Finite Temperature Tunneling and Phase Transitions in $SU(2)$-Gauge Theory

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Abstract

A pure Yang-Mills theory extended by addition of a quartic term is considered in order to study the transition from the quantum tunneling regime to that of classical, i.e. thermal, behaviour. The periodic field configurations are found, which interpolate between the vacuum and sphaleron field configurations. It is shown by explicit calculation that only smooth second order transitions occur for all permissible values of the parameter Λ introduced with the quartic term. The theory is one of the rare cases which can be handled analytically.

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1. Introduction.

One of the amazing phenomena of quantum physics is the barrier penetration due to tunneling processes. The occurrence of such processes in different areas of physics (solid state physics, high energy multiparticle scattering with baryon number violation, low-temperature physics, nuclear reactions, the formation of the Universe etc.) does not leave any doubt that they do actually take place. The theory of tunneling has been studied in many ways. It has become evident that such processes are due to classical configurations, which are solutions of the classical equation of motion with Euclidean time, namely stable vacuum configurations, now called instantons, which are responsible for transitions between topologically distinct vacua (relevant in explaining the high energy multiparticle collisions accompanied by baryon number violation) [1-4] and unstable periodic configurations called periodic instantons and periodic bounces, which determine the decay of metastable physical systems [5-7]. On the basis of the latter a theory of barrier penetration at finite energies has been developed [8-12]. In particular it was shown, that the transition from temperature assisted tunneling to thermal activation can be considered as a phase transition which takes place as the temperature (or energy) of the system increases. Dissipative forces do not affect the general features of transitions. At zero temperature the barrier penetration is determined by tunneling with a rate controlled by vacuum instantons and is proportional to $\exp(-S)$ where S is the action. As the temperature increases the tunneling process (temperature assisted tunneling) begins to be suppressed and at sufficiently high energies (comparable with the height of the potential barrier) the system overrides the barrier (by thermal activation) and the penetration is governed by the Boltzman factor $\exp(-E_0/kT)$ where E_0 is the energy of the system, corresponding to a particle sitting at the top of the barrier (the sphaleron). The configurations, which interpolate between these two processes are the periodic instantons [13-15]. It was recently discovered, that depending on the shape of the potential barrier a transition of the first order (i.e. a sharp transition) is also possible [16]. In the context of the Higgs model with some effective potential this type of transition was known before [17]. In refs. [18,19] criteria for the occurrence of the transition of the first order have been derived and examined for various quantum mechanical models. Furthermore it has been shown [20-21], that the periodic instantons may have bifurcations, which qualitatively change the behaviour of the phase transitions at finite temperature.

Recently large-spin systems turned out to be of increased interest, as these exhibit the first order phase transitions [22,23]. The specific feature of these systems is a nonlinearity of the kinetic term. This could hint at the existence of sharp first order transitions in σ-models. One should keep in mind, that although the general theory of tunneling is well understood , only few examples are known, which make it possible to analyse the problem of phase transitions by explicit analytical calculations. In field theory the problem is even more complicated. In spite of these drawbacks one has succeeded to investigate the problem in some field-theoretical models numerically or by reducing the problem to a quantum mechanical one [24-29]. In the present work we consider a nonabelian globally $SU(2)$-invariant theory in four dimensions and show analytically, that the phase transitions are of second order. The model we consider here is one of the very few in more than two dimensions which can be treated analytically and therefore deserves particular attention even in spite of its idealisation.

2. The model

The model we consider is described by the Euclidean action

$$S = \int d^4x \left\{ \frac{1}{4} F_{\mu\nu a} F_{\mu\nu a} - \frac{Ag^2}{12} (A_{\mu a} A_{\mu a})^2 \right\} \tag{1}$$

in which as usual

$$F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + g\varepsilon_{abc} A_{\mu b} A_{\nu c}. \tag{2}$$

Although the additonal quartic term breaks the local gauge invariance the model is still globally $SU(2)$-invariant and is of interest from a field-theoretical point of view. The model admits pseudoparticle-antipseudoparticle classical field configurations[30]. We present below periodic field configurations, which are responsible for quantum-classical phase transitions. In what follows we shall work in the framework of a field-theoretical approach [31], which is constructed on the sphere S^3 embedded in a 4-dimensional Euclidean space. This approach is especially convenient for conformally invariant theories, as the system can be considered to evolve along the radius of S^3, in which case the operator of scale transformations becomes an evolution operator. Thus the radius r, namely the parameter $\tau = \ln r$, is a proper time of the physical system and the operator of scale transformations is considered as the “scaled energy” of the system.

We shall illustrate this by considering the simplest example of the scalar field $\Phi(x)$. The general conformal group contains a dilatation operator D defined in terms of the field $\Phi(x)$ as

$$D = \int d^3 x x_\nu T_{4\nu}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. The scaling transformation for the field is

$$i[D, \Phi(x)] = x_\mu \partial_\mu \Phi(x) + \Phi(x).$$

We introduce spherical coordinates in 4-dimensional Euclidean space by setting

$$x_\mu = r n_\mu \tag{3}$$

with a unit vector

$$n_\mu = (\sin\psi \sin\theta \cos\phi,\ \sin\psi \sin\theta \sin\phi,\ \sin\psi \cos\theta), \tag{4}$$

and define the field $\chi(r, n_\mu) = r\Phi(x)$. In terms of the new coordinates and the field $\chi(r, n_\mu)$ the transformation law reads:

$$i[D, \chi(r, n_\mu)] = \frac{\partial \chi(r, n_\mu)}{\partial \ln r},$$

which makes the role of D as the evolution operator evident. One can also find the following integral representation for D:

$$D = \int d\Omega \left\{ \frac{\partial \mathcal{L}}{\partial \partial\chi/\partial \ln r} \frac{\partial \chi}{\partial \ln r} - \mathcal{L} \right\},$$

which is the Legendre-transform of the Lagrangean $\mathcal{L}$ integrated over the angles. The action of the system is then

$$S = \int d\ln r d\Omega \mathcal{L}.$$

Thus we conclude, that in field theory constructed on S^3 the energy is replaced by the eigenvalues of the operator of the scaling transformations. The temperature is defined as the inverse of the period of periodic field configurations expressed in terms of the proper time $\ln r$. The period is determined as the derivative of the action with respect to the "scaled energy".

We now return to our model. We look for periodic field configurations of period $P(E)$, where E is a "scaled energy" of the system, which interpolate between the vacuum and the unstable field configuration named sphaleron as the proper time τ varies from $-P(E)/2$ to $P(E)/2$. We make the Ansatz

$$A_{\mu a} = \frac{1}{g}\eta_{a\mu\sigma} n_\sigma \frac{\phi(r)}{r} \tag{5}$$

with the spherically symmetric function $\phi(r)$ to be determined. Performing the integration over the angle variables we obtain (with $\tau = \ln r$):

$$S = \frac{6\pi^2}{g^2}\int_{-P(E)/2}^{P(E)/2} d\tau \left\{ \frac{1}{2}\left(\frac{\partial\phi(\tau)}{\partial\tau}\right)^2 + V(\phi(\tau)) \right\}, \tag{6}$$

in which $V(\phi(\tau))$ is an effective potential in terms of the function $\phi(\tau)$:

$$V(\phi(\tau)) = \frac{(2-\Lambda)}{4}\phi^4(\tau) - 2\phi^3(\tau) + 2\phi^2(\tau). \tag{7}$$

The shape of the potential depends on the parameter Λ. The potential $V(\phi(\tau))$ is quartic except for $\Lambda = 2$ in which case it becomes cubic. For nonzero values of Λ the potential is asymmetric, indeed for $0 < \Lambda < 2$ there are two unequal minima at

$$\phi_{1min} = 0, \qquad \phi_{2min} = \frac{3+\sqrt{1+4\Lambda}}{2-\Lambda} > 0 \tag{8}$$

and one maximum at

$$\phi_{max} = \frac{3-\sqrt{1+4\Lambda}}{2-\Lambda} > 0. \tag{9}$$

For $\Lambda > 2$ the shape of the potential is changed and there are two maxima at

$$\phi_{lmax} = \frac{-3-\sqrt{1+4\Lambda}}{\Lambda-2} < 0, \ \phi_{smax} = \frac{-3+\sqrt{1+4\Lambda}}{\Lambda-2} > 0, \tag{10}$$

with the small and large barriers V_{smax} and V_{lmax} and one minimum at

$$\phi_{min} = 0. \tag{11}$$

In the case of $\Lambda = 2$ the potential is cubic with the minimum at $\phi_c = 0$ and maximum at $\phi_c = 2/3$. The case of $\Lambda = 0$ is just the gauge theory with local $SU(2)$-symmetry and the effective potential is symmetric in this case with two minima at $\phi = 0, 2$ and one maximum at $\phi = 1$. All these cases are shown in Fig. 1. The points of intersection of the line $E = const$ with the potential determine the turning points of the periodic motion.

3. Quantum-classical transitions.

1. The case $0 < \Lambda < 2$.

The equation of motion for nonzero constant of integration E reads:

$$\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 = V(\phi(\tau)) - E. \tag{12}$$

The solution, which satisfies the periodicity condition $\phi(\tau + P(E)) = \phi(\tau)$ is found to be:

$$\phi(\tau) = \frac{c(b-d) - d(b-c)\mathrm{sn}^2(B(\Lambda, E)\tau + K(k), k)}{(b-d) - (b-c)\mathrm{sn}^2(B(\Lambda, E)\tau + K(k), k)}, \tag{13}$$

in which

$$B(\Lambda, E) = \sqrt{\frac{(2-\Lambda)}{8}}\sqrt{(b-d)(a-c)}, \tag{14}$$

and k is the modulus of the Jacobian elliptic function

$$k = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}. \tag{15}$$

The function $K(k)$ is the complete elliptic integral of the first kind. The quantities $a,\ b,\ c,\ d$ with $a > b > c > d$ are turning points of the motion (see Appendix). The solution corresponds to the periodic motion from the point c via the maximum of the potential to b and back. The period of motion is

$$P(E) = \frac{2K(k)}{B(\Lambda, E)}, \tag{16}$$

so that

$$\phi(-P(E)/2) = \phi(P(E)/2) = c \quad \text{and} \quad \phi(0) = b.$$

The period $P(E)$ is a monotonous function of the energy, as shown in Fig. 2. This indicates that the quantum-classical phase transition is of second order. Substituting the solution into the action and integrating over the period gives the following expression:

$$S(\Lambda, E) = \frac{6\pi^2}{g^2}\Big\{ EP(E) + D(\Lambda, E)K(k) + G(\Lambda, E)E(k) + H(\Lambda, E)\Pi(\alpha^2, k)\Big\}, \tag{17}$$

