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Is it possible to construct exactly solvable models?

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Abstract

We develop a constructive method to derive exactly solvable quantum mechanical models of rational (Calogero) and trigonometric (Sutherland) type. This method starts from a linear algebra problem: finding eigenvectors of triangular finite matrices. These eigenvectors are transcribed into eigenfunctions of a selfadjoint Schrödinger operator. We prove the feasibility of our method by constructing an "AG_3 model" of trigonometric type (the rational case was known before from Wolfes 1975). Applying a Coxeter group analysis we prove its equivalence with the B_3 model. In order to better understand features of our construction we exhibit the F_4 rational model with our method.

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Dedicated to Professor Jan Lopuszański on the occasion of his 75-th birthday

1 Introduction

The completely integrable models are traditionally characterized by their relation with simple Lie algebras A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8. This relation is the starting point of the Hamiltonian reduction method exploited by Olshanetsky and Perelomov [1]. These models possess as limiting cases the trigonometric (Sutherland) and rational (Calogero) models that are exactly soluble, i.e. their eigenvalues and eigenvectors can be derived by elementary methods.

This exact solvability has been shown to follow from the fact that the Schrödinger operators can, after a "gauge transformation", be rewritten as a quadratic form of Lie algebra operators. These Lie algebra operators are represented as differential operators acting on polynomial spaces. This program was formulated in [2] and successfully applied first to the A_n series in [3]. Then it was carried over to the other sequences B_n, C_n, D_n and G_2 and even to corresponding supersymmetric models [4, 5].

Our aim was to turn the arguments around and to develop an algorithm which may allow us to construct new exactly soluble models. First investigations were presented in [6]. The program contains two major and separate issues, to render a second order differential operator curvature free and to find a first order differential operator satisfying an integrability constraint. In this paper we present our algorithm in the following version. We start from a standard flat Laplacian and introduce Coxeter (or Weyl) group invariants as new coordinates. If the Coxeter group contains a symmetric group as subgroup, these invariants are built from elementary symmetric polynomials. The second order differential operators obtained this way are curvature free by construction, and act on polynomial spaces of these Coxeter invariants that form a flag. This flag is defined by means of a characteristic vector ($\vec{p}$-vector).

Then we solve the integrability constraints by constructing "prepotentials" with a fixed algorithm. These prepotentials define the gauge transformation alluded to above which renders the differential operator the form of a standard Schrödinger operator of N particles in 1-dimensional space with a potential. Each prepotential contributes an additive term to this potential with a free (real) coupling constant. Finally the prepotentials define the ground state wave function of the Schrödinger operator which originates from the trivial polynomial in the flag and thus contains no further information. Except a possible oscillator prepotential in the translation invariant cases, the prepotentials are in one-to-one relation with the orbits of the Coxeter group.

We show that all known exactly soluble models can be obtained this way (at present we have to make an exemption with respect to E_6, E_7, E_8, but this will soon be overcome). Applying the method of constructing the Coxeter invariants of A_2 [4] to A_3, we obtain an "AG_3 model". Its Coxeter diagram is that of the affine Coxeter group $\hat{B}_3$, which possesses the same invariants as the Coxeter group B_3. This leads to an explicit proof of the equivalence of the AG_3 model with the B_3 model. Thus a translation invariant four–particle model after separation of the c.m. motion is shown to be equivalent with a translation non–invariant three-particle model. In this paper we also discuss F_4 from the view point of our algorithm. The Schrödinger operator obtained (only the rational case) deviates slightly from the one given in [1] (probably due to a simple printing error in [1]).

Thus our method shifts the centre of interest from the simple Lie algebras and their homogeneous spaces to the corresponding Weyl groups and by generalization to the Coxeter groups. On the other hand, the differential operators acting on polynomial spaces of Coxeter invariants define Lie algebras of their own, but at present these algebras are only of marginal interest.

2 The constructive program

We are interested here in the bound state spectrum of Schrödinger operators. The whole analysis is therefore performed in real spaces. Consider a flag of polynomial spaces $V_N(\vec{p})$, $N \in \mathbb{Z}_{\geq}$, $\vec{p} \in \mathbb{N}^n$

$$V_N(\vec{p}) = \text{span}\,\{z_1^{r_1} z_2^{r_2} ... z_n^{r_n} | r_1 p_1 + r_2 p_2 + ... + r_n p_n \leq N\} \quad (p_i \in \mathbb{N}) \tag{2.1}$$

We consider differential operators of first order

$$D^{(1)}_{[\vec{\alpha};a]} = z^{[\vec{\alpha}]} \frac{\partial}{\partial z_a} \tag{2.2}$$

($\vec{\alpha}$ a multi-exponent)
and of second order

$$D^{(2)}_{[\vec{\alpha};a,b]} = z^{[\vec{\alpha}]} \frac{\partial^2}{\partial z_a \partial z_b} \tag{2.3}$$

that leave each space $V_N(\vec{p})$ invariant. If

$$\vec{p} = (1, 1, ..., 1) \tag{2.4}$$

then the operators (2.2) generate the full linear (inhomogeneous) group of $\mathbb{R}_n$ and the operators of second order (2.3) can be obtained as products from the first order operators, i.e. in (2.2)

$$\vec{\alpha} = e^{(c)},\ e_b^{(c)} = \delta_b^c \quad \text{or} \quad \vec{\alpha} = 0 \tag{2.5}$$

and in (2.3)

$$\vec{\alpha} = e^{(c)} + e^{(d)} \quad \text{or} \quad \vec{\alpha} = \mathrm{e}^{(c)} \quad \text{or} \quad \vec{\alpha} = 0 \tag{2.6}$$

Now we consider a candidate for a future Schrödinger operator

$$\begin{aligned} D &= -\sum_{\vec{\alpha},a,b} g_{[\vec{\alpha};a,b]} D^{(2)}_{[\vec{\alpha};a,b]} \\ &+ \sum_{\vec{\beta},c} h_{[\vec{\beta};c]} D^{(1)}_{[\vec{\beta};c]} \end{aligned} \tag{2.7}$$

The eigenvectors and values of D in V_N can be calculated easily by finite linear algebra methods. Let

$$U_N = V_N / V_{N-1} \tag{2.8}$$

and the diagonal part of D on U_N be defined as D_N

$$D_N U_N = D U_N \cap U_N \tag{2.9}$$

If the eigenvalues of D_N are all different, the number of eigenvectors equals $\dim U_N$. But if some eigenvalues coincide (this is true in the generic case!) the number of eigenvectors is smaller. Then the Hilbert space on which the final selfadjoint Schrödinger operator is acting is not an L^2 -space. The missing eigenfunctions can be described. For more details see [6].

