ON SIMPSON MODULI SPACES OF STABLE SHEAVES ON $\mathbb{P}_2$ WITH LINEAR HILBERT POLYNOMIAL

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Preprint – Universität Kaiserslautern

Abstract. In this short note we prove some general results on semi-stable sheaves on $\mathbb{P}_2$ and $\mathbb{P}_3$ with arbitrary linear Hilbert polynomial. Using Beilinson's spectral sequence, we compute free resolutions for this class of semi-stable sheaves and deduce that if μ and χ are coprime the smooth moduli spaces $M_{\mu m+\chi}(\mathbb{P}_2)$ and $M_{\mu\, m+(\mu-\chi)}(\mathbb{P}_2)$ are birationally equivalent.

1. Introduction

Moduli of torsionfree semi-stable sheaves on $\mathbb{P}_2$ and $\mathbb{P}_3$ with fixed Hilbert polynomial were introduced by Maruyama and others. They have been intensively studied during the last decades. In 1994, Simpson [9] showed that the family of *arbitrary* semi-stable sheaves with fixed Hilbert Polynomial P on a smooth projective variety X is bounded. Using this, he proved the existence of a projective scheme $M_P(X)$ corepresenting the moduli functor $\mathcal{M}_P(X)(S)$ of S-flat coherent sheaves over $X \times S$ with semi-stable fibers $\boldsymbol{\mathcal{F}}_s$ and $P_{\boldsymbol{\mathcal{F}}_s} = P$. For $\dim(X) \geq 2$ and linear Hilbert polynomial $P(m) = \mu m + \chi$, id est if all the sheaves in $M_P(X)$ have torsion and are supported on degree μ curves, there is not much known about these spaces.

LePotier [7] proved that the coarse moduli spaces $M_{\mu m+\chi}(\mathbb{P}_2)$ are irreducible, locally factorial projective varieties of dimension $\mu^2 + 1$. They are rational at least if $\chi \equiv \pm 1 \pmod{\mu}$, $\chi \equiv \pm 2 \pmod{\mu}$ and for small multiplicities $\mu \leq 4$.

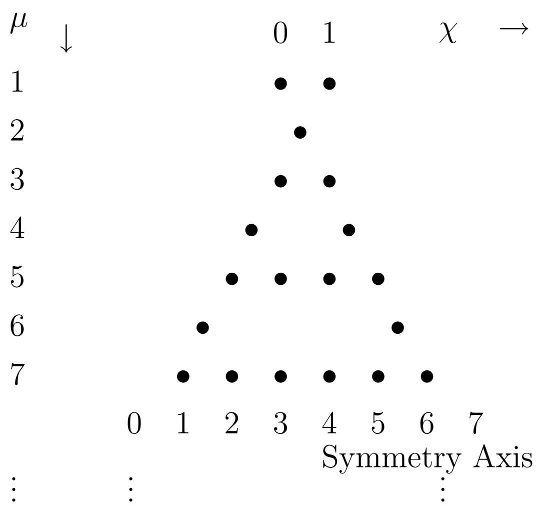
Furthermore, he described for $\mu \leq 4$ the geometrical properties of $M_{\mu m+\chi}(\mathbb{P}_2)$ and the birational map [6] to the Maruyama scheme $\mathcal{M}_{\mathbb{P}_2^\vee}(\mu; 0, \mu)$ of semi-stable, torsionfree rank μ sheaves with second Chern class μ on the dual projective plane $\mathbb{P}_2^\vee$.

We investigated in [1], [2] the geometry of $M_{3m+1}(\mathbb{P}_3)$ which has two smooth, rational components of dimension 12 and 13 intersecting each other transversally along an 11-dimensional smooth subvariety. It is in some sense the "smallest" example for a reducible Simpson space and plays a role similar to $\mathrm{Hilb}_{3m+1}(\mathbb{P}_3)$ in the case of Hilbert schemes.

Doing this, we noted as in [7] that in the planar case $M_{3m+1}(\mathbb{P}_2)$ and $M_{3m+2}(\mathbb{P}_2)$ are *both* isomorphic to the universal cubic $\mathcal{C} \longrightarrow \mathbb{P}_2$. This is not an accident and turned out to be part of a more general "symmetry" result which is the subject of this short note.

Date: 10/21/2001.

Figure 1. Schematic Picture. Each box corresponds to an $M_{\mu m+\chi}(\mathbb{P}_2)$.



•: The moduli space is fine.

: These two spaces in each row are isomorphic.

Theorem 1. *Let $P(m) = \mu m + \chi$, $0 < \chi \leq \mu$, μ and χ coprime, be a linear polynomial* [1], *and define its "dual" by $P^\nabla(m) := \mu m + \mu - \chi$. Denote by $N \subset M_P(\mathbb{P}_2)$ and $N^\nabla \subset M_{P^\nabla}(\mathbb{P}_2)$ respectively the closed subvarieties of isomorphism classes of sheaves with non-vanishing first cohomology. Then there is a natural isomorphism*

$$\Phi : M_P(\mathbb{P}_2) \setminus N \xrightarrow{\approx} M_{P^\nabla}(\mathbb{P}_2) \setminus N^\nabla.$$

Thus, the moduli spaces $M_P(\mathbb{P}_2)$ and $M_{P^\nabla}(\mathbb{P}_2)$ are birationally equivalent. Moreover, the spaces $M_{\mu m+1}(\mathbb{P}_2)$ and $M_{\mu m+\mu-1}(\mathbb{P}_2)$ are isomorphic.

Finally, we can extend LePotier's result cited above in a way certainly known to him:

Theorem 2. *If μ and χ are coprime, the fine Simpson moduli spaces $M_{\mu m+\chi}(\mathbb{P}_2)$ are* **smooth** *projective varieties of dimension $\mu^2 + 1$.*

The author would like to thank Günther Trautmann for useful discussions.

2. Preliminaries

We call the a projective scheme over an algebraically closed field k a *variety*. One can equip the support of a coherent sheaf $\mathcal{F}$ on a smooth variety X in several ways with the structure

[1]Note that $M_{\mu m+\tau}(\mathbb{P}_2) \cong M_{\mu m+\chi}(\mathbb{P}_2)$ if $\tau \equiv \chi \pmod{\mu}$ since the Hilbert polynomial involved is linear.

of a (not necessarily reduced) variety. One is using the annihilator ideal sheaf $\mathrm{Ann}(\mathcal{F}) \subset \mathcal{O}_X$. We write $Z_a(\mathcal{F}) := (\mathrm{Supp}(\mathcal{F}), \mathcal{O}_X/\mathrm{Ann}(\mathcal{F}))$. Another way is the following: Let

$$\bigoplus_{\mu=1}^{r} \mathcal{O}_X(-b_\mu) \xrightarrow{A} \bigoplus_{\nu=1}^{s} \mathcal{O}_X(-a_\nu) \to \mathcal{F} \to 0$$

be an arbitrary presentation of $\mathcal{F}$ and denote by $\mathrm{Fitt}_i(\mathcal{F}) \subset \mathcal{O}_X$ the ideal sheaf generated by the $(s-i)\times(s-i)$-minors of the homogeneous matrix A. Due to Fitting's lemma, the sheaf $\mathrm{Fitt}_i(\mathcal{F})$ does not depend on the choice of the presentation. Furthermore, one has

$$\mathrm{Fitt}_0(\mathcal{F}) \subset \mathrm{Ann}\,\mathcal{F} \quad \text{and} \quad (\mathrm{Ann}\,\mathcal{F})\,\mathrm{Fitt}_i(\mathcal{F}) \subset \mathrm{Fitt}_{i-1}(\mathcal{F}) \quad \forall\, i > 0$$

Now define

$$Z_f(\mathcal{F}) := (\,\mathrm{Supp}(\mathcal{F}), \mathcal{O}_X/\mathrm{Fitt}_0(\mathcal{F})\,) \hookrightarrow (\,X, \mathcal{O}_X\,)$$

$Z_a(\mathcal{F})$ is obviously a subvariety of $Z_f(\mathcal{F})$ and $Z_a(\mathcal{F})_{red} = Z_f(\mathcal{F})_{red} = \mathrm{Supp}(\mathcal{F})$.