with

$$\alpha^2 = \frac{b-c}{b-d} > 0$$

The functions $D(E)$, $G(E)$, $H(E)$ are defined in the Appendix. The functions $K(k)$, $E(k)$ and $\Pi(\alpha^2, k)$ are complete elliptic integrals of the first, second and third kind respectively. In semiclassical approximation the solution describes the quantum tunneling process. As the solution responsible for the thermal activation we choose the constant solution of the equation of motion, namely $\phi(\tau) = \phi_{max}$ which corresponds to the "particle" with the maximal energy $E = V_{max} = V(\phi_{max})$ sitting at the top of the potential barrier. The action of this configuration is

$$S_0(T) = \frac{6\pi^2}{g^2}\frac{V_{max}}{T}, \tag{18}$$

where T is the temperature. In Fig. 3 we display the action-versus-temperature plot (with $P(E) = 1/T(E)$ in (17)). One can see, that the transition is of the second order with the transition temperature $T_{cr} = 0.24$.

The case of $\Lambda = 0$ (theory with local symmetry) is included in our formulae for $0 < \Lambda < 2$ by setting $\Lambda = 0$. Nevertheless we give the exact expressions for the solution of the problem. The turning points for the periodic motion with finite"energy" as the solutions of the equation

$$V(\phi) = \frac{1}{2}\phi^4 - 2\phi^3 + \phi^2 = E$$

are:

$$1-\sqrt{1+\sqrt{E}},\ 1-\sqrt{1-\sqrt{E}},\ 1+\sqrt{1-\sqrt{E}},\ 1+\sqrt{1+\sqrt{E}}.$$

The periodic field configuration, which describes the motion between the points $1-\sqrt{1-\sqrt{E}}$ and $1-\sqrt{1+\sqrt{E}}$ is

$$\phi(\tau)=1+\sqrt{1-\sqrt{E}}\mathrm{sn}\sqrt{1+\sqrt{E}}(\tau+P_0(E)/2) \tag{19}$$

and satisfies the condition $\phi(-P_0(E)/2)=\phi(P_0(E)/2)$, in which

$$P_0(E)=\frac{4}{\sqrt{1+\sqrt{E}}}K(k_0)$$

is a period of motion with modulus

$$k_0=\sqrt{\frac{1-\sqrt{E}}{1+\sqrt{E}}}.$$

The values of the actions corresponding to the periodic motion and the static field configuration $\phi=1/2$ (at the top of the barrier) are then

$$S_0(E)=\frac{6\pi^2}{g^2}\left\{\frac{E}{T(E)}+\frac{4}{3}(1+\sqrt{E})\sqrt{1-\sqrt{E}}\left[(1+k_0^2)E(k_0)-(1-k_0^2)K(k_0)\right]\right\},$$

$$S_{0st}=\frac{3\pi^2}{g^2T}.$$

with temperarute $T_0(E)=P_0^{-1}(E)$. The period is a decreasing function of the energy and the phase transition is of second order. Finally we mention, that although we have restricted ourselves to positive values of the parameter Λ, some negative values may also be allowed, in particular one sees from the expressions of the extrema of the potential $V(\phi(\tau))$ that the maximum exists for $-1/4<\Lambda$. For the values of $\Lambda\leq -1/4$ the potential $V(\phi(\tau))$ does not have a maximum any more.

2. The case $\Lambda>2$.

In the case of $\Lambda>2$ the "particle" sitting in the potential well with minimal energy can move in the direction of either a bigger or smaller barrier.

The solutions of the field equations in both cases will be presented in the form, which is different from that given by (13). The turning points $a_s > b_s > a_l > b_l$ are given by

$$a_s = m_s + n_s,\ b_s = m_s - n_s,\ a_l = m_l + n_l,\ b_l = m_l - n_l,$$

where $= m_s,\ n_s,\ m_l,\ n_l$ are determined in the Appendix. We consider first the motion to the small barrier. In this case the periodic field configuration is:

$$\phi_s(\tau) = \frac{q_s b_s + p_s a_s - (q_s b_s - p_s a_s)\mathrm{cn}\Big(\sqrt{\frac{(\Lambda-2)p_s q_s}{2}}(\tau + P_s(E)/2), k_s\Big)}{q_s + p_s - (q_s - p_s)\mathrm{cn}\Big(\sqrt{\frac{(\Lambda-2)p_s q_s}{2}}(\tau + P_s(E)/2), k_s\Big)}, \quad (20)$$

with the conditions

$$\phi_s(-P_s(E)/2) = \phi_s(P_s(E)/2)) = a,$$

$$\phi_s(-P_s(E)/4) = \phi_s(P_s(E)/4) = V_{\mathrm{lmax}},\ \phi_s(0) = b.$$

The real quantities $q_s,\ p_s$ and the modulus k_s of the Jacobian elliptic functions are defined as

$$q_s = (m_l - a_s)^2 - n_l^2, \qquad p_s^2 = (m_l - b_s)^2 - n_l^2,$$

$$k_s = \frac{1}{2}\sqrt{\frac{4n_s^2 - (p_s - q_s)^2}{p_s q_s}}.$$

The period of the motion

$$P_s(E) = 4\sqrt{\frac{2}{(\Lambda-2)p_s q_s}}K(k_s) \quad (21)$$

is again a monotonically decreasing function of the energy E. The action integrated out over the period is a linear combination of complete elliptic integrals:

$$S_s(\Lambda, E) = \frac{6\pi^2}{g^2}\Big\{EP_s(E) + D_s(\Lambda, E)K(k_s) + G_s(\Lambda, E)E(k_s) +$$

$$H_s(\Lambda, E)\Pi(\frac{a_s^2}{\alpha_s^2 - 1}, k_s)\Big\}, \quad (22)$$

with

$$\alpha_s^2 = \frac{p_s - q_s}{p_s + q_s}$$

The coefficients $D_s(E)$, $G_s(E)$, $H_s(E))$ are given in the Appendix. Fig. 4 a) shows the action-versus-temperature plot, in which

$$S_{s0}(T) = \frac{6\pi^2}{g^2}\frac{V_{smax}}{T}$$

is an action corresponding to the top of the small barrier.

