ON SIMPSON MODULI SPACES OF STABLE SHEAVES ON $\mathbb{P}_2$ WITH LINEAR HILBERT POLYNOMIAL

HANS GEORG FREIERMUTH

Preprint – Universität Kaiserslautern

Abstract. In this short note we prove some general results on semi-stable sheaves on $\mathbb{P}_2$ and $\mathbb{P}_3$ with arbitrary linear Hilbert polynomial. Using Beilinson’s spectral sequence, we compute free resolutions for this class of semi-stable sheaves and deduce that if $\mu$ and $\chi$ are coprime the smooth moduli spaces $M_{\mu m+\chi}(\mathbb{P}_2)$ and $M_{\mu m+(\mu-\chi)}(\mathbb{P}_2)$ are birationally equivalent.

1. Introduction

Moduli of torsionfree semi-stable sheaves on $\mathbb{P}_2$ and $\mathbb{P}_3$ with fixed Hilbert polynomial were introduced by Maruyama and others. They have been intensively studied during the last decades. In 1994, Simpson [9] showed that the family of arbitrary semi-stable sheaves with fixed Hilbert Polynomial $P$ on a smooth projective variety $X$ is bounded. Using this, he proved the existence of a projective scheme $M_P(X)$ corepresenting the moduli functor $M_P(X)(S)$ of $S$-flat coherent sheaves over $X \times S$ with semi-stable fibers $\mathcal{F}_s$ and $P_{\mathcal{F}_s} = P$. For $\dim(X) \geq 2$ and linear Hilbert polynomial $P(m) = \mu m + \chi$, id est if all the sheaves in $M_P(X)$ have torsion and are supported on degree $\mu$ curves, there is not much known about these spaces.

LePotier [7] proved that the coarse moduli spaces $M_{\mu m+\chi}(\mathbb{P}_2)$ are irreducible, locally factorial projective varieties of dimension $\mu^2 + 1$. They are rational at least if $\chi \equiv \pm 1 \pmod{\mu}$, $\chi \equiv \pm 2 \pmod{\mu}$ and for small multiplicities $\mu \leq 4$.

Furthermore, he described for $\mu \leq 4$ the geometrical properties of $M_{\mu m+\chi}(\mathbb{P}_2)$ and the birational map [6] to the Maruyama scheme $\mathcal{M}_{\mathbb{P}^2}(\mu; 0, \mu)$ of semi-stable, torsionfree rank $\mu$ sheaves with second Chern class $\mu$ on the dual projective plane $\mathbb{P}^2_Y$.

We investigated in [1], [2] the geometry of $M_{3m+1}(\mathbb{P}_3)$ which has two smooth, rational components of dimension 12 and 13 intersecting each other transversally along an 11-dimensional smooth subvariety. It is in some sense the “smallest” example for a reducible Simpson space and plays a role similar to Hilb$_{3m+1}(\mathbb{P}_3)$ in the case of Hilbert schemes.

Doing this, we noted as in [7] that in the planar case $M_{3m+1}(\mathbb{P}_2)$ and $M_{3m+2}(\mathbb{P}_2)$ are both isomorphic to the universal cubic $\mathcal{C} \longrightarrow \mathbb{P}_2$. This is not an accident and turned out to be part of a more general “symmetry” result which is the subject of this short note.

Date: 10/21/2001.
**Figure 1.** Schematic Picture. Each box corresponds to an $M_{\mu m+\chi}(\mathbb{P}_2)$.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>●</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>●</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>●</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>●</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>●</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>●</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>●</td>
</tr>
</tbody>
</table>

: The moduli space is fine.

: These two spaces in each row are isomorphic.

*Symmetry Axis*

**Theorem 1.** Let $P(m) = \mu m + \chi$, $0 < \chi \leq \mu$, $\mu$ and $\chi$ coprime, be a linear polynomial, and define its “dual” by $P^\nabla(m) := \mu m + \mu - \chi$. Denote by $N \subset M_P(\mathbb{P}_2)$ and $N^\nabla \subset M_{P^\nabla}(\mathbb{P}_2)$ respectively the closed subvarieties of isomorphism classes of sheaves with non-vanishing first cohomology. Then there is a natural isomorphism

$$\Phi : M_P(\mathbb{P}_2) \setminus N \cong M_{P^\nabla}(\mathbb{P}_2) \setminus N^\nabla.$$

Thus, the moduli spaces $M_P(\mathbb{P}_2)$ and $M_{P^\nabla}(\mathbb{P}_2)$ are birationally equivalent. Moreover, the spaces $M_{\mu m+1}(\mathbb{P}_2)$ and $M_{\mu m+\mu-1}(\mathbb{P}_2)$ are isomorphic.

Finally, we can extend LePotier's result cited above in a way certainly known to him:

**Theorem 2.** If $\mu$ and $\chi$ are coprime, the fine Simpson moduli spaces $M_{\mu m+\chi}(\mathbb{P}_2)$ are smooth projective varieties of dimension $\mu^2 + 1$.

The author would like to thank Günther Trautmann for useful discussions.

2. Preliminaries

We call the a projective scheme over an algebraically closed field $k$ a variety. One can equip the support of a coherent sheaf $\mathcal{F}$ on a smooth variety $X$ in several ways with the structure

\[^1\text{Note that } M_{\mu m+\tau}(\mathbb{P}_2) \cong M_{\mu m+\chi}(\mathbb{P}_2) \text{ if } \tau \equiv \chi \pmod{\mu} \text{ since the Hilbert polynomial involved is linear.}\]
of a (not necessarily reduced) variety. One is using the annihilator ideal sheaf $\text{Ann}(\mathcal{F}) \subset \mathcal{O}_X$. We write $Z_a(\mathcal{F}) := (\text{Supp}(\mathcal{F}), \mathcal{O}_X/\text{Ann}(\mathcal{F}))$. Another way is the following: Let

$$
\bigoplus_{\mu=1}^{r} \mathcal{O}_X(-b_{\mu}) \xrightarrow{A} \bigoplus_{\nu=1}^{s} \mathcal{O}_X(-a_{\nu}) \to \mathcal{F} \to 0
$$

be an arbitrary presentation of $\mathcal{F}$ and denote by $\text{Fitt}_i(\mathcal{F}) \subset \mathcal{O}_X$ the ideal sheaf generated by the $(s - i) \times (s - i)$-minors of the homogeneous matrix $A$. Due to Fitting’s lemma, the sheaf $\text{Fitt}_i(\mathcal{F})$ does not depend on the choice of the presentation. Furthermore, one has

$$\text{Fitt}_0(\mathcal{F}) \subset \text{Ann} \mathcal{F} \quad \text{and} \quad (\text{Ann} \mathcal{F}) \text{Fitt}_i(\mathcal{F}) \subset \text{Fitt}_{i-1}(\mathcal{F}) \quad \forall \ i > 0$$

Now define

$$Z_f(\mathcal{F}) := (\text{Supp}(\mathcal{F}), \mathcal{O}_X/\text{Fitt}_0(\mathcal{F})) \hookrightarrow (X, \mathcal{O}_X)$$

$Z_a(\mathcal{F})$ is obviously a subvariety of $Z_f(\mathcal{F})$ and $Z_a(\mathcal{F})_{\text{red}} = Z_f(\mathcal{F})_{\text{red}} = \text{Supp}(\mathcal{F})$.

