Report in Wirtschaftsmathematik

Anwendungen

Stochastik   Optimierung

Fachbereich Mathematik ◆ Universität Kaiserslautern

Continuous Location of Dimensional Structures

J. M. Díaz-Báñez [*]   J.A. Mesa [†]   A. Schöbel [‡]

April 20, 2001

Abstract

A natural extension of point facility location problems are those problems in which facilities are extensive, i.e. those that can not be represented by isolated points but as some dimensional structures, such as straight lines, segments of lines, polygonal curves or circles. In this paper a review of the existing work on the location of extensive facilities in continuous spaces is given. Gaps in the knowledge are identified and suggestions for further research are made.

1   Introduction

Given a set of demand points, the goal of classical location problems is to find one or several points for placing new facilities such that they optimize one or several possibly constrained objective functions. Usually the objective functions depend on the interactions among demand points and new facilities. When the new facilities cannot be represented as points but as some kind of dimensional sets then extensive facility location problems arise. Loosely speaking these problems consist of choosing an element in a class of (geometric) sets representing the candidate facilities, that best fits the set of demand points according to some specified criterion. In particular, the location of straight lines, line segments, hyperplanes, spheres and some types of polygonal curves has been studied to some extent. In this way, the location of dimensional structures can be considered as a natural extension of point location. However, as a consequence of the

[*]Dept. de Matemática Aplicada II, E. U. Politécnica, Universidad de Sevilla, Spain (dbanez@cica.es).

[†]Dept. de Matemática Aplicada II, E. S. Ingenieros, Universidad de Sevilla, Spain (jmesa@cica.es).

[‡]Fachbereich   Mathematik,   Universität   Kaiserslautern,   Germany (schoebel@mathematik.uni-kl.de).

different nature of the candidate elements some specific clarifications should be remarked on.

First, the shape and the features of the facilities has to be specified (according to the problem instance) in order to choose the family of geometrical structures to represent them. Secondly, some clarifications should be provided in order to distinguish the field of extensive facility location from others such as routing or network design. While classical location problems are trivially solvable when all the given demand points can be directly covered by the point facilities, this is in general not true when the facilities are represented by dimensional structures. This means that problems consisting in directly covering all the demand points by the facility, with the Euclidean **TSP** being a paradigmatic example, arise in the area of placing or choosing a dimensional structure among a given set of candidates. In order to make a coherent extension of the concept of facility location problem from points to dimensional structures, the inherent features must be maintained. Thus, a problems will be categorized as *extensive facility location* if a function dependent on the location of both the given points and the facility is given in order to measure the (spacial) interaction between demand points and facility. This function often is given through a metric (or a more general distance function) measuring the distance from a demand point to the set of new facilities.

Continuous location of dimensional structures can in general be described as follows: Given a set of existing facilities or demand points $\mathcal{P} = \{p_1, \ldots, p_M\}$ in a n-dimensional space $\mathcal{E}$, a weight set $\mathcal{W} = \{w_1, \ldots, w_M\}$, a distance d in $\mathcal{E}$, a family $\mathcal{S}$ of subsets of $\mathcal{E}$ and an objective function $\mathcal{F}$, find the subset(s) $S \in \mathcal{S}$ such that the function

$$\mathcal{F}(d(p_1, S), \ldots, d(p_M, S))$$

is minimized, where the distance between a point p_m and a set S is given by $d(p_m, S) = \min_{x \in S} d(p_m, x)$.

Note that the elements in $\mathcal{S}$ are given by means of a common geometric property, possibly with some contraints on the size.

Since $\mathcal{P}, \mathcal{E}, \mathcal{W}, d$ are fixed data for each instance, the problem may be summarized as

$$\text{optimize}_{S \in \mathcal{S}} \mathcal{F}(S)$$

As a consequence, extensive facility location, however, overlaps (when a physical interpretation of the facility is not required but only a mathematical one) with other classical areas like curve fitting of a set of points and metrology of

geometrical tolerancing. Taking into account the ample variety of location problems that arises when considering different objective functions, constraints, and metrics occurring in the various applications, the inclusion of these problems within the area of extensive facility location leads to a wider point of view that provides insight and allows to define new problems with potential applications in the fields from which they come.

The literature about locating dimensional facilities is quite heterogeneous. One reason is that results in this area are strongly dependent on the type of curve representing the facility. For the location of straight lines or circles, good characterization results for the optimal elements are available. However, due to the higher degree of freedom of polygonal shaped facilities, no such results have been published for the location of polygonal curves. Hence, in this area, a lot of research aims to directly obtain efficient algorithms. Another reason is due to the fact that extensive facility location problems arise in different mathematical disciplines, meaning that sometimes different approaches have been used in proofs of the same result. Furthermore, also the suggested algorithms show a wide variety, since they are based on different techniques, among them case analysis, convex analysis, computational geometry, and linear and dynamic programming as the most usual approaches in this field.

Although the problems dealt with in this paper have a continuous formulation they can be classified as discrete ones, in the usual sense that a finite procedure for finding an optimal solution exists. For this reason discrete characterizations of the optimal solutions and exact algorithms have been stressed, and only a few iterative algorithms will be briefly mentioned.

The study has been divided according to the type of facility to be located. Since a lot of research has been done in the field of locating *linear* facilities, Sections 2, 3 and 4 present results for line location in the plane, some extensions and the location of other linear structures, respectively. In Section 5, circle and, more general, sphere location is described and literature dealing with such problems is reviewed. Section 6 is dedicated to polygonal curve location problems. Only in special cases structural properties for locating a polygonal curve exist. Therefore, the section emphasizes the research on algorithmic results. Finally, in Section 7 some comments on locating other structures — almost not dealt with in the literature so far — are given, including some conclusions and suggestions for further research.

2 Locating lines in the plane

Given a set of points (also called existing facilities) $\mathcal{P} = \{p_1, p_2, \ldots, p_M\}$ in the plane the *line location problem* is to find a straight line l which is as close as possible to the points in $\mathcal{P}$, i.e., it is a good linear approximation for the set $\mathcal{P}$. The following two objective functions are based on the L_1 and the L_∞ norm and are of particular interest for location theory. Let d be a distance measure and $w_m \geq 0$ be nonnegative weights, then the *median objective* for a line l can be formulated as

$$f(l) = \sum_{m \in \mathcal{M}} w_m d(p_m, l),$$

while the *center objective function* is given by

$$g(l) = \max_{m \in \mathcal{M}} w_m d(p_m, l).$$

A line l minimizing f is called a *median line* and an optimal line with respect to g is a *center line*.

Line location problems have applications not only in operations research and location theory, but also in robust statistics and in computational geometry, e.g., in pattern recognition and in cartography. In statistics, line location problems are known as regression line problems, or as L_1-fit or L_∞-fit problems; in computational geometry mainly the generalization to hyperplane location problems is interesting (see Section 4.3).

A line l in the plane can formally be defined as

$$l_{s_1, s_2, b} = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : \mathrm{x}_1 \mathrm{s}_1 + \mathrm{x}_2 \mathrm{s}_2 + \mathrm{b} = 0 \right\}.$$

Note that the slope s of the line $l_{s_1, s_2, b}$ is given by $-\frac{s_1}{s_2}$.

The main focus of this section is to discuss line location problems with a distance d derived from a norm (i.e., there exists a norm γ such that d is given by $d(x, y) = \gamma(y - x)$). But before turning our attention to norms, we mention the following simple case of, namely line location problems with respect to the vertical distance d_v. For two points $p = (p_1, p_2), q = (q_1, q_2) \in \mathbb{R}^2$, the vertical distance is defined as follows:

$$d_v(p, q) = \begin{cases} |p_2 - q_2| & \text{if } p_1 = q_1 \\ \infty & \text{otherwise} \end{cases}$$

The vertical distance between a point $p = (p_1, p_2)$ and a non-vertical line $l_{s, -1, b}$ can hence be calculated by

$$d_v(p, l) = |p_1 s - p_2 + b|.$$

In fact, if we assume that not all points of $\mathcal{P}$ lie on the same vertical line, we can restrict ourselves to non-vertical lines, since there exists a point $p \in \mathcal{P}$ such that $d_v(p, l) = \infty$ if l is a vertical line. This means that we can assume that both objective functions f and g are piecewise linear and convex for the vertical distance. (Convexity of the problem is not true any more if d is a distance derived from a norm, even for the rectangular distance that property does not hold.) Consequently, line location problems with the vertical distance are easy to solve. Fortunately, many results for the vertical distance can be transferred to any distance d derived from a norm by using the following basic result (see [126]), leading to proofs for most of the results presented in this section.

Lemma 1 *Let d be the vertical distance or a distance derived from a norm γ and let $s \in \mathbb{R}$. Then there exists a constant $C = C(\gamma, s)$ such that for any line l with slope s and for any point $p \in \mathbb{R}^2$ the following holds.*

$$d_v(p, l) = C d(p, l)$$

Given a line l let $\mathcal{B}_l^+$ and $\mathcal{B}_l^-$ denote the two open halfplanes separated by the line l and let $W = \sum_{m \in \mathcal{M}} w_m$ be the sum of all weights. Then we can formulate the first theorem for the line location problem with median objective function.

Theorem 2 *Let d be a distance derived from a norm. Then for any median line l^* the following hold.*

(i) $\sum_{p_m \in \mathcal{B}_{l^*}^-} w_m \leq \frac{1}{2} W$

(ii) $\sum_{p_m \in \mathcal{B}_{l^*}^+} w_m \leq \frac{1}{2} W$

Note that (i) and (ii) are equivalent to

$$(iii) \qquad \left| \sum_{p_m \in \mathcal{B}_{l^*}^-} w_m - \sum_{p_m \in \mathcal{B}_{l^*}^+} w_m \right| \leq \sum_{p_m \in l} w_m .$$

A line satisfying this property will be called *pseudo-halving*.
The proof of Theorem 2 for the Euclidean distance can be found in [145, 106, 103, 81]. The general versions for the vertical distance, for all distances derived from norms can be found in [128]. The following is the main result for finding a median line.

Theorem 3 *For the vertical distance and for all distances d derived from norms there exists a median line passing through at least two of the points in $\mathcal{P}$.*

This has first been shown for d_2, i.e. for the Euclidean distance, by [145]. Proofs for the same result can also be found in [106, 103, 90] and [81]. Most of these proofs use Theorem 2 and that $f(l)$ is concave when rotating the line about one of the points in $\mathcal{P}$. The minima of this function are attained when the line touches another point in $\mathcal{P}$.

For the rectangular distance, Theorem 3 has been shown in [146, 103] by separating the problem into two problems with vertical distance. A similar generalization makes it possible to prove Theorem 3 for all distances derived from block norms (see [124]).

Finally, by using a reduction to the vertical distance with the help of Lemma 1 it has been shown in [126] that Theorem 3 is valid for all distances derived from norms.