In the case of motion to the large barrier the solution is defined by formula (21) with appropriately replaced coefficients, in particular

$$\phi_l(\tau) = \frac{q_l b_l + p_l a_l - (q_l b_l - p_l a_l)\mathrm{cn}\left(\sqrt{\frac{(\Lambda-2)p_l q_l}{2}}(\tau + P_l(E)/2), k_l\right)}{q_l + p_l - (q_l - p_l)\mathrm{cn}\left(\sqrt{\frac{(\Lambda-2)p_l q_l}{2}}(\tau + P_l(E)/2), k_l\right)}, \qquad (23)$$

This solution corresponds to periodic motion starting at the point a_l, going to b_l and back to a_l. The quantities p_l, q_l are given by

$$q_l = (m_s - a_l)^2 - n_s^2, \qquad p_l^2 = (m_s - b_l)^2 - n_s^2.$$

The period $P_l(E)$ is a monotonically decreasing function of E and is given by (20) with k_s replaced as

$$k_s \to k_l = \frac{1}{2}\sqrt{\frac{n_l^2 - (p_l - q_l)^2}{p_l q_l}}$$

The action $S_l(\Lambda, E)$ is also defined by (22) with different coefficients $D_l(E)$, $G_l(E)$, $H_l(E)$ (see Appendix) and the parameter

$$\alpha_l^2 = \frac{p_l - q_l}{p_l + q_l}$$

instead of α_s^2. The action of the sphaleron in this case is expressed through the value V_{lmax}:

$$S_{l0} = \frac{6\pi^2}{g^2}\frac{V_{lmax}}{T}.$$

The corresponding diagrams, again showing the existence of the smooth second order phase transition, are shown in Fig. 4 b). The critical temperatures are $T_{lcr} = 0.58$ and $T_{scr} = 0.38$. One can see, that tunneling through the small potential barrier dominates that of the large barrier ($S_l(\Lambda, E) > S_s(\Lambda, E)$ for a given "energy" E). With increase of the parameter Λ this difference is diminished. In Fig. 5 we display the behaviour of $f(\Lambda) = S_l(\Lambda, E) - S_s(\Lambda, E)$ as a function of Λ at the "energy" $E = 0.1$.

3. The case $\Lambda = 2$.

In the case of $\Lambda = 2$ the finite "energy" periodic solution of the equation of motion is

$$\phi_c(\tau) = \frac{b_c(a_c - d_c) - d_c(a_c - b_c)\mathrm{sn}^2(\sqrt{\frac{a_c - d_c}{2}}(\tau + \frac{P(E)}{2}))}{(a_c - d_c) - (a_c - b_c)\mathrm{sn}^2(\sqrt{\frac{a_c - d_c}{2}}(\tau + \frac{P(E)}{2}))}, \tag{24}$$

in which

$$k_c = \sqrt{\frac{a_c - b_c}{a_c - d_c}}$$

is the modulus of the elliptic functions and the quantities $a_c > b_c > d_c$ are turning points given by the solutions of the cubic equation

$$-2\phi^3 + 2\phi^2 = E. \tag{25}$$

The solution describes a motion from the point b_c to a_c and back. The period of motion

$$P(E) = \frac{2\sqrt{2}K(k_c)}{\sqrt{a_c - d_c}}$$

is a monotonic function of E. The action integrated over the period $P(E)$ reads:

$$\begin{aligned} S_c(\Lambda, E) &= \frac{6\pi^2}{g^2}\left\{\frac{E}{T(E)} + \frac{8(b_c - d_c)^2}{5}\sqrt{\frac{a_c - d_c}{2}}\right. \\ &\times \left.\left[\left(\frac{2(2 - k_c^2)}{3(1 - k_c^2)^2} - \frac{2}{1 - k_c^2}\right)E(k_c) - \frac{(2 - k_c^2)}{3(1 - k_c^2)}K(k_c)\right]\right\}. \end{aligned} \tag{26}$$

The maximum of the potential is $8/27$ and thus the action of the static field ϕ_c is $S_{c0} = 16\pi^2/9T$. The"energy" dependence of the period and the action-versus-temperature diagrams confirm the existence of the phase transitions of the second order and we do not reproduce the corresponding diagrams.

Conclusions

Above we demonstrated that in the model considered the smooth phase transitions of the second order take place. This is not surprising, since it is known from general arguments that in theories with cubic and quartic terms the phase transitions of the second order occur. Although our considerations are explicit they are based on the Ansatz (5), which singles out a class of periodic field configurations (which are nonselfdual) reducing the problem to the one-dimensional one. In order to check the criteria for occurrence of a transition of the first order [19] one has to investigate fluctuations around a static (in our model constant) configuration. The nonselfdual character of the solutions complicates the second order fluctuation differential equations for derivation of analytical solutions considerably, so that this is not attempted here.