Let $X$ be a variety and $S$ be a Noetherian (base-)scheme of finite type over $k$ and call the projections from $X \times_k S$ to the first and second factor by $q$ and $p$ respectively. If $\mathcal{F} \in \text{Coh}(X)$, $\mathcal{G} \in \text{Coh}(S)$ and $\mathcal{H} \in \text{Coh}(X \times S)$ are coherent sheaves, we will write $\mathcal{F} \boxtimes \mathcal{G} := q^* \mathcal{F} \otimes p^* \mathcal{G}$, $\mathcal{F}(m) \boxtimes \mathcal{O}_S := q^* \mathcal{F}(m)$, $\mathcal{H}(m) := \mathcal{H} \otimes q^* \mathcal{O}_X(m)$. A purely 1-dimensional coherent sheaf $\mathcal{F}$ with linear Hilbert polynomial $P(m) = \mu m + \chi$ on a smooth variety $X$ is called semi-stable resp. stable if for all proper coherent submodules $0 \neq \mathcal{F}' \subset \mathcal{F}$

$$\frac{\chi(\mathcal{F}')}{\mu(\mathcal{F}')} \leq \frac{\chi}{\mu} \quad \text{resp.} \quad \frac{\chi(\mathcal{F}')}{\mu(\mathcal{F}')} < \frac{\chi}{\mu}$$

$\mu(\mathcal{F})$ is called the multiplicity and $p(\mathcal{F}) := \frac{\chi}{\mu}$ the slope of the sheaf $\mathcal{F}$.

We collect now some properties of (semi-)stable sheaves supported on curves in the projective plane or projective space in the following theorem:

**Theorem 3.** Let $\mathcal{F}$ be a semi-stable sheaf on $\mathbb{P}_n$, $n = 2, 3$, with linear Hilbert polynomial $P_\mathcal{F}(m) = \mu m + \chi$, $0 \leq \chi < \mu$ and $C := Z_a(\mathcal{F})$ be its support.

1. $\mathcal{F}$ is Cohen-Macaulay, or equivalently: $\mathcal{F}$ has no zero-dimensional torsion.
2. If $C$ is smooth then $\mathcal{F}$ is locally free. If $C$ is integral $\mathcal{F}$ is still locally free on an open dense subset $U = C \setminus \{p_1, \ldots, p_r\}$.
3. Let $n = 2$. Then $(r; c_1, c_2) = (0; \mu, \frac{n(\mu+3)}{2} - \chi)$. If $n = 3$, we have $(r; c_1, c_2, c_3) = (0; 0, -\mu, 2\chi - 4\mu)$ In both cases, $r = rk_{\mathbb{P}_n}(\mathcal{F})$ denotes the rank and $c_i = c_i(\mathcal{F})$ are the Chern classes w.r.t. $\mathbb{P}_n$.
4. The not necessarily reduced curve $C \subset \mathbb{P}_n$ has no zero-dimensional components and no embedded points.
5. $\mu = \chi(\mathcal{F}|_H)$ where $H = Z(l) \in |\mathcal{O}_{\mathbb{P}_n}(1)|$ is $\mathcal{F}$-regular. Thus,

$$\mu = h^0(\mathcal{F}|_H) = \sum_{p \in \text{Ch}_H} \dim_k(\mathcal{F}_p)$$
6. \( \mu(O_{C_{\text{red}}}) \leq \mu(O_C) \leq \mu \) and \( \mu(F \otimes O_{C_{\text{red}}}) \leq \mu \)
7. If \( \chi > 0 \) and \( (\chi, \mu) = Z \) then \( F \) is stable.
8. There are the following bounds for the cohomology and the Castelnuovo-Mumford regularity of the sheaf \( F \):
   - \( \chi \leq h^0F \leq \mu - 1 \).
   - \( 0 \leq h^1F \leq \mu - \chi - 1 \).
   - \( \text{reg}(F) \leq \mu - \chi \), in particular \( H^1F(i) = 0 \) for all \( i \geq \mu - \chi - 1 \).

Proof. Cf. [1]. The only part which is not obvious is 8.: Let \( H \) be a \( F \)-regular hyperplane. Then \( 0 \to F(-1) \to F \to F|_H \to 0 \) induces an exact sequence

\[
0 \to H^0F(n-1) \to H^0F(n) \xrightarrow{f_n} k^\mu \to H^1F(n-1) \to H^1F(n) \to 0 \quad \forall n \in \mathbb{Z}
\]

This implies that \( n \mapsto h^1F(n) \) is decreasing and \( \chi \leq h^0F \leq h^0F(-1) + \mu \). But \( \text{Hom}(O_C(1), F) \) vanishes because of the semi-stability, and thus \( \chi \leq h^0F \leq \mu \).

Now assume that \( f_n \) is surjective. The commutative diagram

\[
\begin{array}{ccc}
H^0F(n) \otimes H^0O(1) & \xrightarrow{f_n \otimes \text{id}} & k^\mu \otimes H^0O(1) \to 0 \\
\downarrow & & \downarrow \\
H^0F(n+1) & \xrightarrow{f_{n+1}} & k^\mu \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

implies that \( f_{n+1} \) is also a surjection. Therefore we get

\[
H^1F(n-1) \cong H^1F(n) \cong H^1F(n+1) \cong \cdots \cong 0
\]

by Serre’s theorem B. If \( f_n \) is not surjective, then we see from the sequence (1) that \( h^1F(n-1) > h^1F(n) \). Thus, the function \( n \mapsto h^1F(n) \) is strictly decreasing until it reaches 0.

Next, we show that \( h^0F \leq \mu - 1 \). Suppose \( h^0(F) = \mu \). Then the injective (!) map \( f_0 \) is an isomorphism and \( \mu - \chi = h^1F(-1) = 0 \). Contradiction.

