An economic approach to discretization of nonstationary iterated Tikhonov method

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Abstract

An adaptive discretization scheme of ill-posed problems is used for nonstationary iterated Tikhonov regularization. It is shown that for some classes of operator equations of the first kind the proposed algorithm is more efficient compared with standard methods.

1 Introduction

Let X be a real Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$ generated by it. Consider the equation

$$Ax = f, \quad f \in \mathrm{Range}(A), \tag{1}$$

with a compact linear operator $A : X \to X$, $\|A\| \leq 1$. Suppose that only an approximation $f_\delta \in X$ to f is available such that $\|f - f_\delta\| \leq \delta$, where $\delta > 0$ is a known error bound. We shall study the problem of efficient finite-dimensional approximation to the minimal norm least-squares solution $x^\dagger$ of (1) under the assumption that $x^\dagger$ lies in the range of $(A^*A)^{\nu/2}$, $\nu > 0$, i.e.

$$x^\dagger \in \mathcal{M}_{\nu,\rho}(A) = \{u : \ u = |A|^\nu v, \ \|v\| \leq \rho\}, \quad |A| = (A^*A)^{1/2}, \quad \rho \geq 1,$$

where A^* is the adjoint of A. It is well known (see, for example, [9], [11]) that in this case for any approximate method the best possible accuracy is $\rho^{1/(\nu+1)}\delta^{\nu/(\nu+1)}$.

Our aim is to construct an algorithm that realizes optimal order of accuracy $O(\delta^{\nu/(\nu+1)})$ and requires substantially less discrete information compared with standard methods. Note that the problem of economical use of discrete information within the framework of the Galerkin method was considered in the paper by R.Plato and G.Vainikko [8]. The subsequent investigations (see, for example, [3]) show that the standard Galerkin method is not the best in the sense of information expences. It was found that the problem of constructing efficient finite-dimensional algorithms for solving some classes of equations (1) is connected with the adaptive approach to discretization. In this connection it should be pointed out that the idea of adaptive discretization of ill-posed problems was expressed for the first time by E.Schock in [10]. The first adaptive discretization scheme was constructed [3] for ordinary Tikhonov method, which guarantees the best order of accuracy only for $0 < \nu \leq 1$. It is known that for $\nu > 1$ the so-called saturation comes for the ordinary Tikhonov method, i.e. the level of accuracy remains constant in increasing ν. To remove this deficiency it is necessary to employ an iteration regularization method. The main difficulty in constructing such algorithms consists in the following. Up till now in discretizing the method of generating functions is used. But for analysis of adaptivly discretized iterative methods such approach is completely useless, because each iteration step is connected with own generating function. So, there is no one generating function for such methods. Therefore, to take our investigations it is necessary to find a radically new scheme for corresponding analysis. The mentioned above problem has been remedied by means of Lemma 2.

An approach to adaptive discretization of iterated Tikhonov method was proposed in [3]. But the advantages of such algorithm were demonstrated in the indicated paper only by means of one interesting (but partial) test example. At the same time the strick teoretical background of such advantages would allow to develop the theory of optimal discretization for ill-posed problems. To this end, following the recommendation of [3], as a regularizator let us make use the nonstationary iterated Tikhonov method

$$x_0 = 0,$$
$$x_l = \alpha_l (A^*A + \alpha_l I)^{-1} x_{l-1} + (A^*A + \alpha_l I)^{-1} A^* f, \quad l = 1, 2, \dots, \tag{2}$$

where $\alpha = \alpha_l$ is a positiv regularization parameter and I is the identity operator. The first variant of (2) for $\alpha_l \neq$ const has been investigated in [1], [10]. A special case of (2), where

$$\alpha_l = \frac{\|A^*(Ax_{l-1} - f)\|^2}{\|Ax_{l-1} - f\|^2},$$

has been analyzed by H.Brakhage (see, for example, [7]). Finally, M.Hanke and C.W.Groetsch [2] have established the convergence rate $O(\delta^{\nu/(\nu+1)})$ for (2) under general conditions on the parameter $\alpha = \alpha_l$. In the present paper, we shall employ a geometric choice of α, i.e.

$$\alpha_l = \alpha_0 q^l$$

with fixed $\alpha_0 > 0$ and $0 < q < 1$. The regularization method (2) is generated by the function

$$g_l(\lambda) = (1 - r_l(\lambda))/\lambda, \qquad r_l(\lambda) = \prod_{j=1}^{l} \alpha_j/(\alpha_j + \lambda),$$

for which the following estimates [2]

$$\max_{0 \le \lambda < \infty} g_l(\lambda) = \sigma_l, \qquad \max_{0 \le \lambda < \infty} \lambda g_l(\lambda) \le 1,$$
$$\max_{0 \le \lambda < \infty} \lambda^\nu (1 - \lambda g_l(\lambda)) \le \chi_\nu \sigma_l^{-\nu}, \qquad 0 < \nu \le l, \tag{3}$$

are true, where

$$\chi_\nu = \begin{cases} O(\nu^\nu), & 0 < \nu \le 1, \\ O(c^{\nu^\nu}), & \nu > 1 \end{cases}, \qquad \sigma_l := \sum_{j=1}^{l} 1/\alpha_j.$$

2 Several auxiliary assertions

Denote by $\{\varphi_k, \psi_k, \lambda_k\}$ the singular value decomposition for A, where $\varphi_k, \psi_k \in X$ are the singular vectors and $\lambda_k > 0$ are the singular values, i.e.

$$A = \sum_k \lambda_k \varphi_k(\cdot, \psi_k).$$

Then

$$Ax_l - f = \sum_k \lambda_k |\lambda_k|^\nu \varphi_k(v, \psi_k) \prod_{j=1}^{l} \alpha_j/(\alpha_j + \lambda_k^2), \quad (4)$$

$$x^\dagger - x_l := (I - g_l(A^*A)A^*A)x^\dagger = \sum_k |\lambda_k|^\nu (v, \psi_k) \prod_{j=1}^{l} \alpha_j/(\alpha_j + \lambda_k^2). \quad (5)$$