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Appendix

We here give some explicit formulae referred to above for the 3 possible domains of the parameter Λ.

a) $0 < \Lambda < 2$.

The turning points of the periodic motion as solutions of the equation

$$\frac{(2-\Lambda)}{4}\phi^4 - 2\phi^3\phi^2 = E$$

are:

$$\begin{aligned}
a &= \frac{1}{2}(r + \sqrt{r^2 - 2r + y}) + \frac{1}{2}\sqrt{2(r^2 - r - y) + 2r\frac{r^2 - 2r}{\sqrt{r^2 - 2r + y}}},\\
b &= \frac{1}{2}(r + \sqrt{r^2 - 2r + y}) - \frac{1}{2}\sqrt{2(r^2 - r - y) + 2r\frac{r^2 - 2r}{\sqrt{r^2 - 2r + y}}},\\
c &= \frac{1}{2}(r - \sqrt{r^2 - 2r + y}) + \frac{1}{2}\sqrt{2(r^2 - r - y) - 2r\frac{r^2 - 2r}{\sqrt{r^2 - 2r + y}}},
\end{aligned}$$

$$d = \frac{1}{2}(r - \sqrt{r^2 - 2r + y}) - \frac{1}{2}\sqrt{2(r^2 - r - y) - 2r\frac{r^2 - 2r}{\sqrt{r^2 - 2r + y}}},$$

in which

$$r = \frac{4}{2 - \Lambda},$$

$$y = \frac{1}{3}r + 2\sqrt{\frac{1}{3}\left(\frac{r^2}{3} - rE\right)} \cos\left(\frac{1}{3}\arctan\frac{2\sqrt{-\frac{2}{27}E(E - V_{max})(E - V_{2min})}}{-\frac{2}{27}r^3 + \frac{1}{2}r^2(r - \frac{4}{3}E)}\right).$$

One checks, that the quantities a, b, c, d are real for "energies" E in $0 < E < V_{max}$. The coefficients in the expression for the action $S(\Lambda, E)$ are defined as

$$D(\Lambda, E) = \frac{4B(\Lambda, E)}{\alpha^2}(b-c)^2\left(\frac{\alpha^2 k^2 + \alpha^2 - 3k^2}{3(1 - \alpha^2)} + \frac{(2\alpha^2 k^2 + 2\alpha^2 - \alpha^4 - 3k^2)^2}{8(1 - \alpha^2)^2(k^2 - \alpha^2)}\right.$$
$$\left. + \frac{k^2(2\alpha^2 k^2 + 2\alpha^2 - \alpha^4 - 3k^2)}{12(1 - \alpha^2)(k^2 - \alpha^2)}\right),$$

$$G(\Lambda, E) = \frac{4B(\Lambda, E)(b - c)^2}{k^2 - \alpha^2}\left(\frac{\alpha^2 k^2 + \alpha^2 - 3k^2}{3(1 - \alpha^2)} + \frac{(2\alpha^2 k^2 + 2\alpha^2 - \alpha^4 - 3k^2)^2}{8(1 - \alpha^2)^2(k^2 - \alpha^2)}\right),$$

$$H(\Lambda, E) = \frac{4B(\Lambda, E)(b - c)^2}{\alpha^2}\left(-k^2 + \right.$$
$$\left.\frac{(\alpha^2 k^2 + \alpha^2 - 3k^2)(2\alpha^2 k^2 + 2\alpha^2 - \alpha^4 - 3k^2)}{(1 - \alpha^2)(k^2 - \alpha^2)} + \frac{(2\alpha^2 k^2 + 2\alpha^2 - \alpha^4 - 3k^2)^3}{8(1 - \alpha^2)^2(k^2 - \alpha^2)^2}\right).$$

b) $\Lambda > 2$.

The quantities m_s, n_s, m_l, n_l which determine the turning points in this case, are given by

$$\begin{aligned}
m_s &= \frac{1}{2}(-r + \sqrt{r^2 + 2r + y}),\\
n_s &= \frac{1}{2}\sqrt{4r(r + 1) - (r - \sqrt{r^2 + 2r + y})^2 + 4\sqrt{y^2 - rE}},\\
m_l &= \frac{1}{2}(-r - \sqrt{r^2 + 2r + y}),\\
n_l &= \frac{1}{2}\sqrt{4r(r + 1) - (r + \sqrt{r^2 + 2r + y})^2 - 4\sqrt{y^2 - rE}},
\end{aligned}$$

with

$$r = \frac{4}{\Lambda - 2}$$

and

$$y = -\frac{1}{3}r + 2\sqrt{\frac{1}{3}\left(\frac{r^2}{3} + rE\right)} \cos\left(\frac{1}{3}\arctan\frac{2\sqrt{-\frac{2}{27}E(E - V_{min})(E - V_{\max})}}{\frac{2}{27}r^3 + \frac{1}{2}r^2(r + \frac{4}{3}E)}\right).$$

