Mathematical Aspects of Decoherence[1]

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Abstract

The aim of this lecture is to investigate the mathematical frame for environment induced superselection rules. Some exactly soluble models are dicussed in detail and norm estimates on decoherence effects are derived.

1 Introduction

One of the puzzles of quantum mechanics is the question, how classical objects can arise in quantum theory. Quantum mechanics is a statistical theory, but its statistics differs on a fundamental level from the statistics of classical objects. It is known since a long time that the statistical results of quantum mechanics become consistent with a classical statistics of *facts*, if the superposition principle is reduced to *superselection sectors*, i.e. coherent orthogonal subspaces of the full Hilbert space. The mathematical structure of quantum mechanics and of quantum field theory provides us with only a few *superselection rules*, the most important being the charge superselection rule related to gauge invariance, see e.g. [4] [14] and the references given therein. But there are definitively not enough of these superselection rules to understand classical properties in quantum theory. A possible solution of this problem is the emergence of effective superselection rules due to decoherence caused by the interaction with the environment. A detailed investigation of these problems together with an extensive list of references up to 1996 is given in [6].

The aim of this lecture is to present the mathematical context of decoherence and induced superselection rules. For that purpose some exactly soluble models of open systems are investigated. The dynamics of the total system – including the open system and the environment – is always determined by a semibounded Hamiltonian. Norm estimates of the off–diagonal parts of the statistical operators intertwining between different superselection sectors are discussed in detail.

The lecture is organized as follows. After a short introduction to superselection rules and to the dynamics of subsystems in Sects. 2 and 3, several models are presented in Sect. 4. For a class of simple models, given in Sect. 4.1, the transition between the induced superselection sectors is suppressed uniformly in trace norm. A more realistic example with a quantum field as environment is investigated in Sect. 4.2. In this case the infrared behaviour of the environment is of essential importance for the emergence of induced superselection rules. Uniform estimates, which persist for arbitrary times, are only possible in the limit of infrared divergence.

For the models of Sects. 4.1 and 4.2 the projection operators onto the induced superselection sectors commute with the Hamiltonian. By adding a scattering potential we obtain in Sect. 4.3 a model, which still has induced superselection sectors, and for which the projection operators onto these sectors do not commute with the Hamiltonian. In Sect 4.4 we present a spin model which has induced superselection sectors only in an approximate sense.

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2 Superselection rules

We start with a few mathematical notations. Let $\mathcal{H}$ be a separable Hilbert space, then the following spaces of linear operators are used.

$\mathcal{B}(\mathcal{H})$: The $\mathbf{R}$–linear space of all bounded self–adjoint operators A. The norm of this space is the operator norm $\|A\|$.

$\mathcal{T}(\mathcal{H})$: The $\mathbf{R}$–linear space of all self–adjoint nuclear operators A. These operators have a pure point spectrum $\alpha_i \in \mathbf{R}$, $i = 1, 2, ...$, with $\sum_i |\alpha_i| < \infty$. The natural norm of this space is the trace norm$\|A\|_1 = \mathrm{tr}\sqrt{A^+A} = \sum_i |\alpha_i|$. Another norm, used in the following sections, is the Hilbert–Schmidt norm $\|A\|_2 = \sqrt{\mathrm{tr}\, A^+A}$. These norms satisfy the inequalities $\|A\| \leq \|A\|_2 \leq \|A\|_1$.

$\mathcal{D}(\mathcal{H})$: The set of all statistical operators, i.e. positive nuclear operators W with a normalized trace $\mathrm{tr}\, W = 1$.

$\mathcal{P}(\mathcal{H})$: The set of all rank one projection operators P^1.

These sets satisfy the obvious inclusions $\mathcal{P}(\mathcal{H}) \subset \mathcal{D}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$.

Any state of a quantum system is represented by a statistical operator $W \in \mathcal{D}(\mathcal{H})$, the elements of $\mathcal{P}(\mathcal{H})$ thereby correspond to the pure states. Any (bounded) observable is represented by an operator $A \in \mathcal{B}(\mathcal{H})$, and the expectation of the observable A in the state W is the trace $\mathrm{tr}\, WA$. Without additional knowledge about the structure of the system we have to assume that the set of all states corresponds exactly to $\mathcal{D}(\mathcal{H})$, and the set of all (bounded) observables is $\mathcal{B}(\mathcal{H})$. The state space $\mathcal{D}(\mathcal{H})$ has an essential property: it is a convex set, i.e. $W_1, W_2 \in \mathcal{D}(\mathcal{H})$ implies $\lambda_1 W_1 + \lambda_2 W_2 \in \mathcal{D}(\mathcal{H})$ if $\lambda_{1,2} \geq 0$ and $\lambda_1 + \lambda_2 = 1$. Any statistical operator $W \in \mathcal{D}(\mathcal{H})$ can be decomposed into pure states $W = \sum_n w_n P_n^1$ with $P_n^1 \in \mathcal{P}(\mathcal{H})$ and probabilities $w_n \geq 0$, $\sum_n w_n = 1$. An explicit example is the spectral decomposition of W. But there are many other possibilities. It is exactly this arbitrariness that does not allow a classical interpretation of quantum probability. A more detailed discussion of the state space of quantum mechanics can be found in [10].