Let X be a variety and S be a Noetherian (base-)scheme of finite type over k and call the projections from $X \times_k S$ to the first and second factor by q and p respectively. If $\mathcal{F} \in \mathrm{Coh}(X)$, $\mathcal{G} \in \mathrm{Coh}(S)$ and $\boldsymbol{\mathcal{H}} \in \mathrm{Coh}(X \times S)$ are coherent sheaves, we will write $\mathcal{F} \boxtimes \mathcal{G} := q^*\mathcal{F} \otimes p^*\mathcal{G}$, $\mathcal{F}(m) \boxtimes \mathcal{O}_S := q^*\mathcal{F}(m)$, $\boldsymbol{\mathcal{H}}_s := \boldsymbol{\mathcal{H}}|_{X\times\{s\}}$ and $\boldsymbol{\mathcal{H}}(m) := \boldsymbol{\mathcal{H}} \otimes q^*\mathcal{O}_X(m)$.

A purely 1-dimensional coherent sheaf $\mathcal{F}$ with linear Hilbert polynomial $P(m) = \mu\, m + \chi$ on a smooth variety X is called *semi-stable* resp. *stable* if for all proper coherent submodules $0 \neq \mathcal{F}' \subset \mathcal{F}$

$$\frac{\chi(\mathcal{F}')}{\mu(\mathcal{F}')} \le \frac{\chi}{\mu} \;\text{ resp. }\; \frac{\chi(\mathcal{F}')}{\mu(\mathcal{F}')} < \frac{\chi}{\mu}$$

$\mu(\mathcal{F})$ is called the *multiplicity* and $p(\mathcal{F}) := \frac{\chi}{\mu}$ the *slope* of the sheaf $\mathcal{F}$.

We collect now some properties of (semi-)stable sheaves supported on curves in the projective plane or projective space in the following theorem:

Theorem 3. *Let $\mathcal{F}$ be a semi-stable sheaf on $\mathbb{P}_n$, $n = 2, 3$, with linear Hilbert polynomial $P_{\mathcal{F}}(m) = \mu m + \chi$, $0 \le \chi < \mu$ and $C := Z_a(\mathcal{F})$ be its support.*

1. *$\mathcal{F}$ is Cohen-Macaulay, or equivalently: $\mathcal{F}$ has no zero-dimensional torsion.*
2. *If C is smooth then $\mathcal{F}$ is locally free. If C is integral $\mathcal{F}$ is still locally free on an open dense subset $U = C \setminus \{p_1, \dots p_r\}$.*
3. *Let $n = 2$. Then $(r; c_1, c_2) = (0; \mu, \frac{\mu\,(\mu+3)}{2} - \chi)$. If $n = 3$, we have $(r; c_1, c_2, c_3) = (0; 0, -\mu, 2\chi - 4\mu)$ In both cases, $r = rk_{\mathbb{P}_n}(\mathcal{F})$ denotes the rank and $c_i = c_i(\mathcal{F})$ are the Chern classes w.r.t. $\mathbb{P}_n$.*
4. *The not necessarily reduced curve $C \subset \mathbb{P}_n$ has no zero-dimensional components and no embedded points.*
5. *$\mu = \chi(\mathcal{F}|_H)$ where $H = Z(l) \in |\,\mathcal{O}_{\mathbb{P}_n}(1)\,|$ is $\mathcal{F}$-regular. Thus,*

$$\mu = h^0(\mathcal{F}|_H) = \sum_{p \in C\cap H} \dim_k(\mathcal{F}_p)$$

6. $\mu(\mathcal{O}_{C_{red}}) \leq \mu(\mathcal{O}_C) \leq \mu$ *and* $\mu(\mathcal{F} \otimes \mathcal{O}_{C_{red}}) \leq \mu$
7. *If* $\chi > 0$ *and* $(\chi, \mu) = \mathbb{Z}$ *then* $\mathcal{F}$ *is stable.*
8. *There are the following bounds for the cohomology and the Castelnuovo-Mumford regularity of the sheaf* $\mathcal{F}$*:*
   - $\chi \leq h^0\mathcal{F} \leq \mu - 1$.
   - $0 \leq h^1\mathcal{F} \leq \mu - \chi - 1$.
   - *reg*$(\mathcal{F}) \leq \mu - \chi$*, in particular* $H^1\mathcal{F}(i) = 0$ *for all* $i \geq \mu - \chi - 1$.

Proof. Cf. [1]. The only part which is not obvious is 8.: Let H be a $\mathcal{F}$-regular hyperplane. Then $0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}|_H \to 0$ induces an exact sequence

$$0 \to H^0\mathcal{F}(n-1) \to H^0\mathcal{F}(n) \xrightarrow{f_n} k^\mu \to H^1\mathcal{F}(n-1) \to H^1\mathcal{F}(n) \to 0 \quad \forall n \in \mathbb{Z} \tag{1}$$

This implies that $n \mapsto h^1\mathcal{F}(n)$ is decreasing and $\chi \leq h^0\mathcal{F} \leq h^0\mathcal{F}(-1)+\mu$. But $\mathrm{Hom}(\,\mathcal{O}_C(1)\,,\,\mathcal{F}\,)$ vanishes because of the semi-stability, and thus $\chi \leq h^0\mathcal{F} \leq \mu$.
Now assume that f_n is surjective. The commutative diagram

$$\begin{array}{ccccc} H^0\mathcal{F}(n) \otimes H^0\mathcal{O}(1) & \xrightarrow{f_n \otimes \mathrm{id}} & k^\mu \otimes H^0\mathcal{O}(1) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ H^0\mathcal{F}(n+1) & \xrightarrow{f_{n+1}} & k^\mu & & \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

implies that f_{n+1} is also a surjection. Therefore we get

$$H^1\mathcal{F}(n-1) \cong H^1\mathcal{F}(n) \cong H^1\mathcal{F}(n+1) \cong \cdots \cong 0$$

by Serre's theorem B. If f_n is not surjective, then we see from the sequence (1) that $h^1\mathcal{F}(n-1) > h^1\mathcal{F}(n)$. Thus, the function $n \mapsto h^1\mathcal{F}(n)$ is *strictly* decreasing until it reaches 0.
Next, we show that $h^0\mathcal{F} \leq \mu - 1$. Suppose $h^0(\mathcal{F}) = \mu$. Then the injective (!) map f_0 is an isomorphism and $\mu - \chi = h^1\mathcal{F}(-1) = 0$. Contradiction.
Since $h^0\mathcal{F} < \mu$ the homomorphism f_0 cannot be surjective. The situation is then the following:

$h^1\mathcal{F}(n)$

$3\mu - \chi$

$2\mu - \chi$

$\mu - \chi$

worst case...

n

-5 -2 -1 $\mu - \chi - 1$

This implies that $\mathrm{reg}(\mathcal{F}) \leq \mu - \chi$. $\square$

3. The Resolutions

The key idea in the proof of theorem 1 is to find a common free resolution for all sheaves in an open subset of the moduli space $M_{\mu m+\chi}(\mathbb{P}_2)$ and then to dualize this resolution. An appropriate tool for this are the Beilinson complexes:

Given a coherent sheaf $\mathcal{F}$ on $\mathbb{P}_n$, one has the following two complexes

$$0 \longrightarrow \mathcal{B}_{-n} \longrightarrow \cdots \longrightarrow \mathcal{B}_{-1} \longrightarrow \mathcal{B}_0 \longrightarrow \mathcal{B}_1 \longrightarrow \cdots \longrightarrow \mathcal{B}_n \longrightarrow 0$$

where

$$\mathcal{B}_p = \bigoplus_{q=0}^{n} H^q(\mathbb{P}_n, \mathcal{F}(p-q)) \otimes_k \Omega^{q-p}_{\mathbb{P}_n}(q-p), \quad p \in \mathbb{Z}$$

and

$$0 \longrightarrow \mathcal{C}_{-n} \longrightarrow \cdots \longrightarrow \mathcal{C}_{-1} \longrightarrow \mathcal{C}_0 \longrightarrow \mathcal{C}_1 \longrightarrow \cdots \longrightarrow \mathcal{C}_n \longrightarrow 0$$

with

$$\mathcal{C}_p = \bigoplus_{q=0}^{n} H^{q+p}(\mathbb{P}_n, \mathcal{F} \otimes \Omega^q_{\mathbb{P}_n}(q)) \otimes_k \mathcal{O}_{\mathbb{P}_n}(-q), \quad p \in \mathbb{Z}$$

They are exact except at $\mathcal{B}_0$ resp. $\mathcal{C}_0$, where the homology is $\mathcal{F}$, and can be obtained from the Beilinson I/II spectral sequences. For example the second complex comes from the sequence with E_1-term

$$E_1^{rs} := H^r(\,\mathbb{P}_n, \mathcal{F} \otimes \Omega^{-s}_{\mathbb{P}_n}(-s)\,) \otimes_k \mathcal{O}_{\mathbb{P}_n}(s)$$

which converges to $E_\infty^i = \begin{cases} \mathcal{F}, & \text{for } i=0 \\ 0, & \text{otherwise} \end{cases}$. More detailed: $E_\infty^{rs} = 0$ for $r = -s$ and $\bigoplus_{r=0}^n E_\infty^{-r,r}$ is the associated graded sheaf of a filtration of $\mathcal{F}$. For more details on the Beilinson sequence we refer for example to [8].

Applying this technique to semi-stable sheaves in $\mathbb{P}_2$, we get:

Theorem 4. *Let $\mathcal{F}$ be a semi-stable sheaf on $\mathbb{P}_2$ with linear Hilbert polynomial $P(m) = \mu m + \chi$, $0 \le \chi < \mu$. Furthermore, let $a := h^0(\mathbb{P}_2, \mathcal{F} \otimes \Omega^1_{\mathbb{P}_2}(1))$.*

(i) *There are complexes*

$$0 \to (2\,\mu - \chi)\mathcal{O}_{\mathbb{P}_2}(-1) \longrightarrow H^0\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2} \oplus (\mu - \chi)\,\Omega^1_{\mathbb{P}_2}(1) \longrightarrow H^1\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2} \to 0$$

and

$$0 \to a\mathcal{O}_{\mathbb{P}_2}(-1) \oplus (\mu - \chi)\mathcal{O}_{\mathbb{P}_2}(-2) \to H^0\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2} \oplus (a + \mu - 2\chi)\,\mathcal{O}_{\mathbb{P}_2}(-1) \to H^1\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2} \to 0$$

which are exact with exception of the homology sheaf in the middle which is isomorphic to $\mathcal{F}$. In particular, if $H^1(\mathcal{F}) \cong 0$ we have free resolutions

$$0 \to (2\mu - \chi)\mathcal{O}_{\mathbb{P}_2}(-1) \longrightarrow \chi\,\mathcal{O}_{\mathbb{P}_2} \oplus (\mu - \chi)\Omega^1_{\mathbb{P}_2}(1) \longrightarrow \mathcal{F} \to 0 \tag{2}$$

and

$$0 \to a\,\mathcal{O}_{\mathbb{P}_2}(-1) \oplus (\mu - \chi)\mathcal{O}_{\mathbb{P}_2}(-2) \longrightarrow \chi\,\mathcal{O}_{\mathbb{P}_2} \oplus (a + \mu - 2\chi)\mathcal{O}_{\mathbb{P}_2}(-1) \longrightarrow \mathcal{F} \to 0. \tag{3}$$

(ii) *If $\mu(\mathcal{O}_C) < 4 - \frac{2\chi}{\mu}$ then $h^1\mathcal{F} = 0$.*

Proof. In our case, all the $\mathcal{B}_p$ resp. $\mathcal{C}_p$ vanish if $p \neq -2, -1, 0, 1$. Using the facts that $h^0\mathcal{F}(-j) = 0$ for all $j > 0$ because of the semi-stability and $\Omega^2(2) = \mathcal{O}_{\mathbb{P}_2}(-1)$, we obtain

$$\begin{aligned}
\mathcal{B}_1 &= H^1\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2} \\
\mathcal{B}_0 &= H^0\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2} \oplus H^1\mathcal{F}(-1) \otimes \Omega^1(1) = H^0\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2} \oplus (\mu - \chi)\,\Omega^1(1) \\
\mathcal{B}_{-1} &= H^0\mathcal{F}(-1) \otimes \Omega^1(1) \oplus H^1\mathcal{F}(-2) \otimes \Omega^2(2) = (2\mu - \chi)\,\mathcal{O}_{\mathbb{P}_2}(-1) \\
\mathcal{B}_{-2} &= H^0\mathcal{F}(-2) \otimes \Omega^2(2) = 0
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{C}_1 &= H^1\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2} \\
\mathcal{C}_0 &= H^0\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2} \oplus H^1(\mathcal{F} \otimes \Omega^1(1)) \otimes \mathcal{O}_{\mathbb{P}_2}(-1) \\
\mathcal{C}_{-1} &= H^0(\mathcal{F} \otimes \Omega^1(1)) \otimes \mathcal{O}_{\mathbb{P}_2}(-1) \oplus H^1(\mathcal{F} \otimes \Omega^2(2)) \otimes \mathcal{O}_{\mathbb{P}_2}(-2) = a\,\mathcal{O}_{\mathbb{P}_2}(-1) \oplus (\mu - \chi)\,\mathcal{O}_{\mathbb{P}_2}(-2) \\
\mathcal{C}_{-2} &= H^0(\mathcal{F} \otimes \Omega^2(2)) \otimes \mathcal{O}_{\mathbb{P}_2}(-2) = 0.
\end{aligned}$$