If the symmetry property of a norm is not satisfied, i.e., $\gamma(x) = \gamma(-x)$ does not hold, γ is called a *gauge* ([105]). For gauges, Theorem 3 does in general not hold. But it can be shown that there exists a median line containing at least one of the points in $\mathcal{P}$ (see [128, ?]). For metrics and mixed norms, however, it even can happen that none of the optimal lines contains a point of $\mathcal{P}$ (for examples we refer to Chapter 5 of [128]).

Now we turn to the center problem.

Theorem 4 *For the vertical distance and for all distances d derived from norms or gauges there exists a center line which is at maximum distance from at least three of the points in $\mathcal{P}$.*

Formally, Theorem 4 states that there exists an optimal line l^* and a subset $\mathcal{M}^0 \subseteq \mathcal{M}$ with cardinality at least 3 such that

$$g(l^*) = w_k d(p_k, l^*) = \max_{m \in \mathcal{M}} w_m d(p_m, l^*) \text{ for all } k \in \mathcal{M}^0.$$

This has been proven first by [130] for the vertical distance d_v, and later by [107] for the rectangular and for the Euclidean distance. The generalization to all distances derived from norms can be found in [126], the generalization to distances derived from gauges is obtained by geometric methods in [128].

Note that, in contrast to the incidence property stated in Theorem 3 for median line location problems the above Theorem 4 can be transferred to distances derived from gauges, even if we allow different gauges d_m for each of the points in $\mathcal{P}$. Also for strictly monotone metrics this property can be verified, but for arbitrary metrics it is in general not true. For details we refer to Chapter 5 of [128].

From an algorithmic point of view, the following property dealing with the convex hull $CH(\mathcal{P})$ of the set of existing facilities, is also interesting for line location problems with center objective.

Theorem 5 *For the vertical distance and for all distances d derived from norms or gauges the following holds. If the weights are all equal there exists a center line which is parallel to a facet of CH(P).*

For the vertical and the rectangular distance this has been shown in [130]; the generalization to all distances derived from norms or gauges can be found in [126] and [128], respectively.

In the next result the set of *all* optimal lines is studied. Recall that a *smooth* norm is a norm such that each point on the boundary of its unit ball $B = \{x \in \mathbb{R}^n : \gamma(x) \leq 1\}$ is supported by exactly one hyperplane (see e.g., [111]).

Theorem 6 *Let d be a distance derived from a norm.*

(i) *Any median line with respect to any weighted set $\mathcal{P}$ with weights w_m and distance d passes through at least two of the points in $\mathcal{P}$ if and only if d has been derived from a smooth norm.*

(ii) *Any center line with respect to any weighted set $\mathcal{P}$ with weights w_m and distance d is at maximum distance from at least three of the points in $\mathcal{P}$ if and only if d has been derived from a smooth norm.*

Theorem 6 has been shown in [128], where it has been used to derive an algorithm for calculating *all* optimal lines — even in the case of non-smooth norms.

Algorithms for line location problems

Using Theorems 3 and 4, a simple enumeration of all lines passing through two of the points in $\mathcal{P}$ in the median case, or being at maximum distance from three points in $\mathcal{P}$ in the center case is possible. This can be done in polynomial time, for any distance d derived from a norm (assuming that a norm evaluation can be done in constant time). However, for some distance measures, there exist the following better algorithms.
Again, we first discuss the case of median lines.

- For the Euclidean distance, [106] proved Theorem 3 and proposed an enumeration algorithm with running time in $O(M^3)$. In [103] this algorithm was improved to $O(M^2 \log M)$ and finally [86] succeeded in an $O(M^2)$ time

7

algorithm for the median problem with Euclidean distance. However, for the problem with identical weights [148] derived an algorithm with time complexity of $O(M^{\frac{3}{2}} \log^2 M)$ by using that the number of pseudo-halving lines is bounded by $M^{\frac{3}{2}}$. Using a sharper bound (by [109]) for the number of pseudo-halving lines, [82] could derive an $O(M^{\frac{3}{2}} \log^{2-\frac{1}{100}} M)$ time approach. Recently, [26] showed that the number of pseudo-halving lines is at most $O(M^{\frac{4}{3}})$ in the unweighted case. This suggests a further reduction of the worst-case complexity of the Euclidean line-location problem. The question about a time optimal algorithm remains open yet. The known lower bound is $\Omega(M \log M)$, proven in [148] by reduction from the uniform gap on a circle problem.

- Line location problems with the rectangular distance can be approached by solving two line location problems with vertical distance. This can be done in $O(M)$ time, see [149] and later also [75], both using the linear programming methods of Megiddo ([100, 101]). Earlier approaches include an $O(Mlog^2 M)$ time algorithm by [103] and a polynomial approach by [132, 116] evaluating all breakpoints of the function. In older statistics literature, e.g., in [78], infinite iterative processes were applied to find a median line minimizing the sum of weighted vertical distances. Another approach used in statistics is formulated by [21, 54, 143], but also here no polynomial bounds on the running time are given. In [107] a similar approach based on solving a small linear program is used.

- For solving line location problems with respect to block norms, [124] shows that this can be done in $O(GM)$ time, when G is the number of extreme points of the block norm. The idea of the algorithm is to decompose the problem into G line location problems with respect to the vertical distance and for them use the linear approach of [149] mentioned above.

For finding center lines, we mention the following specialized procedures.

- For the vertical and for the rectangular distance the problem can easily be solved in linear time by linear programming (using the methods of Megiddo, see [100, 101]).

- For block norms, the same decomposition as in the median case has been proposed in [128], also leading to an algorithm which runs in $O(GM)$ time.

- In the case of the Euclidean distance [27] used Theorems 4 and 5 to present an $O(M \log M)$ time algorithm for the unweighted problem, while [?] presented an $O(M \log M)$ time algorithm for computing the width of a set

8

— a problem in computational geometry which is equivalent to finding a center line in the unweighted case. The optimality of these algorithms was shown in [89]. In this paper, also the weighted case has been discussed and an O($M^2 log M$) time algorithm for this case has been suggested. This approach could be improved to the optimal running time of O($M \log M$) in [49].

There is another objective function, which is sometimes referred to in the literature, namely to find a straight line maximizing the minimum distance to the existing facilities. For the Euclidean distance, this problem has been studied in $\mathrm{I\!R}^2$ ([70]) and in $\mathrm{I\!R}^3$ ([56, 55]). In $\mathrm{I\!R}^2$ the presented algorithm runs in time O(M^2), while in $\mathrm{I\!R}^3$ the time complexity has been improved from O($M log^5 M$) ([56]) to O($M log^4 M$) ([55]). In the case of polyhedral norms the problem has been discussed in [69], where the existing facilities in $\mathcal{P}$ are given not by points, but by polygonal sets which must not be intersected by the line.

3 Extensions of line location problems

3.1 Locating more than one line

In [102] it has been shown that for general r it is NP-hard to decide wether or not there exist r straight lines $l_1, \ldots, l_r$ in the plane such that

$$\mathcal{P} \subseteq \bigcup_{i=1,\ldots,r} l_i,$$

where $\mathcal{P}$ is a given set of M points (the so called *line cover problem*). Consequently, it is NP-hard to locate r lines in the plane such that the sum of distances or the maximum distance between the points in $\mathcal{P}$ and the set of lines is minimized.

In Section 2 we have shown that for $r = 1$ both problems are polynomially solvable, if the distance is derived from a norm. For $r > 1$ only little research has been done so far. Apart from [97] who solved the location of r lines heuristically as an intermediate step in their procedure, the only exception we are aware of refers to the *Euclidean 2-line-center problem*: Given a finite set $\mathcal{P}$ of (unweighted) points in the plane, find two lines l_1 and l_2, such that they minimize the maximal distance from the given points to their closest line, i.e. minimize

$$\max_{p \in \mathcal{P}} \min\{d_2(p, l_1), d_2(p, l_2)\}.$$

For this problem, in [6] an an O($M^2 log^5 M$) time algorithm was developed, using the parametric searching technique of Megiddo [99]. The corresponding

9

fixed-size problem which is used is the following: Given a width w determine wether $\mathcal{P}$ can be covered by two strips of width w each. In [6] it is shown that this problem can be solved in $O(M^2 log^3 M)$ time (by using the results of [5] about the off-line dynamic maintenance of the width of a planar point set), while in [77] this time bound is improved to $O(M^2 log M)$ time, leading directly to an $O(M^2 log^3 M)$ algorithm for the 2-line-center problem. Other approaches for solving the 2-line-center problem have been provided in [79] and [59], both resulting in $O(M^2 log^4 M)$ time procedures.

The best known approach has been presented by [77]. It leads to an $O(M^2 log^2 M)$ algorithm, using a data structure called *anchored chains* for efficiently updating rotated feasible solutions. Note that this algorithm allows to find all optimal solution of the 2-line-center problem within the same time complexity of $O(M^2 log^2 M)$.

3.2   Restricted line location problems

When applying location theory to model real-life problems one often has to take into account restrictions for the set of feasible solutions. Such additional constraints often change the whole structure of the model, and thus many theoretical results may be useless in practice. Therefore, in classical (point-shaped) location theory many papers deal with different types of restrictions. One kind of restriction that is often used is the introduction of a *forbidden region R* (also called a *restricting set*) in the interior of which the new facilities cannot be located. For the case of line location, a given forbidden region R must not be intersected by the new line-shaped facility.

Instead of looking at a forbidden set one might wish to look at an *enforced* set F, i.e. we require that $l \cap F \neq \emptyset$. For the case that the line is forced to pass through one specified point p_0 the *anchored line location problem* has first been considered by [107] for the median objective function and the Euclidean distance. They have shown the following theorem.

Theorem 7 *All optimal lines for the anchored line location problem with Euclidean distance and median objective pass through at least one point in $\mathcal{P}$.*

In [82] the anchored line location problem appears as a subprocedure for the unrestricted line location problem with median objective function, and has been solved in $O(M log M)$ time for the Euclidean distance. Recently, Theorem 7 has been generalized to all distances derived from norms (see [129]).

Now we turn to restricted line location problems with a forbidden sets R through the interior of which the new line is not allowed to pass. As possible applications

for restricted line location problems one can think of a highway that is not allowed to pass through a natural habitat, or a conveyor belt that must not be within a dangerous area. To formalize the *restricted line location problem* let R be a given area in the plane that must not be intersected by any line. Then the problem is the following.

$$\min f(l) \qquad\qquad or \qquad \min g(l)$$
$$s.t. \ \ l \cap int(R) = \emptyset \qquad\qquad s.t. \ \ l \cap int(R) = \emptyset, \ \text{respectively.}$$

Among the few problems of this type which have been mentioned in the literature of linear facility location so far is the location of a center hyperplane with respect to the Euclidean distance where the forbidden region is given by $CH(\mathcal{P})$, see [118].

In a more general context, [127] investigated restricted line location problems with polygonal forbidden sets R for the vertical distance d_v by looking at the dual version of the problem (in which lines have been transformed to points and points have been transformed to lines, see e.g., [119, 67, 127]). The result which was obtained by transforming the forbidden set R to dual space could be adapted first to block norm distances and then also to all distances d derived from norms. The result is the following (see [127] for details).