The arbitrariness of the decomposition of W originates in the superposition principle. The superposition principle can be restricted by superselection rules. Here we do not want to discuss the arguments to establish such rules, for that purpose see e.g. [4][14] and also Chap. 6 of [6], or to refute them, see e.g. [12]. We only refer to some important consequences for the structure of the state space. In a theory with discrete superselection rules like the charge superselection rule, the Hilbert space $\mathcal{H}$ splits into orthogonal superselection sectors $\mathcal{H}_m$, $m \in \mathbf{M}$, such that $\mathcal{H} = \oplus_m \mathcal{H}_m$. Pure states with charge m (in appropriate normalization) are then represented by vectors in $\mathcal{H}_m$, and superpositions of vectors with different charges have no physical interpretation. The projection operators P_m onto the orthogonal subspaces $\mathcal{H}_m$ satisfy $P_m P_n = \delta_{mn}$ and $\sum_m P_m = I$. The set of states is reduced to those statistical operators which satisfy $P_m W = W P_m$ for all projection operators $P_m, m \in \mathbf{M}$. The state space of the system is then

$$\mathcal{D}^S = \left\{ W \in \mathcal{D}(\mathcal{H}) | W = \sum_m P_m W P_m \right\}. \tag{1}$$

An equivalent statement is that all observables of such a system have to commute with the projection operators P_m, $m \in \mathbf{M}$, and the set of observables of the system is given by

$$\mathcal{B}^S = \left\{ A \in \mathcal{B}(\mathcal{H}) \mid A = \sum_m P_m A P_m \right\}. \tag{2}$$

The projection operators $\{P_m \mid m \in \mathbf{M}\}$ are observables, which commute with all observables of the system, and they can be interpreted as classical observables.

In theories with continuous superselection rules the finite or countable set of projection operators $\{P_m, m \in \mathbf{M}\}$ is substituted by a (weakly continuous) family of projection operators $P(\Delta)$ indexed by measurable subsets $\Delta \subset \mathbf{R}$, see e.g. [1]. These projection operators have to satisfy

$$\begin{array}{l} P(\Delta_1 \cup \Delta_2) = P(\Delta_1) + P(\Delta_2) \quad \text{for all intervalls } \Delta_1, \Delta_2 \\ P(\Delta_1)P(\Delta_2) = O \quad \text{if } \Delta_1 \cap \Delta_2 = \emptyset, \quad \text{and } P(\emptyset) = O,\ P(\mathbf{R}) = 1. \end{array} \tag{3}$$

The set of observables is now given by
$\mathcal{B}^S = \{A \in \mathcal{B}(\mathcal{H}) \mid AP(\Delta) = P(\Delta)A,\ \Delta \subset \mathbf{R}\}$, but there is no formulation of the corresponding set of states within the class of nuclear statistical operators.

Remark 1 *The statements about superselection rules are statements about the mathematical structure of a theory. The strict superselection rules considered so far are rather unstable against slight modifications of the theory. Take e.g. the quantum field theory of elementary particles with the superselection rules of baryonic charge. If we add to the Hamiltonian a small term, which allows the decay of the proton (as done in grand unified theories), the superselection rule of baryonic charge is spoiled. What remains is a strong quantitative suppression of matrix elements of observables which intertwine between different sectors.*

The importance of superselection rules for the transition from quantum probability to classical probability is obvious. But there remains an essential problem: Only very few superselection rules can be found in quantum mechanics and quantum field theory that are compatible with the mathematical structure and with experiment. A satisfactory solution to the problem of classical observables is the emergence of effective superselection rules induced by the interaction with the environment.

3 Dynamics of subsystems

In the following we consider an *open system*, i.e. a system S which interacts with an *environment* E, such that the total system $S+E$ satisfies the usual Hamiltonian dynamics. The Hilbert space $\mathcal{H}_{S+E}$ of the total system $S+E$ is the tensor space $\mathcal{H}_S \otimes \mathcal{H}_E$ of the Hilbert spaces for S and for E. Let $W \in \mathcal{D}(\mathcal{H}_{S+E})$ be the state of the total system and $A \in \mathcal{B}(\mathcal{H}_S)$ be an observable of the system S, then the expectation $\mathrm{tr}_{S+E} W(A \otimes I_E)$ satisfies the identity $\mathrm{tr}_{S+E}\,(A \otimes I_E)W = \mathrm{tr}_S\, A\rho$ with the reduced statistical operator $\rho = \mathrm{tr}_E W \in \mathcal{D}(\mathcal{H}_S)$. We shall refer to $\rho = \mathrm{tr}_E W$ as the *state* of the subsystem.