Now consider the Euler sequence tensored with $\mathcal{F}$

$$0 \longrightarrow \Omega^1(1) \otimes \mathcal{F} \longrightarrow 3\mathcal{F} \longrightarrow \mathcal{F}(1) \longrightarrow 0$$

in order to see that $h^1(\mathcal{F} \otimes \Omega^1(1)) = a + \chi(\mathcal{F}(1)) - 3\chi(\mathcal{F}) = a + \mu - 2\chi$.
To show (ii), let $C := Z_a(\mathcal{F})$. Then $H^0(C, \mathcal{F} \otimes \Omega^1_{\mathbb{P}_2}(1)) \cong \mathrm{Hom}(\mathcal{O}_C(-1) \otimes (\Omega^1)^\vee, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{O}_C(2) \otimes \Omega^1, \mathcal{F})$. $\mathcal{O}_C$ is stable and thus p-stable. Ω^1 is p-stable, too. The stability of

$\mathcal{O}_C(2) \otimes \Omega^1$ implies the vanishing of $H^0(\mathcal{F} \otimes \Omega^1(1))$ if $p(\Omega^1 \otimes \mathcal{O}_C(2)) > p(\mathcal{F})$. But a straightforward computation using the exact sequence

$$0 \longrightarrow \Omega^1 \otimes \mathcal{O}_C(2) \longrightarrow 3\mathcal{O}_C(1) \longrightarrow \mathcal{O}_C(2) \longrightarrow 0$$

and $p_a(C) = \frac{1}{2}(\deg(C) - 1)(\deg(C) - 2)$ gives $p(\Omega^1 \otimes \mathcal{O}_C(2)) = 2 - \frac{\mu(\mathcal{O}_C)}{2}$ and consequently the result. □

Remark: The inequality $\mu(\mathcal{O}_C) < 4 - \frac{2\chi}{\mu}$ or $H^1(\mathcal{F}) = 0$ is for example fullfilled in the following cases:

$P(m)$	Resolution
m	$0 \to \mathcal{O}_{\mathbb{P}_2}(-2) \to \mathcal{O}_{\mathbb{P}_2}(-1) \to \mathcal{F} \to 0$
$2m$	$0 \to 2\,\mathcal{O}_{\mathbb{P}_2}(-2) \to 2\,\mathcal{O}_{\mathbb{P}_2}(-1) \to \mathcal{F} \to 0$
$2m+1$	$0 \to \mathcal{O}_{\mathbb{P}_2}(-2) \to \mathcal{O}_{\mathbb{P}_2} \to \mathcal{F} \to 0$
$3m$	$0 \to 3\,\mathcal{O}_{\mathbb{P}_2}(-2) \to 3\,\mathcal{O}_{\mathbb{P}_2}(-1) \to \mathcal{F} \to 0$
$3m+1$	$0 \to 2\,\mathcal{O}_{\mathbb{P}_2}(-2) \to \mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}_2}(-1) \to \mathcal{F} \to 0$
$3m+2$	$0 \to \mathcal{O}_{\mathbb{P}_2}(-2) \oplus \mathcal{O}_{\mathbb{P}_2}(-1) \to 2\,\mathcal{O}_{\mathbb{P}_2} \to \mathcal{F} \to 0$

For these resolutions, one can verify that the space of matrices occuring in the resolutions modulo automorphisms is isomorphic to the corresponding moduli space $M_P(\mathbb{P}_2)$. This helps getting a more explicit description of the spaces: $M_m(\mathbb{P}_2)$ is clearly isomorphic to $\mathbb{P}_2$ since $\mathcal{F} \cong \mathcal{O}_L(-1)$ for some line L. Leopold [5] showed that $M_{2m}(\mathbb{P}_2) \cong M_{2m+1}(\mathbb{P}_2) \cong \mathbb{P}_5$. In [1] or [7] one can find a proof for $M_{3m+1}(\mathbb{P}_2) \cong M_{3m+2}(\mathbb{P}_2) \cong \mathcal{C}$, where $\mathcal{C} \xrightarrow{\pi} \mathbb{P}_2$ denotes the universal cubic on the projective plane. One problem occuring here is that the groups $\mathrm{Aut}(2\,\mathcal{O}_{\mathbb{P}_2}(-2) \times \mathrm{Aut}(\mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}_2}(-1))$ and $\mathrm{Aut}(\mathcal{O}_{\mathbb{P}_2}(-2) \oplus \mathcal{O}_{\mathbb{P}_2}(-1)) \times \mathrm{Aut}(2\,\mathcal{O}_{\mathbb{P}_2})$ divided out are not reductive.

□

Now we assume for the moment $H^1\mathcal{F} = 0$. One would like to determine $a = h^0(\mathcal{F} \otimes \Omega^1(1))$ in the theorem above in terms of the integers μ and χ. For this, we consider the following diagram where the second column is induced by the Koszul resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{\alpha} 3\,\mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{\beta} \Omega^1_{\mathbb{P}_2}(1) \longrightarrow 0$$

of the twisted cotangent bundle $\Omega^1_{\mathbb{P}_2}(1)$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
0 & \longrightarrow & (2\mu-\chi)\,\mathcal{O}_{\mathbb{P}_2}(-1) & \longrightarrow & \chi\,\mathcal{O}_{\mathbb{P}_2}\oplus(\mu-\chi)\,\Omega^1_{\mathbb{P}_2}(1) & \longrightarrow & \mathcal{F} & \longrightarrow & 0\\
 & & \| & & \uparrow{\scriptstyle id\times\beta} & & & & \\
 & & (2\mu-\chi)\mathcal{O}_{\mathbb{P}_2}(-1) & & \chi\,\mathcal{O}_{\mathbb{P}_2}\oplus 3\,(\mu-\chi)\,\mathcal{O}_{\mathbb{P}_2}(-1) & & & & \\
 & & & & \uparrow{\scriptstyle\alpha} & & & & \\
 & & & & (\mu-\chi)\,\mathcal{O}_{\mathbb{P}_2}(-2) & & & & \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

An application of the mapping cone lemma yields the exact sequence

$$0\to(2\mu-\chi)\,\mathcal{O}_{\mathbb{P}_2}(-1)\oplus(\mu-\chi)\,\mathcal{O}_{\mathbb{P}_2}(-2)\xrightarrow{B}\chi\,\mathcal{O}_{\mathbb{P}_2}\oplus 3\,(\mu-\chi)\,\mathcal{O}_{\mathbb{P}_2}(-1)\to\mathcal{F}\to 0 \tag{4}$$

where the blockmatrix B has the shape

$$B=\left(\begin{array}{c|c} L_1 & C\\ \hline Q & L_2\end{array}\right).$$

$Q\in\mathrm{Mat}(\,\mu-\chi,\chi,k[Z_0,Z_1,Z_2]_2\,)$is a matrix of quadratic forms, L_1 and L_2 are matrices of linear forms and $C\in\mathrm{Mat}(\,2\mu-\chi,3\,\mu-3\,\chi,k\,)$.

This resolution is in fact not minimal. Using the semi-stability of the sheaf $\mathcal{F}$ we can prove the following lemma:

Lemma 1. $rk(C)=r\,':=\min\{\,2\mu-\chi,\,3\mu-3\chi\,\}$.