From (4) we obtain

$$\|Ax_l - f\|^2 = \sum_k \lambda_k^{2(\nu+1)} (v, \psi_k)^2 \prod_{j=1}^{l} \left(\frac{\alpha_j}{\alpha_j + \lambda_k^2} \right)^2.$$

Present the last equality as

$$\|Ax_l - f\|^2 = \sigma_l^{-(\nu+1)} |d_{\nu,l}(v)|^2, \quad (6)$$

where

$$|d_{\nu,l}(v)|^2 := \sigma_l^{\nu+1} \sum_k \lambda_k^{2(\nu+1)} (v, \psi_k)^2 r_l^2(\lambda_k^2). \quad (7)$$

Next, by means of (5) we find

$$\|x^\dagger - x_l\|^2 = \sum_k \lambda_k^{2\nu} (v, \psi_k)^2 \prod_{j=1}^{l} \left(\frac{\alpha_j}{\alpha_j + \lambda_k^2} \right)^2.$$

This equation may be written in the equivalent form

$$\|x^\dagger - x_l\|^2 = \sigma_l^{-\nu} |c_{\nu,l}(v)|^2, \quad (8)$$

where

$$|c_{\nu,l}(v)|^2 := \sigma_l^\nu \sum_k \lambda_k^{2\nu} (v, \psi_k)^2 r_l^2(\lambda_k^2). \quad (9)$$

Lemma 1. *For any $l = 1, 2, \ldots$ it happens*

$$|c_{\nu,l}(v)| \leq |d_{\nu,l}(v)|^{\nu/(\nu+1)} \|v\|^{1/(\nu+1)},$$

$$|d_{\nu,l}(v)| \leq \sqrt{\chi_{\nu+1}} \|v\|, \quad \nu \leq l-1.$$

Proof. To estimate the value $|c_{\nu,l}(v)|$ we shall use Hölder's inequality. Taking (7) and (9) into account, we have

$$|c_{\nu,l}(v)|^2 = \sum_k \left(\sigma_l^{\nu+1} \lambda_k^{2(\nu+1)} r_l^2(\lambda_k^2)(v, \psi_k)^2 \right)^{\frac{\nu}{\nu+1}} \left(r_l^2(\lambda_k^2)(v, \psi_k)^2 \right)^{\frac{1}{\nu+1}} \leq$$

$$\leq |d_{\nu,l}(v)|^{\frac{2\nu}{\nu+1}} \left(\sum_k r_l^2(\lambda_k^2)(v, \psi_k)^2 \right)^{\frac{1}{\nu+1}} \leq |d_{\nu,l}(v)|^{\frac{2\nu}{\nu+1}} \|v\|^{\frac{2}{\nu+1}}.$$

Using (3) and (7), we find

$$|d_{\nu,l}(v)|^2 \leq \sigma_l^{\nu+1} \|v\|^2 \sup_{\lambda \in [0,\infty)} r_l(\lambda^2) \sup_{\lambda \in [0,\infty)} (\lambda^2)^{\nu+1} r_l(\lambda^2) \leq \chi_{\nu+1} \|v\|^2.$$

Thus the lemma 1 is proved.

Now write a discretized variant of regularization method (2):

$$\hat{x}_0 = 0,$$
$$\hat{x}_l = \alpha_l (A_l^* A_l + \alpha_l I)^{-1} \hat{x}_{l-1} + (A_l^* A_l + \alpha_l I)^{-1} A_l^* f_\delta, \quad l \geq 1,$$

where A_l is a finite-dimensional approximation to A that is constructed for the l-th iteration step. Then the approximate solution $\hat{x}_l$ becomes

$$\hat{x}_l = \sum_{k=0}^{l-1} \frac{1}{\alpha_{l-k}} \left(\prod_{j=0}^{k} \alpha_{l-j} (A_{l-j}^* A_{l-j} + \alpha_{l-j} I)^{-1} \right) A_{l-k}^* f_\delta.$$

Here, as usual, the operator multiplication is written as

$$\prod_{j=N}^{M} A_j := A_N A_{N+1} \ldots A_M.$$

In case where $M < N$ by $\prod_{j=N}^{M} A_j$ we mean the identity operator I.

Introduce an auxiliary element

$$x_l^\delta := g_l(A^*A)A^*f_\delta = \sum_{k=0}^{l-1} \frac{1}{\alpha_{l-k}} \Big(\prod_{j=0}^{k} \alpha_{l-j}(A^*A + \alpha_{l-j}I)^{-1} \Big) A^* f_\delta.$$

Write down the error presentation

$$x_l^\delta - \hat{x}_l := B_l f_\delta, \tag{10}$$

where

$$B_l := \sum_{k=0}^{l-1} \frac{1}{\alpha_{l-k}} \left(\Big(\prod_{j=0}^{k} \alpha_{l-j}(A^*A + \alpha_{l-j}I)^{-1} \Big) A^* - G_{l,k} A_{l-k}^* \right),$$

$$G_{l,k} = \prod_{j=0}^{k} \alpha_{l-j}(A_{l-j}^* A_{l-j} + \alpha_{l-j}I)^{-1}. \tag{11}$$

