA}$, $\overline{PB}$ and $\overline{PC}$ are always 120°. This has to be proved:

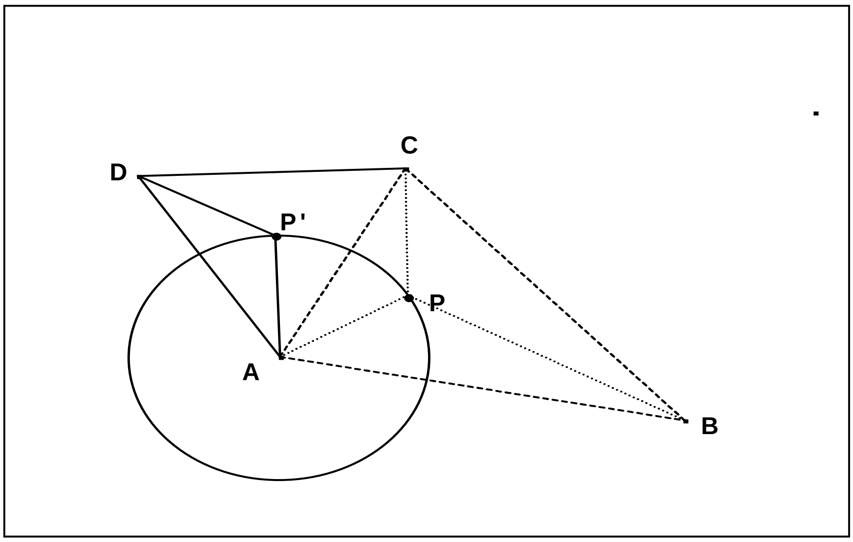


Figure 15: Construction to the 2nd proof of the Fermat point

Assumption: Figure and construction of P’ like above (conf. fig. 15).

The points P, P', B, D lie on a straight line:

Claim: $\sphericalangle CPA = \sphericalangle APB = \sphericalangle BPC = 120°$

Proof: 1. ΔAPP' is equilateral (conf. point 2 above) and $\sphericalangle P'PA = 60°$.

Thus yields: $\sphericalangle APB = 180° - \sphericalangle P'PA = 180° - 60° = 120°$.

2. $\sphericalangle CPA = \sphericalangle DP'A$ (point 1 above)

Thus yields: $\sphericalangle CPA = \sphericalangle DP'A = 180° - \sphericalangle AP'P = 180° - 60° = 120°$

3. Now holds: $\sphericalangle BPC = 360° - 120° - 120° = 120°$.

qed.