Particle with torsion on $3d$ null-curves

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We consider a $(2+1)$-dimensional mechanical system with the Lagrangian linear in the torsion of a light-like curve. We give Hamiltonian formulation of this system and show that its mass and spin spectra are defined by one-dimensional nonrelativistic mechanics with a cubic potential. Consequently, this system possesses the properties typical of resonance-like particles.

1. Introduction

The search of Lagrangians, describing spinning particles, both massive and massless, has a long story. The conventional approach in this direction consists in the extension of the initial space-time by auxiliary odd/even coordinates which equip a system with spinning degrees of freedom. There is another, less developed approach, where the spinning particle systems are described by the Lagrangians, which are formulated in the initial space-time, but depend on higher derivatives. The aesthetically attractive point of the latter approach is that spinning degrees of freedom are encoded in the geometry of trajectories. The Poincaré and reparametrization invariance require actions to be of the form

$$\mathcal{S} = \int \mathcal{L}(k_1, ..., k_N)|d\mathbf{x}|, \quad N \leq D-1, \qquad (1)$$

where k_I denote the reparametrization invariants (extrinsic curvatures) of curves,

$$k_I = \frac{\sqrt{\det \hat{g}_{I+1} \det \hat{g}_{I-1}}}{\det \hat{g}_I}, \qquad (2)$$

where $(g_I)_{ij} \equiv \partial_{(i)}\mathbf{x}\partial_{(j)}\mathbf{x}$, $i,j = 1,\ldots,I$.

It was shown by M.Plyushchay that a four-dimensional system of this sort with Lagrangian $\mathcal{L} = ck_1$ describes a massless particle with the helicity c [1]. This model has W_3 gauge symmetry [2]: its classical trajectories are *space-like plane curves* with arbitrary first curvature. Higher-dimensional generalization of this model is given by the action [3]

$$\mathcal{S} = c\int k_N|d\mathbf{x}|, \quad N \leq [(D-2)/2]. \qquad (3)$$

This system has the following interesting properties:

- This is the only model that leads to an irreducible representation of the Poincaré group. It describes massless particles specified by the coinciding weights of $SO(N)$ group.
- It has $N+1$ gauge degrees of freedom: the classical solution of this model is a space-like curve specified by the relations: $k_1, \ldots k_N$ are arbitrary; $k_{N+a} = k_{N-a}$, $k_{2N} = 0$, $a = 1,\ldots,N-1$ (W_{N+2} gauge symmetry? [4]).

All massive models with an action (1) correspond to reducible representations of the Poincaré group. Nevertheless, these models can be useful in planar physics, where a value of spin can be arbitrary. Extensive studies in this direction were inspired by the remarkable work of Polyakov [5] where the CP^1 model with the Chern-Simons term was investigated. Evaluating

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the effective action for the charged solitonic excitation he found that it is of the type (1), where $\mathcal{L} = m_0 + \frac{c^2}{m_0}k_2$. Later it was shown [6] that though the trajectories of the system are *time-like*, it has not only massive, but also tachionic and massless sectors, with mass and spin related by the Majorana condition

$$\text{Spin} \times \text{Mass} = c^2, \tag{4}$$

while $|\text{Mass}| \geq m_0$.

Adding, to the initial Lagrangian, the term proportional to k_1 modifies the spin-mass relation but preserves the basic properties of the initial model [7].

In a $(2+1)$-dimensional space one can consider actions of another sort defined on the light-like (or null) curves

$$\mathcal{S} = \int \mathcal{L}(K)d\sigma, \quad d\sigma = |\ddot{\mathbf{x}}|^{1/2}du, \quad \dot{\mathbf{x}}^2 = 0, \tag{5}$$

where $K = |d^3\mathbf{x}/d\sigma^3|^2$ is the torsion for light-like curves.

In Ref. [9], the simplest system of this sort was considered given by the action

$$\mathcal{S} = 2c\int d\sigma. \tag{6}$$

It was found that it describes the anyons with Majorana-like spectrum (4), while its classical solutions are null-helices.

In the present paper, we consider a more complicated three-dimensional system, associated with null-curves

$$\mathcal{S} = 2c\int(\epsilon + K)d\sigma. \tag{7}$$

where ϵ is a constant, and K is the torsion of a null-curve.

We show that this system has a much reacher structure than the previous one and is related with nonrelativistic mechanics

$$d\pi \wedge dq, \quad \pi^2 + q^3 - 2\epsilon q^2 + \frac{ms}{c^2}q + \frac{m^2}{c^2} = 0,$$

where m and s denote the mass and spin of the system, while $q = \epsilon - K/c$.