Lemma 2. *For any $l = 2, 3, \ldots$ it holds*

$$B_l = \sum_{k=0}^{l-1} \frac{1}{\alpha_{l-k}} \left(\prod_{j=0}^{k} \alpha_{l-j}(A^*A + \alpha_{l-j}I)^{-1} \right) (A^* - A_{l-k}^*) - \sum_{k=1}^{l} F_k,$$

where

$$F_k = \sum_{j=0}^{l-k} \left(\prod_{i=0}^{j-1} \alpha_{l-i}(A^*A + \alpha_{l-i}I)^{-1} \right) T_{j,k}, \quad k = \overline{1, l},$$

$$T_{j,1} = (A_{l-j}^* A_{l-j} + \alpha_{l-j}I)^{-1}(A^*A - A_{l-j}^* A_{l-j}) \sum_{k=j}^{l-1} \left(\prod_{s=j}^{k} R_s \right) \frac{A_{l-k}^*}{\alpha_{l-k}},$$

$$T_{j,k} = (A_{l-j}^* A_{l-j} + \alpha_{l-j}I)^{-1}(A^*A - A_{l-j}^* A_{l-j}) \sum_{i=j+1}^{l-k+1} \left(\prod_{s=j}^{i-1} R_s \right) T_{i,k-1}, \quad k \geq 2,$$

$$R_j = \alpha_{l-j}(A^*A + \alpha_{l-j}I)^{-1}$$

Proof. It is not difficult to make sure that for any $j = 1, 2, \dots$

$$(A_j^* A_j + \alpha_j I)^{-1} = (A^* A + \alpha_j I)^{-1} + (A_j^* A_j + \alpha_j I)^{-1}(A^* A - A_j^* A_j)(A^* A + \alpha_j I)^{-1}.$$

To reduce the subsequent computations we introduce the following notation

$$H_j = (A_{l-j}^* A_{l-j} + \alpha_{l-j} I)^{-1}(A^* A - A_{l-j}^* A_{l-j}) R_j.$$

Then

$$\alpha_{l-j}(A_{l-j}^* A_{l-j} + \alpha_{l-j} I)^{-1} = H_j + R_j. \tag{12}$$

Now introduce into consideration a special operation for operator summation. So, let there be given two operator sequences $\{D_i\}$ and $\{C_i\}$, $i = 1, 2, \dots$. Denote the mentioned operation by

$$D \bigoplus_N^M C^{(p)},$$

where $M \geq N \geq 1$, $p \leq M - N + 1$. This operation is performed on the multiplication of $M - N + 1$ operators $D_N D_{N+1} \dots D_M$ and its main point consists in the following. We replace all combinations of arbitrary p operators $D_{i_1}, \dots, D_{i_p}$, $i_k \neq i_l$, $k \neq l$, by operators $C_{i_1}, \dots, C_{i_p}$ respectively such that the order of another multiplicators is preserved. As the result of this operation we finnaly obtain a sum of all replaced in such manner operators. The indicated operation possesses some properties that will be used repeatedly in the further reasoning. Namely,

$$D \bigoplus_N^M C^{(M-N+1)} = \prod_{j=N}^M C_j,$$

$$D \bigoplus_N^M C^{(M-N)} = \sum_{q=0}^{M-N} \left(\prod_{i=0}^{q-1} C_{i+N} \right) D_{N+q} \left(\prod_{s=q+1}^{M} C_{s+N} \right),$$

$$D \bigoplus_N^M C^{(p)} = \sum_{q=0}^{p} \left(\prod_{i=0}^{q-1} C_{i+N} \right) D_{N+q} \left(D \bigoplus_{N+q+1}^{M} C^{(p-q)} \right), \quad p < M - N.$$

Next, using (12) let us transform (11)

$$G_{l,k} := \prod_{j=0}^{k} \alpha_{l-j}(A_{l-j}^* A_{l-j} + \alpha_{l-j} I)^{-1} = \prod_{j=0}^{k} (H_j + R_j) =$$

$$= \sum_{i=0}^{k+1} \left(H \bigoplus_{0}^{k} R^{(k-i+1)} \right) = \prod_{j=0}^{k} R_j + \sum_{i=1}^{k+1} S_{k,i},$$

where

$$S_{k,i} = H \bigoplus_{0}^{k} R^{(k-i+1)}.$$

Then

$$B_l = \sum_{k=0}^{l-1} \frac{1}{\alpha_{l-k}} \left(\prod_{j=0}^{k} R_j \right) (A^* - A_{l-k}^*) - \sum_{k=0}^{l-1} \sum_{i=1}^{k+1} S_{k,i} \frac{A_{l-k}^*}{\alpha_{l-k}} =$$

$$= \sum_{k=0}^{l-1} \frac{1}{\alpha_{l-k}} \left(\prod_{j=0}^{k} R_j \right) (A^* - A_{l-k}^*) - \sum_{j=1}^{l} \hat{F}_j,$$

where

$$\hat{F}_j = \sum_{k=j-1}^{l-1} S_{k,j} \frac{A_{l-k}^*}{\alpha_{l-k}}.$$

We must prove that $\hat{F}_j = F_j$, $j = \overline{1,l}$. Consider first the case $j = 1$. Thus,

$$\hat{F}_1 := \sum_{k=0}^{l-1} S_{k,1} \frac{A_{l-k}^*}{\alpha_{l-k}} = \sum_{k=0}^{l-1} \left(H \bigoplus_{0}^{k} R^{(k)} \right) \frac{A_{l-k}^*}{\alpha_{l-k}} =$$

$$= \sum_{k=0}^{l-1} \left(\sum_{q=0}^{k} \left(\prod_{i=0}^{q-i} R_i \right) H_q \left(\prod_{s=q+1}^{k} R_s \right) \right) \frac{A_{l-k}^*}{\alpha_{l-k}} =$$

$$= \{\text{the interchange of the order of summation: } k \rightleftharpoons q\} =$$

$$= \sum_{q=0}^{l-1} \Big(\prod_{i=0}^{q-1} R_i \Big) H_q \sum_{k=q}^{l-1} \Big(\prod_{s=q+1}^{k} R_s \Big) \frac{A_{l-k}^*}{\alpha_{l-k}} = \sum_{j=0}^{l-1} \Big(\prod_{i=0}^{j-1} R_i \Big) T_{j,1},$$

where

$$T_{j,1} = (A_{l-j}^* A_{l-j} + \alpha_{l-j} I)^{-1} (A^* A - A_{l-j}^* A_{l-j}) \sum_{k=j}^{l-1} \Big(\prod_{s=j}^{k} R_s \Big) \frac{A_{l-k}^*}{\alpha_{l-k}}.$$