Theorem 8 *Let R be a polygon and d be a distance derived from a norm.*

- *For the restricted line location problem with median objective there exists an optimal line which*

  - *contains a facet of R or*
  - *passes through one of the points in $\mathcal{P}$ and through a vertex of R or*
  - *passes through two of the points in $\mathcal{P}$.*

- *For the restricted line location problem with center objective there exists an optimal line which*

  - *contains a facet of R or*
  - *is at maximum distance from two of the points in $\mathcal{P}$ and passes through a vertex of R or*
  - *is at maximum distance from three of the points in $\mathcal{P}$.*

By enumerating all candidate lines of Theorem 8, polynomial time algorithms are possible. Note that Theorem 7 appears as a special case in the theory developed in [127] and can easily be generalized to all distances d derived from norms.

3.3 Linear Approximation of Simple Objects

In the following generalization of line location problems we consider a set $\mathcal{P}$ which does not consist of points, but of M polygonal convex sets. The *line stabbing problem* or *line transversal problem* is to find out, if there exists a line l which intersects all objects in $\mathcal{P}$. Such problems can be solved in optimal in $O(M \log M)$ time by the algorithm of Hershberger [66, 10], even if the sets in $\mathcal{P}$ are convex, but not polygonal sets. For the case of M polygonal sets $p_1, \ldots, p_M$ with G_m extreme points, $m = 1, 2, \ldots, M$ the time complexity can be improved to $O(\sum_{m \in \mathcal{M}} G_m \log (\sum_{m \in \mathcal{M}} G_m))$, see [66]. If all $p \in \mathcal{P}$ are translates of one polyhedral set with G extreme points, the line stabbing problem can be solved in $O(GM)$ time, see [49]. Finally, M rectangles can be stabbed in $O(M)$ time, see also [49].

In the case that no stabbing line exists, the goal is to find a line l which is close to a line transversal in the following sense:

- In the case of the median objective, the goal is to find a line, minimizing the weighted sum of distances to the given objects in $\mathcal{P}$.

- the center problem consists of finding a line minimizing the weighted maximal distance to the objects in $\mathcal{P}$.

For the Euclidean distance both problems are discussed extensively in [119]. The methods used are the dual interpretation already mentioned in Section 3.2 and well-known techniques from computational geometry. The main results are the algorithms for finding median and center lines, if the given set $\mathcal{P}$ consists of M polygonal sets with a total of G vertices. For determining a median line an $O(MG \log M)$ time algorithm is suggested, while the proposed procedure for solving the center line problem runs in $O(G^2 \log G)$ time. In the case of circles and line segments, better procedures can be obtained. Many other complexity results for constrained versions of both problems, and even for finding a hyperplane in $\mathbb{R}^3$ are given. We refer to [119] for details.

4 Locating other linear facilities

4.1 Locating line segments

Probably the first paper in which line segment problems have been mentioned is due to McKinnon and Barber [97]. In the context of designing a transportation network, the authors try to find r line segments such that the sum of distances from each point in $\mathcal{P}$ to its nearest line segment is minimized. They propose a

heuristic iterative approach, in which they first partition the set $\mathcal{P}$ into "apparently linear" subsets and then approximate a median line for each of the subsets (without knowing any of the exact algorithms of Section 2), before re-allocating each point to its closest facility and repeating the procedure. Their heuristic has been applied in the area of Southern Ontario and Quebec (for M=55).

While for general distance measures very few research has been done in the field of locating line segments, there are some papers about the *segment center problem* in the Euclidean case. This is specified as follows.
Given a planar and finite set of points $\mathcal{P}$ and $l \geq 0$ find a line segment S of length l such that
$$\max_{p \in \mathcal{P}} d_2(S, p)$$
is minimized.
The first solution approach for this problem has been given in [76] with a running time of $O(M^4 log M)$. Using the parametric searching technique of Megiddo ([99]) [4] improved this approach to $O(M^2 \alpha(M) log^3 M)$ time, where $\alpha(M)$ is the inverse Ackermann function. Finally, [50] presented an algorithm that solves the problem in nearly linear time, also using Megiddo's parametric searching technique. To apply this technique the authors solved the following fixed-size problem: Given $\mathcal{P}$, l and g^0 determine if there exists a line segment S with length l such that all points of $\mathcal{P}$ lie within distance g^0 from S.
As has been already mentioned in [76], this problem is equivalent to determine, if a given polygonal set $CH(\mathcal{P})$ can be located within a hippodrome of size l and g^0, i.e., within a rectangle of dimension $l \times 2g^0$ with two semicircles with radius g^0 attached to its sides. In [50], the number of *critical* placements of such a polygon within the hippodrome could be bounded, leading to a $O(M^{1+\epsilon})$ algorithm for locating a center line segment.

It should be mentioned that there exists one case in which segment location problems can be solved efficiently: If the distance considered is the vertical distance d_v, it has been shown in [128] that the location of a line segment with given length l can be transformed to a restricted line location problem which can be solved in $O(M)$ time.

Finally, we discuss the following extension of the line-segment problem. Instead of fixing the length l of the new line segment facility, a bicriterial approach is possible, i.e., the goal is to minimize not only the median or the center objective function, but also the length $l(S)$ of the new facility S, that gives an estimation for the costs of building up and operating the facility. This approach has been discussed in Chapter 6 of [128]. The main result is a characterization of the efficient line segments in the case that the distance d is derived from a strictly

convex norm. Let l_{crit} denote the smallest length l to obtain the optimal objective value of the corresponding line location problem with a line segment. Then all optimal solutions of the fixed length line segment location problem with given length l are efficient, if $l \leq l_{crit}$.

4.2 Location of half-lines

The *half-line location problem* is the following. Given a set of points $\mathcal{P}$ find a half-line h emanating from a given point p_0 such that

$$\sum_{p \in \mathcal{P}} w_p d(p, h) \qquad \text{or} \qquad \max_{p \in \mathcal{P}} w_p d(p, h)$$

is minimized, respectively.

In [107] the half-line location problem has been considered for the Euclidean distance and the median objective function f and the following result has been shown.

Theorem 9 *For the half-line location problem with median objective and Euclidean distance, the following holds:*
All optimal half-lines pass through at least one point in $\mathcal{P}$.

In [89] the center objective function is considered for the half-line location problem with Euclidean distance, and an O($M \log M$) algorithms has been suggested. The case of the rectangular and the Chebychef distance has been discussed in [32]. For the center objective function, an O(M^2) time algorithms is proposed, which can be improved to an O(M) algorithm, if $p_0 \notin CH(\mathcal{P})$, i.e. in the case that the ray should emanate from a point outside of the convex hull of the given points in $\mathcal{P}$.

Another interesting objective function which has been considered mainly for halflines is the maxmin objective. In this case the goal is to find a new facility, such that the minimal distance to the points in $\mathcal{P}$ is maximized. The resulting problem is called the *maxmin half-line location problem*. This problem has been studied in the plane by [55] and can be solved in O($M \log M$) optimal time. Its three-dimensional conterpart is to find a ray emanating from a given point p_0 such that the minimal distance to a point in the finite set $\mathcal{P} \in \mathbb{R}^3$ is maximized. This problem is motivated by applications in neurosurgery where a line shaped instrument should be intruded into the patient's brain to remove a sample from a specific point p_0 without damaging critical brain areas (see [55]). In [56] an O($M \log^5 M$) time algorithm is suggested to find a half-line in $\mathbb{R}^3$ with Euclidean distance and maxmin objective, which has been improved to O($M \log^4 M$) time

in [55]. In this paper, also extensions are studied, e.g., to locate a line or to consider a point set $\mathcal{P}$ not consisting of points, but of lines, or of line segments.

4.3   Locating hyperplanes in normed spaces

Most of the results of Section 2 can be generalized to hyperplane location problems. Since the used methods are very similar to those used for line location in the plane, we only mention the results here and refer to [82, 72] for an extensive discussion of hyperplane location problems with Euclidean and rectangular distance and to [94] for a recent survey on hyperplane location problems in normed spaces. For the details of the proofs in this section, see [128, 95].
A hyperplane $H_{\overline{n},b}$ can be described as

$$H_{\overline{n},b} = \{x \in \mathrm{I\!R}^n : \langle \mathrm{x}, \overline{\mathrm{n}} \rangle + \mathrm{b} = 0\}.$$

Hyperplane location problems can now be defined as follows. Given a set of n-dimensional points $\mathcal{P} \subseteq \mathrm{I\!R}^n$ one wants to find a hyperplane H that minimizes

$$f(H) = \sum_{m \in \mathcal{M}} w_m d(p_m, l), \text{ or}$$

$$g(H) = \max_{m \in \mathcal{M}} w_m d(p_m, l), \text{ respectively.}$$

The optimal solutions are called *median hyperplane* and *center hyperplane*, respectively. Using the same notation as in Section 2 we can summarize the main results for hyperplane location problems.

Theorem 10 *For hyperplane location problems in* $\mathrm{I\!R}^n$ *and all distances d derived from norms the following hold.*

1. *There exists a median hyperplane which passes through n affinely independent points in* $\mathcal{P}$.

2. *All median hyperplanes are pseudo-halving.*

3. *There exists a center hyperplane which is at maximum distance from* $n+1$ *affinely independent points in* $\mathcal{P}$.

To prove this theorem one first shows all three results for the convex case of the vertical distance in $\mathrm{I\!R}^n$. To prove (3.) one can use a result of [39] (based on Hellys Theorem, see [65]) which states that for convex minmax problems

$$\min \max_{m \in \mathcal{M}} g_m \text{ with } g_m = w_m d_v(p_m, H)$$

15

there always exists a solution which is only determined by $n + 1$ different functions g_m. The generalization from the vertical distance to all distances d derived from norms works analogously to the methods in the plane.

Using a result of [19] of multicriteria optimization problems and that the distance from a point p to a hyperplane $H_{\overline{n},b}$ can be given by the following formula

$$d(p, H_{\overline{n},b}) = \frac{|\langle p, \overline{n} \rangle - b|}{\gamma^0(\overline{n})}$$

where γ^0 denotes the dual norm, [114] gave another proof for the first property in Theorem 10. Additionally, they showed the following result for distances d derived from gauges.

Theorem 11 *If d has been derived from a gauge, then there exists a median hyperplane which passes through at least $M - 1$ points in $\mathcal{P}$.*

The extension of Theorem 6 to normed spaces leads to the following characterization of smooth norms (see [93]).

Theorem 12 *Let d be a distance derived from a norm γ. Then γ is a smooth norm, if and only if for all instances of the hyperplane location problem one of the following two (equivalent) conditions holds.*

(i) All median hyperplanes are passing through n affinely independent points in $\mathcal{P}$.

(ii) All center hyperplanes are at maximum distance from $n + 1$ points in $\mathcal{P}$.