As mentioned above we assume the usual Hamiltonian dynamics for the total system, i.e. $W \to W(t) = U(t)WU^+(t) \in \mathcal{D}(\mathcal{H}_{S+E})$ with the unitary group $U(t) = \exp(-iH_{S+E}t)$ generated by the total Hamiltonian H_{S+E}. Except for the trivial case that S and E do not interact, the dynamics of the reduced statistical operator

$$\rho(t) = \mathrm{tr}_E\, U(t)WU^+(t) \tag{4}$$

is no longer unitary, and it is exactly this dynamics which can produce effective superselection sectors. More explicitly, the Hamiltonian of the total system can provide a family of projection operators $\{P_m,\ m \in \mathbf{M}\}$ which are independent from the initial state, such that the statistical operator behaves like

$$\rho(t) \cong \sum_m P_m \rho(t) P_m \tag{5}$$

in sufficiently short time. An equivalent statement is that the superpositions between vectors of different sectors $P_m\mathcal{H}_S$ are strongly suppressed, $P_m\rho(t)P_n \to 0$ for $t \to \infty$ if $m \neq n$. Any mechanism, which leads to this effect, will be called *decoherence*.

In the case of induced continuous superselection rule the asymptotics is more appropriately described in the Heisenberg picture, as stated above. But the decoherence effect is also seen in the Schrödinger picture:
$P(\Delta_1)\rho(t)P(\Delta_2) \to 0$ for $t \to \infty$ if Δ_1 and Δ_2 have a positive distance.

The investigation of the models in the following sections allows to give a more precise meaning to the statement (5). All models (with partial exception of the spin model in Sect. 4.4) have superselection sectors in the following sense

Definition 1 *The subspaces $P_m\mathcal{H}_S$, $m \in \mathbf{M}$, are denoted as induced superselection sectors of the dynamics (4), if for all observables $A \in \mathcal{B}(\mathcal{H}_S)$, which have no diagonal matrix elements, i.e. $P_mAP_m = O$, $m \in \mathbf{M}$, and for all initial states $W \in \mathcal{D}_1 \subset \mathcal{D}(\mathcal{H}_{S+E})$, where $\mathcal{D}_1$ is a dense subset of $\mathcal{D}(\mathcal{H}_{S+E})$, the trace*

$$\mathrm{tr}_{S+E}(A \otimes I_E)U(t)WU^+(t) = \mathrm{tr}_S A\rho(t)$$

vanishes if $t \to \infty$.

Since only weak convergence is used, $\mathrm{tr}_{S+E}(A \otimes I_E)U(t)WU^+(t) \to 0$ for $W \in \mathcal{D}_1 \subset \mathcal{D}(\mathcal{H}_{S+E})$ implies that this trace vanishes for all $W \in \mathcal{D}(\mathcal{H}_{S+E})$. Definition 1 is therefore equivalent to a definition with $W \in \mathcal{D}_1 \subset \mathcal{D}(\mathcal{H}_{S+E})$ substituted by $W \in \mathcal{D}(\mathcal{H}_{S+E})$, and the superselection sectors do not depend on the choice of the dense subset $\mathcal{D}_1 \subset \mathcal{D}(\mathcal{H}_{S+E})$.

This definition is so far rather useless, since it does not specify the time scale of decoherence. The investigation of the models will clarify the dependence of this time scale on the initial state, especially its matrix elements related to the environment.

4 Models of decoherence

In the following we present models for which the Hamiltonian of the total system provides a family of projection operators $\{P_m,\, m \in \mathbf{M}\}$ such that the off–diagonal elements $P_m\rho(t)P_n$, $m \neq n$, of the statistical operator of the reduced dynamics (4) can be estimated with the trace norm. Especially we are interested to derive a uniform decrease

$$\|P_m\rho(t)P_n\|_1 \to 0 \text{ for } t \to \infty \text{ if } m \neq n \tag{6}$$

for arbitrary initial states $\rho(0) \in \mathcal{D}(\mathcal{H}_S)$. But such a result is only possible for the simple models of Sect. 4.1. In Sects. 4.2 and 4.3 we shall see that the decoherence of more realistic models yields more complicated results.