Since \( h^0F < \mu \) the homomorphism \( f_0 \) cannot be surjective. The situation is then the following:
\begin{align*}
    h^1 \mathcal{F}(n) \\
    3\mu - \chi \\
    2\mu - \chi \\
    \mu - \chi \\
    \text{worst case...}
\end{align*}

\[-5 -2-1 \quad \mu - \chi - 1 \quad n\]

This implies that $\text{reg}(\mathcal{F}) \leq \mu - \chi$. \qed

3. The Resolutions

The key idea in the proof of theorem 1 is to find a common free resolution for all sheaves in an open subset of the moduli space $M_{\mu,m+\chi}(\mathbb{P}_2)$ and then to dualize this resolution. An appropriate tool for this are the Beilinson complexes:

Given a coherent sheaf $\mathcal{F}$ on $\mathbb{P}_n$, one has the following two complexes

\[
0 \rightarrow B_{-n} \rightarrow \cdots \rightarrow B_{-1} \rightarrow B_0 \rightarrow B_1 \rightarrow \cdots \rightarrow B_n \rightarrow 0
\]

where

\[
B_p = \bigoplus_{q=0}^{n} H^q(\mathbb{P}_n, \mathcal{F}(p-q)) \otimes_k \Omega^{q-p}_{\mathbb{P}_n}(q-p), \quad p \in \mathbb{Z}
\]

and

\[
0 \rightarrow C_{-n} \rightarrow \cdots \rightarrow C_{-1} \rightarrow C_0 \rightarrow C_1 \rightarrow \cdots \rightarrow C_n \rightarrow 0
\]

with

\[
C_p = \bigoplus_{q=0}^{n} H^{q+p}(\mathbb{P}_n, \mathcal{F} \otimes \Omega^q_{\mathbb{P}_n}(q)) \otimes_k \mathcal{O}_{\mathbb{P}_n}(-q), \quad p \in \mathbb{Z}
\]

They are exact except at $B_0$ resp. $C_0$, where the homology is $\mathcal{F}$, and can be obtained from the Beilinson I/II spectral sequences. For example the second complex comes from the sequence with $E_1$-term

\[
E_1^{rs} := H^r(\mathbb{P}_n, \mathcal{F} \otimes \Omega^{s}_{\mathbb{P}_n}(-s)) \otimes_k \mathcal{O}_{\mathbb{P}_n}(s)
\]
which converges to $E^i_\infty = \{ \mathcal{F}, \text{ for } i=0 \} \oplus \{ 0, \text{ otherwise} \}$. More detailed: $E^r_\infty = 0$ for $r = -s$ and $\bigoplus_{r=0}^{n} E^{-r,r}_\infty$ is the associated graded sheaf of a filtration of $\mathcal{F}$. For more details on the Beilinson sequence we refer for example to [8].

Applying this technique to semi-stable sheaves in $\mathbb{P}_2$, we get:

**Theorem 4.** Let $\mathcal{F}$ be a semi-stable sheaf on $\mathbb{P}_2$ with linear Hilbert polynomial $P(m) = m^2 + \chi$, $0 \leq \chi < \mu$. Furthermore, let $a := h^0(\mathbb{P}_2, \mathcal{F} \otimes \Omega^1_{\mathbb{P}_2}(1))$.

(i) There are complexes

$$0 \rightarrow (2\mu - \chi) \mathcal{O}_{\mathbb{P}_2}(-1) \rightarrow H^0 \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2} \oplus (\mu - \chi) \Omega^1_{\mathbb{P}_2}(1) \rightarrow H^1 \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2} \rightarrow 0$$

and

$$0 \rightarrow a \mathcal{O}_{\mathbb{P}_2}(-1) \oplus (\mu - \chi) \mathcal{O}_{\mathbb{P}_2}(-2) \rightarrow H^0 \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2} \oplus (a + \mu - 2\chi) \mathcal{O}_{\mathbb{P}_2}(-1) \rightarrow H^1 \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2} \rightarrow 0$$

which are exact with exception of the homology sheaf in the middle which is isomorphic to $\mathcal{F}$. In particular, if $H^1(\mathcal{F}) \cong 0$ we have free resolutions

$$(2) \quad 0 \rightarrow (2\mu - \chi) \mathcal{O}_{\mathbb{P}_2}(-1) \rightarrow \chi \mathcal{O}_{\mathbb{P}_2} \oplus (\mu - \chi) \Omega^1_{\mathbb{P}_2}(1) \rightarrow \mathcal{F} \rightarrow 0$$

and

$$(3) \quad 0 \rightarrow a \mathcal{O}_{\mathbb{P}_2}(-1) \oplus (\mu - \chi) \mathcal{O}_{\mathbb{P}_2}(-2) \rightarrow \chi \mathcal{O}_{\mathbb{P}_2} \oplus (a + \mu - 2\chi) \mathcal{O}_{\mathbb{P}_2}(-1) \rightarrow \mathcal{F} \rightarrow 0.$$ 

(ii) If $\mu(\mathcal{O}_C) < 4 - \frac{2\chi}{\mu}$ then $h^1 \mathcal{F} = 0$.

**Proof.** In our case, all the $\mathcal{B}_p$ resp. $\mathcal{C}_p$ vanish if $p \neq -2, -1, 0, 1$. Using the facts that $h^0 \mathcal{F}(-j) = 0$ for all $j > 0$ because of the semi-stability and $\Omega^2(2) = \mathcal{O}_{\mathbb{P}_2}(-1)$, we obtain

$$\mathcal{B}_1 = H^1 \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2}$$

$$\mathcal{B}_0 = H^0 \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2} \oplus H^1 \mathcal{F}(-1) \otimes \Omega^1(1) = H^0 \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2} \oplus (\mu - \chi) \Omega^1(1)$$

$$\mathcal{B}_{-1} = H^0 \mathcal{F}(-1) \otimes \Omega^1(1) \oplus H^1 \mathcal{F}(-2) \otimes \Omega^2(2) = (2\mu - \chi) \mathcal{O}_{\mathbb{P}_2}(-1)$$

$$\mathcal{B}_{-2} = H^0 \mathcal{F}(-2) \otimes \Omega^2(2) = 0$$

and

$$\mathcal{C}_1 = H^1 \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2}$$

$$\mathcal{C}_0 = H^0 \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2} \oplus H^1(\mathcal{F} \otimes \Omega^1(1)) \otimes \mathcal{O}_{\mathbb{P}_2}(-1)$$

$$\mathcal{C}_{-1} = H^0(\mathcal{F} \otimes \Omega^1(1)) \otimes \mathcal{O}_{\mathbb{P}_2}(-1) \oplus H^1(\mathcal{F} \otimes \Omega^2(2)) \otimes \mathcal{O}_{\mathbb{P}_2}(-2) = a \mathcal{O}_{\mathbb{P}_2}(-1) \oplus (\mu - \chi) \mathcal{O}_{\mathbb{P}_2}(-2)$$

$$\mathcal{C}_{-2} = H^0(\mathcal{F} \otimes \Omega^2(2)) \otimes \mathcal{O}_{\mathbb{P}_2}(-2) = 0.$$ 

Now consider the Euler sequence tensored with $\mathcal{F}$

$$0 \longrightarrow \Omega^1(1) \otimes \mathcal{F} \longrightarrow 3 \mathcal{F} \longrightarrow \mathcal{F}(1) \longrightarrow 0$$

in order to see that $h^1(\mathcal{F} \otimes \Omega^1(1)) = a + \chi(\mathcal{F}(1)) - 3\chi(\mathcal{F}) = a + \mu - 2\chi$.