We conclude the Introduction with some basic facts from the geometry of three-dimensional null-curves to be used in this paper.

For the description of null curves it is convenient to use the moving frame $(\mathbf{e}_\pm, \mathbf{e}_1)$:

$$\mathbf{e}_\pm\mathbf{e}_\pm = \mathbf{e}_\pm\mathbf{e}_1 = 0, \quad \mathbf{e}_+\mathbf{e}_- = -\mathbf{e}_1^2 = 1, \tag{8}$$

with the vector product $\times$ defined as follows

$$\mathbf{e}_+ \times \mathbf{e}_- = \mathbf{e}_1, \quad \mathbf{e}_\pm \times \mathbf{e}_1 = \pm\mathbf{e}_\pm. \tag{9}$$

In this notation pseudoarch-length $d\sigma \equiv \tilde{\sigma}du$ and the torsion K are defined via the Frenet equations [8]:

$$\begin{gathered} \mathbf{x}' = \mathbf{e}_+, \quad \mathbf{e}_+' = \mathbf{e}_1, \\ \mathbf{e}_1' = K\mathbf{e}_+ + \mathbf{e}_-, \quad \mathbf{e}_-' = K\mathbf{e}_1, \end{gathered} \tag{10}$$

where $' \equiv d/d\sigma$.
Hence,

$$\tilde{\sigma} = -\dot{\mathbf{e}}_+\mathbf{e}_1, \quad 2K = \mathbf{e}_1'^{\,2}. \tag{11}$$

2. Hamiltonian formulation

Prior giving the Hamiltonian formulation of the system (7), let us present, for completeness, the Hamiltonian system describing (6) [9].

The system (6) is described by the Hamiltonian structure

$$\begin{gathered} \omega = d\mathbf{p} \wedge d\mathbf{x} + c d\mathbf{e}_+ \wedge d\mathbf{e}_1, \\ \mathcal{H} = \frac{\tilde{\sigma}}{2c}\left[c^2\mathbf{e}_1^2 + (\mathbf{p}\mathbf{e}_+ - 2c)^2 + \mathbf{p}^2\mathbf{e}_+^2\right] \end{gathered} \tag{12}$$

and the constraints

$$\begin{cases} \mathbf{e}_1^2 + 1 \approx 0, \quad \mathbf{e}_+^2 \approx 0, \quad \mathbf{e}_1\mathbf{e}_+ \approx 0, \\ \mathbf{p}\mathbf{e}_+ - c \approx 0, \quad \mathbf{p}\mathbf{e}_1 \approx 0. \end{cases} \tag{13}$$

Its Lorentz generator is of the form

$$\mathbf{J} = \mathbf{p} \times \mathbf{x} + c\mathbf{e}_+, \tag{14}$$

from which the relation (4) follows immediately.
Introducing

$$K = -\mathbf{p}^2/2c^2, \quad \mathbf{e}_- = \mathbf{p}/c + K\mathbf{e}_+, \tag{15}$$

we reduce the equations of motion to the Frenet formulae, while the effective coordinate reads

$$\mathbf{X} \equiv \mathbf{x} - \frac{c}{\mathbf{p}^2}\mathbf{p}_+ \; : \; \ddot{\mathbf{X}} = 0.$$

Thus, massive (tachionic) solutions of (6) correspond to the light-like helices with negative (positive) torsion.

Now let us give the Hamiltonian formulation of (7). Taking into account the Frenet equations (10), we can replace the initial Lagrangian depending on third derivatives by the classically equivalent one depending on first derivatives only

$$L = \tilde{\sigma}\left[2c\epsilon + c{\mathbf{e}'}_1^2 + \mathbf{p}(\mathbf{x}' - \mathbf{e}_+) + \right. \\ \left. + \mathbf{p}_+(\mathbf{e}'_+ - \mathbf{e}_1) - \textstyle\sum_{i,j} d_{ij}(\mathbf{e}_i\mathbf{e}_j - \eta_{ij})\right], \tag{16}$$

where $\mathbf{x}, \mathbf{p}, \mathbf{p}_+, \mathbf{e}_i, d_{ij}, \tilde{\sigma}$ are independent variables, $i, j = +, 1$; $\eta_{++} = \eta_{1+} = 0, \eta_{11} = -1$.