Proof. By contradiction. Suppose $r:=\mathrm{rk}(C)<r\,'$. After deleting the appropriate rows and columns of the matrix B with the Gauß algorithm, we get

$$0\to(2\mu-\chi-r)\,\mathcal{O}_{\mathbb{P}_2}(-1)\oplus(\mu-\chi)\,\mathcal{O}_{\mathbb{P}_2}(-2)\xrightarrow{B\,'}\chi\,\mathcal{O}_{\mathbb{P}_2}\oplus(3\mu-3\chi-r)\,\mathcal{O}_{\mathbb{P}_2}(-1)\to\mathcal{F}\to 0$$

with

$$B\,'=\left(\begin{array}{c|c} L_1' & 0\\ \hline Q\,' & L_2'\end{array}\right)$$

where we identify the isomorphic cokernels $\mathcal{F}$ and $\mathrm{Coker}(B')$ by abuse of notation. Thus, let us investigate the diagram

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & 0 & & \mathcal{K}_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (2\mu-\chi-r)\,\mathcal{O}_{\mathbb{P}_2}(-1) & \longrightarrow & \mathcal{L}_1 & \longrightarrow & (\mu-\chi)\,\mathcal{O}_{\mathbb{P}_2}(-2) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle L_1'} & & \downarrow{\scriptstyle B'} & & \downarrow{\scriptstyle L_2'} & & \\
0 & \longrightarrow & \chi\,\mathcal{O}_{\mathbb{P}_2} & \longrightarrow & \mathcal{L}_0 & \longrightarrow & (3\mu-3\chi-r)\,\mathcal{O}_{\mathbb{P}_2}(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathcal{C}_1 & \xrightarrow{f} & \mathcal{F} & \longrightarrow & \mathcal{C}_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Here we write $\mathcal{L}_1 := (2\mu-\chi-r)\,\mathcal{O}_{\mathbb{P}_2}(-1)\oplus(\mu-\chi)\,\mathcal{O}_{\mathbb{P}_2}(-2)$, $\mathcal{L}_0 := \chi\,\mathcal{O}_{\mathbb{P}_2}\oplus(3\mu-3\chi-r)\,\mathcal{O}_{\mathbb{P}_2}(-1)$ and $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{K}_2$ for the cokernels respectively kernels of L_1' and L_2'. The snake lemma implies $\mathrm{Ker}(f) \cong \mathcal{K}_2$ and the injectivity of the map L_1'. The latter also implies forces $2\,\mu - r + \chi \le \chi$ and consequently we obtain the following bounds for r:

$$
2\,(\mu-\chi) \le r < \min\{\,2\,\mu-\chi, 3\,(\mu-\chi)\,\} \tag{5}
$$

If $\chi = 0$, we get the contradiction. Suppose now $0 < \chi < \mu$. After taking $\Lambda^{2\mu-\chi-r}(\,\bullet\,)$ of the map L_1' in the first column and after dualizing and twisting, we obtain an exact sequence:

$$
0 \xrightarrow{\;!\;} \binom{\chi}{2\mu-\chi-r}\,\mathcal{O}_{\mathbb{P}_2}(r+\chi-2\mu) \longrightarrow \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{O}_{Z_f(\mathcal{C}_1)} \longrightarrow 0
$$

where $Z_f(\mathcal{C}_1) \subset \mathbb{P}_2$ denotes the Fitting support of $\mathcal{C}_1$. Thus

$$
P_{Z_f(\mathcal{C}_1)}(m) = \frac{1}{2}\left[1-\binom{\chi}{2\,\mu-\chi-r}\right] m^2 + \cdots
$$

This forces the binomial coefficient $\binom{\chi}{2\,\mu-\chi-r}$ to be 0 or 1. Using the inequalities in (5), we deduce that $r = 2\,(\mu-\chi)$. The diagram above simplifies now to

$$
\begin{array}{ccccccccccc}
 & & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
 & & & & 0 & & 0 & & \mathcal{K}_2 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \chi\,\mathcal{O}_{\mathbb{P}_2}(-1) & \longrightarrow & \mathcal{L}_1 & \longrightarrow & (\mu-\chi)\,\mathcal{O}_{\mathbb{P}_2}(-2) & \longrightarrow & 0 \\
 & & & & {\scriptstyle L_1'}\downarrow & & {\scriptstyle B'}\downarrow & & {\scriptstyle L_2'}\downarrow & & \\
 & & 0 & \longrightarrow & \chi\,\mathcal{O}_{\mathbb{P}_2} & \longrightarrow & \mathcal{L}_0 & \longrightarrow & (\mu-\chi)\,\mathcal{O}_{\mathbb{P}_2}(-1) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{C}_1 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{C}_2 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}
\quad .
$$

Since $Z_a(\mathcal{C}_2) \subset Z_a(\mathcal{F})$ is zero- or one-dimensional, it follows from

$$1 = \text{exp.codim}_{\,\mathbb{P}_2} Z_f(\mathcal{C}_2) \geq \text{codim}_{\,\mathbb{P}_2} Z_f(\mathcal{C}_2) = \text{codim}_{\,\mathbb{P}_2} Z_a(\mathcal{C}_2) \geq 1$$

that $\mathcal{C}_2$ is supported on a curve and that the morphism L_2' is regular. Therefore the kernel sheaf $\mathcal{K}_2$ vanishes. An easy computation shows that the subsheaf $\mathcal{C}_1 \subset \mathcal{F}$ has Hilbert polynomial $P_{\mathcal{C}_1}(m) = \chi\,m + \chi$. Thus we have found a 1-dimensional subsheaf of the semi-stable sheaf $\mathcal{F}$ with

$$1 = \frac{\chi}{\chi} = \frac{\chi(\mathcal{C}_1)}{\mu(\mathcal{C}_1)} \leq \frac{\chi}{\mu} < 1.$$

Contradiction. Thus, $r = \text{rk}(C) = \min\{\,2\mu-\chi,\, 3\mu-3\chi\,\}$. □

Corollary 1. *Let* $[\mathcal{F}] \in M_{\mu\,m+\chi}(\mathbb{P}_2)$, $0 \leq \chi < \mu$ *with* $H^1\mathcal{F} = 0$. *Then* $\mathcal{F}$ *has one of the following two minimal free resolutions:*

$$0 \longrightarrow (\mu-\chi)\,\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{(\,Q\,|\,L_2\,)} \chi\,\mathcal{O}_{\mathbb{P}_2} \oplus (\mu-2\,\chi)\,\mathcal{O}_{\mathbb{P}_2}(-1) \longrightarrow \mathcal{F} \longrightarrow 0, \tag{6}$$

if $\chi \leq \frac{\mu}{2}$.

$$0 \longrightarrow (2\,\chi-\mu)\,\mathcal{O}_{\mathbb{P}_2}(-1) \oplus (\mu-\chi)\,\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{\binom{L_1}{Q}} \chi\,\mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{F} \longrightarrow 0, \tag{7}$$

if $\chi \geq \frac{\mu}{2}$.
Furthermore,

$$a = h^0(\,\mathbb{P}_2,\, \mathcal{F} \otimes \Omega^1_{\mathbb{P}_2}(1)\,) = \begin{cases} 0 & , \quad \chi \leq \frac{\mu}{2} \\ 2\chi-\mu & , \quad \chi > \frac{\mu}{2} \end{cases}$$

Proof. Consider the blockmatrix $B = \left(\begin{array}{c|c} L_1 & C \\ \hline Q & L_2 \end{array}\right)$ in the exact sequence (4). Lemma 1 says that $\mathrm{rk}(C) = \min\{\, 2\mu - \chi,\, 3\mu - 3\chi\,\}$. Therefore, the resolution (6) can be obtained by deleting the last $3\,\mu - 3\,\chi$ columns of B if $\mathrm{rk}(C) = 3\,\mu - 3\,\chi$. Similarly, one gets (7) by killing the first $2\,\mu - \chi$ rows of B with Gauß' algorithm in case of $\mathrm{rk}(C) = 2\,\mu - \chi$. Comparing (6) and (7) with the resolution (3) in theorem 4.(i), we also obtain the value for $a = h^0(\mathcal{F} \otimes \Omega^1_{\mathbb{P}_2}(1))$. □