Recently, the *anchored hyperplane location problem* has been considered. In this context, not only the weighted set $\mathcal{P}$ of existing facilities, but also another set of points, Q, is given. The objective is to find a hyperplane minimizing the distances to the points in $\mathcal{P}$, but passing through all points in Q. In [129] the following result has been shown.

Theorem 13 *Let d be a distance derived from a norm γ, and let $k \leq n$ be the number of affinely independet points in $\mathcal{P}$. Furthermore, let $Q \subseteq \mathbb{R}^n$ be a given set.*

(i) There exists an anchored median hyperplane passing through at least $n - k$ affinely independent points of Q.

(ii) There exists an anchored center hyperplane which is at maximum distance from at least $n - k + 1$ affinely independent points of Q.

16

(iii) If γ is a smooth norm, **all** *anchored median hyperplanes are passing through at least $n - k$ affinely independent points of Q, and* **all** *anchored center hyperplanes are at maximum distance from at least $n - k + 1$ affinely independent points of Q.*

Note that Theorem 13 contains the incidence properties for unrestricted line- and hyperplane location problems in the special cases that $Q = \emptyset$.

A further generalization of hyperplane location problems should briefly be mentioned. Given a set $\mathcal{P}$ of M points in $\mathbb{R}^n$, find a k-dimensional plane ($k < n$) such to minimize the sum of distances or the maximum distance to the given point set $\mathcal{P}$. This problem has been considered in [92], where it has been shown that for the median objective function it is at least as difficult as the location of a point in $\mathbb{R}^{n-k}$ minimizing the median objective function i.e. as the classical Weber problem in $n - k$ dimensions. In the special case $k = 1, n = 3$, i.e. for locating a line in $\mathbb{R}^3$, the problem with center objective function has been previously studied in [71] and also in [133]. For the median objective function, some first models and solution approaches have recently been developed by [17].

Algorithms for hyperplane location

As in the planar case, Theorem 10 shows that polynomial algorithms are possible for solving hyperplane location problems when the distance d is derived from a norm. Some special case algorithms should also be mentioned for hyperplane location problems.

- For the vertical distance or the rectangular norm the algorithms of [149, 101] run in linear time, also for hyperplane location problems for each fixed dimension n, resulting in $\mathrm{O}(M)$ time algorithms for the median and for the center problem.

- For block norms this can be used to derive an $\mathrm{O}(GM)$ time algorithm for any fixed dimension n (where G as usual denotes the number of extreme points of the block norm), see [95].

- In the case of the Euclidean distance, both [82] and [72] suggest $\mathrm{O}(M^n)$ time algorithms for solving the median problem. For the center problem [72] succeeded to develop an $\mathrm{O}(M^{\lfloor \frac{n}{2}+1 \rfloor})$ time algorithm for all fixed dimensions $n \geq 3$.

  For $n = 3$, however, further improvements are possible. Since finding a center hyperplane in $\mathbb{R}^3$ is equivalent to computing the width of a point

set in $\mathbb{R}^3$ it is possible to use the $O(M^2)$ time algorithm of [71]. In [22] its complexity has been improved to $O(M^{\frac{8}{5}+\epsilon})$, and further improvements up to $O(M^{\frac{17}{11}+\epsilon})$ have been suggested by [3]. Finally, [7] succeeded in developing a randomized algorithm for the 3-dimensional set width problem with an expected running time of $O(M^{\frac{3}{2}+\epsilon})$.

- To solve center hyperplane location problems in the case of an arbitrary norm, algorithms from transversal theory can be used (see [128]). In particular the algorithm developed in [66] for stabbing arbitrary convex sets can be applied together with a binary search to find a center hyperplane even if we allow different metrics d_m for each of the points in $\mathcal{P}$. Note that, on the other hand, algorithms for the location of a center hyperplane can be used to solve transversal problems (see [96]).

5   Sphere location

The majority of research carried out has been orientated towards solving problems in a planar setting. After describing the above mentioned, some extensions to higher dimensions will be briefly described.

Given a set of points $\mathcal{P} \subset \mathbb{R}^2$ circle location problems consist of finding a circle C such that it minimizes some function of the distance from the given points to the circle. As far as the authors are aware the only metric used for distances in these problems is the (orthogonal) Euclidean.

In the field of facility location, an example of a potential application of these models, is to determine the alignment of a rapid transit underground line when the points in $\mathcal{P}$ (demand points) are the centroids of the census tracks and weights represent the corresponding populations. In particular, in already constructed metro networks, and for the purpose of improving the mobility of the citizens, the introduction of a circle line contributes to improving several effectivity measures of the network [85].

The approximation criteria applied are the corresponding to L_1, L_∞ and L_2^2, which lead, respectively, to the following objective functions:

$$f(c, R) = \sum_{m \in \mathcal{M}} |d(p_m, c) - R|,$$

$$g(c, R) = \max_{m \in \mathcal{M}} |d(p_m, c) - R|.$$

$$h(c, R) = \sum_{n \in \mathcal{M}} |d(p_m, c) - R|^2.$$

in which $c = (x, y)$ is the center of the circle, R is the radius and $d(\cdot, \cdot)$ is the Euclidean distance. Note that $|d(p_m, c) - R| = d(p_m, \delta C)$, i.e. it gives the distance from p_m to the boundary of C.

The corresponding non-convex minimization problems give rise to the concept of median, center and least-squares circles, respectively. One of the reasons for including the last one is the frequent application in practice as an alternative to the center problem.

The following characterizations of the optimal solutions have been described.

Theorem 14 *The center of the optimal circle solving the Least Squares Problem is the point at which the variance of the distances to the points is minimized. The optimal R is the average of all distances.*

This result can be found in [43] and shows that this problem is equivalent to that of locating a point that minimizes the variance in the plane with Euclidean

distances, for which no algorithm is known for finding an exact solution. Therefore, only iterative algorithms ([147], [43]) have been proposed. Now we turn to the median objective.

Theorem 15   *(i) If $M = 3$ and the points are affinely independent then there exists exactly one circle containing the points. Otherwise, i.e. they are colinear, no optimal circle with finite radius exists. If $M \geq 3$ then there always exists an median circle that contains one of the given points. Furthermore, all median circles are pseudo-halving (2).*

   (ii) There exists a median circle whose center is the point at which the mean absolute deviation of the distances to the points is minimized, and the radius is the median of all distances.

The first part of the theorem can be found in [128], while (ii) in [43].

Let us note, that the these results are not sufficient to construct an algorithm for finding the exact solution of the median circle problem, since there is no similar known algorithm for the minimum absolute deviation problem in the plane. For this reason, up to now, only iterative algorithms have been devised. The center circle problem is equivalent to finding a minimum radius annulus enclosing all the points, and is also known as the out-of-roundness problem in Computational Metrology. In this area the problem of determining whether or not a manufactured object fits to the previously prescribed shape arises. For this, a sample of points in the boundary of the object is obtained and, according to a specified criterium, the best curve of the family is fitted in order to evaluate the quality of the object. Sample points can be interpreted as demand points in location models and the curve to be fitted as the facility structure to be located. For instance when producing circle planar pieces the problem known as roundness inspection arises [147]. This problem has been treated with Statistics and Computational Geometry techniques. In several areas of production, such as car or precision machinery manufacturing the problem of sphericity inspection of produced ball bearings appears [?].

For the center planar problem or equivalently the minimum width enclosing annulus the main discretization results are the following.

Theorem 16 *If $M \geq 4$ then at least two are on the inner circle and two are on the outer circle. Furthermore, the points on the inner circle interlace angle-wise with the points on the outer circle as viewed from the center of the annulus.*

This theorem was first stated by [117] and its application leads to an O(M^5) time naive algorithm ([142]). However, the annulus can degenerate to a slab ([134]) which can be viewed as an annulus centered at ∞.

In order to obtain a non trivial algorithm the following result is crucial.

Theorem 17 *The center of the concentric circles providing the enclosing annulus of minimum radius is in the set of vertices of the farthest-point (nearest-point) Voronoi diagram lying in vertices or edges of the nearest-point (farthest-point) Voronoi diagram.*

This theorem was proved in two steps which are contained, respectively, in [48], [144] and [47], and leads to the first non-trivial algorithm with $O(M \log M + I)$ time, where I is the number of candidate points in the above theorem and can be cuadratic in the worst case.

Subsequently applying Megiddo's parametric search, $O(M^{\frac{8}{5}+\epsilon})$ [?], and $O(M^{\frac{17}{11}+\epsilon})$ [2] time theoretical algorithm have been obtained, while in [?] an algorithm with expected running time $O(M^{\frac{3}{2}+\epsilon})$ has been proposed.

Round about the same time, some efforts have been given to clarify the implementation of the practical algorithm based on the Voronoi diagrams ([121], [122]).

Further research has been done on the application of the characterization of local minima. When the points in $\mathcal{P}$ are given in circular order, as happens when computing roundness, there is at most one local minimum and it can be computed in $O(M \log M)$ time ([57]). The corresponding problem to the special case in which the points of $\mathcal{P}$ are in convex position only has a local minimum inside the convex hull, and can be computed in $\Theta(M)$ ([57]).

The problem in which the existing facility is a simple polygon, has been studied by [?], who obtained an $O(M \log M + I)$ time algorithm, where I is the quoted number of intersections, and an $O(M)$ time algorithm has been devised by [135] for the special case in which the polygon is convex containing the center of the annulus.

A related but simpler problem is that of finding an annulus of minimum area that contains $\mathcal{P}$. This problem can be formulated as a linear programming problem in $\mathbb{R}^4$ and, therefore can be solved in $O(M)$ time ([101]). In [?] this model, called the algebraic Chebychev problem of fitting circles, is considered as an approximation to the geometric Chebychev or zone problem, i.e. the above called the circle center problem. Also the constrained version of the center circle problem in which the radius is given has been under research and an $O(M \log M)$ time algorithm provided [28].

The extension of the circle center problem to higher dimensions consists in finding a hyperspherical surface such that it minimizes the orthogonal distances from the given points. The essence of the characterization of local minima remains true for dimensions higher than two ([57]). In the recent past, an

21

iterative algorithm based on a combination of polyhedral outer approximation, branch-and-bound and cutting plane techniques ([25]) and an $O(M^{3-\frac{1}{19}+\epsilon})$ time algorithm ([2]) for dimension three have been described. Finally, for the harder problem of finding a cylindrical shell (a region enclosed between two concentric cylinders) of smallest width (difference between radii), which contains the set $\mathcal{P}$, no solution improving the brute-force technique is known.

6   Location of polygonal curves

The natural extension of the linear facility problem in the plane is that of locating a polygonal facility which provides in some sense, the best approximation for a point set.

The optimal location of a polygonal route among a set of points in the plane has been dealt with from different points of view according to the objectives and applications which have suggested them.

In this sense, two optimization criteria have been applied. On the one hand, when we ask for a route for distributing goods to a set of customers represented by points in the plane, the application of the min-max criterion has been taken into account. On the contrary, when designing an obnoxious route (with potential undesirable effects, for instance a pipeline) or in the robotic field, the max-min criterion has to be applied in order to locate the facility as far as possible from the population. Note that although the min-sum criterion is common in facility location theory, has not been considered for locating polygonal curves yet.

consequently, we limit our attention to the location of a monotone polygonal curve connecting two points (the endpoints) using the min-max and the max-min criteria.

