Beyond the Thouless energy

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The distribution and the correlations of the small eigenvalues of the Dirac operator are described by random matrix theory (RMT) up to the Thouless energy $E_c \propto 1/\sqrt{V}$, where V is the physical volume. For somewhat larger energies, the same quantities can be described by chiral perturbation theory (chPT). For most quantities there is an intermediate energy regime, roughly $1/V < E < 1/\sqrt{V}$, where the results of RMT and chPT agree with each other. We test these predictions by constructing the connected and disconnected scalar susceptibilities from Dirac spectra obtained in quenched SU(2) and SU(3) simulations with staggered fermions for a variety of lattice sizes and coupling constants. In deriving the predictions of chPT, it is important to take into account only those symmetries which are exactly realized on the lattice.

The theoretical understanding of the Dirac eigenvalue spectrum in a finite volume has improved considerably in recent years. The smallest Dirac eigenvalues are described by universal functions which can be computed most easily in chiral RMT [1,2]. The agreement persists up to the so-called Thouless energy E_c which scales like $1/L^2$, where $V = L^4$ [3–5]. Beyond this energy, the Dirac spectrum can be described by chPT [6]. This has been discussed in the continuum theory in Ref. [7]. On a coarse lattice, the situation is different, and one should take into account only the lattice symmetries. Here, we present an analysis appropriate for staggered fermions at relatively strong coupling and compare our predictions to SU(2) and SU(3) lattice gauge data. For details of the SU(2) analysis, we refer to Ref. [8].

We are interested in the connected and disconnected scalar susceptibilities defined by

$$\chi_{\rm lat}^{\rm conn}(m) = -\frac{1}{N}\left\langle \sum_{k=1}^{N} \frac{1}{(i\lambda_k + m)^2} \right\rangle , \qquad (1a)$$

$$\chi_{\rm lat}^{\rm disc}(m) = \frac{1}{N}\left\langle \sum_{k,l=1}^{N} \frac{1}{(i\lambda_k + m)(i\lambda_l + m)} \right\rangle - \frac{1}{N}\left\langle \sum_{k=1}^{N} \frac{1}{i\lambda_k + m} \right\rangle^2 , \qquad (1b)$$

respectively, where the $i\lambda_k$ are the Dirac eigenvalues and m is a valence quark mass. Most of the RMT-predictions for these quantities are given in Refs. [8,9]. The corresponding chPT-predictions can be derived from an effective partition function Z by differentiating with respect to the quark masses [8]. We consider N_v generations of valence quarks of mass m_v and N_s generations of sea quarks of mass m_s (corresponding to $4N_v$ valence quarks and $4N_s$ sea quarks in the continuum limit). Our starting point is the following expression for the free energy,

$$\ln Z(m_v, m_s, L) \propto VS(m_v, m_s) - \frac{1}{2}\sum_Q K_Q \sum_p \ln\left[\hat{p}^2 + m_Q^2(m_v, m_s)\right] , \qquad (2)$$

where $S(m_v, m_s)$ is the saddle-point contribution, and the double sum represents the one-loop contribution coming from light composite bosons. The sum runs over the allowed momenta p ($p_\mu = 2\pi n_\mu/L$ with integer n_μ) and over particle type Q (with multiplicity K_Q and mass m_Q). We use $\hat{p}^2 \equiv 2\sum_\mu(1-\cos p_\mu)$.

The main task is to determine the K_Q and m_Q for our particular problem. Consider first gauge group SU(3). The symmetry in the chiral limit is $\mathrm{SU}(N_v+N_s)\times\mathrm{U}(1)\times\mathrm{SU}(N_v+N_s)\times\mathrm{U}(1)$ which is spontaneously broken to $\mathrm{SU}(N_v+N_s)\times\mathrm{U}(1)$.

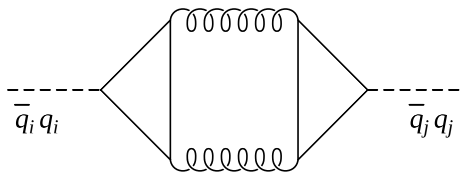

Figure 1. Annihilation diagram for the "flavor-diagonal" mesons.