To show (ii), let $C := Z_a(\mathcal{F})$. Then $H^0(C, \mathcal{F} \otimes \Omega^1_{\mathbb{P}_2}(1)) \cong \text{Hom}(\mathcal{O}_C(-1) \otimes (\Omega^1)^\vee, \mathcal{F}) \cong \text{Hom}(\mathcal{O}_C(2) \otimes \Omega^1, \mathcal{F})$. $\mathcal{O}_C$ is stable and thus $p$-stable. $\Omega^1$ is $p$-stable, too. The stability of
\( \mathcal{O}_C(2) \otimes \Omega^1 \) implies the vanishing of \( H^0(\mathcal{F} \otimes \Omega^1(1)) \) if \( p(\Omega^1 \otimes \mathcal{O}_C(2)) > p(\mathcal{F}) \). But a straightforward computation using the exact sequence

\[
0 \longrightarrow \Omega^1 \otimes \mathcal{O}_C(2) \longrightarrow 3\mathcal{O}_C(1) \longrightarrow \mathcal{O}_C(2) \longrightarrow 0
\]

and \( p_\text{a}(C) = \frac{1}{2}(\text{deg}(C) - 1)(\text{deg}(C) - 2) \) gives \( p(\Omega^1 \otimes \mathcal{O}_C(2)) = 2 - \frac{\mu(\mathcal{O}_C)}{2} \) and consequently the result. \( \square \)

**Remark:** The inequality \( \mu(\mathcal{O}_C) < 4 - \frac{2\chi}{\mu} \) or \( H^1(\mathcal{F}) = 0 \) is for example fullfilled in the following cases:

<table>
<thead>
<tr>
<th>( P(m) )</th>
<th>Resolution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>( 0 \rightarrow \mathcal{O}<em>{\mathbb{P}^2}(-2) \rightarrow \mathcal{O}</em>{\mathbb{P}^2}(-1) \rightarrow \mathcal{F} \rightarrow 0 )</td>
</tr>
<tr>
<td>( 2m )</td>
<td>( 0 \rightarrow \mathcal{O}<em>{\mathbb{P}^2}(-2) \rightarrow 2\mathcal{O}</em>{\mathbb{P}^2}(-1) \rightarrow \mathcal{F} \rightarrow 0 )</td>
</tr>
<tr>
<td>( 2m + 1 )</td>
<td>( 0 \rightarrow \mathcal{O}<em>{\mathbb{P}^2}(-2) \rightarrow \mathcal{O}</em>{\mathbb{P}^2} \rightarrow \mathcal{F} \rightarrow 0 )</td>
</tr>
<tr>
<td>( 3m )</td>
<td>( 0 \rightarrow 3\mathcal{O}<em>{\mathbb{P}^2}(-2) \rightarrow 3\mathcal{O}</em>{\mathbb{P}^2}(-1) \rightarrow \mathcal{F} \rightarrow 0 )</td>
</tr>
<tr>
<td>( 3m + 1 )</td>
<td>( 0 \rightarrow \mathcal{O}<em>{\mathbb{P}^2}(-2) \rightarrow \mathcal{O}</em>{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow \mathcal{F} \rightarrow 0 )</td>
</tr>
<tr>
<td>( 3m + 2 )</td>
<td>( 0 \rightarrow \mathcal{O}<em>{\mathbb{P}^2}(-2) \oplus \mathcal{O}</em>{\mathbb{P}^2}(-1) \rightarrow 2\mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{F} \rightarrow 0 )</td>
</tr>
</tbody>
</table>

For these resolutions, one can verify that the space of matrices occuring in the resolutions modulo automorphisms is isomorphic to the corresponding moduli space \( M_P(\mathbb{P}^2) \). This helps getting a more explicit description of the spaces: \( M_m(\mathbb{P}^2) \) is clearly isomorphic to \( \mathbb{P}^2 \) since \( \mathcal{F} \cong \mathcal{O}_L(-1) \) for some line \( L \). Leopold [5] showed that \( M_{2m}(\mathbb{P}^2) \cong M_{2m+1}(\mathbb{P}^2) \cong \mathbb{P}^5 \). In [1] or [7] one can find a proof for \( M_{3m+1}(\mathbb{P}^2) \cong M_{3m+2}(\mathbb{P}^2) \cong \mathcal{C} \), where \( \mathcal{C} \longrightarrow \mathbb{P}^2 \) denotes the universal cubic on the projective plane. One problem occuring here is that the groups \( \text{Aut}(2\mathcal{O}_{\mathbb{P}^2}(-2) \times \text{Aut}([\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)]) \) and \( \text{Aut}([\mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)]) \times \text{Aut}(2\mathcal{O}_{\mathbb{P}^2}) \) divided out are not reductive. \( \square \)

Now we assume for the moment \( H^1(\mathcal{F}) = 0 \). One would like to determine \( a = h^0(\mathcal{F} \otimes \Omega^1(1)) \) in the theorem above in terms of the integers \( \mu \) and \( \chi \). For this, we consider the following diagram where the second column is induced by the Koszul resolution

\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2) \overset{\alpha}{\longrightarrow} 3\mathcal{O}_{\mathbb{P}^2}(-1) \overset{\beta}{\longrightarrow} \Omega^1_{\mathbb{P}^2}(1) \longrightarrow 0
\]
of the twisted cotangent bundle $\Omega^1_{\mathbb{P}^2}(1)$:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & (2\mu - \chi) \mathcal{O}_{\mathbb{P}^2}(-1) & \longrightarrow & \chi \mathcal{O}_{\mathbb{P}^2} \oplus (\mu - \chi) \Omega^1_{\mathbb{P}^2}(1) & \longrightarrow & \mathcal{F} \longrightarrow 0 \\
\downarrow & & \downarrow & id \times \beta & & \downarrow & \\
(2\mu - \chi) \mathcal{O}_{\mathbb{P}^2}(-1) & \longrightarrow & \chi \mathcal{O}_{\mathbb{P}^2} \oplus 3(\mu - \chi) \mathcal{O}_{\mathbb{P}^2}(-1) & \longrightarrow & (\mu - \chi) \mathcal{O}_{\mathbb{P}^2}(-2) & \longrightarrow & 0
\end{array}
\]

An application of the mapping cone lemma yields the exact sequence

(4) \quad 0 \to (2\mu - \chi) \mathcal{O}_{\mathbb{P}^2}(-1) \oplus (\mu - \chi) \mathcal{O}_{\mathbb{P}^2}(-2) \xrightarrow{B} \chi \mathcal{O}_{\mathbb{P}^2} \oplus 3(\mu - \chi) \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{F} \to 0

where the blockmatrix $B$ has the shape

\[
B = \begin{pmatrix} L_1 & C \\ Q & L_2 \end{pmatrix}.
\]

$Q \in \text{Mat}(\mu - \chi, \chi, k[Z_0, Z_1, Z_2])$ is a matrix of quadratic forms, $L_1$ and $L_2$ are matrices of linear forms and $C \in \text{Mat}(2\mu - \chi, 3\mu - 3\chi, k)$.