If we want completely integrable models we must make sure that a complete set of involutive differential operators exists. For this task Lie algebraic methods may be very helpful.

Given a differential operator (2.7) one can characterize the vector $\vec{p}$ in (2.1) by inequalities

$$g_{[\vec{\alpha};a,b]} \neq 0 \quad \Rightarrow \quad \vec{p}\vec{\alpha} - p_a - p_b \leq 0 \tag{2.10}$$

$$h_{[\vec{\beta};c]} \neq 0 \quad \Rightarrow \quad \vec{p}\vec{\beta} - p_c \leq 0 \tag{2.11}$$

There should be enough equality signs in (2.10),(2.11) for a chosen $\vec{p}$ so that $D_N \neq 0$. It turns out that there exists a minimal $\vec{p}$-vector $\vec{p}_{\mathrm{min}}$ so that the $V_N(\vec{p}_{\mathrm{min}})$ spaces are maximal: For each $N, \vec{p}$ there is N' so that

$$V_N(\vec{p}) \subset V_{N'}(\vec{p}_{\mathrm{min}}) \tag{2.12}$$

It is convenient to work only with this minimal $\vec{p}$-vector.

The first step in transforming D into a Schrödinger operator is to write it symmetrically

$$D = -\sum_{a,b} \frac{\partial}{\partial z_a} g^{-1}_{ab}(z) \frac{\partial}{\partial z_b} + \sum_a r_a(z) \frac{\partial}{\partial z_a} \tag{2.13}$$

where

$$g^{-1}_{ab} = \sum_{\vec{\alpha}} g_{[\vec{\alpha};a,b]} z^{[\vec{\alpha}]} \tag{2.14}$$

We write g_{ab}^{-1} because this is the inverse of a Riemann tensor. The Riemann tensor g_{ab} is assumed to be curvature free. The task to make it so will not arise in this work. But we mention that we developed a minimal algorithm to solve this issue.

Following the notations of [6] we "gauge" the polynomial eigenfunctions φ of D by

$$\psi(z) = e^{-\chi(z)}\varphi(z) \tag{2.15}$$

so that

$$e^{-\chi}De^{+\chi} = -\frac{1}{\sqrt{g}}\sum_{a,b}\frac{\partial}{\partial z_a}(\sqrt{g}g_{ab}^{-1})\frac{\partial}{\partial z_b} + W(z) \tag{2.16}$$

$(g = (\det g^{-1})^{-1})$.
This is possible if and only if

$$\sum_b g_{ab}^{-1}(z)\frac{\partial}{\partial z_b}[2\chi - \ln\sqrt{g}] = r_a(z) \tag{2.17}$$

which implies integrability constraints on the functions $\{r_a(z)\}$. If they are fulfilled we obtain a "prepotential"

$$\rho = \ln P \tag{2.18}$$

so that

$$\rho = 2\chi - \ln\sqrt{g} \tag{2.19}$$

In most cases studied, we found solutions for ρ as follows. Let

$$\det g^{-1}(z) = \prod_{i=1}^{r} P_i(z) \tag{2.20}$$

where $\{P_i(z)\}$ are different real polynomials. Then

$$\rho(z) = \sum_{i=1}^{r}\gamma_i \ln P_i(z) \tag{2.21}$$

with free parameters γ_i solves the requirement that $\{r_a(z)\}$ (2.17) belong to differential operators leaving each V_N invariant. In particular

$$r_a^{(i)}(z) = \frac{1}{P_i(z)}\sum_b g_{ab}^{-1}(z)\frac{\partial P_i}{\partial z_b} \tag{2.22}$$

are polynomials. Inserting (2.20), (2.21) in (2.19) we obtain finally

$$\chi = \frac{1}{2}\sum_{i=1}^{r}(\gamma_i - \frac{1}{2})\ln P_i \tag{2.23}$$

We will later see that in the case of the models of Calogero type a term

$$\gamma_0 \ln P_0 \tag{2.24}$$

can be added to ρ, where

$$P_0(z) = e^{z_1} \tag{2.25}$$

is not contained in $\det g^{-1}$ as a factor. This prepotential gives rise to the oscillator potential.

Finally we mention that $e^{-\chi}$ is the ground state wave function of the Schrödinger operator, as follows from (2.15).

The expression [6], (6.17) for the potential $W(z)$ contains a term linear in χ

$$-\sum_{a,b} \frac{\partial}{\partial z_a}\left(g_{ab}^{-1}\frac{\partial \chi}{\partial z_b}\right) = -\frac{1}{2}\sum_{i=1}^{r}(\gamma_i - \frac{1}{2})\sum_a \frac{\partial}{\partial z_a} r_a^{(i)} \tag{2.26}$$

Each divergence

$$\sum_a \frac{\partial}{\partial z_a} r_a^{(i)}(z) = C^{(i)} \tag{2.27}$$

ought to be a constant. From now on we shall dismiss all constant terms in $W(z)$.

We can then write the potential as

$$W(z) = \sum_{i,j} \gamma_{ij} R_{ij}(z) \tag{2.28}$$

$$R_{ij} = \sum_{a,b} g_{ab}^{-1} \frac{\partial \ln P_i}{\partial z_a}\frac{\partial \ln P_j}{\partial z_b} \tag{2.29}$$

$$\gamma_{ij} = \frac{1}{4}(\gamma_i\gamma_j - \frac{1}{4}) \quad (i, j \neq 0). \tag{2.30}$$

In the cases of this article

$$R_{ij} = \text{const if } i \neq j \tag{2.31}$$

If we then set

$$\gamma_i = -\nu_i + \frac{1}{2} \quad (i \neq 0) \tag{2.32}$$

we obtain

$$W(z) = \sum_{i=1}^{r} \gamma_{ii} R_{ii}(z) \tag{2.33}$$

with

$$\gamma_{ii} = \frac{1}{4}\nu_i(\nu_i - 1) \tag{2.34}$$

As stated in the Introduction the variables $\{z_i\}$ appearing in this section are identified with Coxeter invariants formed from root space coordinates

$\{x_n\}$ or $\{y_n\}$. These invariants are either polynomial or trigonometric. Finally we return from the invariant coordinates $\{z_i\}$ to the root space coordinates $\{x_n\}$ in the Schrödinger operator (2.16). Each contribution

$$R_{ii} = \frac{Q_{ii}}{P_i} \tag{2.35}$$

admits a partial fraction decomposition due to the factorization of the prepotentials P_i (Section 5). The label $i = 1$ is always reserved to a "Vandermonde prepotential", i.e.

$$P_1 \sim \prod_{i<j} (x_i - x_j)^2 \quad \text{or} \quad \prod_{i<j} (\sin(x_i - x_j))^2 \tag{2.36}$$

or alike.