Furthermore, for $j \geq 2$ we have

$$\hat{F}_j := \sum_{k=j-1}^{l-1} S_{k,j} \frac{A_{l-k}^*}{\alpha_{l-k}} = \{\text{the change of variables: } p = k - j + 1\} =$$

$$= \sum_{p=0}^{l-j} S_{p+j-1,j} \frac{A_{l-p-j+1}^*}{\alpha_{l-p-j+1}} = \sum_{p=0}^{l-j} \Big(H \bigoplus_{0}^{p+j-1} R^{(p)} \Big) \frac{A_{l-p-j+1}^*}{\alpha_{l-p-j+1}} =$$

$$= \sum_{p=0}^{l-j} \Bigg(\sum_{q=0}^{p} \Big(\prod_{i=0}^{q-1} R_i \Big) H_q \Big(H \bigoplus_{q+1}^{p+j-1} R^{(p-q)} \Big) \Bigg) \frac{A_{l-p-j+1}^*}{\alpha_{l-p-j+1}} = \{p \rightleftharpoons q\} =$$

$$= \sum_{q=0}^{l-j} \Big(\prod_{i=0}^{q-1} R_i \Big) H_q \sum_{p=q}^{l-j} \Big(H \bigoplus_{q+1}^{p+j-1} R^{(p-q)} \Big) \frac{A_{l-p-j+1}^*}{\alpha_{l-p-j+1}} = \sum_{k=0}^{l-j} \Big(\prod_{i=0}^{k-1} R_i \Big) \hat{T}_{k,j},$$

where

$$\hat{T}_{k,j} := H_k \sum_{p=k}^{l-j} \Big(H \bigoplus_{k+1}^{p+j-1} R^{(p-k)} \Big) \frac{A_{l-p-j+1}^*}{\alpha_{l-p-j+1}}.$$

It remains to prove that $\hat{T}_{k,j} = T_{k,j}$ for $j \geq 2$. Let $j = 2$, $k = \overline{0, l-2}$. Then,

$$\hat{T}_{k,2} := H_k \sum_{p=k}^{l-2} \Big(H \bigoplus_{k+1}^{p+1} R^{(p-k)} \Big) \frac{A_{l-p-1}^*}{\alpha_{l-p-1}} =$$

$$= H_k \sum_{p=k}^{l-2} \sum_{q=0}^{p-k} \Big(\prod_{j=k+1}^{k+q} R_j \Big) H_{k+q+1} \Big(\prod_{s=k+q+2}^{p+1} R_s \Big) \frac{A_{l-p-1}^*}{\alpha_{l-p-1}} = \{p \rightleftharpoons q\} =$$

$$= H_k \sum_{q=0}^{l-k-2} \Big(\prod_{j=k+1}^{k+q} R_j\Big) H_{k+q+1} \sum_{p=k+q}^{l-2} \Big(\prod_{s=k+q+2}^{p+1} R_s\Big) \frac{A^*_{l-p-1}}{\alpha_{l-p-1}} = \{q = i-k-1\} =$$

$$= H_k \sum_{i=k+1}^{l-1} \Big(\prod_{j=k+1}^{i-1} R_j\Big) H_i \sum_{p=i-1}^{l-2} \Big(\prod_{s=i+1}^{p+1} R_s\Big) \frac{A^*_{l-p-1}}{\alpha_{l-p-1}} = \{m = p+1\} =$$

$$= H_k \sum_{i=k+1}^{l-1} \Big(\prod_{j=k+1}^{i-1} R_j\Big) H_i \sum_{m=i}^{l-1} \Big(\prod_{s=i+1}^{m} R_s\Big) \frac{A^*_{l-m}}{\alpha_{l-m}} =$$

$$= (A^*_{l-k}A_{l-k} + \alpha_{l-k} I)^{-1}(A^*A - A^*_{l-k}A_{l-k}) \sum_{i=k+1}^{l-1} \Big(\prod_{j=k}^{i-1} R_j\Big) T_{i,1}.$$

Finally, for $j \geq 3$, $k = \overline{0, l-j}$, it holds

$$\hat{T}_{k,j} = H_k \sum_{p=k}^{l-j} \Big(H \bigoplus_{k+1}^{p+j-1} R^{(p-k)}\Big) \frac{A^*_{l-p-j+1}}{\alpha_{l-p-j+1}}$$

$$= H_k \sum_{p=k}^{l-j} \sum_{q=0}^{p-k} \Big(\prod_{s=k+1}^{k+q} R_s\Big) H_{k+q+1} \Big(H \bigoplus_{k+q+2}^{p+j-1} R^{(p-k-q)}\Big) \frac{A^*_{l-p-j+1}}{\alpha_{l-p-j+1}} = \{p \rightleftharpoons q\}$$

$$= H_k \sum_{q=0}^{l-k-j} \Big(\prod_{s=k+1}^{k+q} R_s\Big) H_{k+q+1} \sum_{p=q+k}^{l-j} \Big(H \bigoplus_{k+q+2}^{p+j-1} R^{(p-k-q)}\Big) \frac{A^*_{l-p-j+1}}{\alpha_{l-p-j+1}} = \{q = i-k-1\}$$

$$= H_k \sum_{i=k+1}^{l-j+1} \Big(\prod_{s=k+1}^{i-1} R_s\Big) H_i \sum_{p=i-1}^{l-j} \Big(H \bigoplus_{i+1}^{p+j-1} R^{(p-i+1)}\Big) \frac{A^*_{l-p-j+1}}{\alpha_{l-p-j+1}} = \{m = p+1\}$$

$$= H_k \sum_{i=k+1}^{l-j+1} \Big(\prod_{s=k+1}^{i-1} R_s\Big) H_i \sum_{m=i}^{l-j+1} \Big(H \bigoplus_{i+1}^{m+j-2} R^{(m-i)}\Big) \frac{A^*_{l-m-j+2}}{\alpha_{l-m-j+2}}$$

$$= (A^*_{l-k}A_{l-k} + \alpha_{l-k} I)^{-1}(A^*A - A^*_{l-k}A_{l-k}) \sum_{i=k+1}^{l-j+1} \Big(\prod_{s=k}^{i-1} R_s\Big) \hat{T}_{i,j-1}.$$

This completes the proof of lemma.