The coefficients which appear in the action S_s read:

$$D_s(\Lambda, E) = \frac{8p_s^2 q_s^2 (a_s - b_s)^2}{3\alpha_s^4 (p_s + q_s)^4}\sqrt{\frac{(\Lambda - 2)p_s q_s}{2}}\left[-4k_s^2 - \frac{2(6k_s^2 + \alpha_s^2 - 2\alpha_s^2 k_s^2)}{(\alpha_s^2 - 1)(\alpha_s^2 + k_s^2 - \alpha_s^2 k_s^2)}\right.$$

$$\left. - \frac{3(2\alpha_s^2 k_s^2 - \alpha_s^2 - 2k_s^2)^2}{(\alpha_s^2 - 1)^2(\alpha_s^2 + k_s^2 - \alpha_s^2 k_s^2)^2} + \frac{2k_s^2(2\alpha_s^2 k_s^2 - \alpha_s^2 - 2k_s^2)}{(\alpha_s^2 - 1)(\alpha_s^2 + k_s^2 - \alpha_s^2 k_s^2)}\right],$$

$$G_s(\Lambda, E) = \frac{8p_s^2 q_s^2 (a_s - b_s)^2}{3\alpha_s^4 (p_s + q_s)^4}\sqrt{\frac{(\Lambda - 2)p_s q_s}{2}}\left[\frac{2\alpha_s^2(6k_s^2 + \alpha_s^2 - 2\alpha_s^2 k_s^2)}{(\alpha_s^2 - 1)(\alpha_s^2 + k_s^2 - \alpha_s^2 k_s^2)} + \right.$$

$$\left. \frac{3\alpha_s^2(2\alpha_s^2 k_s^2 - \alpha_s^2 - 2k_s^2)}{(\alpha_s^2 - 1)^2(\alpha_s^2 + k_s^2 - \alpha_s^2 k_s^2)^2}\right],$$

$$H_s(\Lambda, E) = \frac{8p_s^2 q_s^2 (a_s - b_s)^2}{3\alpha_s^4 (p_s + q_s)^4}\sqrt{\frac{(\Lambda - 2)p_s q_s}{2}}\left(\frac{12k_s^2}{1 - \alpha_s^2} + \right.$$

$$\left. \frac{3(2\alpha_s^2 k_s^2 - \alpha_s^2 - 2k_s^2)^3}{(\alpha_s^2 - 1)^3(\alpha_s^2 + k_s^2 - \alpha_s^2 k_s^2)^2} + \frac{3(2\alpha_s^2 k_s^2 - \alpha_s^2 - 2k_s^2)(6k_s^2 + \alpha_s^2 - 2\alpha_s^2 k_s^2)}{(\alpha_s^2 - 1)^2(\alpha_s^2 + k_s^2 - \alpha_s^2 k_s^2)}\right).$$

The corresponding quantities in $S_l(\Lambda, E)$ are obtained by the following replacements:

$$a_s,\ b_s \to a_l,\ b_l,\ \alpha_s \to \alpha_l\ \ k_s \to k_l,\ p_s,\ q_s \to p_l,\ q_l.$$

c) $\Lambda = 2$.

The turning points in this case are the solutions of the cubic equation (25), in particular:

$$a_c = \frac{1}{3} + \cos\frac{\phi}{3},\ b_c = \frac{1}{3} + \cos\frac{\phi + 2\pi}{3} +,\ d_c = \frac{1}{3} + \cos\frac{\phi + 4\pi}{3},$$

in which the angle ϕ is defined by

$$\tan\frac{\phi}{3} = -\frac{2\sqrt{-\frac{1}{4}(-\frac{2}{27} + \frac{E}{2})^2 + \frac{1}{27}}}{-\frac{2}{27} + \frac{E}{2}}.$$

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Figure captions

FIG. 1. Different shapes of the effective potential for different values of Λ.

FIG. 2. Energy dependence of the period for $\Lambda = 1/8$, which is typical for all Λ.

FIG. 3. The action-versus-temperature diagram: The solid line represents the action of the periodic instanton and the dashed line the action of the sphaleron $\phi(\tau) = \phi_{\max}$ for $\Lambda = 1/8$.

FIG. 4. The action-versus-temperature diagrams: a) For the small barrier the solid line represents the action $S_s(\Lambda, E)$ of the periodic instanton and the dashed line the action S_{s0} of the the sphaleron $\phi(\tau) = \phi_{smax}$ at $\Lambda = 14$. b) For the large barrier the solid line represents the action $S_l(\Lambda, E)$ of the periodic instanton and the dashed line the action S_{l0} of the sphaleron $\phi(\tau) = \phi_{lmax}$ at $\Lambda = 14$.

FIG. 5. The difference of the actions $S_l(\Lambda, E)$ and $S_s(\Lambda, E)$ as a function of the parameter Λ at the "energy" $E = 0.1$.

Fig. 2.