The models of Sects. 4.1 and 4.2 have the following structure. The total Hamiltonian on $\mathcal{H}_{S+E} = \mathcal{H}_S \otimes \mathcal{H}_E$ has the form

$$H_{S+E} = H_S \otimes I_E + I_S \otimes H_E + A \otimes B \tag{7}$$

where H_S is the Hamiltonian of $\mathcal{S}$, H_E is the Hamiltonian of $\mathcal{E}$, and $A \otimes B$ is the interaction term between $\mathcal{S}$ and $\mathcal{E}$ with self–adjoint operators A on $\mathcal{H}_S$ and B on $\mathcal{H}_E$. This Hamiltonian can be written as
$H_{S+E} = \left(H_S - \frac{1}{2}A^2\right) \otimes I_E + \frac{1}{2}\left(A \otimes I_E + I_S \otimes B\right)^2 + I_S \otimes \left(H_E - \frac{1}{2}B^2\right)$. It is semibounded from below, if the operators $H_S - \frac{1}{2}A^2$ and $H_E - \frac{1}{2}B^2$ are semibounded operators on $\mathcal{H}_S$ or $\mathcal{H}_E$ respectively. We make the following additional assumptions

1) The operators H_S and A commute, $[H_S, A] = O$, hence
$[H_S \otimes I_E, A \otimes B] = O$.

2) The operator B has an absolutely continuous spectrum.

The assumption 1) is a rather severe restriction, which will be modified only in Sects. 4.3 by adding a scattering potential V, which has not to commute with any of the other operators. The investigation of the spin model in Sect. 4.4 shows that at least a reduced form of this assumption must hold in order to retain induced superselection rules. The assumption 2) has more technical reasons. It implies that estimates can be derived in the limit $t \to \infty$ (without recurrences) in agreement with Definition 1.

For the purpose of this paper it is sufficient to consider operators A which have a pure point spectrum

$$A = \sum_m \lambda_m P_m. \tag{8}$$

In the following we shall see that exactly the projection operators of this spectral decomposition determine the induced superselection sectors. If we allow a continuous spectrum for A, then the projection operators $\{P_m, m \in \mathbf{M}\}$ are substituted by a weakly continuous family of projection operators $P(\Delta)$ as introduced in (3)), and continuous superselection sectors will emerge. Instead of trace norm estimates like (6) we obtain Hilbert–Schmidt norm estimates for $\|P(\Delta)\rho(t)P(\Delta')\|_2$ if the sets Δ and Δ' have a finite distance, see Sect. 7.6 of [6].

As a consequence of assumption 1) we have $[H_S, P_m] = O$ for all $m \in \mathbf{M}$. The Hamiltonian (7) has therefore the form

$$H_{S+E} = H_S \otimes I_E + \sum_m P_m \otimes \Gamma_m \text{ with } \Gamma_m = H_E + \lambda_m B. \tag{9}$$

The calculation of the reduced dynamics (4) leads to

$$P_m \rho(t) P_n = P_m \mathrm{e}^{-\imath H_S t} \left(\mathrm{tr}_E \mathrm{e}^{-i\Gamma_m t} W \mathrm{e}^{i\Gamma_n t} \right) \mathrm{e}^{\imath H_S t} P_n, \tag{10}$$

where the operators P_n are the projection operators of the spectral representation (8) of A, see e.g. Sect. 7.6 of [6]. For a factorizing initial state $W = \rho \otimes \omega$ with $\rho \in \mathcal{D}(\mathcal{H}_S)$ and a reference state $\omega \in \mathcal{D}(\mathcal{H}_E)$ of the environment, the operator (10) simplifies to $P_m \rho(t) P_n = P_m e^{-iH_S t} \rho e^{iH_S t} P_n \, \chi_{m,n}(t)$ with

$$\chi_{m,n}(t) = \mathrm{tr}_E \left(\mathrm{e}^{i\Gamma_n t} \mathrm{e}^{-i\Gamma_m t} \omega \right), \tag{11}$$

and the emergence of dynamically induced superselection rules depends on an estimate of this trace.

4.1 The Araki–Zurek models

The first soluble models for the investigation of the reduced dynamics have been given by Araki [1] and Zurek [15], and the following construction is essentially based on these papers. In addition to the specifications made above, we demand that

3) the Hamiltonian H_E and the potential B commute, $[H_E, B] = O$.