3 Translation invariant models

3.1 Relative coordinates

The Laplacian for an Euclidean space $\mathbb{R}_N$

$$\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \tag{3.1}$$

is translation invariant. We introduce relative coordinates by

$$y_i = x_i - \frac{1}{N} X \tag{3.2}$$

$$X = \sum_{i=1}^{N} x_i \tag{3.3}$$

They separate the Laplacian such that

$$\Delta = N \frac{\partial^2}{\partial X^2} + \sum_{i=1}^{N} \frac{\partial^2}{\partial y_i^2} - \frac{1}{N} \left(\sum_{i=1}^{N} \frac{\partial}{\partial y_i} \right)^2 \tag{3.4}$$

We use all $\{y_i\}_{i=1}^N$ as coordinates on the plane

$$\sum_{i=1}^{N} y_i = 0 \tag{3.5}$$

in order to maintain permutation symmetry.

3.2 Elementary symmetric polynomials

Elementary symmetric polynomials of N variables $\{q_i\}_{i=1}^N$ are defined by a generating function

$$\sum_{n=0}^{N} p_n(q)t^n = \prod_{i=1}^{N}(1+q_i t) \tag{3.6}$$

They are invariant under the symmetric group S_N. For each $g \in S_N$ we have a sector (simplex) $E_g \subset \mathbb{R}_N$

$$E_g = \{q_{i_1} < q_{i_2} < \ldots < q_{i_N}; \quad i_n = g(n)\} \tag{3.7}$$

so that

$$\mathbb{R}_N = \bigcup_{g\in S_N} \bar{E}_g \tag{3.8}$$

Inside E_g we can use the $\{p_n\}_{n=1}^N$ as coordinates since

$$\mathcal{M}_{ni} = \frac{\partial p_n}{\partial q_i} \tag{3.9}$$

$$\det \mathcal{M} = (-1)^{[\frac{N}{2}]} V(q_1, q_2, \ldots q_N) \tag{3.10}$$

where V is the Vandermonde determinant.

3.3 The A_{N-1} series

The root system of A_{N-1} and the corresponding Weyl group possess elementary symmetric polynomials as invariants. We express the Laplacian in each sector E_g (3.7) intersected with the plane (3.5) in terms of these polynomials

$$\tau_n(y_1, \ldots, y_N) = p_n(q)|_{q_i = y_i \text{ all } i} \tag{3.11}$$

The dynamics will be bounded to such sectors by corresponding potential walls automatically.

Then (see [3]) it results

$$\begin{aligned} &\sum_{i=1}^{N} \frac{\partial^2}{\partial y_i^2} - \frac{1}{N}\left(\sum_{i=1}^{N}\frac{\partial}{\partial y_i}\right)^2 \\ &= \sum_{n,m=2}^{N} g_{nm}^{-1} \frac{\partial^2}{\partial \tau_n \partial \tau_m} + \sum_{n=2}^{N} h_n \frac{\partial}{\partial \tau_n} \end{aligned} \tag{3.12}$$

with

$$g_{nm}^{-1}(\tau) = \frac{1}{N}(m-1)(N-n+1)\tau_n\tau_m - T_{n-1,m-1}(\tau) \tag{3.13}$$

and

$$T_{nm}(\tau) = \sum_{l\geq 1}(2l+n-m)\tau_{n+l}\tau_{m-l} \tag{3.14}$$

Here it is understood that

$$\begin{aligned}\tau_0 &= 1\\ \tau_1 &= 0\\ \tau_n &= 0 \text{ for } n<0,\ n>N \end{aligned} \tag{3.15}$$

In this case $\det g^{-1}$ is indecomposable as a polynomial, so we set

$$P_0 = e^{\omega\tau_2} \tag{3.16}$$

$$P_1 = \det g^{-1} = C_N V(y_1,...,y_N)^2 \tag{3.17}$$

The resulting vectors $\{r_a\}_2^N$ are

$$r^{(0)} = (-2\tau_2, -3\tau_3, ..., -N\tau_N) \tag{3.18}$$

$$r^{(1)}:\ \text{explicit formulas known only for } N \leq 4 \tag{3.19}$$

and the potential is

$$\frac{1}{2}W(x) = \frac{1}{2}\omega^2\sum_{i=1}^N x_i^2 + g\sum_{1\leq i<j\leq N}(x_i-x_j)^{-2} \tag{3.20}$$

The corresponding Sutherland models are obtained as follows. We use as coordinates a system $\{\sigma_n\}_{n=2}^N$ defined by (these differ from those in [3])

$$\sigma_0 = \prod_{i=1}^N \cos y_i \tag{3.21}$$

and

$$\sigma_n = \sigma_0 \cdot p_n(q)|_{q_i=\tan y_i} \tag{3.22}$$

The identity

$$\begin{aligned} 1 &= \exp\left(i\sum_{j=1}^N y_j\right)\\ &= \prod_{j=1}^N(\cos y_j + i\sin y_j)\\ &= \sum_{n=0}^N i^n\sigma_n(y) \end{aligned} \tag{3.23}$$

allows us to eliminate σ_0 and σ_1 in terms of the remaining $\{\sigma_n\}_{n=2}^N$ so that polynomials go into polynomials.

The Laplacian is expressed correspondingly as

$$\sum_{i=1}^{N} \frac{\partial^2}{\partial y_1^2} - \frac{1}{N}\left(\sum_{i=1}^{N} \frac{\partial}{\partial y_i}\right)^2 =$$
$$= \sum_{n,m=2}^{N} g_{nm}^{-1} \frac{\partial^2}{\partial\sigma_n \partial\sigma_m} + \sum_{n=2}^{N} h_n \frac{\partial}{\partial\sigma_n} \tag{3.24}$$

$$\begin{aligned} g_{nm}^{-1}(\sigma) \;\; = \;\; & -T_{n+1,m+1}(\sigma) - T_{n+1,m-1}(\sigma) \\ & -T_{n-1,m+1}(\sigma) - T_{n-1,m-1}(\sigma) \\ & +\frac{1}{N}[(m+1)\sigma_{m+1} + (m-1)\sigma_{m-1}] \\ & \times[(N-n-1)\sigma_{n+1} + (N-n+1)\sigma_{n-1}] \end{aligned} \tag{3.25}$$

with T_{nm} as in (3.14).