Remark: In the case $\chi = \mu - 1$ one has $H^1\mathcal{F} = 0$ for all $[\mathcal{F}] \in M_{\mu\, m+\mu-1}(\mathbb{P}_2)$ since $\mathrm{reg}(\mathcal{F}) \leq 1$ according to theorem 3.(8). The resolution is therefore in this case:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-2) \oplus (\mu-2)\,\mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow[A]{} (\mu-1)\,\mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{F} \longrightarrow 0$$

M. Maican used this free resolution in order to prove that the moduli spaces $M_{\mu\, m+\mu-1}(\mathbb{P}_2)$ can be described as geometric quotients of maps A by the non-reductive group

$$G := \mathrm{Aut}(\,(\mu-2)\,\mathcal{O}_{\mathbb{P}_2}(-2) \oplus \mathcal{O}_{\mathbb{P}_2}(-1)\,) \,\times\, \mathrm{Aut}(\,(\mu-1)\,\mathcal{O}_{\mathbb{P}_2})$$

using a suitable polarization.

□

We also need a "relative version" of corollary 1 for families. As in the absolute case, there exists for any $\boldsymbol{\mathcal{F}} \in \mathrm{Coh}(\mathbb{P}_n \times S)$ a Beilinson-type spectral sequence with E_1-term

$$E_1^{rs} = \mathcal{O}_{\mathbb{P}_2}(r) \boxtimes R^s p_*(\boldsymbol{\mathcal{F}} \otimes \Omega^{-s}_{\mathbb{P}_n \times S/S}(-s))$$

which converges to $E_\infty^i = \begin{cases} \boldsymbol{\mathcal{F}}, & \text{for } i{=}0 \\ 0, & \text{otherwise} \end{cases}$, i.e. $E_\infty^{rs} = 0$ for $r+s \neq 0$ and $\bigoplus_{r=0}^n E_\infty^{-r,r}$ is the associated graded sheaf of a filtration of $\boldsymbol{\mathcal{F}}$ (cf. [8], p.306). Again, the spectral sequence gives rise to a complex

$$0 \longrightarrow \mathcal{C}_{-n} \longrightarrow \cdots \longrightarrow \mathcal{C}_{-1} \longrightarrow \mathcal{C}_0 \longrightarrow \mathcal{C}_1 \longrightarrow \cdots \longrightarrow \mathcal{C}_n \longrightarrow 0$$

with

$$\mathcal{C}_p = \bigoplus_{q=0}^{n} \mathcal{O}_{\mathbb{P}_n}(-q) \boxtimes R^{q+p} p_*(\boldsymbol{\mathcal{F}} \otimes \Omega^q_{\mathbb{P}_n \times S/S}(q))$$

which is exact everywhere with exception of $\mathcal{C}_0$, where the homology is $\boldsymbol{\mathcal{F}}$.

Now let $\boldsymbol{\mathcal{F}} \in \mathrm{Coh}(\mathbb{P}_2 \times S)$ be a family of semi-stable sheaves $\boldsymbol{\mathcal{F}}_s$ with Hilbert polynomial $P_{\boldsymbol{\mathcal{F}}_s}(m) = \mu\, m + \chi$ and $H^1(\mathbb{P}_2, \boldsymbol{\mathcal{F}}_s) = 0$ for all $s \in S$. Using the base change theorem and exactly the same arguments as in the proof of theorem 4,(i), we obtain a non-minimal (!) exact sequence

$$\begin{aligned} 0 \longrightarrow\ & [\,\mathcal{O}_{\mathbb{P}_2}(-1) \boxtimes p_*(\boldsymbol{\mathcal{F}} \otimes \Omega^1(1))\,] \oplus [\,\mathcal{O}_{\mathbb{P}_2}(-2) \boxtimes R^1 p_* \boldsymbol{\mathcal{F}}(-1)\,] \xrightarrow{B_s} \\ \xrightarrow{B_s}\ & [\,\mathcal{O}_{\mathbb{P}_2} \boxtimes p_* \boldsymbol{\mathcal{F}}\,] \oplus [\,\mathcal{O}_{\mathbb{P}_2}(-1) \boxtimes R^1 p_*(\boldsymbol{\mathcal{F}} \otimes \Omega^1(1))\,] \longrightarrow \boldsymbol{\mathcal{F}} \longrightarrow 0 \end{aligned}$$

Proof. To give a flavour of how to proceed, we show for example why $p_*(\boldsymbol{\mathcal{F}} \otimes \Omega^2_{\mathbb{P}_2\times S/S}(2)) = 0$ (and consequently $\mathcal{C}_{-2} = 0$):

Since all the sheaves $\boldsymbol{\mathcal{F}}_s$ are supported on curves one has $H^2(\mathbb{P}_2, \boldsymbol{\mathcal{F}}_s(-1)) = 0$. The base change theorem implies that $R^1p_*\boldsymbol{\mathcal{F}}(-1)(s) \xrightarrow{\approx} H^1(\mathbb{P}_2, \boldsymbol{\mathcal{F}}_s(-1))$ for all $s \in S$. Therefore $R^1p_*\boldsymbol{\mathcal{F}}(-1)$ is locally free. Another application of the base change theorem yields $p_*\boldsymbol{\mathcal{F}}(-1)(s) \cong H^0(\mathbb{P}_2, \boldsymbol{\mathcal{F}}_s(-1))$. But then

$$0 = \mathrm{Hom}(\mathcal{O}_{\mathbb{P}_2}, \boldsymbol{\mathcal{F}}_s(-1)) \cong H^0(\mathbb{P}_2, \boldsymbol{\mathcal{F}}_s(-1)) \quad \forall\, s \in S,$$

due to the semi-stability of $\boldsymbol{\mathcal{F}}_s$. Thus, $p_*(\boldsymbol{\mathcal{F}} \otimes \Omega^2_{\mathbb{P}_2\times S/S}(2)) \cong p_*\boldsymbol{\mathcal{F}}(-1) = 0$. □

By looking at the rank of the constant block in the family of matrices $(B_s)_{s\in S}$ as we did it for the absolute case in lemma 1, we can simplify the resolution and obtain the analogon to corollary 1:

Theorem 5. *Let* $[\,\boldsymbol{\mathcal{F}}\,] \in \mathcal{M}_{\mu\, m+\chi}(\mathbb{P}_2)(S)$, $0 \le \chi < \mu$ *with* $H^1(\mathbb{P}_2, \boldsymbol{\mathcal{F}}_s) = 0$ *for all* $s \in S$. *Then* $\boldsymbol{\mathcal{F}}$ *has one of the following two minimal free resolutions:*

(8)
$$0 \to \mathcal{O}_{\mathbb{P}_2}(-2) \boxtimes R^1p_*\boldsymbol{\mathcal{F}}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}_2} \boxtimes p_*\boldsymbol{\mathcal{F}} \oplus \mathcal{O}_{\mathbb{P}_2}(-1) \boxtimes R^1p_*(\boldsymbol{\mathcal{F}} \otimes \Omega^1_{\mathbb{P}_2\times S/S}(1)) \longrightarrow \boldsymbol{\mathcal{F}} \to 0,$$

if $\chi \le \frac{\mu}{2}$.