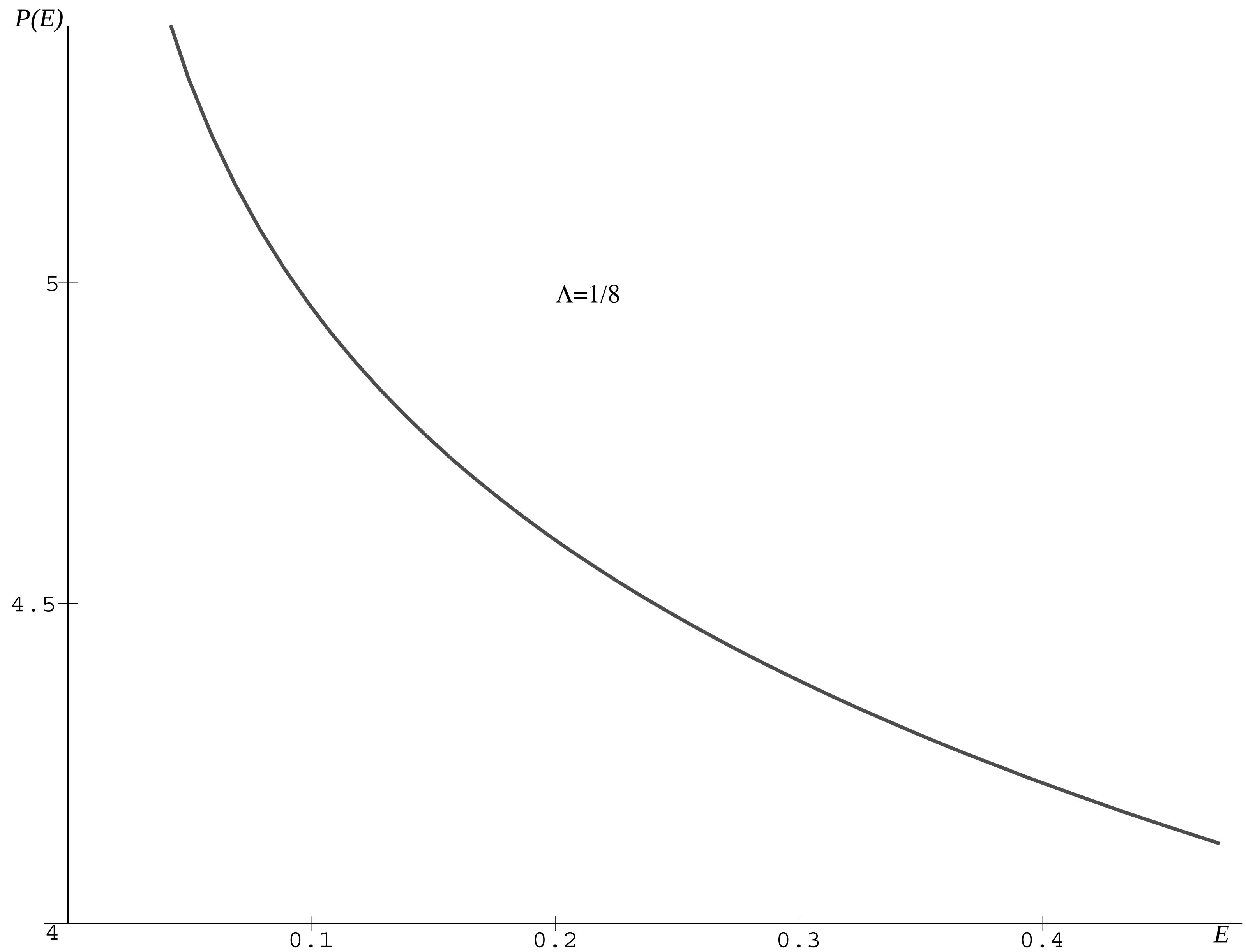

Fig. 3.