Once again $\det g^{-1}$ is indecomposable, so we set

$$P_1 = \det g^{-1} = C'_N \tilde{V}(y_1, ..., y_N)^2 \tag{3.26}$$

where

$$\tilde{V}(y_1, ..., y_N) = \prod_{i<j} \sin(y_i - y_j) \tag{3.27}$$

has the symmetry of the Vandermonde determinant (translations and permutations). The vector $r^{(1)}$ is known only up to $N = 4$. Finally we obtain as potential

$$\frac{1}{2} W(x) = g \sum_{1 \le i<j \le N} \sin(x_i - x_j)^{-2} \tag{3.28}$$

In each case A_{N-1} the minimal p-vector is $(1, 1, ..., 1) \in \mathbb{N}^{N-1}$.

3.4 The G_2 and AG_3 models

The models G_2 and AG_3 belong also to the domain of translation invariant models [4]. For G_2 we start from A_2 and extend its Weyl group by a $\mathbb{Z}_2$ group

$$y_i \to -y_i$$

As invariant variables we use [4]

$$\lambda_2 = \tau_2 \tag{3.29}$$

$$\lambda_3 = \tau_3^2 \tag{3.30}$$

In these variables

$$\sum_{i=1}^{3} \frac{\partial^2}{\partial y_i^2} - \frac{1}{3}\left(\sum_{i=1}^{3} \frac{\partial}{\partial y_i}\right)^2 =$$
$$= \sum_{a,b=2}^{3} g^{-1}_{ab} \frac{\partial^2}{\partial\lambda_a \partial\lambda_b} + \sum_{a=2}^{3} h_a \frac{\partial}{\partial\lambda_a} \tag{3.31}$$

We find

$$g^{-1}(\lambda) = \begin{pmatrix} -2\lambda_2, & -6\lambda_3 \\ -6\lambda_3, & +\frac{8}{3}\lambda_2^2\lambda_3 \end{pmatrix} \tag{3.32}$$

so that

$$\det g^{-1} = -\frac{4}{3}\lambda_3(4\lambda_2^3 + 27\lambda_3) \tag{3.33}$$

Thus as ansatz for the prepotentials we use

$$P_0 = e^{\omega\lambda_2} \tag{3.34}$$
$$P_1 = 4\lambda_2^3 + 27\lambda_3 \tag{3.35}$$
$$P_2 = \lambda_3 \tag{3.36}$$

The r-vectors (justifying this ansatz) are

$$r^{(0)} = (-2\lambda_2, -6\lambda_3) \tag{3.37}$$
$$r^{(1)} = (-6, 0) \tag{3.38}$$
$$r^{(2)} = (-6, +\frac{8}{3}\lambda_2^2) \tag{3.39}$$

The minimal $\vec{p}$-vector is

$$\vec{p} = (1, 2) \tag{3.40}$$

The potential is

$$\begin{aligned} \frac{1}{2}W(x) &= \frac{1}{2}\omega^2 \sum_{i=1}^{3} x_i^2 \\ &+ g_1 \sum_{1\le i<j\le 3} (x_i - x_j)^{-2} + g_2 \sum_{i<j, k\notin(i,j)} (x_i + x_j - 2x_k)^{-2} \end{aligned} \tag{3.41}$$

with

$$\begin{aligned} g_1 &= \nu_1(\nu_1 - 1) \\ g_2 &= 3\nu_2(\nu_2 - 1) \end{aligned} \tag{3.42}$$

If

$$\nu_2 = 0 \text{ or } \nu_2 = 1 \tag{3.43}$$

we return to the A_2 model.

In the Sutherland case we use as variables

$$\begin{aligned}\mu_2 &= \sigma_2 & (3.44)\\ \mu_3 &= \sigma_3^2 & (3.45)\end{aligned}$$

leading to the inverse Riemann tensor

$$g^{-1} = \begin{pmatrix} -2\mu_2 - 2\mu_2^2 + \frac{2}{3}\mu_3, & -\mu_3(6 + \frac{16}{3}\mu_2) \\ -\mu_3(6 + \frac{16}{3}\mu_2), & \frac{8}{3}\mu_2^2\mu_3 - 8\mu_3^2 \end{pmatrix} \tag{3.46}$$

Now $\det g^{-1}$ is decomposable with

$$\det g^{-1} = -\frac{4}{3}\mu_3 P_1(\mu) \tag{3.47}$$

and

$$P_1(\mu) = 4\mu_3^2 + \mu_3(8\mu_2^2 + 36\mu_2 + 27) + 4\mu_2^3(1 + \mu_2) \tag{3.48}$$

$$P_2(\mu) = \mu_3 \tag{3.49}$$

The r-vectors are

$$\begin{aligned} r^{(1)} &= (-6 - 8\mu_2, -16\mu_3) & (3.50)\\ r^{(2)} &= (-6 - \frac{16}{3}\mu_2, \frac{8}{3}\mu_2^2 - 16\mu_3) & (3.51)\end{aligned}$$

The resulting potential is

$$\begin{aligned}\frac{1}{2}W(x) &= g_1 \sum_{1\le i<j\le 3} \sin(x_i - x_j)^{-2} \\ &+ \frac{1}{9}g_2 \sum_{i<j, k\notin(i,j)} \sin\frac{1}{3}(x_i + x_j - 2x_k)^{-2} \end{aligned} \tag{3.52}$$

In the case of the A_2 models the spaces V_N decompose into even and odd subspaces in τ_3 (or σ_3) which are left invariant separately under action of the Laplacian. In the case of the odd spaces we can factor $\tau_3(\sigma_3)$ and leave an even space as well. In each case we obtain a polynomial space in the variables $\lambda_2, \lambda_3 = \tau_3^2$ ($\mu_2, \mu_3 = \sigma_3^2$). Thus starting from such polynomial space and multiplying with $\tau_3^{\nu_2}$ ($\sigma_3^{\nu_2}$) we obtain the A_2 model if $\nu_2 = 0$ or $\nu_2 = 1$ but a new potential in all other cases.

It is plausible that a similar procedure works for A_3 but not for A_{N-1}, $N \geq 5$. In the latter models we have two or more odd variables $\tau_3, \tau_5, ...(\sigma_3, \sigma_5, ...)$ and there is no factorization of the odd invariant subspaces. Let us sketch the A_3 model whose extension leads to the AG_3 model [8].