We first investigate $P_m \rho(t) P_n$ for a factorizing initial state $W = \rho \otimes \omega$. Under the assumption 3) the trace (11) simplifies to $\chi_{m,n}(t) = \mathrm{tr}_E \left(\mathrm{e}^{-i(\lambda_m - \lambda_n)Bt} \omega \right)$. Let $B = \int_{\mathbf{R}} \lambda P_E(d\lambda)$ be the spectral representation of the operator B. Then, as a consequence of assumption 2), for any $\omega \in \mathcal{D}(\mathcal{H}_E)$ the measure $d\mu(\lambda) := \mathrm{tr}_E \left(P_E(d\lambda) \, \omega \right)$ is absolutely continuous with respect to the

Lebesgue measure, and the function $\chi(t) := \operatorname{tr}\left(e^{-\imath Bt}\omega\right) = \int_{\mathbf{R}} e^{-i\lambda t}\, d\mu(\lambda)$ vanishes if $t \to \infty$. But to have a decrease which is effective in sufficiently short time, we need an additional smoothness condition on ω (which does not impose restrictions on the statistical operator $\rho \in \mathcal{D}(\mathcal{H}_S)$ of the system $\mathcal{S}$). If the integral operator, which represents ω in the spectral representation of B, is a sufficiently differentiable function (vanishing at the boundary points of the spectrum) we can derive estimates like $|\chi(t)| \leq C_\gamma(1+|t|)^{-\gamma}$ with arbitrarily large values of γ. Such an estimate leads to the upper bound for the norm (6)

$$\|P_m\rho(t)P_n\|_1 \leq |\chi_{m,n}(t)| \leq C_\gamma(1+\delta\,|t|)^{-\gamma} \tag{12}$$

if $|\lambda_m - \lambda_n| \geq \delta > 0$. The constants $\gamma > 0$, $\delta > 0$ and $C_\gamma > 0$ do not depend on the initial state $\rho(0) \equiv \rho \in \mathcal{D}(\mathcal{H}_S)$. Moreover one can achieve large values of γ and/or small values of the constant C_γ if the reference state $\omega \in \mathcal{D}(\mathcal{H}_E)$ is sufficiently smooth.

These results depend on the reference state ω only via the decrease of $\chi(t)$. We could have chosen a more general initial state $W \in \mathcal{D}(\mathcal{H}_{S+E})$

$$W = \sum_\mu c_\mu\, \rho_\mu \otimes \omega_\mu \tag{13}$$

with $\rho_\mu \in \mathcal{D}(\mathcal{H}_S)$, $\omega_\mu \in \mathcal{D}(\mathcal{H}_E)$ and numbers $c_\mu \in \mathbf{R}$ which satisfy $\sum_\mu |c_\mu| < \infty$ and $\sum_\mu c_\mu = \operatorname{tr} W = 1$. The set (13) is dense in $\mathcal{D}(\mathcal{H}_{S+E})$. With the arguments given above for factorizing initial states the statement of Definition 1 can be derived for all initial states (13), and the sectors $P_n\mathcal{H}_S$ are induced superselection sectors in the sense of this definition. Moreover, assuming that the components of the statistical operator W affiliated to the environment are sufficiently smooth functions in the spectral representation of B, the sum $\sum_\mu \left| c_\mu \operatorname{tr}_E\left(e^{-\imath(\lambda_m-\lambda_n)Bt}\omega_\mu\right)\right|$ satisfies a uniform estimate (12). Hence the time scale of the decoherence can be as short as we want without restriction on $\rho(0) = \operatorname{tr}_E W = \sum_\mu c_\mu\, \rho_\mu$.

4.2 The interaction with a Boson field

In this section we present a model without the restriction 3) on the Hamiltonian. As specific example we consider an environment given by a Boson field. Such models can be calculated explicitly, and they have often been used as the starting point for Markov approximations.

As Hilbert space $\mathcal{H}_E$ we choose the Fock space based on the one particle space $\mathcal{H}^{(1)} = \mathcal{L}^2(\mathbf{R}_+)$ with inner product $\langle f \mid g\rangle = \int_0^\infty \overline{f(k)}g(k)dk$. The one–particle Hamilton operator, denoted by $\widehat{\varepsilon}$, is the positive multiplication operator $(\widehat{\varepsilon}f)(k) := \varepsilon(k)f(k)$ with the positive energy function $\varepsilon(k) = c\cdot k$, $c > 0$, $k \in \mathbf{R}_+$, defined for all functions f with $(1+\varepsilon(k))f(k) \in \mathcal{L}^2(\mathbf{R}_+)$. The creation/annihilation operators a_k^+ and a_k are normalized to $\left[a_k, a_{k'}^+\right] = \delta(k-k')$. The Hamiltonian of the environment is then

$$H_E = \int_0^\infty \varepsilon(k)a_k^+ a_k dk.$$

With $a^+(f) = \int_0^\infty f(k)a_k^+ dk$ and $a(f) = \int_0^\infty f(k)a_k dk$ we define field operators by $\Phi(f) := 2^{-\frac{1}{2}}\left(a^+(f) + a(f)\right)$ for real functions $f \in \mathcal{L}^2(\mathbf{R}_+)$. The interaction potential is chosen as $B = \Phi(f)$ with

$$\left(1+\widehat{\varepsilon}^{-1}\right) f \in \mathcal{L}^2(\mathbf{R}_+) \text{ and } \left\|\widehat{\varepsilon}^{-\frac{1}{2}} f\right\| < 2^{-\frac{1}{2}}. \tag{14}$$