This resolution is in fact not minimal. Using the semi-stability of the sheaf $\mathcal{F}$ we can prove the following lemma:

**Lemma 1.** $\text{rk}(C) = r' := \min\{2\mu - \chi, 3\mu - 3\chi\}$.

**Proof.** By contradiction. Suppose $r := \text{rk}(C) < r'$. After deleting the appropriate rows and columns of the matrix $B$ with the Gauß algorithm, we get

\[
0 \to (2\mu - \chi - r) \mathcal{O}_{\mathbb{P}^2}(-1) \oplus (\mu - \chi) \mathcal{O}_{\mathbb{P}^2}(-2) \xrightarrow{B'} \chi \mathcal{O}_{\mathbb{P}^2} \oplus (3\mu - 3\chi - r) \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{F} \to 0
\]

with

\[
B' = \begin{pmatrix} L'_1 & 0 \\ Q' & L'_2 \end{pmatrix}.
\]
where we identify the isomorphic cokernels \( \mathcal{F} \) and \( \text{Coker}(B') \) by abuse of notation. Thus, let us investigate the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & (2\mu - \chi - r) \mathcal{O}_{\mathbb{P}^2}(-1) & \rightarrow & \mathcal{L}_1 & \rightarrow & (\mu - \chi) \mathcal{O}_{\mathbb{P}^2}(-2) & \rightarrow & 0 \\
& & \downarrow L'_1 & & \downarrow \mu' & & \downarrow L'_2 & & \\
0 & \rightarrow & \chi \mathcal{O}_{\mathbb{P}^2} & \rightarrow & \mathcal{L}_0 & \rightarrow & (3\mu - 3\chi - r) \mathcal{O}_{\mathbb{P}^2}(-1) & \rightarrow & 0 \\
& & \downarrow C_1 & \rightarrow & \mathcal{F} & \rightarrow & \mathcal{C}_2 & \rightarrow & 0 \\
& & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

Here we write \( \mathcal{L}_1 := (2\mu - \chi - r) \mathcal{O}_{\mathbb{P}^2}(-1) \oplus (\mu - \chi) \mathcal{O}_{\mathbb{P}^2}(-2) \), \( \mathcal{L}_0 := \chi \mathcal{O}_{\mathbb{P}^2} \oplus (3\mu - 3\chi - r) \mathcal{O}_{\mathbb{P}^2}(-1) \) and \( \mathcal{C}_1, \mathcal{C}_2, \mathcal{K}_2 \) for the cokernels respectively kernels of \( L'_1 \) and \( L'_2 \). The snake lemma implies \( \text{Ker}(f) \cong \mathcal{K}_2 \) and the injectivity of the map \( L'_1 \). The latter also implies forces \( 2\mu - r + \chi \leq \chi \) and consequently we obtain the following bounds for \( r \):

\[
2 (\mu - \chi) \leq r < \min\{ 2\mu - \chi, 3 (\mu - \chi) \}
\]

If \( \chi = 0 \), we get the contradiction. Suppose now \( 0 < \chi < \mu \). After taking \( \Lambda^{2\mu - \chi - r}(\bullet) \) of the map \( L'_1 \) in the first column and after dualizing and twisting, we obtain an exact sequence:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \left( \frac{\chi}{2\mu - \chi - r} \right) \mathcal{O}_{\mathbb{P}^2}(r + \chi - 2\mu) & \rightarrow & \mathcal{O}_{\mathbb{P}^2} & \rightarrow & \mathcal{O}_{Z_f(C_1)} & \rightarrow & 0 \\
\end{array}
\]

where \( Z_f(C_1) \subset \mathbb{P}^2 \) denotes the Fitting support of \( C_1 \). Thus

\[
P_{Z_f(C_1)}(m) = \frac{1}{2} \left[ 1 - \left( \frac{\chi}{2\mu - \chi - r} \right) \right] m^2 + \cdots
\]
This forces the binomial coefficient \( \binom{\mu - \chi - r}{2} \) to be 0 or 1. Using the inequalities in (5), we deduce that \( r = 2(\mu - \chi) \). The diagram above simplifies now to

Since \( Z_a(C_2) \subset Z_a(F) \) is zero- or one-dimensional, it follows from

\[
1 = \exp \text{codim}_{P_2} Z_f(C_2) \geq \text{codim}_{P_2} Z_f(C_2) = \text{codim}_{P_2} Z_a(C_2) \geq 1
\]

that \( C_2 \) is supported on a curve and that the morphism \( L'_2 \) is regular. Therefore the kernel sheaf \( K_2 \) vanishes. An easy computation shows that the subsheaf \( C_1 \subset F \) has Hilbert polynomial \( P_{C_1}(m) = \chi m + \chi \). Thus we have found a 1-dimensional subsheaf of the semi-stable sheaf \( F \) with

\[
1 = \frac{\chi}{\chi} = \frac{\chi(C_1)}{\mu(C_1)} \leq \frac{\chi}{\mu} < 1.
\]

Contradiction. Thus, \( r = \text{rk}(C) = \min\{2\mu - \chi, 3\mu - 3\chi\} \).

**Corollary 1.** Let \([F] \in M_{\mu - \chi}(\mathbb{P}_2), 0 \leq \chi < \mu \) with \( H^1F = 0 \). Then \( F \) has one of the following two minimal free resolutions:

(6) \[
0 \longrightarrow (\mu - \chi)O_{P_2}(-2) \overset{(Q|L_2)}{\longrightarrow} \chi O_{P_2} \oplus (\mu - 2\chi)O_{P_2}(-1) \longrightarrow F \longrightarrow 0,
\]

if \( \chi \leq \frac{\mu}{2} \).

(7) \[
0 \longrightarrow (2\chi - \mu)O_{P_2}(-1) \oplus (\mu - \chi)O_{P_2}(-2) \overset{(L_1)}{\longrightarrow} \chi O_{P_2} \longrightarrow F \longrightarrow 0,
\]

if \( \chi \geq \frac{\mu}{2} \).