In this case the variables are chosen as in (3.29), (3.30), (3.44), (3.45)

$$\lambda_2 = \tau_2,\ \lambda_3 = \tau_3^2,\ \lambda_4 = \tau_4 \tag{3.53}$$

The inverse Riemann tensor is

$$g^{-1} = \begin{pmatrix} -2\lambda_2, & -6\lambda_3, & -4\lambda_4 \\ -6\lambda_3, & 4\lambda_3(\lambda_2^2 - 4\lambda_4), & \lambda_2\lambda_3 \\ -4\lambda_4, & +\lambda_2\lambda_3, & -2\lambda_2\lambda_4 + \frac{3}{4}\lambda_3 \end{pmatrix} \tag{3.54}$$

The determinant is decomposable as

$$\det g^{-1} = \lambda_3 P_1(\lambda) \tag{3.55}$$

and the ansatz for the prepotentials is

$$\begin{aligned} P_0(\lambda) &= e^{\omega\lambda_2} && (3.56) \\ P_1(\lambda) &= 27\lambda_3^2 - 256\lambda_4^3 + 128\lambda_2^2\lambda_4^2 && (3.57) \\ &\quad -16\lambda_2^4\lambda_4 + 4\lambda_2^3\lambda_3 - 144\lambda_2\lambda_3\lambda_4 \\ P_2(\lambda) &= \lambda_3 && (3.58) \end{aligned}$$

The r-vectors come out as

$$\begin{aligned} r^{(0)} &= (-2\lambda_2, -6\lambda_3, -4\lambda_4) && (3.59) \\ r^{(1)} &= (-12, 0, -2\lambda_2) && (3.60) \\ r^{(2)} &= (-6, 4(\lambda_2^2 - 4\lambda_4), \lambda_2) && (3.61) \end{aligned}$$

The potential for this Calogero type model is

$$\begin{aligned} \frac{1}{2}W(x) &= \frac{1}{2}\omega^2 \sum_{i=1}^{4} x_i^2 && (3.62) \\ &\quad + g_1 \sum_{1 \le i < j \le 4} (x_i - x_j)^{-2} + g_2 \sum_{\text{3 terms}} (x_i + x_j - x_k - x_l)^{-2} \end{aligned}$$

with

$$g_1 = \nu_1(\nu_1 - 1), \; g_2 = 2\nu_2(\nu_2 - 1) \tag{3.63}$$

It was discovered first by Wolfes, [7].

The Sutherland model is obtained in the same fashion. With

$$\mu_2 = \sigma_2, \; \mu_3 = \sigma_3^2, \; \mu_4 = \sigma_4 \tag{3.64}$$

the inverse Riemann tensor is

$$\begin{aligned} g_{22}^{-1} &= -2\mu_2 - 2\mu_2^2 - 8\mu_4 + 2\mu_3 + 8\mu_2\mu_4 + 8\mu_4^2 && (3.65) \\ g_{23}^{-1} &= -6\mu_3 - 4\mu_2\mu_3 && (3.66) \\ g_{24}^{-1} &= -4\mu_4 - 6\mu_2\mu_4 + \mu_3 + 4\mu_4^2 && (3.67) \\ g_{33}^{-1} &= 4\mu_3[-4\mu_4 + \mu_2^2 - 4\mu_2\mu_4 + 4\mu_4^2 - 2\mu_3] && (3.68) \\ g_{34}^{-1} &= \mu_2\mu_3 - 6\mu_3\mu_4 && (3.69) \\ g_{44}^{-1} &= -2\mu_2\mu_4 + \frac{3}{4}\mu_3 && (3.70) \end{aligned}$$

Its determinant decomposes

$$\det g^{-1} = -\mu_3 P_1(\mu) \tag{3.71}$$

$$P_1(\mu) = 256\mu_4^6 + 32 \text{ further terms} \tag{3.72}$$

$$\text{(equ. (A.2) from [8])}$$

$$P_2(\mu) = \mu_3 \tag{3.73}$$

and the r-vectors are

$$r^{(1)} = (-16\mu_2 - 12, -24\mu_3, -12\mu_4 - 2\mu_2) \tag{3.74}$$

$$r^{(2)} = (-4\mu_2 - 8, 16\mu_4^2 - 16\mu_4\mu_2 + 4\mu_2^2 - 8\mu_3 - 16\mu_4, -6\mu_4 + \mu_2) \tag{3.75}$$

The factorization of σ_3 which is necessary in this case is

$$\sigma_3 = -\prod_{1\le i<j\le 3} \sin(y_i + y_j) \tag{3.76}$$

implying

$$\frac{Q_{22}}{P_2} = 4 \sum_{1\le i<j\le 3} (\sin(y_i + y_j))^{-2} \tag{3.77}$$

This gives the potential

$$\begin{aligned} \frac{1}{2}W(x) &= g_1 \sum_{1\le i<j\le 4} (\sin(x_i - x_j))^{-2} \\ &\quad + \frac{1}{4} g_2 \sum_{3\,\text{cases}} (\sin \frac{1}{2}(x_i + x_j - x_k - x_l))^{-2} \end{aligned} \tag{3.78}$$

The discussion of this AG_3 model is resumed in Section 5.

4 Translation non-invariant models

4.1 The BC_N and D_N models

As we shall see there is only one series with two (Calogero) and three (Sutherland) independent coupling constants. For any such model we use as Cartesian coordinates $\{x_i\}_{i=1}^N$ and require permutation symmetry S_N and reflection symmetry $(\mathbb{Z}_2)^N$ $x_i \to -x_i$ for each i separately. Then the natural coordinates invariant under these group actions are [5]

$$\lambda_n(x) = p_n(q)|_{q_i = x_i^2,\, \text{all}\, i} \tag{4.1}$$

There is a bilinear relation with the $\{p_n(x)\}_{n=1}^N$

$$\lambda_n(x) = \sum_{k=0}^{2n} (-1)^{n-k} p_{2n-k}(x) p_k(x) \tag{4.2}$$

The inverse Riemann tensor for the full Laplacian (3.1) is then

$$g^{-1}_{nm}(\lambda) = 4M_{nm}(\lambda) \tag{4.3}$$

where we introduce the shorthand

$$M_{nm}(\lambda) = \sum_{l\geq 0}(2l+n-m+1)\lambda_{n+l}\lambda_{m-1-l} \tag{4.4}$$