The restriction $\left\|\widehat{\varepsilon}^{-\frac{1}{2}} f\right\| < 2^{-\frac{1}{2}}$ implies that $H_E - \frac{1}{2}\Phi^2(f)$ is bounded from below, a necessary condition to have a semibounded total Hamiltonian (7), see [11]. An example for the total

Hamiltonian is given by a single particle coupled to the quantum field with velocity coupling

$$\begin{aligned} H_{S+E} &= \frac{1}{2}P^2 \otimes I_E + P \otimes \Phi(f) + I_S \otimes H_E \\ &= \frac{1}{2}\left(P \otimes I_E + I_S \otimes \Phi(f)\right)^2 + I_S \otimes \left(H_E - \frac{1}{2}\Phi^2(f)\right) \end{aligned}$$

Since the particle is coupled to the field with $A = P$, the reduced dynamics yields in this case continuous superselection sectors for the momentum P of the particle.

The operators (9) Γ_m are substituted by $H_\lambda := H_E + \lambda\Phi(f),\ \lambda \in \mathbf{R}$, which are Hamiltonians of the van Hove model [8], see also [3] and [5]. The restriction $\left(1 + \widehat{\varepsilon}^{-1}\right) f \in \mathcal{L}^2(\mathbf{R}_+)$ is necessary to guarantee that all operators H_λ, $\lambda \in \mathbf{R}$, are unitarily equivalent and defined on the same domain. To derive induced superselection sectors we have to estimate the time dependence of the traces $\chi_{\alpha\beta}(t) := \mathrm{tr}_E U_{\alpha\beta}(t)\omega$, $\alpha \neq \beta$, where the unitary operators $U_{\alpha\beta}(t)$ are given by

$$U_{\alpha\beta}(t) := \exp(iH_\alpha t)\exp(-iH_\beta t),$$

see (11). In [11] the following results are derived for states ω which are mixtures of coherent states.

a) Under the restrictions (14) the traces $\chi_{\alpha\beta}(t)$ do not vanish for $t \to \infty$.

b) If $\Phi(f)$ has contributions at arbitrarily small energies, the traces $\chi_{\alpha\beta}(t), \alpha \neq \beta$, can nevertheless strongly decrease within a very long time interval $0 \leq t \leq T$. Estimates like (12) are substituted by $\|P_m\rho(t)P_n\|_1 \leq \vartheta(t)$ or $\|P(\Delta)\rho(t)P(\Delta')\|_2 \leq \vartheta(t)$. But in contrast to (12) the function $\vartheta(t)$ increases again if $t > T$ for some large T.

c) For fixed $\alpha \neq \beta$ a limit $\chi_{\alpha\beta}(t) \to 0$ for $t \to \infty$ is possible if $\widehat{\varepsilon}^{-1} f \in \mathcal{L}^2(\mathbf{R}_+)$ is violated, i.e. in the case of infrared divergence.

A large infrared contribution is therefore essential for the emergence of induced superselection sectors of the system S. This model has of course not the complexity of quantum electrodynamics, where infrared divergence of the electromagnetic field is related to the classical static limit of the fields. But nevertheless this model indicates that also the classical appearance of the matter might be related to the infrared divergence of the electromagnetic field.

The role of infrared divergence can be illustrated by another singular limit of the model: the coupling to a free particle. We can restrict the one–particle space $\mathcal{H}^{(1)}$ to the one dimensional space $\mathbf{C}$, and the free field becomes a harmonic oscillator of frequency $\widehat{\varepsilon} = \varepsilon > 0$. In the (singular) limit $\varepsilon \to 0$ we obtain the Hamiltonian with coupling to a free particle

$$H_E = \frac{1}{2}P^2 \text{ with coupling } B = Q. \tag{15}$$

Then $\mathrm{tr}_E U_{\alpha\beta}(t)\omega$ can be calculated by standard methods, see the article [13] of Pfeifer, who has used this model to discuss the measurement process of a spin. With (15) we can easily derive $\mathrm{tr}_E U_{\alpha\beta}(t)\omega \to 0$ if $t \to \infty$ and $\alpha \neq \beta$ for all statistical operators ω of the free particle. But the Hamiltonian (7) of the total system is unbounded from below – corresponding to infrared divergence in the field theoretic model.

As in Sect. 4.1 the choice of the initial state W of the total system can be extended to (13) with $\rho_\mu \in \mathcal{D}(\mathcal{H}_S)$ and mixtures of coherent states $\omega_\mu \in \mathcal{D}(\mathcal{H}_E)$. This class of states is again dense in $\mathcal{D}(\mathcal{H}_{S+E})$, and, at least in the infrared divergent case, we obtain induced superselection sectors in the sense of Definition 1.