Note that for a general polygonal structure do not exist any characterization properties for a solution of these optimization problems. Therefore, in this section we will only sketch the general and conceptual methods for solving the problems efficiently. However, there are certain cases where we will also show some nice properties.

Hereafter we will call $\mathcal{P} = \{a = p_0, p_1, \ldots, p_M, p_{M+1} = b\}$ the given set of points and will denote the x-coordinate of a point q by $x(q)$. Furthermore, we will assume that the points in $\mathcal{P}$ are given in lexicographical order. The points a and b are *the endpoints* (can be fixed or not), and are assumed to satisfy $x(a) < x(p_1)$ and $x(p_n) < x(b)$. These endpoints have been added in the notation to the point set to be approximated, which is really $\mathcal{P} \setminus \{a, b\}$, for the sake of making later descriptions easier. Then, denoting the polygonal curve facility by C and by d the distance measure from a point to a polygonal facility, the two problems are, respectively, minimizing the function

$$g(C) = \max_{p_i \in P} d(p_i, C)$$

and maximizing the function

$$h(C) = \min_{p_i \in P} d(p_i, C).$$

6.1   k-Bend polygonal curves

The optimization of the function $g(C)$ has a trivial solution. In fact, the polygonal curve passing through all the points minimizes $g(C)$, but leads to very high costs for building it. Thus it is necessary to consider constrained problems. In this scene, the usual constraints on the route to be constructed arise from two factors. First, the number of bends (vertices or corners) of the polygonal curve plays an important role in the design of paths in robotics or electronic design problems. On the other hand, in the routing context, the length of the polygonal route could be more important than the number of vertices. Therefore, we will refer to the Bend(Length)-Constrained Min-max Problem:

$$\min_{C} \max_{p_i \in \mathcal{P}} d(p_i, C) \quad s.t. \quad b(C) \leq k \quad (l(C) \leq l_0)$$

A great number of such problems has been solved from the Computational Geometry viewpoint. In fact, a problem closely related to the search of bend-constrained min-max polygonal facilities is that of the approximation of polygonal curves. In various situations and applications, images of a scene have to be represented at different resolutions. A topic studied in Computational Geometry and applied to approximate boundaries of complicated figures in cartography, pattern recognition and graphic design [110, 18, 23], is that of approximating piecewise linear curves by more simple ones.

Among the research undertaken in these fields, [74, 140, 104, 20, 62] can be selected, in which the problem of approximating a given polygonal curve by another has been studied. In these papers, the vertices of the new curve are assumed to have either the same abscissas as the given vertices in P or they consist of a subset of the vertices of the original polygonal curve.

In fact two types of approximation problems have been solved,

Min-# problem: Given $\epsilon \geq 0$, find an approximating polygonal curve C with minimum number of bends whose error is not greater than ϵ.

Min-ϵ problem: Given k, find an approximating polygonal curve C minimizing the error among those with a bend number not greater than k.

This type of problems admits several variants that arise when imposing constraints on the location of the vertices or considering various types of errors [74]. In order to remain focused on facility location, two approximation errors are considered, $e_2(C) = \max d_2(p_i, C)$ and $e_v(C) := \max d_v(p_i, C)$ when $d_2(p_i, C)$ and $d_v(p_i, C)$ are respectively, the distance from the point to the polygonal curve induced by the Euclidean distance and the vertical distance.

Therefore, from the point of view of Facility Location Analysis, the Min-ϵ problem with d_2 and d_v distances becomes an *Euclidean min-max problem* and a *Fitting min-max problem*, respectively. The idea proposed in [30] is to use the known methods in Computational Geometry to approximate the polygonal chain $\{a = p_0, p_1, \ldots, p_M, p_{M+1}\}$ whose vertices are the demand points. In this way, since the set $\mathcal{P}$ is given in lexicographical order, a x-monotone polygonal facility can be found.

The general method for solving the Bend-Constrained Min-max Problem is proposed in [74] and works as follows: We first generate a set Γ of candidate error values, in such a way that the polygonal curve $\mathcal{C}^*$ we are looking for has one of these values as error. To each candidate e in the set we can associate the minimal number of bends of a polygonal curve $\mathcal{C}(e)$ that can be constructed with error at most e (the solution of a Min-# problem). Finally, we look for the smallest $e \in \Gamma$ whose associated length is not greater than k, and so get $\mathcal{C}^* = \mathcal{C}(e)$.

The most efficient algorithms for the Euclidean distance problems were devised by Chan and Chin [20]. They give an $O(M^2)$ and an $O(M^2 log M)$ time complexity algorithm for the Min-# and Min-ϵ problem, respectively. They further show that if the curve to be approximated forms part of a convex polygon, the two problems can solved in $O(M)$ time. Note that the Euclidean case has only been solved when the vertices or bends of new polygonal curve are a subset of the original set of points. This is the so-called *discrete k-bend polygonal curve*.

On the other hand, the problem with respect to the vertical distance appears in several and important disciplines with applications. Not only in operational research and location theory, but also in Statistics, computer to graphics or artificial intelligence the vertical distance is considered. In this case we refer to fitting a polygonal curve to a point set.

The k-bend constrained min-max problem with the vertical distance was posed in [63]. They solve two variants of the problem: when C is required to have its vertices on points in $\mathcal{P}$, the *discrete problem*, and when its bends can be on any point in the plane, the so-called *free problem*. For both of them, they have devised $O(M^2 log M)$ time algorithms which do not work in the presence of degeneracies, i.e. they do not admit points with the same x-coordinate. The approach is similar to the general method used in [74].

However, in the context of Facility Location, the points of $\mathcal{P}$ (potential users) are to be found in any position. For this reason, a dynamic programming procedure is applied in [33] to remove the non-degeneracy assumption in the discrete case. Besides, a nice observation made here is that the algorithm to

solve the Discrete k-Bend Constrained Min-Max problem can be adapted to find the length constrained min-max problem.

The location of a special type of a polygonal chain is considered in [30]. A rectilinear path is a path consisting of only vertical or horizontal segments. Generally, this kind of path appears in problems involving transportation routing design with applications such as floor planning, manufacturing environment design, VLSI layout design, robot moving, etc. In this sense, a Computational Geometry topic is that of finding shortest paths in the presence of obstacles. Even though these problems cannot be considered as location problems, they are useful as tools.

There exists a characterization property of the k-Bend Constrained Problem with vertical distance, proved in [36].

Theorem 18 *There exists a solution to the rectilinear polygonal center problem constrained by the bend number, such that each horizontal segment is a segment center of its allocation set and for each vertical segment there exists a point $p \in \mathcal{P}$ with the same abscissa.*

By enumerating all candidate rectilinear paths of Theorem 18, a polynomial time (but not efficient) algorithm is possible.

However, in [37], this property is used to find efficient algorithms for two instances of the min-max problem. In fact, min-max problems of location a monotone rectilinear route with constraints both on the bend number $b(R)$ and on the length $l(R)$ when using the vertical distance have been studied:

$$\min_{C} \max_{p_i \in \mathcal{P}} d_v(p_i, R) \quad s.t. \quad b(R) \leq b_0 \quad (l(R) \leq l_0).$$

In the second case (with length constraint), an $O(M^2)$ time algorithm based on geometric properties of an optimum route has been designed. For the second problem, the property of centrality for the horizontal segments of a solution of Theorem 18 is crucial and the search on a finite set of candidate routes devises an $O(M^2 log M)$ time algorithm. Finally, both problems can be solved in $O(M log^2 M)$ time [31], by using the *Parametric Search Technique* of Megiddo [99].

Now, we turn our attention to maximize the function $h(C)$. This concerns the location of obnoxious routes that has recently been considered by the Facility Location community. Most of the papers deal with models within an underlying discrete space. Therefore, when the path has to be integrated into a network, shortest path [11], multiobjective [1] and other problems considering

26

the possibility of road accidents [11], [15] have been studied. However, in the continuous case, in which the path can be located anywhere, the research has made very little progress. First of all, it is necessary to consider a restriction on the path, otherwise the problem admits a trivial solution. Two problems arise when considering constraints either on the spacial situation or on the length of the polygonal curve, namely:

- *the region-constrained problem:* given a polygonal region $\mathcal{R}$ containing the point set $\mathcal{P}$, finding a polygonal curve path C within $\mathcal{R}$ maximizing $h(C)$, i.e.,

$$\max_{C \subset \mathcal{R}} \min_{p_i \in \mathcal{P}} d(p_i, C).$$

- *the length-constrained problem:* given a positive value l_0, finding a polygonal curve path C with length-bound l_0 maximizing $h(C)$, i.e.,

$$\max_{l(C) \leq l_0} \min_{p_i \in \mathcal{P}} d(p_i, C).$$

Note that the number of bends (k) is not fixed in the input of such problems. For the Euclidean region-constrained problem, Drezner and Wesolowsky [44] provide an approximate algorithm for calculating a polygonal curve route C in a polygonal region with entry and exit segments. A similar approximation procedure with rectilinear distance was studied by Hinojosa [68]. However, in a recent paper [35], Díaz-Báñez and Hurtado devise an exact $O(M \log M)$ time algorithm for this problem by using Voronoi Diagrams.

The *length-constrained problem* can be solved with an approximation algorithm for general polygonal curves and by an $O(M \log M)$ time algorithm when the polygonal has only one free-corner [35].

6.2   1-Bend polygonal curves

A particular and important case of polygonal curve location is worth mentioning. The approximation of a point set by a 1-bend polygonal curve has interesting applications in approximation theory and in statistics. For instance, we can think of min-max approximation by two anchored lines, assuming that the whole population is split into two unknown groups with distinct characteristics.

First, we refer to the use of the Euclidean distance. The so-called double-ray center problem is defined as follows. Given a set of M points $\mathcal{P}$ in the plane, we want to find a configuration, $\mathcal{C} = (O, r_1, r_2)$, consisting of a point O in the plane and two rays, r_1, r_2, emanating from O, such that the Hausdorff distance

from $\mathcal{P}$ to $\mathcal{C}$ is minimized. The Hausdorff distance from $\mathcal{P}$ to $\mathcal{C}$ is defined by

$$h(\mathcal{P}, \mathcal{C}) = \max_{p \in \mathcal{P}} \min[d_2(p, r_1), d_2(p, r_2)]$$

where $d_2(p, r)$ denotes the Euclidean distance between the point p and the ray r. The distance between a point p and a ray r starting at point O is defined by the distance between p and the line l through r if the perpendicular line to l through p intersects r, it is the distance between O and p otherwise.

In the paper [60] an algorithm that finds a double-ray center of $\mathcal{P}$ is proposed. (Note that the double-ray configuration is not necessarily unique). The main theoretical result allows to modify a given double-ray configuration $\mathcal{C}$ with distance $d = h(\mathcal{P}, \mathcal{C})$ to a special double-ray configuration with distance d.