The momentum conjugate to $\mathbf{e}_1$ reads

$$\mathbf{p}_1 = \frac{\partial L}{\partial \dot{\mathbf{e}}_1} = 2c{\mathbf{e}'}_1.$$

Thus, we get the primary constraints

$$\mathbf{p}_1\mathbf{e}_+ - 2c \approx 0, \quad \mathbf{p}_1\mathbf{e}_1 \approx 0.$$

Performing the Legendre transformation, after some work, we obtain the following Hamiltonian system:

$$\omega = d\mathbf{p} \wedge d\mathbf{x} + d\mathbf{p}_+ \wedge d\mathbf{e}_+ + d\mathbf{p}_1 \wedge d\mathbf{e}_1 \\ \mathcal{H} = \tilde{\sigma}\left[\phi_0 + \lambda\phi_1 + \textstyle\sum_{i,j=+,1} d_{ij}u_{ij}\right] \tag{17}$$

with constraints

$$\left\{ \begin{array}{c} \phi_0 = {\mathbf{p_1}}^2/4c + \mathbf{p}\mathbf{e}_+ + \mathbf{p}_+\mathbf{e}_1 - 2c\epsilon \approx 0; \\ \phi_1 = \mathbf{p}_1\mathbf{e}_+ - 2c \approx 0, \\ \phi_2 = \mathbf{p}_1\mathbf{e}_1 \approx 0, \\ \phi_3 = \mathbf{p}_+\mathbf{e}_+ \approx 0, \\ u_{ij} = \mathbf{e}_i\mathbf{e}_j - \eta_{ij}, \end{array} \right. \tag{18}$$

and the expessions for the Lagrangian multipliers:

$$2d_{11} = \mathbf{p}\mathbf{e}_+ - \mathbf{p}_1^2/2c, \quad 2c\lambda = \mathbf{p}_+\mathbf{e}_1 - \mathbf{p}\mathbf{e}_+. \tag{19}$$

Let us introduce

$$\mathbf{e}_- \equiv \frac{\mathbf{p}_1}{2c} + \frac{1}{2c}(\mathbf{p}\mathbf{e}_+ + \mathbf{p}_+\mathbf{e}_1 - 2c\epsilon)\mathbf{e}_+, \tag{20}$$

which forms, together with $\mathbf{e}_+, \mathbf{e}_1$, the moving frame.

Thus, $\mathbf{p}$ and $\mathbf{p}_+$ are decomposed as follows

$$\mathbf{p}_+ = y_+\mathbf{e}_+ - y_1\mathbf{e}_1, \tag{21}$$

$$\mathbf{p}/c = q\mathbf{e}_- + x\mathbf{e}_+ - \pi\mathbf{e}_1, \tag{22}$$

The equations of motion for $\mathbf{x}, \mathbf{e}_\pm, \mathbf{e}_1$ coincide with (10), if we identify

$$K = \epsilon - \mathbf{p}\mathbf{e}_+/c \equiv \epsilon - q, \tag{23}$$

while the Lorentz generator is of the form

$$\mathbf{J} = \mathbf{p} \times \mathbf{x} + \mathbf{p}_+ \times \mathbf{e}_+ + \mathbf{p}_1 \times \mathbf{e}_1 = \\ = \mathbf{p} \times \mathbf{x} + c(2\epsilon - q)\mathbf{e}_+ - 2c\mathbf{e}_-. \tag{24}$$

Hence, the Poincaré invariants (Casimirs) read

$$\mathbf{p}^2/c^2 = 2qx - \pi^2, \\ \mathbf{p}\mathbf{J}/c^2 = (2\epsilon - q)q - 2x. \tag{25}$$

Therefore, the system under consideration has internal degrees of freedom, so that different classical solutions have the same mass and spin.

Let us reduce the Hamiltonian system, substituting (21) and (22) into (17)-(20). The resulting symplectic one-form reads

$$\mathcal{A} = \mathbf{p}d(\mathbf{x} + \frac{2\mathbf{e}_1}{q}) + \frac{2cd\pi}{q} + \frac{\mathbf{p}\mathbf{J}}{cq}\mathbf{e}_+ d\mathbf{e}_1, \tag{26}$$

while the Lorentz generator is of the form

$$\mathbf{J} = \mathbf{p} \times (\mathbf{x} + \frac{2\mathbf{e}_1}{q}) + \frac{\mathbf{p}\mathbf{J}}{cq}\mathbf{e}_+ \times \mathbf{e}_1. \tag{27}$$