Since for staggered fermions in strong coupling the U(1) symmetry is anomaly-free, we expect $(N_v+N_s)^2$ Goldstone bosons. The bosons made of different quark flavors $\bar{q}_i q_j$ will have a mass given by $m^2 = A(m_i+m_j)/2$. (According to the Gell-Mann–Oakes–Renner relation, $A = 2\Sigma/f_\pi^2$, where $\Sigma = |\langle\bar\psi\psi\rangle|$.) We thus have $N_v^2 - N_v$ mesons of mass Am_v, $N_s^2-N_s$ mesons of mass Am_s, and $2N_vN_s$ mesons of mass $A(m_v+m_s)/2$.

For "flavor-diagonal" mesons we must also consider the annihilation process in Fig. 1. Because of the anomaly-free U(1) symmetry, the amplitude for $\bar{q}_iq_i \to \bar{q}_jq_j$ vanishes for $m_i = 0$ or $m_j = 0$. Therefore, we make the following ansatz for the mass-squared matrix of the states $(\bar{v}_1v_1, \ldots, \bar{v}_{N_v}v_{N_v}, \bar{s}_1s_1, \ldots, \bar{s}_{N_s}s_{N_s})^T$,

$$M^2 = A\,\mathrm{diag}(m_v,\ldots,m_v,m_s,\ldots,m_s) + z\begin{pmatrix} m_v^2 & \cdots & m_v^2 & m_vm_s & \cdots & m_vm_s \\ \vdots & & \vdots & \vdots & & \vdots \\ m_v^2 & \cdots & m_v^2 & m_vm_s & \cdots & m_vm_s \\ m_sm_v & \cdots & m_sm_v & m_s^2 & \cdots & m_s^2 \\ \vdots & & \vdots & \vdots & & \vdots \\ m_sm_v & \cdots & m_sm_v & m_s^2 & \cdots & m_s^2 \end{pmatrix}$$

with an additional parameter z. The eigenvalues of M^2 are Am_v with multiplicity N_v-1, Am_s with multiplicity N_s-1, and $\lambda_\pm$ with multiplicity one (the expression for $\lambda_\pm$ is given in [8]). This completes the determination of the light boson spectrum of the gauge group SU(3) in Table 1.

For the gauge group SU(2), the symmetry in the chiral limit is $\mathrm{U}(2N_v+2N_s)$, spontaneously broken to $\mathrm{O}(2N_v+2N_s)$ [10]. We thus have $(N_v+N_s)(2N_v+2N_s+1)$ Goldstone particles. Some of the baryon (q_iq_j and $\bar{q}_i\bar{q}_j$) states have the same mass as the mesons, $m^2 = A(m_i+m_j)/2$, giving rise to the light particle spectrum in Table 1.

m^2	multiplicity	
	SU(2)	SU(3)
Am_v	$2N_v^2+N_v-1$	N_v^2-1
Am_s	$2N_s^2+N_s-1$	N_s^2-1
$A(m_v+m_s)/2$	$4N_vN_s$	$2N_vN_s$
λ_-	1	1
λ_+	1	1

Table 1
The light particle spectrum for gauge groups SU(2) and SU(3).

Table 1 determines the one-loop contribution to the free energy in Eq. (2). The saddle-point contribution is parameterized by a smooth function of m_v and m_s, independent of the lattice size. Taking appropriate derivatives of $\ln Z$ with respect to the quark masses [8], we obtain the chPT-predictions for the susceptibilities of Eq. (1). In the final results, we take the limits $m_v = m_s = m$, $N_v \to 0$, and $N_s \to 0$. The fit parameters are A, z, and the smooth background. Since Σ can be determined independently by a fit to RMT, our results for the parameter $A = 2\Sigma/f_\pi^2$ also give us an estimate of f_π [8].

Taking the infinite-volume limit of the chPT-expressions, we obtain several terms containing logarithms in the quark mass [8]. Note, however, that the leading term $\propto \ln m$ in the chiral condensate, which is expected in the quenched approximation [6], is absent in our case because of the anomaly-free U(1) symmetry.