Theorem 19 *Let $\mathcal{C}$ be a double-ray configuration $\mathcal{C} = (O, r_1, r_2)$ with distance $d = h(\mathcal{P}, \mathcal{C})$. Then, either*

1. *$\mathcal{P}$ is of width not greater than $4d$, or*

2. *$\mathcal{P}$ can be covered by two parallel strips of width $2d$ each, say (L_1, L_2) and (L_3, L_4), such that both L_1 and L_4 pass through a point of $CH(\mathcal{P})$, and*

   (a) *Either L_1 or L_4 passes through an edge of $CH(\mathcal{P})$, or,*

   (b) *One of the strips has points of $\mathcal{P}$ on both of its boundaries, or*

3. *There exists a configuration $\mathcal{C}' = (O, r_1', r_2')$ with distance d, such that there are four points of $\mathcal{P}$ with distance d to $\mathcal{C}'$.*

Theorem 19 shows that the cardinality of the sequence $\mathcal{D}$ of all possible values of Hausdorff distance d is $O(M^4)$. Hence, we can find the optimal distance d^* by performing a binary search over $\mathcal{D}$. However, in order to produce an efficient algorithm an implicit searching technique for locating d^* by using the Megiddo's parametric search technique can be applied . In [60] it was shown that the sequential version of the decision algorithm runs in time $O(M^3 \alpha(M))$ where $\alpha(M)$ is the inverse Ackermann function and the parallel version runs in $O(\log M)$ time using $O(M^3)$ processors. Thus, by applying parametric search this yields an $O(M^3 \alpha(M) \log^2 M)$ time algorithm for the double-ray center problem.

On the other hand, the most efficient algorithms for finding 1-bend polygonal curves are developed in a recent paper. In fact, the paper [34] corresponds to variations of the problem of fitting 1-bend polygonal chains where vertical distance is used instead of the Euclidean distance. Although the restriction that

the chain must start and end at specified anchor points a and b is imposed, it is shown that the algorithms can be extended to deal with non anchored polygonal curves.

Several problems arise by considering constraints on the bend position or on the endpoints of the chain. In [34], an $O(M log M)$ time algorithm for the 1-bend discrete min-max problem, and an $O(M log M)$ time algorithm for the 1-bend free min-max problem is proposed, none of them making degeneracy assumptions. Besides, the 1-bend discrete case can be solved within the same time bound as the problem in which a and b are not fixed but must both satisfy a feasible set of linear constraints.

Such problems lead to algorithms of quadratic complexity as it is shown in [30]. However, by using more structure and suitable incremental updating, more efficient solutions are possible. A technical result, crucial for the procedure, that uses the convex hull of the point set $CH(\mathcal{P})$ is the following;

Theorem 20 *Let $\mathcal{P}$ be a point set, and let ℓ and r be the leftmost point and the rightmost point in $\mathcal{P}$, respectively. The following properties hold:*

(1) *The maximum distance between $\mathcal{P}$ and the line ℓr is attained at some vertex of the polygon $CH(\mathcal{P})$;*

(2) *furthermore, the distance function to the line ℓr is unimodal along the boundary of the upper hull, and analogously along the lower hull of P.*

(3) *Let r^* be the ray from ℓ which minimizes the maximum distance to the set $\mathcal{P}$: this maximum distance must be attained by one vertex from the upper hull and another one from the lower hull.*

Consequently, if $CH(\mathcal{P})$ is already available in a given suitable data structure, r^* can be computed in $O(\log^2 M)$ time using nested binary search. Nevertheless, a technique that allows to avoid nested binary search in some cases is the *tentative prune-and-search* described in [80]. In this paper, problems that involve searching for a special k-tuple with one element drawn from each of k lists are considered, and general techniques for computing fixed-points of monotone continuous functions efficiently are provided. For one function binary search is enough, for three functions they develop the tentative prune-and-search technique, and for two functions they prove that standard prune-and-search solves the problem. The later result is repeatedly used in [34] for computing r^* in $O(\log M)$ time. This procedure helps to calculate the solutions of 1-bend polygonal curve problems with vertical distance in $O(M log M)$ time.

7 Other problems, further research and conclusions

It can be appreciated that two kinds of problems have not almost been researched. The first one is the planar location of curves different from circles. The papers [?], [?] deal with circle, hyperbola and ellipse fitting problems. However the function to be minimized is the sum of vertical (orthogonal) squared distances and he proposes iterative algorithms.

The second one consist in the location of two-dimensional objects in a planar setting. In this case the shape of the facility may be fixed or not, but some geometric features have to be imposed. For orientated rectangles and non-finite demand sets, in [?] have been considered the problem of locating and determining the dimensions of the rectangle that minimizes the average distance to the given set. By using dynamic Voronoi diagrams, in [?] the problem of locating a convex polygon inside a given polygon (which may be seen as the border of the location of the population affected by the facility) with a convex shape, so that it maximizes the minimum Euclidean distances between pairs of points, one on the polygon to be placed and the other on the given polygon. The time complexity of the resulting algorithm depends on the numbers of edges of both polygons as well as the maximum length of a kind of Davenport-Schinzel sequences.

––––––

For polygonal curves location, different problems occur when the demand is assumed to be continuously distributed in a certain region of $\mathbb{R}^2$. In this case, Voronoi diagrams have been used (Okabe et al. [108]). Indeed, Takeda [136] proposes a computational method for the case that the demand is uniform over a region. The objective function is the average travel cost to the nearest point on the service line and the length of the facility is restricted to be smaller than a given bound.

References

[1] ABKOWITZ, M. AND CHENG, M., Developing a Risk/Cost Framework for Routing Truck Movements of Hazardous Materials. *Accident Analysis and Prevention*, 20, 1988, 39–52.

[2] AGARWAL, P.K., ARONOV, B. AND SHARIR, M., Computing Envelopes in Four Dimensions with Applications. *SIAM Journal on Computing*, 26, 1997, 1714–1732.

[3] AGARWAL, P., ARONOV, B., AND SHARIR, M. Computing lower envelopes in four dimensions with applications. In *Proceedings 10th Annual Symp. on Computational Geometry*, pages 348–358, 1994.

[4] AGARWAL, P.K., EFRAT, A., SHARIR, M. AND TOLEDO, S., Computing a Segment Center for a Planar Point Set. *Journal of Algorithms*, 15, 1993, 314–323.

[5] AGARWAL, P.K. AND SHARIR, M., Off-line dynamic maintenance of the width of a planar point set *Computational Geometry: Theory and Applications*, 1, 1991, 65–78.

[6] AGARWAL, P.K. AND SHARIR, M., Planar Geometric Location Problems. *Algorithmica*, 11, 1994, 185–195.

[7] AGARWAL, P., AND SHARIR, M. Efficient randomized algorithms for some geometric optimization problems. *Discrete Computational Geometry*, 16, 1996, 317–337.

[8] AGARWAL, P.K., SHARIR, M. AND TOLEDO S., Applications of Parametric Searching in Geometric Optimization. *Journal of Algorithms*, 17, 1994, 292–318.

[9] AGARWAL, P.K., SHARIR, M. AND TOLEDO, S., Applications of parametrics searching in geometric optimization. *Journal of Algorithms*, 17, 1994, 292–318.

[10] AVIS, D., ROBERT, J.M., AND WENGER, R. Lower bounds for line stabbing. *Information Processing Letters*, 33, 1989, 59–62.

[11] BATTA, R. AND CHIU, S., Optimal Obnoxious Paths on a Network: Transportation of Hazardous Materials. *Opns. Res.*, 36, 1988, 84–92.

[12] BELLMAN R., *Dynamic Programming*. Princeton University Press, 1957.

[13] BHATTACHARYA B.K. AND TOUSSAINT G.T., On geometric algorithms that use the furthest-point Voronoi-diagram. In *Computational Geometry*, editor G. T. Toussaint, North Holland, 1985, 43–61.

[14] BOFFEY, T.B., COX, M.G., DELVES, L.M., MOHAMED, J.L. AND PURGLOVE, C.J., Fitting Spheres to Data. Technical Report, National Physical Laboratory (UK), 1989.

[15] BOFFEY, B. AND KARKAZIS, J., Optimal Location of Routes for Vehicles: Transporting Hazardous Materials. *European J. Oper. Res.*, 1995, 201–215.

[16] BOISSONNAT, J.D. AND LAUMAND J.P. (EDS.), *Geometry and Robotics*, 391, Lectures Notes in Computer Science, Springer-Verlag, 1989.

[17] BRIMBERG, J., JUEL, H., AND SCHÖBEL, A. Linear Facility Location in Three Dimensions - Models and Solution Methods. *Report in Wirtschaftsmathematik, University of Kaiserslautern 63 submitted*, 2000

[18] BURT, P.J., Fast filter transforms for image processing. *Computer Graphics and Image Processing*, 16, 1979, 20–51.

[19] CARRIZOSA, E., AND PLASTRIA, F., Dominators for quasiconvex multiple objective maximisation. Journal of Global Optimization 18, 2000, 35-58.

[20] CHAN, W. S. AND CHIN, F., Approximation of polygonal curves with minimum number of line segments or minimun error. *International Journal of Computational Geometry & Applications*, 6, 1, 1996, 59–77.

[21] CHARNES, A., COOPER, W.W., AND FERGUSON, R.O., Optimal estimation of executive compensation by linear regression. *Management Science*, 1, 1955, 138–151.

[22] CHAZELLE, B., EDELSBRUNNER, H., GUIBAS, L., AND SHARIR, M. Diameter, width, closest line pair, and parametric searching. *Discrete Computational Geometry*, 10, 1993, 183–196.

[23] CHIN, F., CHOI, A. AND LUO, Y., Optimal Generating Kernel for Image Pyramids by Piecewise Fitting. *IEEE Trans. Pattern Anal. Machine Intell.*, 14, 12, 1992, 1190–1198.

[24] CLARKSON, K. L., KAPOOR, S. AND VAIDYA, P. M., Rectilinear shortest paths through polygonal obstacles in $O(nlog^{\frac{3}{2}}(n))$ time. In *Proccedings of the 3rd Annual ACM Symposium on Computational Geometry* (Waterloo, Ontario, Canada). ACM, New York, 1987, 251–257.

[25] DAI, Y., SHI, J. AND YAMAMOTO, Y., Global Optimization Problem with Multiple Reverse Convex Constraints and its Application to Out-of-roundness Problem. *Journal of the Operational Research Society of Japan*, 39, 1996, 356–371.

[26] DEY, T.K., Improved bounds for planar k-sets and related problems. *Discrete Computational Geometry*, 19, 1998, 373–382.

[27] DOROSHKO, N.N., The solution of two mainline problems in the plane, 1984. Manuscript deposited in VINITY USSR, 14.06.1984, Reg.no. 39614, 15 pages.