Its determinant factorizes

$$\det g^{-1} = (-1)^{\left[\frac{N}{2}\right]}4^N\lambda_N P_1(\lambda) \tag{4.5}$$

where

$$\begin{aligned} P_1(\lambda) &= N^N\lambda_N^{N-1}+... \\ &= D_N V(x_1^2, x_2^2, ...x_N^2)^2 \end{aligned} \tag{4.6}$$

and

$$P_2(\lambda) = \lambda_N \tag{4.7}$$

Both functions P_1, P_2 factorize in a trivial way. In the general case there is no explicit expression for $r^{(1)}$ but

$$r_a^{(2)} = 4(N-a+1)\lambda_{a-1} \tag{4.8}$$

If follows

$$R_{22} = 4\frac{\lambda_{N-1}}{\lambda_N} = 4\sum_{i=1}^{4} x_i^{-2} \tag{4.9}$$

The resulting potential is, including an oscillator potential

$$\begin{aligned} \frac{1}{2}W(x) &= \frac{1}{2}\omega^2\sum_{i=1}^N x_i^2 + g_1\sum_{1\leq i<j\leq N}\left[(x_i-x_j)^{-2}+(x_i+x_j)^{-2}\right] \\ &\quad + g_2\sum_{i=1}^N x_i^{-2} && (4.10) \\ g_1 &= \nu_1(\nu_1-1) && (4.11) \\ g_2 &= \frac{1}{2}\nu_2(\nu_2-1) && (4.12) \end{aligned}$$

In the Sutherland case we use coordinates

$$\mu_0 = \prod_{i=1}^N \cos^2 x_i \tag{4.13}$$

$$\begin{aligned} \mu_n(x) &= \mu_0(x)p_n(q)|_{q_i=\tan^2 x_i,\,\mathrm{all}\,i} \\ &\quad n\in\{1,2,...N\} \end{aligned} \tag{4.14}$$

From the identity

$$\begin{aligned} 1 &= \prod_{i=1}^{N}(\cos^2 x_i + \sin^2 x_i) \\ &= \sum_{n=0}^{N} \mu_n(x) \end{aligned} \tag{4.15}$$

we learn how to eliminate μ_0 in facour of $\{\mu_n\}_{n_1}^N$ so that a polynomial of $\{\mu_n\}_{n=0}^N$ remains a polynomial.

In this case the inverse Riemannian is

$$\begin{aligned} g_{nm}^{-1} &= 4\{M_{n+1,m+1}(\mu) + M_{n,m}(\mu) \\ &\quad -M_{n,m+1}(\mu) - M_{n+1,m}(\mu)\} \end{aligned} \tag{4.16}$$

and the determinant decomposes as

$$\det g^{-1} = 4^N(-1)^{\left[\frac{N}{2}\right]}\mu_0\mu_N P_1(\mu) \tag{4.17}$$

Now the factorization of $P_1(\mu)$ is

$$P_1(\mu) = D'_N \prod_{1\le i<j\le N}(\cos^2 x_i \sin^2 x_j - \sin^2 x_i \cos^2 x_j)^2 \tag{4.18}$$

and we choose

$$P_2(\mu) = \mu_N \tag{4.19}$$

$$P_3(\mu) = \mu_0 \tag{4.20}$$

Again we have no general explicit expression for $r^{(1)}$ but

$$r_a^{(2)} = 4[(N-a+1)\mu_{a-1} - (N-a)\mu_a] \tag{4.21}$$

$$r_a^{(3)} = 4[(a+1)\mu_{a+1} - a\mu_a] \tag{4.22}$$

so that

$$R_{22} = \frac{\mu_{N-1}}{\mu_N} = 4\sum_{i=1}^{N}\cot^2 x_i \tag{4.23}$$

$$R_{33} = \frac{\mu_1}{\mu_0} = 4\sum_{i=1}^{N}\tan^2 x_i \tag{4.24}$$

Thus we end up with a potential

$$\begin{aligned} \frac{1}{2}W(x) &= g_1 \sum_{1\le i<j\le N}[(\sin(x_i - x_j))^{-2} + (\sin(x_i + x_j))^{-2}] \\ &\quad +g_2\sum_{i=1}^{N}(\sin x_i)^{-2} \\ &\quad +g_3\sum_{i=1}^{N}(\cos x_i)^{-2} \end{aligned} \tag{4.25}$$

where $g_{1,2}$ are as in (4.11),(4.12) and

$$g_3 = \frac{1}{2}\nu_3(\nu_3 - 1) \tag{4.26}$$

An alternative form of the potential is obtained from

$$\frac{g_2}{\sin^2 x} + \frac{g_3}{\cos^2 x} = \frac{g_2 - g_3}{\sin^2 x} + \frac{4g_3}{\sin^2 2x} \tag{4.27}$$

If we set $g_2 = g_3$ or $g_3 = 0$ we obtain different samples of the BC_N or D_N series. We mention finally that the minimal p-vector is in all cases

$$\vec{p} = (1, 1, ...1) \in \mathbb{N}^N \tag{4.28}$$

4.2 The F_4 model

The F_4 model belongs also to the translation noninvariant class. The Weyl group of F_4 possesses four basic polynomial invariants

$$I_1(x),\ I_3(x),\ I_4(x), I_6(x) \tag{4.29}$$

(I_n of degree $2n$) which can be expressed as polynomials in the $\{\lambda_n\}_{n=1}^4$ as follows

$$\begin{aligned}
I_1 &= \lambda_1 && (4.30)\\
I_3 &= \lambda_3 - \frac{1}{6}\lambda_1\lambda_2 && (4.31)\\
I_4 &= \lambda_4 - \frac{1}{4}\lambda_1\lambda_3 + \frac{1}{12}\lambda_2^2 && (4.32)\\
I_6 &= \lambda_4\lambda_2 - \frac{1}{36}\lambda_2^3 + \frac{1}{24}\lambda_2^2\lambda_1^2 - \frac{1}{64}\lambda_2\lambda_1^4 && (4.33)
\end{aligned}$$