Our results for gauge group SU(2) and SU(3) are displayed in Figs. 2 and 3. The diamonds represent the lattice data plotted vs. the rescaled valence quark mass $u = mV\Sigma$, the solid lines the (finite-volume) chPT predictions, and the dashed lines the RMT predictions (for topological charge $\nu = 0$), respectively. As expected, for $u < f_\pi^2L^2$ the data are described by RMT. For $u > 1$, they are very well described by our chPT expressions. (chPT breaks down for $u < 1$ since the $p = 0$ modes must be treated non-perturbatively in this region. The deviations between lattice data and

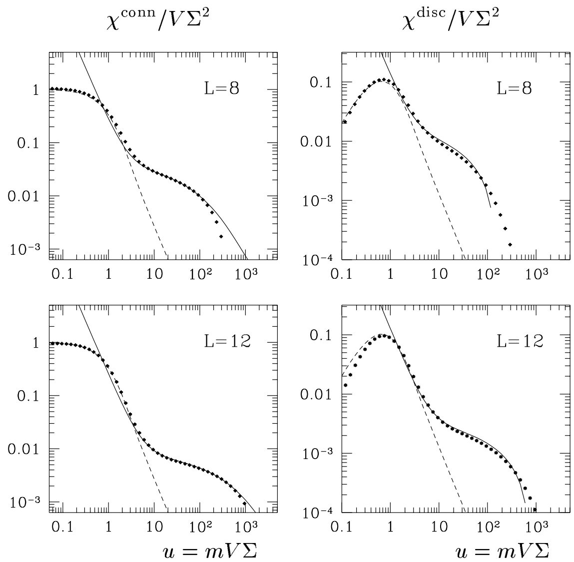

Figure 2. Connected and disconnected scalar susceptibilities versus the rescaled valence quark mass for staggered fermions using gauge group SU(2) at $\beta = 4/g^2 = 2.2$ and $V = 8^4$ and 12^4.

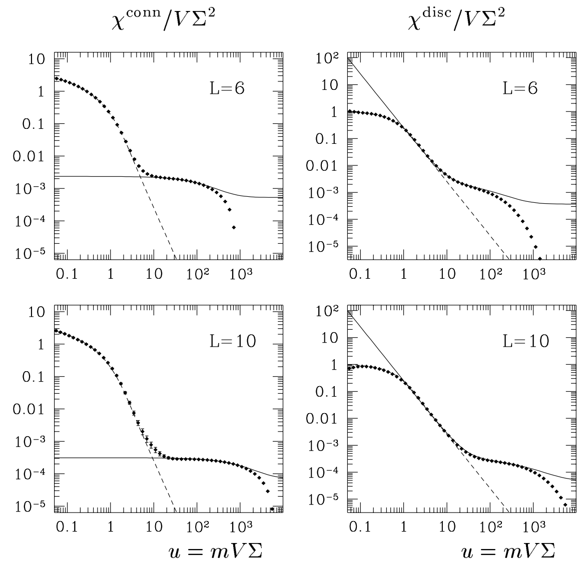

Figure 3. Same as Fig. 2 but for gauge group SU(3) at $\beta = 6/g^2 = 5.4$ and $V = 6^4$ and 10^4.

chPT for very large u are due to the finite lattice.) The domain of common applicability of RMT and chPT, $1 < u < f_\pi^2 L^2$, grows with the lattice size.

In the case of the connected susceptibility in SU(3) (see Fig. 3) we do not see an overlap region of RMT and chPT. The reason is that for this particular quantity (and also for the chiral condensate) the would-be leading terms both in RMT (for large m) and in chPT (for small m) are absent. This is a rather special case caused by the anomaly-free U(1) symmetry and by the fact that $N_s = 0$. As a consequence, the Thouless energy for this quantity scales like $1/L^{8/3}$ instead of like $1/L^2$ so that RMT breaks down for $u \propto L^{4/3}$.

In conclusion, we now have a good theoretical understanding of the finite-volume Dirac spectrum also beyond the Thouless energy. Our analysis was tailored to the case of staggered fermions at strong coupling where the anomaly-free U(1) symmetry causes the light particle spectrum to be different from that of the continuum theory.

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