Now it is convenient to fix the mass m and spin s of the system imposing

$$\mathbf{p}^2 = m^2, \quad \mathbf{p}\mathbf{J} = ms, \tag{28}$$

and to introduce, instead of $\mathbf{e}_+, \mathbf{e}_1$, the new variables

$$\mathbf{E}_1 = \mathbf{e}_1 + \frac{\pi\mathbf{e}_+}{q}, \;\; \mathbf{E}_2 = \frac{m\mathbf{e}_+}{cq} - \frac{\mathbf{p}}{m}, \tag{29}$$

which obey the conditions

$$\mathbf{p}\mathbf{E}_a = 0, \;\; \mathbf{E}_a\mathbf{E}_b = -\delta_{ab}, \;\; a = 1, 2. \tag{30}$$

Then, introducing the complex coordinate

$$\mathbf{z} = \mathbf{E}_1 + i\mathbf{E}_2, \tag{31}$$

one can represent the constraints in the conventional form

$$\mathbf{z}^2 = 0, \quad \mathbf{z}\bar{\mathbf{z}} + 1 = 0, \quad \mathbf{p}\mathbf{z} = 0. \tag{32}$$

In these terms, the symplectic structure is of the form

$$\omega_{red} = d\mathbf{p} \wedge d\mathbf{X} + isd\mathbf{z} \wedge d\bar{\mathbf{z}} + \frac{2cd\pi \wedge dq}{q^2}, \tag{33}$$

while the Lorentz generator reads

$$\mathbf{J} = \mathbf{p} \times \mathbf{X} + is\mathbf{z} \times \bar{\mathbf{z}}, \tag{34}$$

where we introduced the “effective” coordinate

$$\mathbf{X} = \mathbf{x} + \left(\frac{2}{q} + \frac{s}{m}\right)\mathbf{e}_1 + \frac{s\pi}{mq}\mathbf{e}_+. \qquad (35)$$

One may resolve the constraints (32) by noticing that the first two of them imply that $\mathbf{z}$ may be written in terms of a single complex parameter ω as

$$\mathbf{z} = \frac{\alpha}{\imath(\omega - \bar{\omega})}\left(1+\omega^2, 1-\omega^2, 2\omega\right). \qquad (36)$$

From the remaining constraint $\mathbf{pz} = 0$ it follows that

$$\omega = \frac{ip_2 \pm m}{p_0 + p_1}. \qquad (37)$$

So, one can finally write the symplectic structure and Lorentz generator $\mathbf{J}$ solely in terms of $\mathbf{X}$, $\mathbf{p}$ and q, π. It is easy to see that the reduced symplectic structure reads

$$d\mathbf{p} \wedge d\mathbf{X} \pm s\frac{(\mathbf{p} \times d\mathbf{p}) \wedge d\mathbf{p}}{2m^3} + \frac{2cd\pi \wedge dq}{q^2}. \qquad (38)$$

The first two terms in (38) define the symplectic structure of the standard “minimal covariant model” for anyons [10], from which follows that the spin s is not quantized.

To analyze the ”nonrelativistic” part of the system, one can reduce it by $\mathbf{p}$, and get the one- dimensional nonrelativistic mechanics with a cubic potential

$$\pi \wedge dq, \quad \pi^2 + q^3 - 2\epsilon q^2 + \frac{ms}{c^2}q + \frac{m^2}{c^2} = 0 \qquad (39)$$

Thus, the spectrum of the system under consideration contains both massive and tachionic branches, which have no upper/lower bounds, respectively. Nevertheless, this potential has a local minimum, where the so-called “semidiscrete” (“semistationary”) or resonance-like levels can exist, which are responsible for numerous interesting phenomena [11]. Notice that the local minimum of this system (“ground state”) corresponds to the point $q = q_0$ defined by the equation

$$3q_0^2 - 4\epsilon q_0 + \frac{ms}{c^2} = 0,$$

where the mass and spin are related by the expression

$$\left(m^2 + \frac{3\epsilon ms}{4c^2} + \frac{8\epsilon^3}{27}\right)^2 = \frac{4}{3}\left(\frac{ms}{c^2} - \frac{4\epsilon^2}{3}\right)^3.$$

For example, in the simplest case $\epsilon = 0$, the ground state is tachionic, while the “mass” and spin are related as follows

$$m = \frac{4s^3}{9c^6}.$$

Notice, that four-dimensional particle systems, defined by the Lagrangians linear on torsion (third curvature), are also related with one-dimensional non-relativistic mechanics. For example, the symplest four- dimensional system of this sort, formulated on non-isotropic curves, is connected by one-dimensional conformal mechanics [12].

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