We assume that a sequence of discretized operators A_l is constructed such that for any $l = 1, 2, \dots, L$ it happens

$$\|A^*A - A_l^*A_l\| \le \delta/(4\rho\sqrt{\sigma_l}), \quad \|A - A_l\| \le \left(\delta/(4\sqrt{\sigma_l})\right)^{1/2},$$

$$\|(A - A_l)A^*\| \le \delta/(4\rho\sqrt{\sigma_l}), \quad \|(A^* - A_l^*)A\| \le \delta/(4\rho\sqrt{\sigma_l}). \tag{13}$$

In addition, suppose that the general number L of iteration steps is limited by condition

$$\delta\sqrt{\sigma_L} \le 1. \tag{14}$$

Relation (14) does not restrict of generality, because a convergence to the desired solution is possible only at $\delta\sqrt{\sigma_L} \to 0$, $\delta \to 0$.

Lemma 3. *For any $l = 2, 3, \dots, L$ it holds*

$$\left\| \sum_{k=1}^{l} F_k f_\delta \right\| \le c_2\delta/\sqrt{\alpha_l},$$

where

$$c_2 = \frac{1+\sqrt{q}}{1-\sqrt{q}} \cdot \frac{\rho + 1 + 3(1+\sqrt{q})/4}{4\rho - \sqrt{q}}.$$

Proof. Taking (13) and (14) into account, we get

$$\|(A^* - A_{l-k}^*)f_\delta\| \le \|(A^* - A_{l-k}^*)A\|\|x^\dagger\| + \|A - A_{l-k}\|\|f - f_\delta\| \le$$

$$\le \frac{\delta}{4\sqrt{\sigma_{l-k}}} + \frac{\delta^{3/2}}{2\sigma_{l-k}^{1/4}} \le \frac{3\delta}{4\sqrt{\sigma_{l-k}}}.$$

Write down two auxiliary relations

$$\frac{1}{\sqrt{\sigma_{l-j}}\sqrt{\alpha_{l-j}}} = \frac{\sqrt{1-q}}{\sqrt{1-q^{l-j}}},$$

$$\frac{\delta}{\sqrt{\alpha_{l-j}}} \leq q^{j/2}\frac{\sqrt{1-q}}{\sqrt{1-q^{l}}}. \tag{15}$$

Then,

$$\frac{\|(A^* - A^*_{l-k})f_\delta\|}{\alpha_{l-k}} \leq \frac{3\delta}{4\sqrt{\alpha_{l-k}}}, \tag{16}$$

$$\sum_{k=j}^{l-1} \frac{\|(A^* - A^*_{l-k})f_\delta\|}{\alpha_{l-k}} \leq \frac{3\delta}{4\sqrt{\alpha_{l-k}}}\frac{1-\sqrt{q^{l-j}}}{1-\sqrt{q}}, \quad j = \overline{0, l-1}. \tag{17}$$

Since

$$\sum_{k=j}^{l-1} \frac{1}{\alpha_{l-k}} \prod_{i=j}^{k} R_i = g_{l-j}(A^*A),$$

we have

$$T_{j,1}f_\delta = (A^*_{l-j}A_{l-j} + \alpha_{l-j}I)^{-1}(A^*A - A^*_{l-j}A_{l-j})\Big(g_{l-j}(A^*A)A^*Ax^\dagger -$$

$$- g_{l-j}(A^*A)A^*(f - f_\delta) - \sum_{k=j}^{l-1}\Big(\prod_{s=j}^{k} R_s\Big)\frac{(A^* - A^*_{l-k})f_\delta}{\alpha_{l-k}}\Big).$$

Hence, by using (15), (17) and the following estimates

$$\|g_{l-j}(A^*A)A^*A\| \leq 1, \quad \|g_{l-j}(A^*A)A^*\| \leq \sqrt{\sigma_{l-j}}, \quad \|R_s\| \leq 1,$$

we obtain

$$\|T_{j,1}f_\delta\| \leq \frac{\|A^*A - A^*_{l-j}A_{l-j}\|}{\alpha_{l-j}}\Big(\rho\|g_{l-j}(A^*A)A^*A\| + \delta\|g_{l-j}(A^*A)A^*\| +$$

$$+\sum_{k=j}^{l-1}\frac{\|(A^*-A^*_{l-k})f_\delta\|}{\alpha_{l-k}}\Big)\le\frac{\delta}{4\rho\sqrt{\sigma_{l-j}}\alpha_{l-j}}\Big(\rho+\delta\sqrt{\sigma_{l-j}}+\frac{3\delta}{4\sqrt{\alpha_{l-j}}}\cdot\frac{1-\sqrt{q^{l-j}}}{1-\sqrt{q}}\Big)\le$$

$$\le\frac{c_1\delta}{\sqrt{\alpha_{l-j}}},\qquad c_1=1/4+\left(1+3(1+\sqrt{q})/4\right)/(4\rho).$$