[28] DE BERG, M., BOSE, J., BREMMER, D. RAMASWAMI, S. AND WILFONG, G., Computing Constrained Minimum-width Annuli of Point Sets. *Lectures Notes Computer Science*, 1272, 1997, 3–16.

[29] DEREZENDE, P. J., LEE, D.T. AND WU, Y.F., Rectilinear shortest paths with rectangular barriers. *Discrete and Computational Geometry*, 4, 1989, 41–53.

[30] DÍAZ-BÁÑEZ, J.M., *Location of Linear and Piecewise Linear Structures*, Ph.D. Thesis (in Spanish), Universidad de Sevilla. Sevilla. Spain. 1998.

[31] DÍAZ-BÁÑEZ J.M., Approximating a point set by rectilinear paths (in Spanish), *VII Meeting of Computational Geometry*, Castellón, Spain, July, 1999.

[32] DÍAZ-BÁÑEZ, J.M. AND DÍAZ, P., The Half-line Center Problem with $l_\infty(l_1)$ Metrics. *Studies in Locational Analysis*, vol 15, 200, 83–97.

[33] DÍAZ-BÁÑEZ, J.M., GÓMEZ F. AND HURTADO, F., Some Problems on Approximation of Set of Points by Polygonal Curves. *Proc. Tenth Canadian Conference on Computational Geometry (CCCG98)*, Montreal, Canada, 1998, 10–12.

[34] DÍAZ-BÁÑEZ, J.M., GÓMEZ F. AND HURTADO, F., Approximation of Point Sets by 1-Corner Polygonal Chains. *INFORMS Journal on Computing*, vol 12, 4, 2000, 317-323.

[35] DÍAZ-BÁÑEZ J.M. AND HURTADO F., Geometric Approach of Optimal Obnoxious Routes. *Manuscript*, 1999.

[36] DÍAZ-BÁÑEZ J.M. AND MESA J.A., Location of rectilinear center trajectories. *Top*, vol 6, 2, 159–177, 1998.

[37] DÍAZ-BÁÑEZ J.M. AND MESA J.A., Fitting Rectilinear Paths to a Set of Points in the Plane. *European J. Oper. Res.*, vol 130, 1, 2001, 214-222.

[38] DOBKIN, D.P., Computational Geometry and Computer Graphics. *Proc. IEEE,* vol 80, 9, 1992, 1400–1411.

[39] DREZNER, Z., On minimax optimization problems. *Mathematical Programming*, 22, 1982, 227–230.

[40] DREZNER, Z., *Facility Location: A Survey of Applications and Methods.* Editor Zvi Drezner, Springer, 1995.

[41] DREZNER, Z., GAVISH, B., $\epsilon-$approximations for Multidimensional Weighted Location Problem. *Operations Research*, 33, 1985, 772–783.

[42] DREZNER, Z., STEINER, G. AND WESOLOWSKY, G.O., One Facility Location with Rectilinear Tour Distances. *Naval Research Logistics Quarterly*, 32, 1985, 391–405.

[43] DREZNER, Z., STEINER, S. AND WESOLOWSKY, G.O., On the Circle Closest to a Set of Points. *Technical Report, California State University, Dept. of MSIS*, 1996.

[44] DREZNER, Z. AND WESOLOWSKY, G.O., Location of an obnoxious route. *Journal Operational Research Society*, 40, 1989, 1011–1018.

[45] DYER, M.E., Linear time algorithms for two- and three-variable linear programs. *SIAM Journal on Computing*, 13, 1984, 31–45.

[46] EBARA, H., FUKUYAMA, N., NAKANO, H. AND NAKANISHI, Y., Roundness Algorithms Using the Voronoi Diagrams. *First Canadian Conference on Computational Geometry*, 1989.

[47] EBARA, H., FUKUYAMA, N., NAKANO, H. AND NAKANISHI, Y., A Practical Algorithm for Computing the Roundness. *IEICE Transactions on Information and Systems*, E75-D, 1992, 253–257.

[48] EBARA, H., NAKANO, H., NAKANISHI, Y. AND SANADA, T., A Roundness Algorithms Using the Voronoi Diagrams. *Transactions IEICE*, J70-A 1987, 620–624.

[49] EDELSBRUNNER, H. Finding transversals for sets of simple geometric figures. *Theoretical Computer Science*, 35, 1985, 55–69.

[50] EFRAT A. AND SHARIR M., A Near-Linear Algorithm for the Planar Segment-Center Problem. *Discrete and Computational Geometry*, 16, 1996, 239–257.

[51] ELZINGA J. AND HEARN D.W., The Minimum Covering Sphere Problem. *Management Science*, 19, 1972, 96–104.

[52] ERKUT, E. AND NEWMANN S., Analytical Models for Locating Undesirable Facilities. *European J. Oper. Res.*, 40, 1989, 275–291.

[53] ERKUT, E. AND NEWMANN S., Hazardous materials logistics. In *Facility Location: a Survey of Applications and Methods,* (ed. Z. Drezner, Springer, New York, 1995, 467–506.

[54] FISHER, W.D. A note on curve fitting with minimum deviations by linear programming. *Journal Amer. Statist. Association*, 56, 1961, 359–362.

[55] FOLLERT, F. Maxmin location of an anchored ray in 3-space and related problems. *7th Canadian Conference on Computational Geometry, Quebec*, 1995.

[56] FOLLERT, F., SCHÖMER, E., SELLEN, J., SMID, M. THIEL, C. Computing a largest empty anchored cylinder and related problems. *Technical Report MPI-91-1-001, Max-Planck-Institut für Informatik, Saarbrücken*, 1995.

[57] GARCÍA-LÓPEZ, J., RAMOS, P. AND SNOEYINK, J., Fitting a Set of Points by a Circle. *Discret Computational Geometry*, 20, 1998, 389–402.

[58] GAREY, M.R. Y JOHNSON, D.S., *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W.H. Freedman and Company, NJ, 1979.

[59] GLOZMAN, A., KEDEM, K. AND SHPITALNIK, G., On some geometric selection and optimization problems via sorted matrices. *Proceedings 4th Workshop on Algorithms and Data Structures, Lecture Notes in Computer Science* 955, 1995, 26–37.

[60] GLOZMAN, A., KEDEM, K. AND SHPITALNIK, G., Computing a Double-Ray Center for a Planar Point Set. *International Journal of Computational Geometry & Applications*, 9, 2, 1999, 109–124.

[61] GOODMAN, F. AND O'ROURKE J. (EDS.), *Handbook of Discrete and Computational Geometry*, CRC, New York, 1997.

[62] GUIBAS, L.J., HERSHBERGER, J.E., MITCHELL, J.S.B. AND SNOEYINK, J.S., Approximating Polygons and Subdivisions with Minimum- Link Paths. *International Journal of Computational Geometry & Applications*, 3, 1993, 383–415.

[63] HAKIMI, S.L. AND SCHMEICHEL, E.F., Fitting polygonal functions to a set of points in the plane. *Graphical Models y Image Processing*, 53, 2, 1991, 132–136.

[64] HAKIMI, S.L., LABBÉ, M. AND SCHMEICHEL, E.F., The Voronoi Partition of a Network and Its Implications in Location Theory. *ORSA Journal on Computing*, vol 4, 4, 1992, 412–417.

[65] HELLY, E., Über Mengen konvexer Körper mit gemeinschaftlichen Punkten. *Jahrbuch der Deutschen Mathematiker Vereinigung*, 32, 1923, 175–176.

[66] HERSHBERGER, J., Finding the upper envelope of n line segments in O($n \log n$) time. *Information Processing Letters*, 33, 1989, 169–174.

[67] HERSHBERGER, J., AND SURI, S. Finding tailored partitions. *Journal of Algorithms*, 12, 1991, 431–463.

[68] HINOJOSA, Y., Localización de una ruta lineal peligrosa (In Spanish).*Prepublicaciones de la Facultad de Matemáticas de la Universidad de Sevilla*, 9, 1995, 32–43.

[69] HINOJOSA, Y. AND PUERTO, J., The polyhedral norm approach to the problem of locating obnoxious routes. *Studies in Locational Analysis*, 12, 1999, 49–66.

[70] M. HOULE AND A. MACIEL Finding the widest empty corridor through a set of points. *In Snapshots of Computational and Discrete Geometry, Dept. of Computer Science, McGill University* Montreal, Canada, 1988, 201–213.

[71] HOULE, M.E. AND TOUSSAINT, G.T., Computing the width of a set, *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol 10, 5, 1988, 760–765.

[72] HOULE, M.E., IMAI H., IMAI K., ROBERT J.-M. AND YAMAMOTO P. Orthogonal weighted linear l_1 and l_∞ approximation and applications. *Discrete Applied Mathematics*, 43, 1993, 217–232.

[73] IMAI H. AND IRI M., An optimal Algorithm for Approximating a Piecewise Linear Funtion. *Journal of Information Processing*, vol 9, 3, 1986, 159–162.

[74] IMAI H. AND IRI M., Polygonal Approximations of a Curve-Formulations and Algorithms. In *Computational Morphology.* G. T. Toussaint ed., North Holland, 1988.

[75] IMAI, H., KATO, K., AND YAMAMOTO, P. A linear-time algorithm for linear L_1 approximation of points. *Algorithmica*, 4, 1989, 77–96.

[76] IMAI, H., LEE, D. T. AND YANG, CH., 1-Segment Center Problem. *ORSA Journal on Computing*, 4, 1992, 426–434.

[77] JAROMCZYK, J.W. AND KOWALUK, M., The two-line center problem from a polar view: a new algorithm and data structure. *Proceedings 4th Workshop on Algorithms and Data Structures, Lecture Notes in Computer Science* 955, 1995, 13–25.

[78] KARST, O.J., Linear curve fitting using least deviations. *Journal Amer. Statist. Association*, 53, 1958, 118–132.

[79] KATZ, M.J. AND SHARIR, M. Optimal slope via expanders. *Information Processing Letters*, 47, 1993, 115–122.

[80] KIRKPATRICK, D., J. SNOEYINK., Tentative prune-and-search for computing fixed points with applications to geometric computation. *Fundamenta Informaticae*, 22, 1995, 353–370.

[81] KORNEENKO, N.M. AND MARTINI, H., Approximating finite weighted point sets by hyperplanes. *Lecture Notes Computer Science*, 447, 1990, 276–286.

[82] KORNEENKO, N.M. AND MARTINI, H., Hyperplane Approximation and Related Topics. In *New Trends in Discrete and Computational Geometry.* János Pach Ed., chapter 6, pages 135–162. Springer-Verlag, New York, 1993.

[83] KUROZUMI Y. AND DAVIS, W.A., Polygonal Approximation by the Minimax Method. *Computer Graphics and Image Processing*, 19, 1982, 248–264.

[84] LABBÉ M., Location of an Obnoxious Facility on a Network: A Voting Approach. *Network*, 20, 1990, 197–207.

[85] LAPORTE, G., MESA, J.A. AND ORTEGA, F., Assessing Topological Configuration for Rapid Transit Networks. *Studies in Locational Analysis*, 7, 1994, 105–121.