4.3 Models with scattering

For all models presented so far the projection operators onto the induced superselection sectors $P_m \otimes I_S$ commute with the total Hamiltonian $[P_m \otimes I_S, H_{S+E}] = O$. We now modify the Hamiltonian (7) to

$$H = H_{S+E} + V = H_S \otimes I_E + I_S \otimes H_E + A \otimes B + V$$

where the operator V is only restricted to be a *scattering* potential. This property means that the wave operator $\Omega = \lim_{t\to\infty} \exp(iHt)\exp(-iH_{S+E}t)$ exists as strong limit. To simplify the arguments we assume that there are no bound states such that the convergence is guaranteed on $\mathcal{H}_{S+E}$ with $\Omega^+ = \Omega^{-1}$. Then the time evolution $U(t) = \exp(-iHt)$ behaves asymptotically as $U_0(t)\Omega^+$ with $U_0(t) = \exp(-iH_{S+E}t)$. More precisely, we have for all $W \in \mathcal{D}(\mathcal{H}_{S+E})$

$$\lim_{t\to\infty} \left\| U(t)WU^+(t) - U_0(t)\Omega^+ W\Omega U_0^+(t) \right\|_1 = 0 \tag{16}$$

in trace norm. Following Sect. 4.1 the reduced trace $\mathrm{tr}_E U_0(t)\Omega^+ W\Omega U_0^+(t)$ produces the superselection sectors $P_m\mathcal{H}_S$ which are determined by the spectrum (8) of A. The asymptotics (16) then yields (in the sense of Definition 1) the same superselection sectors for $\rho(t) = \mathrm{tr}_E U(t)WU^+(t)$. Moreover we can derive fast decoherence by additional assumptions on the initial state and on the potential. For that purpose we start with a factorizing initial state $W = \rho(0)\otimes\omega$ with smooth ω. To apply the arguments of Sect. 4.1 to the dynamics $U_0(t)\Omega^+ W\Omega U_0^+(t)$ the statistical operator $\Omega^+ (\rho\otimes\omega)\,\Omega$ has to be a sufficiently smooth operator on the tensor factor $\mathcal{H}_E$ for all $\rho \in \mathcal{D}(\mathcal{H}_S)$. That is guaranteed if we choose as scattering potential a smooth potential in the sense of Kato [9]. Then both the limits, (16) and $\lim_{t\to\infty} \left\| P_m \left(\mathrm{tr}_E U_0(t)\Omega^+ W\Omega U_0^+(t)\right) P_n \right\|_1 = 0$, $m \neq n$, are reached in sufficiently short time. Hence $\rho(t)$ can decohere fast into the subspaces $P_m\mathcal{H}_S$ which are determined by the spectrum (8) of A. But in contrast to (12) one does not obtain a uniform bound with respect to the initial state $\rho(0)$, since the limit (16) is not uniform in $W \in \mathcal{D}(\mathcal{H}_{S+E})$.

A final remark should be added. Although the estimates are not uniform in $\rho(0)$, the bounds on decoherence are nevertheless stronger than those derived in the Coleman–Hepp model [7]. For any given initial state $\rho(0) \in \mathcal{D}(\mathcal{H}_S)$ we have an estimate $\|\dot\rho(t)\|_1 \leq c\,(1+|t|)^{-\gamma}$ with some constant $c > 0$. Consequently, for this initial state $\rho(0)$ the expectation values $\mathrm{tr}_S\,\dot\rho(t)\,A$ are uniformly bounded by $|\mathrm{tr}_S\,\dot\rho(t)\,A| \leq c\,\|A\|\,(1+|t|)^{-\gamma}$ for all observables $A \in \mathcal{B}(\mathcal{H}_S)$. Such a uniform estimate does not hold in the Coleman–Hepp model [2].

4.4 A spin model

The following model illustrates that the assumption 1) on the Hamiltonian can be violated only in a restricted sense – e. g. by a scattering potential as seen in Sect. 4.3 – in order to maintain induced superselection sectors. The model of this section has a non–vanishing commutator $[A, H_S]$, and there remain off–diagonal contributions of the statistical operator. But the magnitude of these off-diagonal contributions are limited by the magnitude of $[A, H_S]$.