In these coordinates the inverse Riemannian can be given as

$$\begin{aligned}
g_{1m}^{-1} &= 4mI_m && (4.34)\\
g_{33}^{-1} &= \frac{20}{3}I_4I_1 - \frac{2}{3}I_3I_1^2 && (4.35)\\
g_{34}^{-1} &= 8I_6 - 3I_3^2 - \frac{13}{3}I_4I_1^2 - \frac{3}{4}I_3I_1^3 && (4.36)\\
g_{36}^{-1} &= 16I_4^2 + I_6I_1^2 + 14I_4I_3I_1 + \frac{5}{2}I_3^2I_1^2 - \frac{1}{4}I_4I_1^4 - \frac{5}{32}I_3I_1^5 && (4.37)\\
g_{44}^{-1} &= -4I_4I_3 - 2I_6I_1 + \frac{3}{4}I_4I_1^3 + \frac{3}{4}I_3^2I_1 + \frac{3}{16}I_3I_1^4 && (4.38)\\
g_{46}^{-1} &= 8I_4^2I_1 + 2I_4I_3I_1^2 - \frac{1}{8}I_4I_1^5 && (4.39)\\
g_{66}^{-1} &= 30I_6I_4I_1 + \frac{21}{2}I_6I_3I_1^2 - \frac{3}{32}I_6I_1^5 + 12I_4^2I_3 + 6I_4I_3^2I_1 \\
&\quad -\frac{3}{8}I_4I_3I_1^4 + \frac{3}{4}I_3^3I_1^2 + \frac{3}{1024}I_3I_1^8 - \frac{3}{32}I_3^2I_1^5 && (4.40)
\end{aligned}$$

The determinant decomposes into two factors

$$\det g^{-1} = \frac{1}{3072} P_1(I) P_2(I) \tag{4.41}$$

where $P_1(I)$ is connected with the Vandermonde determinant squared as usual

$$\begin{aligned} P_1(I) \;=\; & -4096 I_4^3 + 432 I_3^4 + 3072 I_6^2 - 2304 I_6 I_4 I_1^2 \\ & -576 I_6 I_3 I_1^3 + 864 I_4 I_3^2 I_1^2 + 216 I_4 I_3 I_1^5 \\ & +432 I_4^2 I_1^4 + 27 I_3^2 I_1^6 - 2304 I_6 I_3^2 + 216 I_3^3 I_1^3 \end{aligned} \tag{4.42}$$

or in factorized form

$$P_1(I) = -16 \prod_{1 \le i < j \le 4} (x_i^2 - x_j^2)^2 \tag{4.43}$$

and $P_2(I)$

$$\begin{aligned} P_2(I) \;=\; & 36864 I_6^2 - 18432 I_6 I_4 I_1^2 - 4608 I_6 I_3 I_1^3 + 32 I_6 I_1^6 \\ & -49152 I_4^3 - 36864 I_4^2 I_3 I_1 + 1536 I_4^2 I_1^4 \\ & +768 I_4 I_3 I_1^5 - 12 I_4 I_1^8 - 9216 I_4 I_3^2 I_1^2 \\ & -768 I_3^3 I_1^3 + 96 I_3^2 I_1^6 - 3 I_3 I_1^9 \end{aligned} \tag{4.44}$$

which factorizes as

$$\begin{aligned} P_2(I) &= -12\lambda_4 (64\lambda_4 - 16\lambda_2^2 + 8\lambda_2\lambda_1^2 - \lambda_1^4)^2 \\ &= -12 x_1^2 x_2^2 x_3^2 x_4^2 \prod_{\nu_2, \nu_3 \nu_4 \in \{1,0\}} (x_1 - \sum_{i=2}^{4} (-1)^{\nu_i} x_i)^2 \end{aligned} \tag{4.45}$$

The r-vectors are

$$r^{(1)} = (48, -2I_1^2, 0, 36 I_4 I_1 + 12 I_3 I_1^2 - \frac{3}{16} I_1^5) \tag{4.46}$$

$$r^{(2)} = (48, -4I_1^2, -12 I_3, 24 I_4 I_1 + 6 I_1^2 I_3 - \frac{3}{8} I_1^5) \tag{4.47}$$

The potential resulting is

$$\begin{aligned} \frac{1}{2} W(x) \;=\; & \frac{1}{2}\omega^2 \sum_{1 \le i \le 4} x_i^2 + g_1 \sum_{1 \le i < j \le 4} [(x_i - x_j)^{-2} + (x_i + x_j)^{-2}] \\ & + g_2 \{ \sum_{\substack{\nu_2, \nu_3, \nu_4 \\ \in \{+1, 0\}}} 4 \left(x_1 - \sum_{i=2}^{4} \nu_i x_i \right)^{-2} + \sum_{i=1}^{4} x_i^{-2} \} \end{aligned} \tag{4.48}$$

where $g_{1,2}$ are as in (4.11),(4.12). The minimal p-vector is

$$\vec{p} = (1, 2, 3, 5) \tag{4.49}$$

5 Coxeter groups, orbits and prepotentials

The prepotentials used in the empirical constructions of sections 3 and 4 necessitate a mathematical interpretation. Let W be a Coxeter group generated by the reflections

$$\{s_\alpha\} \tag{5.1}$$

where α are roots running over a set

$$\Phi = \{\alpha\}_1^M \tag{5.2}$$

The roots span an Euclidian space V. In this space the reflections $\{s_\alpha\}$ act by

$$x \in V : s_\alpha x = x - 2\frac{(\alpha, x)}{(\alpha, \alpha)}\alpha \tag{5.3}$$

If the Coxeter group W is "crystallographic", it is a Weyl group (for more details see [9]).

We denote a set of basic polynomial invariants of W by

$$\{z_1(x), \ldots, z_n(x)\}, \quad n = \dim V \tag{5.4}$$

Invariance means

$$\begin{aligned} z_i(w^{-1}x) &= z_i(x) \\ &= wz_i(x) \end{aligned} \tag{5.5}$$

for all $w \in W$. The Jacobian for the transition $\{x_j\} \to \{z_i\}$

$$J = \det\left\{\frac{\partial z_i}{\partial x_j}\right\} \tag{5.6}$$

can be factorized as follows ([9], Proposition 3.13).

Each reflection s_α leaves a hyperplane H_α in V pointwise fixed, let H_α be given by a linear function l_α

$$l_\alpha(x) = 0 \tag{5.7}$$

Then due to the proposition

$$J = C\prod_{\alpha\in\Phi^+} l_\alpha(x) \tag{5.8}$$

with Φ^+ the set of positive roots. The proof of this proposition is rather elementary.