Furthermore,

$$\|T_{j,2}f_\delta\|\le\frac{\delta}{4\rho\sqrt{\sigma_{l-j}}\alpha_{l-j}}c_1\delta\sum_{i=j+1}^{l-1}\alpha_{l-i}^{-1/2}\le$$

$$\le\frac{c_1\delta}{4\rho\sqrt{\alpha_{l-j}}}\cdot\frac{1-q}{1-\sqrt{q}}\cdot\frac{1-\sqrt{q^{l-j-1}}}{\sqrt{1-q^{l-j}}\sqrt{1-q^l}}\sqrt{q^{j+1}}\le$$

$$\le\frac{c_1\delta\sqrt{q^{j+1}}}{4\rho\sqrt{\alpha_{l-j}}}\cdot\frac{1+\sqrt{q}}{1+\sqrt{q^{l-j}}},$$

$$\|T_{j,3}f_\delta\|\le\frac{\delta}{4\rho\sqrt{\sigma_{l-j}}\alpha_{l-j}}\cdot\frac{c_1\delta(1+\sqrt{q})}{4\rho(1+\sqrt{q^{l-j-1}})}\cdot\sum_{i=j+1}^{l-2}\frac{\sqrt{q^{i+1}}}{\sqrt{\alpha_{l-i}}}\le$$

$$\le\frac{c_1\delta(1+\sqrt{q})}{4^2\rho^2\sqrt{\alpha_{l-j}}}\cdot\frac{1-\sqrt{q^{l-j-1}}}{\sqrt{1-q^{l-j}}\sqrt{1-q^l}}q^{j+3/2}\le\frac{c_1\delta q^{j+3/2}}{4^2\rho^2\sqrt{\alpha_{l-j}}}\cdot\frac{1+\sqrt{q}}{1+\sqrt{q^{l-j}}}.$$

A similar argument yields

$$\|T_{j,k}f_\delta\|\le\frac{c_1\delta}{\sqrt{\alpha_{l-j}}}(\sqrt{q^{j+1}}/(4\rho))^{k-1}\frac{1+\sqrt{q}}{1+\sqrt{q^{l-j}}}$$

for any $k=1,2,\ldots$. Therefore,

$$\Big\|\sum_{k=1}^{l}F_kf_\delta\Big\|:=\Big\|\sum_{k=1}^{l}\sum_{j=0}^{l-k}T_{j,k}f_\delta\Big\|\le\sum_{j=0}^{l-1}\sum_{k=1}^{l-j}\|T_{j,k}f_\delta\|\le$$

$$\le\sum_{j=0}^{l-1}\frac{c_1\delta}{\sqrt{\alpha_{l-j}}}\cdot\frac{1+\sqrt{q}}{1+\sqrt{q^{l-j}}}\cdot\sum_{k=1}^{l-j}(\sqrt{q^{j+1}}/(4\rho))^{k-1}\le$$

$$\le \frac{c_1\delta}{\sqrt{\alpha_l}} \cdot \frac{1+\sqrt{q}}{1-\sqrt{q}} \cdot \frac{1}{1-\sqrt{q}/(4\rho)}.$$

The lemma is proved.

Lemma 4. *For any $l = 2, 3, \dots, L$ the following estimate*

$$\|x^\dagger - \hat{x}_l\| \le |c_{\nu,l}(v)|\sigma_l^{-\nu/2} + \delta\sqrt{\sigma_l} + c_3\delta/\sqrt{\alpha_l} \tag{18}$$

is true, where

$$c_3 = c_2 + \frac{3(1-\sqrt{q^l})}{4(1-\sqrt{q})}.$$

Proof. It is easy to see that

$$x^\dagger - \hat{x}_l = (x^\dagger - x_l) + g_l(A^*A)A^*(f - f_\delta) + (x_l^\delta - \hat{x}_l). \tag{19}$$

Let us estimate separately all terms on the right-hand side of (19).
1. From (8) it follows

$$\|x^\dagger - x_l\| \le |c_{\nu,l}(v)|\sigma_l^{-\nu/2}.$$

2. Taking (3) into account, we get

$$\|g_l(A^*A)A^*(f - f_\delta)\| \le \delta \sup_{0\le\lambda<\infty} \sqrt{\lambda} g_l(\lambda) \le \delta\sqrt{\sigma_l}.$$

To estimate the last term (19), it is necessary, as appears from (10), to calculate the quantity $\|B_l f_\delta\|$. By using lemmas 2,3 and relation (17), we find

$$\|x_l^\delta - \hat{x}_l\| := \|B_l f_\delta\| \le \sum_{k=0}^{l-1} \frac{\|(A^* - A_{l-k}^*)f_\delta\|}{\alpha_{l-k}} + \left\|\sum_{k=1}^{l} F_k f_\delta\right\| \le$$

$$\le \left(c_2 + \frac{3(1-\sqrt{q^l})}{4(1-\sqrt{q})}\right)\delta/\sqrt{\alpha_l}, \tag{20}$$

which implies the required estimate.

Lemma 5. *For any $l = 2, 3, \ldots, L$ it holds*

$$\|Ax_l - f\| \leq \|A_l\hat{x}_l - f\| + c_4\delta,$$

where

$$c_4 = 2 + 1/\rho + c_2 + 3\left(\frac{1}{1-\sqrt{q}} + \left(\frac{1+\sqrt{q}}{1-\sqrt{q}}\right)^{1/2}\right)/8.$$

Proof. It is easy to see that

$$Ax_l - f := Ag_l(A^*A)A^*f - f = \sum_{i=1}^{5} Z_i, \tag{21}$$

where

$$Z_1 = Ag_l(A^*A)A^*(f - f_\delta),$$

$$Z_2 = (A - A_l)A^*g_l(AA^*)f_\delta,$$

$$Z_3 = -(A - A_l)(x_l^\delta - \hat{x}_l),$$

$$Z_4 = A(x_l^\delta - \hat{x}_l),$$

$$Z_5 = A_l\hat{x}_l - f.$$

It is required to estimate the norm of elements Z_1–Z_4.
By using (3), we find

1. $$\|Z_1\| \leq \|f - f_\delta\| \sup_{0\leq\lambda<\infty} \lambda g_l(\lambda) \leq \delta, \tag{22}$$