The Hilbert space of this model is $\mathcal{H}_{S+E} = \mathcal{H}_S \otimes \mathcal{H}_E$ with $\mathcal{H}_S = \mathbf{C}^2$ and $\mathcal{H}_E = \mathcal{L}^2(\mathbf{R})$. The Hamiltonian has the form (7) with the following specifications

$H_S\psi = \frac{1}{2}(\vec{a}\vec{\sigma})\psi$, with $\vec{a} \in \mathbf{R}^3$, $\vec{\sigma}$ Pauli matrices, $\psi \in \mathbf{C}^2$,
$H_E f(x) = bx^2 f(x)$ with a positive constant $b > 0$, $f(x) \in \mathcal{L}^2(\mathbf{R})$,
$A = \frac{\lambda}{2}\sigma_3$, with a real coupling parameter λ,
$B\, f(x) = xf(x)$, $f(x) \in \mathcal{L}^2(\mathbf{R})$.

The total Hamiltonian $H_{S+E} = H_S \otimes I_E + I_S \otimes H_E + A \otimes B$ is bounded from below. The commutator $[H_S, A]$ vanishes only if $\vec{a} = const\, \vec{e}_3$.

The statistical operator of the spin–$\frac{1}{2}$ system is a spin density matrix $\widehat{\rho}\,(\vec{p}) = \frac{1}{2}\,(\mathbf{1} + \vec{p}\vec{\sigma})$ with a polarization vector in the unit ball $\vec{p} \in \mathbf{B}^3 = \{\vec{p} \in \mathbf{R}^3 \mid |\vec{p}| \leq 1\}$. For the total system we assume an initial state $W = \widehat{\rho}\,(\vec{p}) \otimes \omega$ where ω is a statistical operator on $\mathcal{L}^2(\mathbf{R})$ with a smooth integral kernel $\omega(x, y)$. In [10] the reduced dynamics $\rho(t) = \mathrm{tr}_E U(t) W U^+(t)$ with $U(t) = \exp(-iH_{S+E}t)$ is calculated. For $t \to \infty$ the operator $\rho(t)$ converges to the state $\widehat{\rho}\,(\vec{q})$, where the polarization vector $\vec{q}$ is given by the linear mapping

$$\vec{q} = M\vec{p} := \int dx \omega(x, x) \vec{n}(x) \left(\vec{p}\vec{n}(x)\right) \tag{17}$$

with the unit vector $\vec{n}(x) = |\vec{a} - \lambda x \vec{e}_3|^{-1} \left(\vec{a} - \lambda x \vec{e}_3\right)$. Under appropriate conditions for the initial state ω of the environment, the difference $\rho(t) - \widehat{\rho}\,(\vec{q})$ can be uniformly estimated by $\|\rho(t) - \widehat{\rho}\,(\vec{q})\|_1 \leq c(1 + |t|)^{-\gamma}$. The mapping (17) is a symmetric contraction on $\mathbf{R}^3$, with the properties

1. If $\vec{a} \parallel \vec{e}_3$ the mapping (17) reduces to $M\vec{p} = \vec{e}_3\,(\vec{e}_3\vec{p})$, and we obtain the results discussed in Sect. 4.1. We have $|M\vec{p}| = |\vec{p}|$ for $\vec{p} \parallel \vec{e}_3$, and only in this case $\rho(t)$ is not affected by the decoherence.

2. If $\vec{a}$ has components orthogonal to $\vec{e}_3$, also the direction of $M\vec{p}$ depends on $\vec{p}$, and $|M\vec{p}| < |\vec{p}|$ holds for all vectors $\vec{p} \neq \vec{0}$.

In the second case there are no projection operators which commute with all operators $\widehat{\rho}\,(M\vec{p})$, $\vec{p} \in \mathbf{B}^3$ and there are no induced superselection sectors (as defined above).

But if $[H_S, A]$ is very small, more precisely $a_1^2 + a_2^2 \ll a_3^2$, the vector $M\vec{p}$ has very small components orthogonal to $\vec{e}_3$, and the off–diagonal matrix elements of the operators $\widehat{\rho}\,(M\vec{p})$ are negligible for all $\vec{p} \in \mathbf{B}^3$. Hence one can still speak about induced superselection sectors in some approximative sense.

5 Concluding remarks

1. If induced superselection sectors exist, they are fully determined by the interaction in the sense of Definition 1.

2. A fast suppression of the off–diagonal matrix elements $P_m \rho(t) P_n$, $m \neq n$, of the reduced statistical operator is possible, if the initial state $W \in \mathcal{D}(\mathcal{H}_{S+E})$ has "smooth" contributions with respect to the environment, i.e. in the sense discussed at the end of Sect. 4.1.

3. A uniform emergence of superselection sectors with respect to $\rho(0) \in \mathcal{D}(\mathcal{H}_S)$ is consistent with the mathematical rules of quantum theory. But in more realistic models with scattering the estimates are no longer uniform with respect to the initial state of the system.

4. In models with a Bose field as environment a large infrared contribution is important for the emergence of induced superselection sectors.

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