(9)
$$0 \to \mathcal{O}_{\mathbb{P}_2}(-1) \boxtimes p_*(\boldsymbol{\mathcal{F}} \otimes \Omega^1_{\mathbb{P}_2\times S/S}(1)) \oplus \mathcal{O}_{\mathbb{P}_2}(-2) \boxtimes R^1p_*\boldsymbol{\mathcal{F}}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}_2} \boxtimes p_*\boldsymbol{\mathcal{F}} \longrightarrow \boldsymbol{\mathcal{F}} \to 0,$$

if $\chi \ge \frac{\mu}{2}$.
Moreover,

- $p_*\boldsymbol{\mathcal{F}}$ *and* $R^1p_*\boldsymbol{\mathcal{F}}(-1)$ *are locally free of rank* χ *and* $\mu - \chi$ *respectively.*
- $p_*(\boldsymbol{\mathcal{F}} \otimes \Omega^1_{\mathbb{P}_2\times S/S}(1))$ *and* $R^1p_*(\boldsymbol{\mathcal{F}} \otimes \Omega^1_{\mathbb{P}_2\times S/S}(1))$ *are locally free.*
  - *If* $\chi \le \frac{\mu}{2}$ *then* $p_*(\boldsymbol{\mathcal{F}} \otimes \Omega^1_{\mathbb{P}_2\times S/S}(1)) = 0$ *and* $rk\left[R^1p_*(\boldsymbol{\mathcal{F}} \otimes \Omega^1_{\mathbb{P}_2\times S/S}(1))\right] = \mu - 2\,\chi$.
  - *If* $\chi > \frac{\mu}{2}$ *then* $rk\left[p_*(\boldsymbol{\mathcal{F}} \otimes \Omega^1_{\mathbb{P}_2\times S/S}(1))\right] = 2\,\chi - \mu$ *and* $R^1p_*(\boldsymbol{\mathcal{F}} \otimes \Omega^1_{\mathbb{P}_2\times S/S}(1)) = 0$.

Proof. Left to the reader. □

4. Dual Sheaves

We define for a (semi-)stable sheaf $\mathcal{F}$ on $\mathbb{P}_2$ with linear Hilbert polynomial $P(m) = \mu\, m + \chi$ its *dual sheaf* by

$$\mathcal{F}^{\nabla} := \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}_2}}(\mathcal{F}, \omega_{\mathbb{P}_2})(1)$$

$\mathcal{H}om_{\mathcal{O}_{\mathbb{P}_2}}(\mathcal{F}, \omega_{\mathbb{P}_2}) = 0$ since $\mathcal{F}$ is pure with one-dimensional support. Thus, dualizing the minimal free resolution (6) or (7) of $\mathcal{F}$ from the corollary above and twisting by $\bullet \otimes \mathcal{O}_{\mathbb{P}_2}(-2)$

implies that $\mathcal{F}^\nabla$ is (semi-)stable with Hilbert-polynomial $P^\nabla(m) := \mu\, m + (\mu - \chi)$. For example, if $\chi \le \frac{\mu}{2}$ we obtain

$$0 \longrightarrow \chi\, \mathcal{O}_{\mathbb{P}_2}(-2) \oplus (\mu - 2\,\chi)\, \mathcal{O}_{\mathbb{P}_2}(-1) \longrightarrow (\mu-\chi)\, \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{F}^\nabla \xrightarrow{\;!\;} 0$$

by this procedure.

Moreover, one can verify immediately that:

- $\mathcal{F}^{\nabla\,\nabla} \cong \mathcal{F}$
- $H^1\mathcal{F} = 0 \quad \Longleftrightarrow \quad H^1\mathcal{F}^\nabla = 0$

Thus, we get our main result:

Theorem 6. *Let $P(m) = \mu\, m + \chi$ be a linear polynomial with $0 \le \chi < \mu$ and $(\mu, \chi) = \mathbb{Z}$. Denote by $N \subset M_P(\mathbb{P}_2)$ respectively $N^\nabla \subset M_{P^\nabla}(\mathbb{P}_2)$ the closed subvarieties of isomorphism classes of sheaves with non-vanishing first cohomology. Then there is a natural isomorphism*

$$\phi\,:\, M_P(\mathbb{P}_2) \setminus N \xrightarrow{\;\approx\;} M_{P^\nabla}(\mathbb{P}_2) \setminus N^\nabla, \quad [\mathcal{F}] \mapsto [\mathcal{F}^\nabla]$$

Thus, the moduli spaces $M_P(\mathbb{P}_2)$ and $M_{P^\nabla}(\mathbb{P}_2)$ are birationally equivalent.

Proof. Clearly, the remarks above show that ϕ is set-theoretically a bijection. In order to show that ϕ is actually a morphism, note that $M := M_P(\mathbb{P}_2)$ is a fine moduli space with universal family $\mathcal{U} \in \mathcal{M}_P(\mathbb{P}_2)(M)$ since μ and χ are coprime. Without loss of generality, we can assume that $\chi \le \frac{\mu}{2}$. Now consider the minimal free resolution (8) of $\boldsymbol{\mathcal{C}} := \mathcal{U}|_{\mathbb{P}_2 \times M\setminus N}$ from theorem 5:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-2) \boxtimes R^1p_*\boldsymbol{\mathcal{C}}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}_2} \boxtimes p_*\boldsymbol{\mathcal{C}} \oplus \mathcal{O}_{\mathbb{P}_2}(-1) \boxtimes R^1p_*(\boldsymbol{\mathcal{C}} \otimes \Omega^1_{\mathbb{P}_2\times S/S}(1)) \longrightarrow \boldsymbol{\mathcal{C}} \longrightarrow 0.$$

An application of $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}_2\times M\setminus N}}(\,\bullet\,, \mathcal{O}_{\mathbb{P}_2}(-2) \boxtimes \mathcal{O}_{M\setminus N}\,)$ yields:

$$0 \to \mathcal{O}_{\mathbb{P}_2}(-2) \boxtimes [\,p_*\,\boldsymbol{\mathcal{C}}\,]^* \oplus \mathcal{O}_{\mathbb{P}_2}(-1) \boxtimes [\,R^1p_*(\boldsymbol{\mathcal{C}} \otimes \Omega^1_{\mathbb{P}_2\times S/S}(1))\,]^* \longrightarrow \mathcal{O}_{\mathbb{P}_2} \boxtimes [\,R^1p_*\boldsymbol{\mathcal{C}}(-1)\,]^* \longrightarrow \boldsymbol{\mathcal{G}} \to 0,$$

where $\boldsymbol{\mathcal{G}} = \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}_2\times M\setminus N}}(\,\mathcal{C}, \mathcal{O}_{\mathbb{P}_2}(-2) \boxtimes \mathcal{O}_{M\setminus N}\,)$.