2. $$\|Z_2\| \leq \|(A-A_l)A^*\|(\|g_l(AA^*)Ax^\dagger\| + \|g_l(AA^*)\|\|f-f_\delta\|) \leq$$

$$\leq (1 + 1/\rho)\delta. \tag{23}$$

3. By virtue of (15) and (20) we get

$$\|Z_3\| \le \|A - A_l\| \|x_l^\delta - \hat{x}_l\| \le \frac{c_3 \delta^{3/2}}{2\sigma_l^{1/4}\sqrt{\alpha_l}} \le \frac{c_3\delta}{2} \cdot \frac{\sqrt{1-q}}{\sqrt{1-q^l}}. \tag{24}$$

4. To estimate $\|Z_4\|$ let us, once again, make use of (10) and Lemma 2

$$Z_4 := AB_l f_\delta = \sum_{k=0}^{l-1} A\left(\prod_{j=0}^{k} \alpha_{l-j}(A^*A + \alpha_{l-j}I)^{-1}\right)\frac{(A^* - A^*_{l-k})f_\delta}{\alpha_{l-k}} - \sum_{k=1}^{l} AF_k f_\delta.$$

Since, by (16), the following relations

$$\sum_{k=0}^{l-1}\left\| A\Big(\prod_{j=0}^{k} \alpha_{l-j}(A^*A + \alpha_{l-j}I)^{-1}\Big)\frac{(A^* - A^*_{l-k})f_\delta}{\alpha_{l-k}}\right\| \le$$

$$\le \alpha_l \|A(A^*A + \alpha_l I)^{-1}\| \sum_{k=0}^{l-1} \frac{\|(A^* - A^*_{l-k})f_\delta\|}{\alpha_{l-k}} \le \frac{3}{8}\cdot\frac{1-\sqrt{q^l}}{1-\sqrt{q}}\delta,$$

$$\left\|\sum_{k=1}^{l} AF_k f_\delta\right\| \le \frac{\sqrt{\alpha_l}}{2}\sum_{j=0}^{l-1}\sum_{k=1}^{l-j}\|T_{j,k}f_\delta\| \le c_2\delta/2$$

are true, we have

$$\|Z_4\| \le \left(c_2/2 + \frac{3}{8}\cdot\frac{1-\sqrt{q^l}}{1-\sqrt{q}}\right)\delta. \tag{25}$$

Substituting the estimates (22)–(25) into (21), we obtain the assertion of the lemma.

Consider the class $\mathcal{H}^r$, $r = 1, 2, \ldots$, of compact linear operators A, $\|A\| \le 1$, that satisfy for any $m = 1, 2, \ldots$ the condition

$$\|(I - P_m)A\| \le m^{-r}, \qquad \|A(I - P_m)\| \le m^{-r}, \tag{26}$$

where P_m is the orthoprojector onto the linear span of the first m elements of a basis $E = \{e_i\}_{i=1}^{\infty}$ of X. To illustrate the condition (26), we take the integral operator

$$Ax(t) = \int_0^1 h(t,\tau)x(\tau)\ d\tau$$

in the space $X = L_2(0,1)$. In order to satisfy (26), it suffices in this case that $\|\partial^r h(t,\tau)/\partial t^r\|_{L_2} \leq 1$, $\|\partial^r h(t,\tau)/\partial \tau^r\|_{L_2} \leq 1$. As the basis E one can take the Haar orthonormal system (at $r = 1$), the Fourier orthonormal system of trigonometric polynomials (in the periodic case) and the basis of Legendre's polynomials. Some other examples of classes $\mathcal{H}^r$ and corresponding basises E see [6].

Introduce into consideration a discretization scheme that will be used for solving equations (1) with operators $A \in \mathcal{H}^r$. Let $n = n(l)$. By Γ_n we denote the following figure

$$\Gamma_n := \bigcup_{k=1}^{2n(l)} (2^{k-1}, 2^k] \times [1, 2^{2n(l)-k}] \bigcup \{1\} \times [1, 2^{2n(l)}]$$

in the coordinate plane corresponding to the basis E that appears in the definition of $\mathcal{H}^r$. And now we construct some discretized operators A_l, $l = 1, 2, \ldots$, by means of Γ_n

$$A_{n(l)} = A_l := \sum_{k=1}^{2n(l)} (P_{2^k} - P_{2^{k-1}}) A P_{2^{2n(l)-k}} + P_1 A P_{2^{2n(l)}}. \qquad (27)$$

The following lemma characterizes some approximation properties of the operator $A_{n(l)}$.

Lemma 6. *If the parameter $n = n(l)$ is chosen by relation*

$$(1 + 2^{r+3}) n 2^{-2nr} = \delta / (4\rho\sqrt{\sigma_l}),$$

then for $A_{n(l)} = A_l$ (27) and for any $A \in \mathcal{H}^r$ it holds the estimates (13).

This lemma can be proved in the same way as Lemma 1 [3].

Give now the detailed description of proposed algorithm that consists in the combination of iterated Tikhonov regularization and adaptive approach to discretization. As a stopping rule we employ the discrepancy principle [5]. Namely,

1. given date: $A \in \mathcal{H}^r, f_\delta, \delta, \rho$;
2. initialization: $\alpha_0 > 0,\ \ 0 < q < 1,\ \ d > c_4 + \sqrt{2},\ \ \hat{x}_0 = 0$;
3. iteration by $l = 1, 2, \ldots, L$

   a) choosing of regularization parameter

   $$\alpha_l = \alpha_0 q^l; \tag{28}$$

   b) choosing of discretization level $n = n(l)$

   $$(1 + 2^{r+3}) n 2^{-2nr} = \delta/(4\rho\sqrt{\sigma_l}); \tag{29}$$

   c) computation of functionals

   $$\begin{gathered} (f_\delta, e_i), \quad i \in (2^{2n(l-1)}, 2^{2n(l)}], \\ (Ae_j, e_i),\ \ (i,j) \in \Gamma_{n(l)} \setminus \Gamma_{n(l-1)}; \end{gathered} \tag{30}$$

   d) computation of the l-th approximation according to iterated Tikhonov method

   $$\alpha_l \hat{x}_l + A^*_{n(l)} A_{n(l)} \hat{x}_l = \alpha_l \hat{x}_{l-1} + A^*_{n(l)} f_\delta \tag{31}$$

   and by means of discrepancy principle

   $$\begin{gathered} \|A_{n(L)} \hat{x}_L - P_{2^{2n(L)}} f_\delta\| \le d\delta, \\ \|A_{n(l)} \hat{x}_l - P_{2^{2n(l)}} f_\delta\| > d\delta, \quad \forall l < L; \end{gathered} \tag{32}$$

4. approximate solution: $\hat{x}_L$.