[86] LEE, D.T. AND CHING, Y.T., The power of geometric duality revisited. *Inform. Process. Letters*, 21, 1985, 117–122.

[87] LEE D. T. AND LEE V. B., Out-of-roundness problem revisited. *IEEE Transactions on Pattern Analysis and Machinery Intelligence*, 13, 1991, 217–223.

[88] LEE D. T. AND PREPARATA F. P., Euclidean shortest paths in the presence of rectilinear barriers. *Networks*, 14, 1984, 393–410.

[89] LEE, D.T. AND WU, Y.F. Geometric Complexity of Some Location Problems. *Algorithmica*, 1, 1986, 193–211.

[90] LOVE, R.F., MORRIS, J.G., AND WESOLOWSKY, G.O. *Facilities Location*, chapter 3.3, pages 51–60. North-Holland, Amsterdam, 1988.

[91] LOVE, R.F., WESOLOWSKY, G.O. AND KRAEMER, A Multifacility Minimax Location Method for Euclidean Distances. *International Journal of Production Research* 11, 1988, 32–40.

[92] MARTINI, H., Minimsum k-flats of finite point sets in $\mathbb{R}^{\mathrm{d}}$. *Studies in Locational Analysis*, 7, 1994, 123–129.

[93] MARTINI, H. AND SCHÖBEL, A., A characterization of smooth norms. *Geometriae Dedicata*, 77, 1999, 173–183.

[94] MARTINI, H. AND SCHÖBEL, A., Median hyperplanes in normed spaces — a survey. *Discrete Applied Mathematics*, 89, 1998, 181–195.

[95] MARTINI, H. AND SCHÖBEL, A., Median and Center hyperplanes in Minkowski spaces — a unifying approach. *Discrete Mathematics*, 2001, to appear.

[96] MARTINI, H. AND SCHÖBEL,A., Hyperplane transversals of homothetical, centrally symmetric polytopes. *Hungaria Acta Mathematica*, 39, 1999, 73–81.

[97] MCKINNON, R.D., AND BARBER, G.M., A new approach to network generation and map representation: The linear case of the location-allocation problem. *Geographical Analysis*, 4, 1972, 156–168.

[98] MEGIDDO, N., The Weighted Euclidean 1-center Problem. *Mathematics of Operations Research*, 8, 1983, 498–504.

[99] MEGIDDO, N., Applying parallel computation algorithms in the desing of serial algorithms. *J. Assoc. Comput. Mach.*, 30, 1983, 852–865.

[100] MEGIDDO, N., Linear-time Algorithms for Linear Programming in $\mathbb{R}^3$ and Related Problems. *SIAM Journal on Computing*, 12, 1983, 759–776.

[101] MEGIDDO, N., Linear Programming in Linear Time when the Dimension is Fixed. *Journal of the Association of Computing Machinery*, 31, 1984, 114–127.

[102] MEGIDDO, N. AND TAMIR, A., On the Complexity of Locating Linear Facilities in the Plane. *Opns. Res. Letters*, 1, 1982, 194–197.

[103] MEGIDDO, N. AND TAMIR, A., Finding least-distance lines. *SIAM J. on Algebraic and Discrete Methods*, 4(2), 1983, 207–211.

[104] MELKMAN, A. AND O'ROURKE, J., On Polygonal Chain Approximation. *Computational Morphology*. G. T. Toussaint ed., North Holland, 1988.

[105] MINKOWSKI,H., Gesammelte Abhandlungen, Band 2, Chelsea Publishing Company, New York, 1967.

[106] MORRIS, J.G. AND NORBACK, J.P., A simple approach to linear facility location. *Transportation Science*, vol 14, 1, 1980, 1–8.

[107] MORRIS, J.G. AND NORBACK, J.P., Linear Facility Location - Solving Extensions on the Basic Problems. *European Journal of Operational Research*, 12, 1983, 90–94.

[108] OKABE, A., BOOTS, B AND SUGIHARA, K., *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*, John Wiley & Sons, 1992.

[109] PACH, J., STEIGER, W. AND SZEMEREDI, E., An upper bound on the number of planar k–sets. *Discrete Comput. Geometry*, 7, 1992, 109–123.

[110] PAVLIDIS T., Algorithms for shape analysis of contours and waveforms. *IEEE Trans. Pattern Anal. Mach. Intell.,* Pami-2, 1980, 301–312.

[111] R.R. PHELPS, Convex Functions, Monotone Operators and Differentiability, Lecture Notes in Mathematics 1364, Springer, Berlin 1989.

[112] PLASTRIA F., Fully Geometric Solutions to Some Planar Minimax Location Problems. *Studies in Locational Analysis*, 7, 1994, 171–183.

[113] PLASTRIA F., Continuous location problems. *Facility location,* Drezner, Z. ed., Springer Verlag, Berlin, 1995, 225–262.

[114] Plastria, F. and Carrizosa E. Gauge-Distances Median Hyperplanes, *Journal of Optimisation Theory and Applications*, 110, 2001, *to appear.*

[115] PREPARATA, F.P. AND SHAMOS, M.I., *Computational Geometry: An Introduction.* Springer-Verlag, New York, 1985.

[116] RAO, M.R., AND SRINIVASAN, V., A note on Sharpe's algorithm for minimizing the sum of absolute deviations in a simple regression problem. *Management Science*, 19, 1972, 222–225.

[117] RIVLIN, T.J., Approximation by Circles. *Computing*, 1, 1972, 93–104.

[118] ROBERT, J.M. Linear Approximation and Line Transversals. *Ph.D. thesis, School of Computer Sciences, McGill University, Montreal*, 1991

[119] ROBERT, J.M. AND TOUSSAINT, G.T., Linear approximation of simple objects. *Computational Geometry*, 4, 1994, 27–52.

[120] ROBERT, J.M. AND TOUSSAINT, G.T., Computational Geometry and facility location. Technical Report SOCS 90.20, McGill Univ., Montreal, PQ, 1990.

[121] ROY, U. AND ZHANG, X., Establishment of a Pair of Concentric Circles with the Minimum Radial Separation for Assessing Roundness Error. *Computer-aided Design*, 24, 1992, 161–168.

[122] ROY, U. AND ZHANG, X., Development and Application of Voronoi Diagrams in the Assessment of Roundness Error in an Industrial Environment. *Computer Industrial Engineering*, 26, 1994, 11–26.

[123] SCHLOSSMACHER, E.J., An iterative technique for absolute deviations curve fitting. *Journal Amer. Statist. Association*, 68, 1973, 857–859.

[124] SCHÖBEL, A., Locating Least-Distant Lines with Block Norms. *Studies in Locational Analysis*, 10, 1996, 139–150.

[125] SCHÖBEL, A., Locating line segments with vertical distances. *Studies in Locational Analysis*, 11, 1997, 143–158.

[126] SCHÖBEL, A., Locating least distant lines in the plane. *European Journal of Operational Research*, 106, 1998, 152–159.

[127] SCHÖBEL, A., Solving restricted line location problems via a dual interpretation. *Discrete Applied Mathematics*, 93, 1999, 109–125.

[128] SCHÖBEL, A., Locating Lines and Hyperplanes – Theory and Algorithms. *Kluwer*, 1999.

[129] SCHÖBEL, A., Anchored hyperplane location problems. *Report in Wirtschaftsmathematik*, 74, University of Kaiserslautern, 2001.

[130] SHAMOS, M.I., *Computational Geometry.* Ph. D. Thesis, Dept. Computer Sci., Yale Univ., New Haven, 1978.

[131] SHAMOS, M.I. AND HOEY, D., Closest-point problems. *Proc. 16th Annual IEEE Symposium on Foundations of Computer Science, IEEE Computer Society Press, Los Angeles*, 1975, 151–162.

[132] SHARPE, W.G., Mean-absolute deviation characteristic lines for securities and portfolios. *Management Science*, 54, 1959, 206–212.

[133] SCHÖMER, E., SELLEN, J., TEICHMANN, M. AND YAP, CH. Efficient Algorithms for the Smallest Enclosing Cylinders Problem. *Proc. 8th Canadian Conference on Computational Geometry*, 1996, 264–269.

[134] SMID, M. AND JANARDAN, R., On the Width and Roundness of a Set of Points in the PLane. *Proc. 7th Canadian Conference on Computational Geometry*, 1995, 193–198.

[135] SWANSON, K., LEE, D.T. AND WU, V.L., An Optimal Algorithm for Roundness Determination of Convex Polygons. *Computational Geometry: Theory and Applications*, 5, 1995, 225–235.

[136] TAKEDA, S., *On geographical optimization and dynamic facility location problem.* Unpublished Master's Thesis, Depart. of Mathematical Engineering and Information Phisycs, University of Tokio (in Japanese), 1985.

[137] TOUSSAINT G. T., Pattern recognition and geometrical complexity. *Proc. 5th IEEE Internat. Conf. Pattern Recogn.*, 1980, 1324–1347.

[138] TOUSSAINT G. T., Solving geometric problems with the rotating calipers. *Proc. of IEEE MELECON, Athens, Greece*, 1983, A10.02-A10.05.

[139] TOUSSAINT G. T., Computing Largest Empty Circles with Location Constraints. *International Journal of Computer and Information Sciences*, 12, 1983, 347–358.

[140] TOUSSAINT G. T., On the Complexity of Approximating Polygonal Curves in the Plane. *Proc. IASTED, International Symposium on Robotics and Automation, Lugano, Switzerland,* 1985.

[141] VACCARO H., *Alternative Techniques for Modeling Travel Distances.* Thesis in Civil Enginnering, Massachusetts Institute of Technology, 1974.

[142] VENTURA, J.A. AND YERALAN, S., The Minimax Center Estimation Problem for Automated Roundness Inspection. *European Journal of Operational Research*, 41, 1989, 64–72.

[143] WAGNER, H.M., Linear programming techniques for regression analysis. *Journal Amer. Statist. Association*, 54, 1959, 206–212.

[144] WAINSTEIN, A.D., A Non-monotonous Placement Problem in the Plane. Software for Solving Optimal Planning Problems. *9th All-Union Symposium. Minsk. USSR. Symp. Abstracts*, 1986, 70–71.

[145] WESOLOWSKY, G.O., Location of the median line for weighted points. *Environment and Planning A*, 7, 1975, 163–170.

[146] WESOLOWSKY G.O., Rectangular Distance Location Under the Minimax Optimality Criterion. *Transportation Science*, 6, 1972, 103–113.

[147] YERALAN, S. AND VENTURA, J.A., Computerized Roundness Inspection. *International Journal of Production Research*, 26, 1988, 1921–1935.

[148] YAMAMOTO, P., KATO, K., IMAI, K., AND IMAI, H., Algorithms for vertical and orthogonal L_1 linear approximation of points. *Proceedings 4th Ann. Sympos. Comput. Geom.*, 1988, 352–361.

[149] ZEMEL, E., An O(n) algorithm for the linear multiple choice knapsack problem and related problems. *Information Processing Letters*, 18, 1984, 123–128.