For any inverse Riemann tensor $\{g^{-1}\}$ of Sections 3 and 4 we obtain this way

$$\det g_{ab}^{-1} = C^2\prod_{\alpha\in\Phi^+} l_\alpha(x)^2 \tag{5.9}$$

If Φ decomposes into orbits under W

$$\Phi = \bigcup_i \Phi_i \tag{5.10}$$

then

$$P_i = \prod_{\alpha \in \Phi_i^+} l_\alpha(x)^2 \tag{5.11}$$

is an invariant polynomial under action of W and therefore a polynomial in the basic invariants

$$P_i = P_i(z_1, \ldots, z_n) \tag{5.12}$$

These polynomials are the prepotentials constructed in Sections 3 and 4. The factorization of these prepotentials as quoted at the end of Section 2 (eqns. (2.35),(2.36)) and used throughout in Sections 3 and 4 is based on (5.11).

We emphasize that our empirical results of Sections 3 and 4 indicate the validity of further mathematical propositions which could not be traced in the literature:

1. an analogous factorization theorem for the trigonometric invariants;
2. the polynomial properties ("integrability") of the functions $r^{(i)}(z)$ (2.22).

Now we return to the AG_3 model of Section 3. We identify the roots involved in a model using (5.7),(5.9)

$$\begin{aligned} l_\alpha(x) &= (\alpha^\vee, x) \\ (\alpha^\vee &= \frac{2\alpha}{(\alpha,\alpha)}, \text{ the "dual" of } \alpha) \end{aligned} \tag{5.13}$$

and the Sutherland version whose potential is

$$\frac{1}{2}W(x) = \sum_{\text{orbits } i} g_i \sum_{\alpha \in \Phi_i^+} [\sin l_\alpha(x)]^{-2} \tag{5.14}$$

Thus the simple roots of A_3

$$\begin{aligned} \alpha_1 &= e_1 - e_2 \\ \alpha_2 &= e_2 - e_3 \\ \alpha_3 &= e_3 - e_4 \end{aligned} \tag{5.15}$$

are completed by a fourth root in AG_3

$$\alpha_4 = e_3 + e_4 - e_1 - e_2 \tag{5.16}$$

The corresponding Coxeter-diagram is shown in Fig. 1. It belongs to the

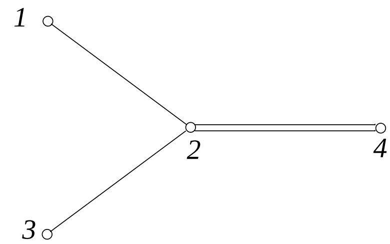

Figure 1: Coxeter diagram of $\hat{B}_3$

affine Coxeter group $\hat{B}_3$ ([9], Figure 1 in Section 2.4).

The coordinates of the $\hat{B}_3$ root space with respect to the standard basis $\{f_i\}_{i=1}^3$ are denoted $\{\xi_i\}_{i=1}^3$, those of AG_3 with respect to the standard basis $\{e_i\}_{i=1}^4$ by $\{x_i\}_{i=1}^4$ as before. The simple roots of B_3 are

$$\beta_1 = f_1 - f_2, \quad \beta_1 = f_2 - f_3, \quad \beta_3 = f_3 \tag{5.17}$$

and $\hat{B}_3$ is obtained by adjoining

$$\beta_4 = -f_1 - f_2 \tag{5.18}$$

It follows that

$$s_4 \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} -\xi_2 \\ -\xi_1 \\ \xi_3 \end{pmatrix} \tag{5.19}$$

leaves the Coxeter invariants of B_3

$$\lambda_1(\xi) = \sum_{1\le i\le 3} \xi_i^2 \tag{5.20}$$

$$\lambda_2(\xi) = \sum_{1\le i<j\le 3} \xi_i^2\xi_j^2 \tag{5.21}$$

$$\lambda_3(\xi) = \xi_1^2\xi_2^2\xi_3^2 \tag{5.22}$$

invariant, too. This suggests the equivalence of the AG_3 and the B_3 models.

An explicit identification of the simple roots

$$f_1 = \frac{1}{2}(e_1 - e_2 - e_3 + e_4) \tag{5.23}$$

$$f_2 = \frac{1}{2}(-e_1 + e_2 - e_3 + e_4) \tag{5.24}$$

$$f_3 = \frac{1}{2}(-e_1 - e_2 + e_3 + e_4) \tag{5.25}$$

gives $(i, j \in \{1, 2, 3\})$

$$x_i - x_j = \xi_i - \xi_j \tag{5.26}$$

$$x_4 - x_j = \sum_{i(\neq j)} \xi_i \tag{5.27}$$

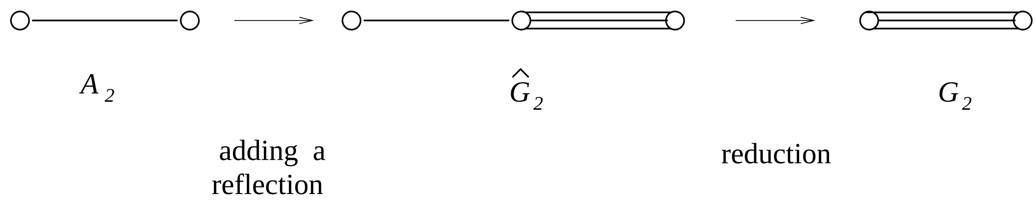

Figure 2: Extending the Coxeter diagram of A_2 to $\hat{G}_2$ and reduction to G_2

It follows

$$\begin{aligned} & g_1 \sum_{1 \le i < j \le 4} [\sin(x_i - x_j)]^{-2} + \frac{1}{4} g_2 \sum_{3\,\mathrm{cases}} [\sin \frac{1}{2}(x_i + x_j - x_k - x_l)]^{-2} \\ = \; & g_1 \sum_{1 \le i < j \le 3} \{[\sin(\xi_i - \xi_j)]^{-2} + [\sin(\xi_i + \xi_j)]^{-2}\} + \frac{1}{4} g_2 \sum_{i=1}^{3} [\sin \xi_i]^{-2} \end{aligned} \tag{5.28}$$

Moreover the rational invariants (3.64) can be identified with the invariants (5.20)–(5.22)

$$\mu_2(x) = -\frac{1}{2}\lambda_1(\xi) \tag{5.29}$$

$$\mu_3(x) = +\frac{1}{4}\lambda_3(\xi) \tag{5.30}$$

$$\mu_4(x) = -\frac{1}{4}\lambda_2(\xi) + \frac{1}{16}\lambda_1(\xi)^2 \tag{5.31}$$

This establishes the equivalence between the two models.

Our method involves a reduction of the affine Coxeter group $\hat{B}_3$ to the Coxeter group B_3 having the same invariants. It may therefore be of interest that the construction performed in [4] is analogous (see Fig. 2).

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