Furthermore,

\[
a = h^0(\mathbb{P}_2, F \otimes \Omega^1_{\mathbb{P}_2}(1)) = \begin{cases} 
0 & \chi \leq \frac{\mu}{2}, \\
2\chi - \mu & \chi > \frac{\mu}{2}
\end{cases}
\]
Proof. Consider the blockmatrix $B = \left( \begin{array}{c} L_1 \\ Q L_2 \end{array} \right)$ in the exact sequence (4). Lemma 1 says that $\text{rk}(C) = \min\{2\mu - \chi, 3\mu - 3\chi\}$. Therefore, the resolution (6) can be obtained by deleting the last $3\mu - 3\chi$ columns of $B$ if $\text{rk}(C) = 2\mu - \chi$. Similarly, one gets (7) by killing the first $2\mu - \chi$ rows of $B$ with Gauß’ algorithm in case of $\text{rk}(C) = 3\mu - 3\chi$. Comparing (6) and (7) with the resolution (3) in theorem 4.(i), we also obtain the value for $a = h^0(\mathcal{F} \otimes \Omega_{\mathbb{P}^2}^1(1))$.

Remark: In the case $\chi = \mu - 1$ one has $H^1\mathcal{F} = 0$ for all $[\mathcal{F}] \in M_{\mu m + \mu - 1}(\mathbb{P}^2)$ since $\text{reg}(\mathcal{F}) \leq 1$ according to theorem 3.(8). The resolution is therefore in this case:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2) \oplus (\mu - 2) \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{A} (\mu - 1) \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{F} \longrightarrow 0$$

M. Maican used this free resolution in order to prove that the moduli spaces $M_{\mu m + \mu - 1}(\mathbb{P}^2)$ can be described as geometric quotients of maps $A$ by the non-reductive group

$$G := \text{Aut}( (\mu - 2) \mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)) \times \text{Aut}( (\mu - 1) \mathcal{O}_{\mathbb{P}^2})$$

using a suitable polarization.

□

We also need a “relative version” of corollary 1 for families. As in the absolute case, there exists for any $\mathcal{F} \in \text{Coh}(\mathbb{P}_n \times S)$ a Beilinson-type spectral sequence with $E_1$-term

$$E_1^{rs} = \mathcal{O}_{\mathbb{P}_2}(r) \boxtimes R^sp_*(\mathcal{F} \otimes \Omega_{\mathbb{P}_n \times S}^{-s}(-s))$$

which converges to $E_\infty^r = \{ \mathcal{F}_i \text{ for } i = 0 \}
\cup \{ 0, \text{ otherwise} \}$, i.e. $E_\infty^{rs} = 0$ for $r + s \neq 0$ and $\bigoplus_{r=0}^n E_\infty^{-r,r}$ is the associated graded sheaf of a filtration of $\mathcal{F}$ (cf. [8], p.306). Again, the spectral sequence gives rise to a complex

$$0 \longrightarrow \mathcal{C}_{-n} \longrightarrow \cdots \longrightarrow \mathcal{C}_{-1} \longrightarrow \mathcal{C}_0 \longrightarrow \mathcal{C}_1 \longrightarrow \cdots \longrightarrow \mathcal{C}_n \longrightarrow 0$$

with

$$\mathcal{C}_p = \bigoplus_{q=0}^n \mathcal{O}_{\mathbb{P}^n}(-q) \boxtimes R^{q+p}p_*(\mathcal{F} \otimes \Omega_{\mathbb{P}_n \times S}^q(q))$$

which is exact everywhere with exception of $\mathcal{C}_0$, where the homology is $\mathcal{F}$.

Now let $\mathcal{F} \in \text{Coh}(\mathbb{P}_2 \times S)$ be a family of semi-stable sheaves $\mathcal{F}_s$ with Hilbert polynomial $P_{\mathcal{F}_s}(m) = \mu m + \chi$ and $H^1(\mathbb{P}_2, \mathcal{F}_s) = 0$ for all $s \in S$. Using the base change theorem and exactly the same arguments as in the proof of theorem 4.(i), we obtain a non-minimal (!) exact sequence

$$0 \longrightarrow [\mathcal{O}_{\mathbb{P}_2}(-1) \boxtimes p_*(\mathcal{F} \otimes \Omega^1(1))] \oplus [\mathcal{O}_{\mathbb{P}_2}(-2) \boxtimes R^1p_*\mathcal{F}(-1)] \xrightarrow{B_s}$$

$$\bigoplus_{B_s} [\mathcal{O}_{\mathbb{P}_2} \boxtimes p_*\mathcal{F}] \oplus [\mathcal{O}_{\mathbb{P}_2}(-1) \boxtimes R^1p_*\mathcal{F} \otimes \Omega^1(1)] \longrightarrow \mathcal{F} \longrightarrow 0$$
Proof. To give a flavour of how to proceed, we show for example why $p_\ast(F \otimes \Omega^2_{\mathbb{P}_2 \times S/S}(2)) = 0$
(and consequently $C_{-2} = 0$):
Since all the sheaves $F_s$ are supported on curves one has $H^2(\mathbb{P}_2, F_s(-1)) = 0$. The base
change theorem implies that $R^1 p_\ast F(-1)(s) \cong H^1(\mathbb{P}_2, F_s(-1))$ for all $s \in S$. Therefore
$R^1 p_\ast F(-1)$ is locally free. Another application of the base change theorem yields $p_\ast F(-1)(s) \cong
H^0(\mathbb{P}_2, F_s(-1))$. But then
\[
0 = \text{Hom}(O_{\mathbb{P}_2}, F_s(-1)) \cong H^0(\mathbb{P}_2, F_s(-1)) \quad \forall s \in S,
\]
due to the semi-stability of $F_s$. Thus, $p_\ast(F \otimes \Omega^2_{\mathbb{P}_2 \times S/S}(2)) \cong p_\ast F(-1) = 0$.

\hspace{12cm} \square

By looking at the rank of the constant block in the family of matrices $(B_s)_{s \in S}$ as we did it
for the absolute case in lemma 1, we can simplify the resolution and obtain the analogon to
corollary 1:

**Theorem 5.** Let $[F] \in M_{\mu, \mu + \chi}(\mathbb{P}_2(S), 0 \leq \chi < \mu$ with $H^1(\mathbb{P}_2, F_s) = 0$ for all $s \in S$. Then $F$
has one of the following two minimal free resolutions:

\begin{align}
(8) & \quad 0 \to O_{\mathbb{P}_2}(-2) \bigotimes R^1 p_\ast F(-1) \to O_{\mathbb{P}_2} \otimes p_\ast F \otimes O_{\mathbb{P}_2}(-1) \bigotimes R^1 p_\ast(F \otimes \Omega^1_{\mathbb{P}_2 \times S/S}(1)) \to F \to 0, \\
(9) & \quad 0 \to O_{\mathbb{P}_2}(-1) \bigotimes p_\ast(F \otimes \Omega^1_{\mathbb{P}_2 \times S/S}(1)) \otimes O_{\mathbb{P}_2}(-2) \bigotimes R^1 p_\ast F(-1) \to O_{\mathbb{P}_2} \otimes p_\ast F \to F \to 0,
\end{align}

if $\chi \leq \frac{\mu}{2}$.