Lemma 7. *Let the discrepancy principle (32) be satisfied for some $L \geq 2$, $d > c_4 + \sqrt{2}$, $A \in \mathcal{H}^r$. Let the discretization parameter $n = n(l)$ be chosen according to (29). Then there exist $d_1, d_2 > 0$ such that*

$$\|Ax_{L-1} - f\| \geq d_1\delta,$$
$$\|Ax_L - f\| \leq d_2\delta.$$

Proof. In accordance with the condition (29) we have for any $l = 1, 2, \ldots, L$ and $f = Ax^\dagger$, $A \in \mathcal{H}^r$, $x^\dagger \in \mathcal{M}_{\nu,\rho}(A)$

$$\|(I - P_{2^{2n(l)}})f\| \leq \delta.$$

Since

$$\|P_{2^{2n(L)}}(f - f_\delta) + (I - P_{2^{2n(L)}})f\|^2 = \|P_{2^{2n(L)}}(f - f_\delta)\|^2 + \|(I - P_{2^{2n(L)}})f\|^2,$$

according to discrepancy principle it holds

$$\|A_L\hat{x}_L - f\| \leq \|A_L\hat{x}_L - P_{2^{2n(L)}}f_\delta\| + \|P_{2^{2n(L)}}(f - f_\delta) + (I - P_{2^{2n(L)}})f\| \leq$$
$$\leq (d + \sqrt{2})\delta.$$

Moreover, by virtue of Lemma 5 we obtain

$$\|Ax_L - f\| \leq (d + c_4 + \sqrt{2})\delta.$$

On the other hand, for the $(L-1)$-th iteration it is fulfilled

$$\|A_{L-1}\hat{x}_{L-1} - P_{2^{2n(L-1)}}f_\delta\| \geq d\delta.$$

From relation (21) for $l = L - 1$ by inverse triangle inequality we find

$$\|Ax_{L-1} - f\| \geq \|Z_5\| - \sum_{j=1}^{4} \|Z_j\| \geq (d - c_4 - \sqrt{2})\delta.$$

Thus, we obtain the assertion of the lemma for

$$d_1 = d - c_4 - \sqrt{2}, \quad d_2 = d + c_4 + \sqrt{2}.$$

3 The main result

Theorem 1. *The algorithm (28)–(32) realizes the optimal order of accuracy* $O(\delta^{\nu/(\nu+1)})$ *on the class of equations (1) with* $A \in \mathcal{H}^r$ *and* $x^\dagger \in \mathcal{M}_{\nu,\rho}(A)$, $0 < \nu \le L-1$.

Proof. Since for each $l \ge 2$ it holds

$$\sigma_l \le (1 + 1/q)\sigma_{l-1},$$

keeping in mind lemmas 1, 7 and relation (6) we have

$$\delta\sqrt{\sigma_L} \le \sqrt{1+1/q}\cdot\delta\left(\frac{|d_{\nu,L-1}(v)|}{\|Ax_{L-1}-f\|}\right)^{\frac{1}{\nu+1}} \le \sqrt{1+1/q}\left(\frac{\rho\sqrt{\chi_{\nu+1}}}{d_1}\right)^{\frac{1}{\nu+1}}\delta^{\frac{\nu}{\nu+1}},$$

$$|c_{\nu,L}(v)|\sigma_L^{-\nu/2} = |c_{\nu,L}(v)|\left(\frac{\|Ax_L - f\|}{|d_{\nu,L}(v)|}\right)^{\frac{\nu}{\nu+1}} \le \rho^{\frac{1}{\nu+1}}(d_2\delta)^{\frac{\nu}{\nu+1}}.$$

Substituting these estimations into (18), we find

$$\begin{aligned}&\|x^\dagger - \hat{x}_L\| \le \xi\delta^{\nu/(\nu+1)},\\ &\xi = \rho^{1/(\nu+1)}\Big((d_2^{\nu/(\nu+1)} + (c_3+1)\sqrt{1+1/q}(\sqrt{\chi_{\nu+1}}/d_1)^{1/(\nu+1)}\Big),\end{aligned} \tag{33}$$

which is what had to be proved.

Remark 1. As is seen from (33), in case where ν has a sufficiently large value the proposed algorithm realizes the optimal order of accuracy only for such parameter ξ that $\xi = O(\sqrt{\chi_{\nu+1}})$. Thus, we obtain an analogue of so-called "principle of indetermination". This principle consists in the impossibillity to strive simultaneausly for improvement of accuracy (by decreasing of ξ) and for expansion of the values ν, for which this accuracy is attained.

Corollary 1. *To guarantee the optimal order of accuracy on the considered class of equations (1) it is required within the algorithm (28)–(32)*

$$O\Big(\delta^{-\frac{\nu+2}{(\nu+1)r}}\log^{1+1/r}(\delta^{-1})\Big) \tag{34}$$

information functionals (30).

Remark 2. As has been shown in [3], the amount of information expences (34) is considerably less in comparison with many known methods (see, for example, [8], [4]). Moreover, we obtain that the general amount of discrete information for all $0 < \nu < \infty$ does not increase compared with the work [3]. In other words, we have verifed the following hypothesis: the interval $(0, 1]$ requires the maximal amount of discrete information among all $\nu > 0$.

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