According to theorem 5, the bundles $[\,p_*\,\boldsymbol{\mathcal{C}}\,]^*$, $[\,R^1p_*(\boldsymbol{\mathcal{C}} \otimes \Omega^1_{\mathbb{P}_2\times S/S}(1))]^*$ and $[\,R^1p_*\boldsymbol{\mathcal{C}}(-1)\,]^*$ have rank χ, $\mu - 2\chi$ and $\mu - \chi$ respectively. Thus, the restriction of the resolution to a fiber $\boldsymbol{\mathcal{G}}_{[\mathcal{F}]}$ is

$$0 \longrightarrow \chi\, \mathcal{O}_{\mathbb{P}_2}(-2) \oplus (\mu - 2\,\chi)\, \mathcal{O}_{\mathbb{P}_2}(-1) \longrightarrow (\mu-\chi)\, \mathcal{O}_{\mathbb{P}_2} \longrightarrow \boldsymbol{\mathcal{G}}_{[\mathcal{F}]} \longrightarrow 0$$

which is exactly the resolution of $\mathcal{F}^\nabla$ obtained above. Therefore $\boldsymbol{\mathcal{G}}_{[\mathcal{F}]} \cong \mathcal{F}^\nabla$. Obviously, the sheaves $\boldsymbol{\mathcal{G}}_{[\mathcal{F}]}$ are stable with Hilbert polynomial $P^\nabla(m) = \mu\, m + (\mu - \chi)$ and $H^1\boldsymbol{\mathcal{G}}_{[\mathcal{F}]} = 0$ for all $[\mathcal{F}] \in M \setminus N$. In other words, $\boldsymbol{\mathcal{G}} \in \mathcal{M}_{P^\nabla}(\mathbb{P}_2)(M \setminus N)$. Per construction, the **morphism**

$$\Phi_{\boldsymbol{\mathcal{G}}} : M \setminus N \longrightarrow M_{P^\nabla}(\mathbb{P}_2)$$

induced by the family $\boldsymbol{\mathcal{G}}$ maps to $M_{P^\nabla}(\mathbb{P}_2) \setminus N^\nabla$ and is indeed equal to the set-theoretical map ϕ. Similarly, one proves that ϕ^{-1} is a morphism. ∎

5. Smoothness

In this section we want to reprove LePotier's result that $M_{\mu m+\chi}(\mathbb{P}_2)$ for coprime coefficients and show that the irreducible moduli space [7] is then indeed smooth.

Theorem 7. *Let $P(m) := \mu m + \chi$ with $(\mu, \chi) = (1)$. Then*

1. *$M := M_P(\mathbb{P}_2)$ is a smooth projective variety of dimension $\mu^2 + 1$.*
2. *The moduli space M is fine with universal family $\mathcal{U} \in \mathcal{M}_P(\mathbb{P}_2)(M)$.*

Proof. Without loss of generality we can assume that $0 \leq \chi < \mu$. By theorem 3.(7), we have that all semi-stable sheaves $\mathcal{F}$ with polynomial P are stable.

1. Serre duality gives $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F}) = \mathrm{Hom}(\mathcal{F}, \mathcal{F} \otimes \omega_{\mathbb{P}_2})^\vee = \mathrm{Hom}(\mathcal{F}, \mathcal{F}(-3))^\vee = 0$ for every $[\mathcal{F}] \in M$. The last equality is due to the stability of $\mathcal{F}$. Id est, there are no obstructions and M is smooth in neighbourhood of $[\mathcal{F}]$. Consequently, M is a smooth projective variety. We are left to compute $\dim M$. Every sheaf in the open, dense subset $M \setminus N = \{ [\mathcal{F}] \in M_P(\mathbb{P}_2) \, : \, H^1\mathcal{F} = 0 \}$ has a resolution (2). If we apply $\mathrm{Hom}(\cdot, \mathcal{F})$ to that sequence, we end up with

$$0 \longrightarrow \mathrm{End}(\mathcal{F}) \longrightarrow \chi\, H^0\mathcal{F} \oplus (\mu - \chi)\, \mathrm{Hom}(\Omega^1_{\mathbb{P}_2}(1), \mathcal{F}) \longrightarrow (2\mu - \chi) H^0\mathcal{F}(1) \longrightarrow \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \longrightarrow$$
$$\cdots \longrightarrow \chi\, H^1\mathcal{F} \oplus (\mu - \chi)\, \mathrm{Ext}^1(\Omega^1_{\mathbb{P}_2}(1), \mathcal{F}) \longrightarrow (2\mu - \chi) H^1\mathcal{F}(1) \longrightarrow \mathrm{Ext}^2(\mathcal{F}, \mathcal{F}) \longrightarrow 0$$

   The stable sheaf $\mathcal{F}$ is simple and therefore $\mathrm{End}(\mathcal{F}) \cong k$. We also have $\mathrm{Hom}(\Omega^1_{\mathbb{P}_2}(1), \mathcal{F}) \cong H^0(\mathcal{F}(-1) \otimes (\Omega^1_{\mathbb{P}_2})^\vee) \cong H^0(\mathcal{F}(2) \otimes \Omega^1_{\mathbb{P}_2})$ and $\mathrm{Ext}^1(\Omega^1_{\mathbb{P}_2}(1), \mathcal{F}) \cong H^1(\mathcal{F}(2) \otimes \Omega^1_{\mathbb{P}_2})$. Using the Euler sequence

$$0 \to \mathcal{F}(2) \otimes \Omega^1_{\mathbb{P}_2} \longrightarrow 3\mathcal{F}(1) \longrightarrow \mathcal{F}(2) \to 0,$$

   we get $\chi(\mathcal{F}(2) \otimes \Omega^1_{\mathbb{P}_2}) = 3\chi(\mathcal{F}(1)) - \chi(\mathcal{F}(2)) = \mu + 2\chi$. But then:

$$\begin{aligned}
\mathrm{ext}^1(\mathcal{F}, \mathcal{F}) &= 1 - \chi\, h^0\mathcal{F} - (\mu - \chi)\, h^0(\mathcal{F}(2) \otimes \Omega^1) + (2\mu - \chi)\, h^0\mathcal{F}(1) + \\
&\quad\ \chi\, h^1\mathcal{F} + (\mu - \chi)\, h^1(\mathcal{F}(2) \otimes \Omega^1) - (2\mu - \chi)\, h^1\mathcal{F}(1) \\
&= 1 - \chi^2 - (\mu - \chi)\chi(\mathcal{F}(2) \otimes \Omega^1) + (2\mu - \chi)\chi(\mathcal{F}(1)) \\
&= 1 - \chi^2 - (\mu - \chi)(\mu + 2\chi) + (2\mu - \chi)(\mu + \chi) \\
&= \mu^2 + 1.
\end{aligned}$$

   Thus $\dim M = \mu^2 + 1$ because $\dim_k T_{[\mathcal{F}]} M = \mu^2 + 1$ for all $[\mathcal{F}] \in M \setminus N$.
2. The existence and construction of the universal family in this case is standard and can be found for example in [3].

∎

Remark 1: Let again $\chi = \mu - 1$, $\mu > 1$. In this case we have $N = \emptyset$. Thus, there is an isomorphism between the smooth, $(\mu^2 + 1)$-dimensional, fine moduli spaces $M_{\mu m+1}(\mathbb{P}_2)$ and $M_{\mu m+\mu-1}(\mathbb{P}_2)$.

Remark 2: [7]. If μ and χ are *not* coprime and $\mu \geq 2$ then the complement of the open subset of stable stable sheaves in $M_{\mu m+\chi}(\mathbb{P}_2)$ has codimension at least $2\mu - 3$, and no matter what open set U in $M_{\mu m+\chi}(\mathbb{P}_2)$ one chooses, there does not exist a universal sheaf over $\mathbb{P}_2 \times U$.

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