Moreover,

- $p_\ast F$ and $R^1 p_\ast F(-1)$ are locally free of rank $\chi$ and $\mu - \chi$ respectively.
- $p_\ast(F \otimes \Omega^1_{\mathbb{P}_2 \times S/S}(1))$ and $R^1 p_\ast(F \otimes \Omega^1_{\mathbb{P}_2 \times S/S}(1))$ are locally free.

- If $\chi \leq \frac{\mu}{2}$ then $p_\ast(F \otimes \Omega^1_{\mathbb{P}_2 \times S/S}(1)) = 0$ and $\text{rk} \left[ R^1 p_\ast(F \otimes \Omega^1_{\mathbb{P}_2 \times S/S}(1)) \right] = \mu - 2 \chi$.

- If $\chi > \frac{\mu}{2}$ then $\text{rk} \left[ p_\ast(F \otimes \Omega^1_{\mathbb{P}_2 \times S/S}(1)) \right] = 2 \chi - \mu$ and $R^1 p_\ast(F \otimes \Omega^1_{\mathbb{P}_2 \times S/S}(1)) = 0$.

\hspace{12cm} \square

**Proof.** Left to the reader.

\section{4. Dual Sheaves}

We define for a (semi-)stable sheaf $F$ on $\mathbb{P}_2$ with linear Hilbert polynomial $P(m) = \mu m + \chi$ its
dual sheaf by
\[
F^\vee := \text{Ext}^1_{\mathbb{O}_{\mathbb{P}_2}}(F, \omega_{\mathbb{P}_2})(1)
\]
$\text{Hom}_{\mathbb{O}_{\mathbb{P}_2}}(F, \omega_{\mathbb{P}_2}) = 0$ since $F$ is pure with one-dimensional support. Thus, dualizing the
minimal free resolution (6) or (7) of $F$ from the corollary above and twisting by $\otimes \mathbb{O}_{\mathbb{P}_2}(-2)$
implies that $\mathcal{F}^\triangledown$ is (semi-)stable with Hilbert-polynomial $P^\triangledown(m) := \mu m + (\mu - \chi)$. For example, if $\chi \leq \frac{\mu}{2}$ we obtain

$$
\begin{array}{c}
0 \longrightarrow \chi \mathcal{O}_{\mathbb{P}^2}(-2) \oplus (\mu - 2 \chi) \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow (\mu - \chi) \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{F}^\triangledown \longrightarrow 0
\end{array}
$$

by this procedure.

Moreover, one can verify immediately that:

- $\mathcal{F}^\triangledown \cong \mathcal{F}$
- $H^1 \mathcal{F} = 0 \iff H^1 \mathcal{F}^\triangledown = 0$

Thus, we get our main result:

**Theorem 6.** Let $P(m) = \mu m + \chi$ be a linear polynomial with $0 \leq \chi < \mu$ and $(\mu, \chi) = \mathbb{Z}$. Denote by $N \subset M_P(\mathbb{P}^2)$ respectively $N^\triangledown \subset M_{P^\triangledown}(\mathbb{P}^2)$ the closed subvarieties of isomorphism classes of sheaves with non-vanishing first cohomology. Then there is a natural isomorphism $[\mathcal{F}] \mapsto [\mathcal{F}^\triangledown]$.

Thus, the moduli spaces $M_P(\mathbb{P}^2)$ and $M_{P^\triangledown}(\mathbb{P}^2)$ are birationally equivalent.

**Proof.** Clearly, the remarks above show that $\phi$ is set-theoretically a bijection. In order to show that $\phi$ is actually a morphism, note that $M := M_P(\mathbb{P}^2)$ is a fine moduli space with universal family $\mathcal{U} \in \mathcal{M}_P(\mathbb{P}^2)(M)$ since $\mu$ and $\chi$ are coprime. Without loss of generality, we can assume that $\chi \leq \frac{\mu}{2}$. Now consider the minimal free resolution (8) of $\mathcal{C} := \mathcal{U}|_{\mathbb{P}^2 \times M \setminus N}$ from theorem 5:

$$
\begin{array}{c}
0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2) \boxtimes R^1 p_* \mathcal{C}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^2} \boxtimes p_* \mathcal{C} \boxplus \mathcal{O}_{\mathbb{P}^2}(-1) \boxtimes R^1 p_* (\mathcal{C} \boxtimes \Omega^1_{\mathbb{P}^2 \times S/\mathbb{P}^2}(1)) \longrightarrow \mathcal{C} \longrightarrow 0.
\end{array}
$$

An application of $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^2 \times M \setminus N}}(\bullet, \mathcal{O}_{\mathbb{P}^2}(-2) \boxtimes \mathcal{O}_{M \setminus N})$ yields:

$$
\begin{array}{c}
0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2) \boxtimes [p_* \mathcal{C}]^* \boxplus \mathcal{O}_{\mathbb{P}^2}(-1) \boxtimes [R^1 p_* (\mathcal{C} \boxtimes \Omega^1_{\mathbb{P}^2 \times S/\mathbb{P}^2}(1))]^* \longrightarrow \mathcal{O}_{\mathbb{P}^2} \boxtimes [R^1 p_* \mathcal{C}(-1)]^* \longrightarrow \mathcal{G} \longrightarrow 0,
\end{array}
$$

where $\mathcal{G} = \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^2 \times M \setminus N}}(\mathcal{C}, \mathcal{O}_{\mathbb{P}^2}(-2) \boxtimes \mathcal{O}_{M \setminus N})$.

According to theorem 5, the bundles $[p_* \mathcal{C}]^*$, $[R^1 p_* (\mathcal{C} \boxtimes \Omega^1_{\mathbb{P}^2 \times S/\mathbb{P}^2}(1))]^*$ and $[R^1 p_* \mathcal{C}(-1)]^*$ have rank $\chi$, $\mu - 2 \chi$ and $\mu - \chi$ respectively. Thus, the restriction of the resolution to a fiber $\mathcal{G}_{[\mathcal{F}]}$ is

$$
\begin{array}{c}
0 \longrightarrow \chi \mathcal{O}_{\mathbb{P}^2}(-2) \oplus (\mu - 2 \chi) \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow (\mu - \chi) \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{G}_{[\mathcal{F}]} \longrightarrow 0
\end{array}
$$

which is exactly the resolution of $\mathcal{F}^\triangledown$ obtained above. Therefore $\mathcal{G}_{[\mathcal{F}]} \cong \mathcal{F}^\triangledown$. Obviously, the sheaves $\mathcal{G}_{[\mathcal{F}]}$ are stable with Hilbert polynomial $P^\triangledown(m) = \mu m + (\mu - \chi)$ and $H^1 \mathcal{G}_{[\mathcal{F}]} = 0$ for all $[\mathcal{F}] \in M \setminus N$. In other words, $\mathcal{G} \in \mathcal{M}_{P^\triangledown}(\mathbb{P}^2)(M \setminus N)$. Per construction, the morphism $\Phi_{\mathcal{G}} : M \setminus N \longrightarrow M_{P^\triangledown}(\mathbb{P}^2)$ induced by the family $\mathcal{G}$ maps to $M_{P^\triangledown}(\mathbb{P}^2) \setminus N^\triangledown$ and is indeed equal to the set-theoretical map $\phi$. Similarly, one proves that $\phi^{-1}$ is a morphism. \qed
5. Smoothness

In this section we want to reprove LePotier’s result that $M_{\mu_m}(\mathbb{P}_2)$ for coprime coefficients and show that the irreducible moduli space [7] is then indeed smooth.

**Theorem 7.** Let $P(m) := \mu m + \chi$ with $(\mu, \chi) = (1)$. Then

1. $M := M_P(\mathbb{P}_2)$ is a smooth projective variety of dimension $\mu^2 + 1$.
2. The moduli space $M$ is fine with universal family $\mathcal{U} \in \mathcal{M}_P(\mathbb{P}_2)(M)$.

**Proof.** Without loss of generality we can assume that $0 \leq \chi < \mu$. By theorem 3.(7), we have that all semi-stable sheaves $\mathcal{F}$ with polynomial $P$ are stable.

1. Serre duality gives $\text{Ext}^2(\mathcal{F}, \mathcal{F}) = \text{Hom}(\mathcal{F}, \mathcal{F} \otimes \omega_{\mathbb{P}_2})^\vee = \text{Hom}(\mathcal{F}, \mathcal{F}(-3))^\vee = 0$ for every $[\mathcal{F}] \in M$. The last equality is due to the stability of $\mathcal{F}$. Id est, there are no obstructions and $M$ is smooth in neighbourhood of $[\mathcal{F}]$. Consequently, $M$ is a smooth projective variety.

We are left to compute $\dim M$. Every sheaf in the open, dense subset $\tilde{M} \setminus N = \{ [\mathcal{F}] \in M_P(\mathbb{P}_2) : H^1(\mathcal{F}) = 0 \}$ has a resolution (2). If we apply $\text{Hom}(\cdot, \mathcal{F})$ to that sequence, we end up with

$$0 \longrightarrow \text{End}(\mathcal{F}) \longrightarrow \chi H^0(\mathcal{F} \oplus (\mu - \chi) \text{Hom}(\Omega^1_{\mathbb{P}_2}(1), \mathcal{F})) \longrightarrow (2\mu - \chi) H^0(\mathcal{F}(1)) \longrightarrow \text{Ext}^1(\mathcal{F}, \mathcal{F}) \longrightarrow \cdots \longrightarrow \chi H^1(\mathcal{F} \oplus (\mu - \chi) \text{Ext}^1(\Omega^1_{\mathbb{P}_2}(1), \mathcal{F})) \longrightarrow (2\mu - \chi) H^1(\mathcal{F}(1)) \longrightarrow \text{Ext}^2(\mathcal{F}, \mathcal{F}) \longrightarrow 0$$

The stable sheaf $\mathcal{F}$ is simple and therefore $\text{End}(\mathcal{F}) \cong k$. We also have $\text{Hom}(\Omega^1_{\mathbb{P}_2}(1), \mathcal{F}) \cong H^0(\mathcal{F}(-1) \otimes (\Omega^1_{\mathbb{P}_2})^\vee) \cong H^0(\mathcal{F}(2) \otimes \Omega^1_{\mathbb{P}_2})$ and $\text{Ext}^1(\Omega^1_{\mathbb{P}_2}(1), \mathcal{F}) \cong H^1(\mathcal{F}(2) \otimes \Omega^1_{\mathbb{P}_2})$. Using the Euler sequence

$$0 \longrightarrow \mathcal{F}(2) \otimes \Omega^1_{\mathbb{P}_2} \longrightarrow 3\mathcal{F}(1) \longrightarrow \mathcal{F}(2) \longrightarrow 0,$$

we get $\chi(\mathcal{F}(2) \otimes \Omega^1_{\mathbb{P}_2}) = 3\chi(\mathcal{F}(1)) - \chi(\mathcal{F}(2)) = \mu + 2\chi$. But then:

$$\text{ext}^1(\mathcal{F}, \mathcal{F}) = 1 - \chi h^0(\mathcal{F} \oplus (\mu - \chi) \mathcal{H}^0(\mathcal{F}(2) \otimes \Omega^1) + (2\mu - \chi) H^0(\mathcal{F}(1))$$

$$\chi h^1(\mathcal{F} \oplus (\mu - \chi) h^1(\mathcal{F}(2) \otimes \Omega^1) - (2\mu - \chi) h^1(\mathcal{F}(1))$$

$$= 1 - \chi^2 - (\mu - \chi) \chi(\mathcal{F}(2) \otimes \Omega^1) + (2\mu - \chi) \chi(\mathcal{F}(1))$$

$$= 1 - \chi^2 - (\mu - \chi) (\mu + 2\chi) + (2\mu - \chi) (\mu + \chi)$$

$$= \mu^2 + 1.$$

Thus $\dim M = \mu^2 + 1$ because $\dim_k T_{[\mathcal{F}]} M = \mu^2 + 1$ for all $[\mathcal{F}] \in \tilde{M} \setminus N$.

2. The existence and construction of the universal family in this case is standard and can be found for example in [3].

\[\square\]

**Remark 1:** Let again $\chi = \mu - 1, \mu > 1$. In this case we have $N = \emptyset$. Thus, there is an isomorphism between the smooth, $(\mu^2 + 1)$-dimensional, fine moduli spaces $M_{\mu m}(\mathbb{P}_2)$ and $M_{\mu m+\mu-1}(\mathbb{P}_2)$. 
Remark 2: [7]. If $\mu$ and $\chi$ are not coprime and $\mu \geq 2$ then the complement of the open subset of stable stable sheaves in $M_{\mu\mu+\chi}(\mathbb{P}_2)$ has codimension at least $2\mu - 3$, and no matter what open set $U$ in $M_{\mu\mu+\chi}(\mathbb{P}_2)$ one chooses, there does not exist a universal sheaf over $\mathbb{P}_2 \times U$.

References


Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027

E-mail address: freiermuth@math.columbia.edu