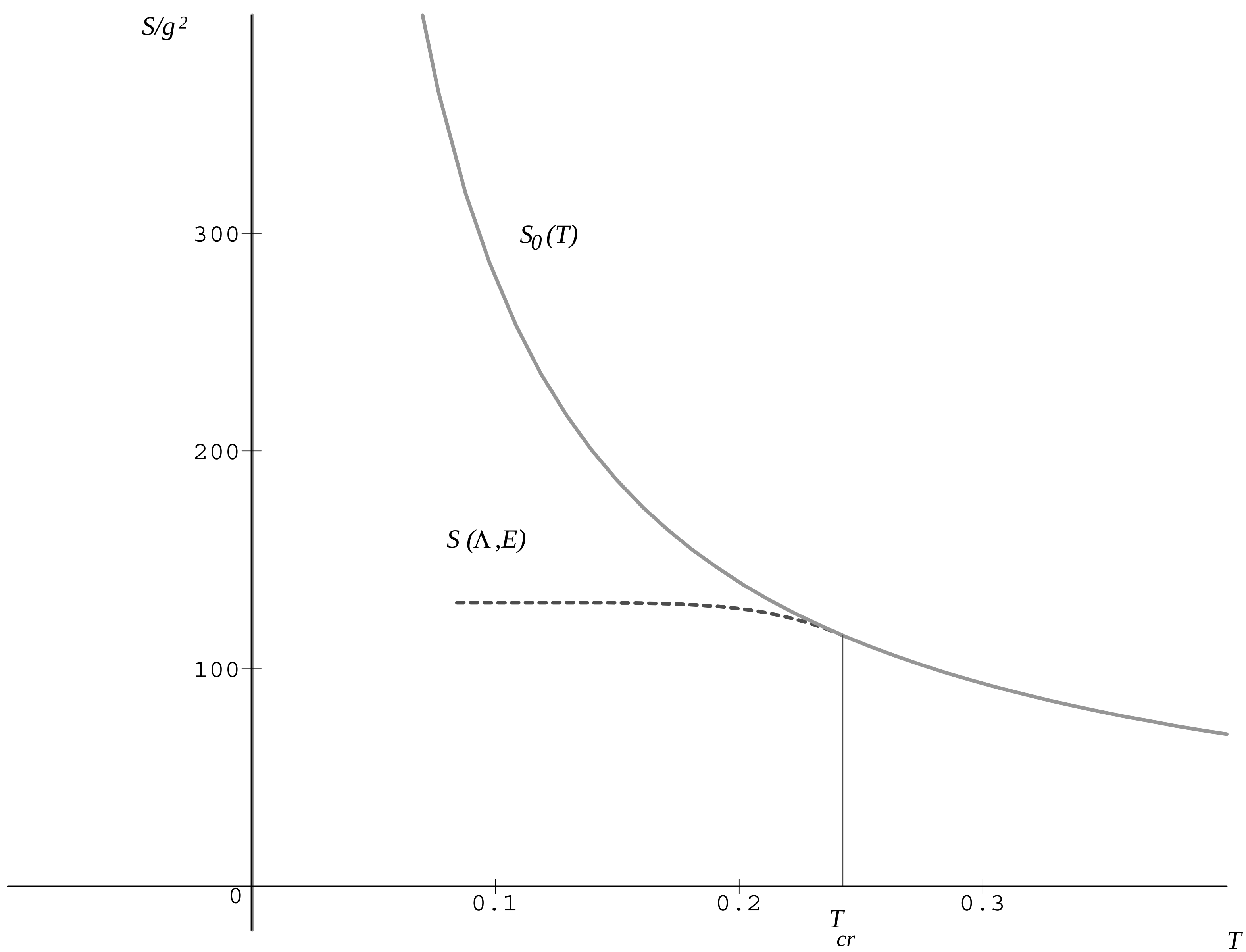

Fig. 1

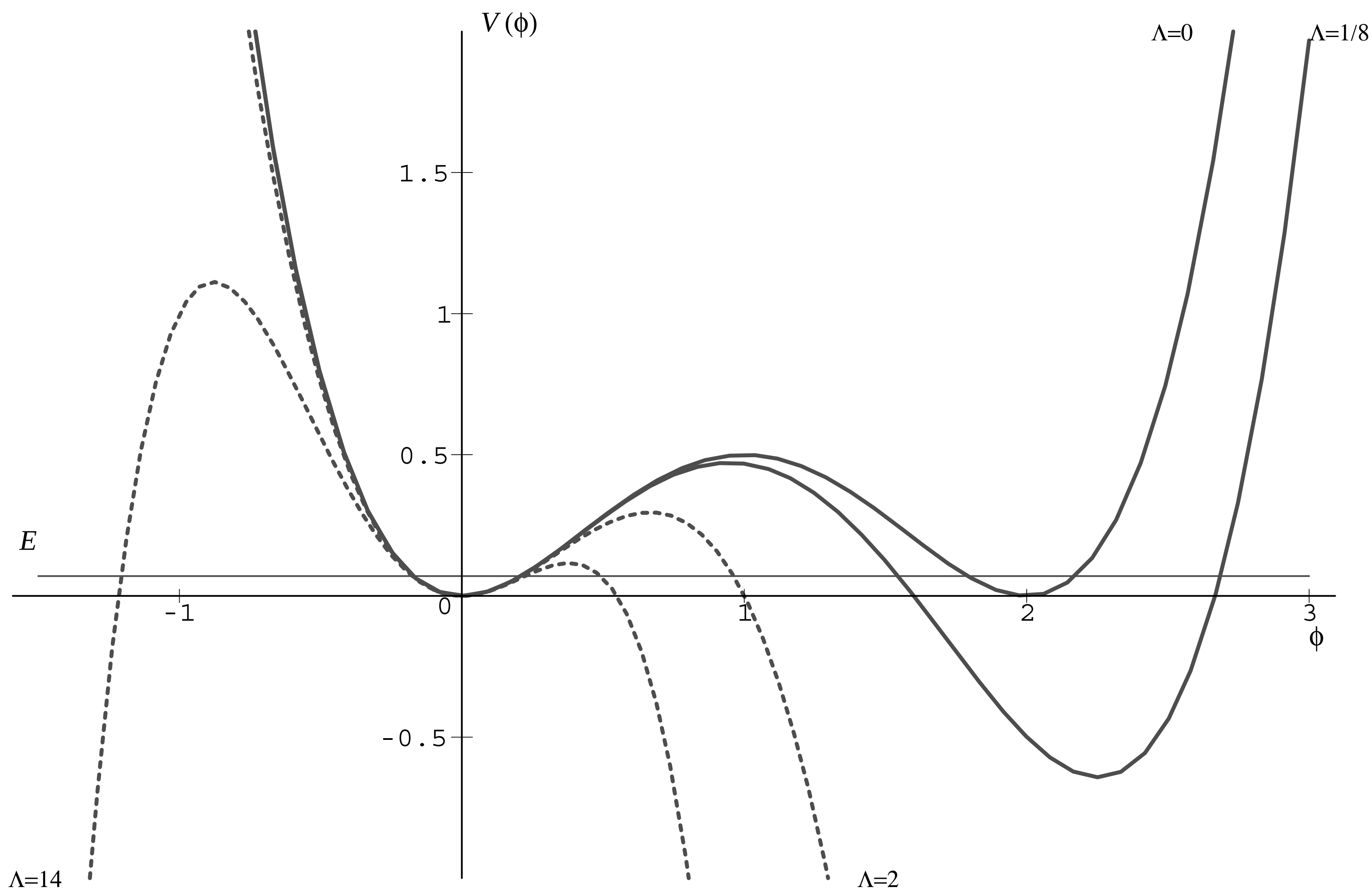

Fig. 4. a)

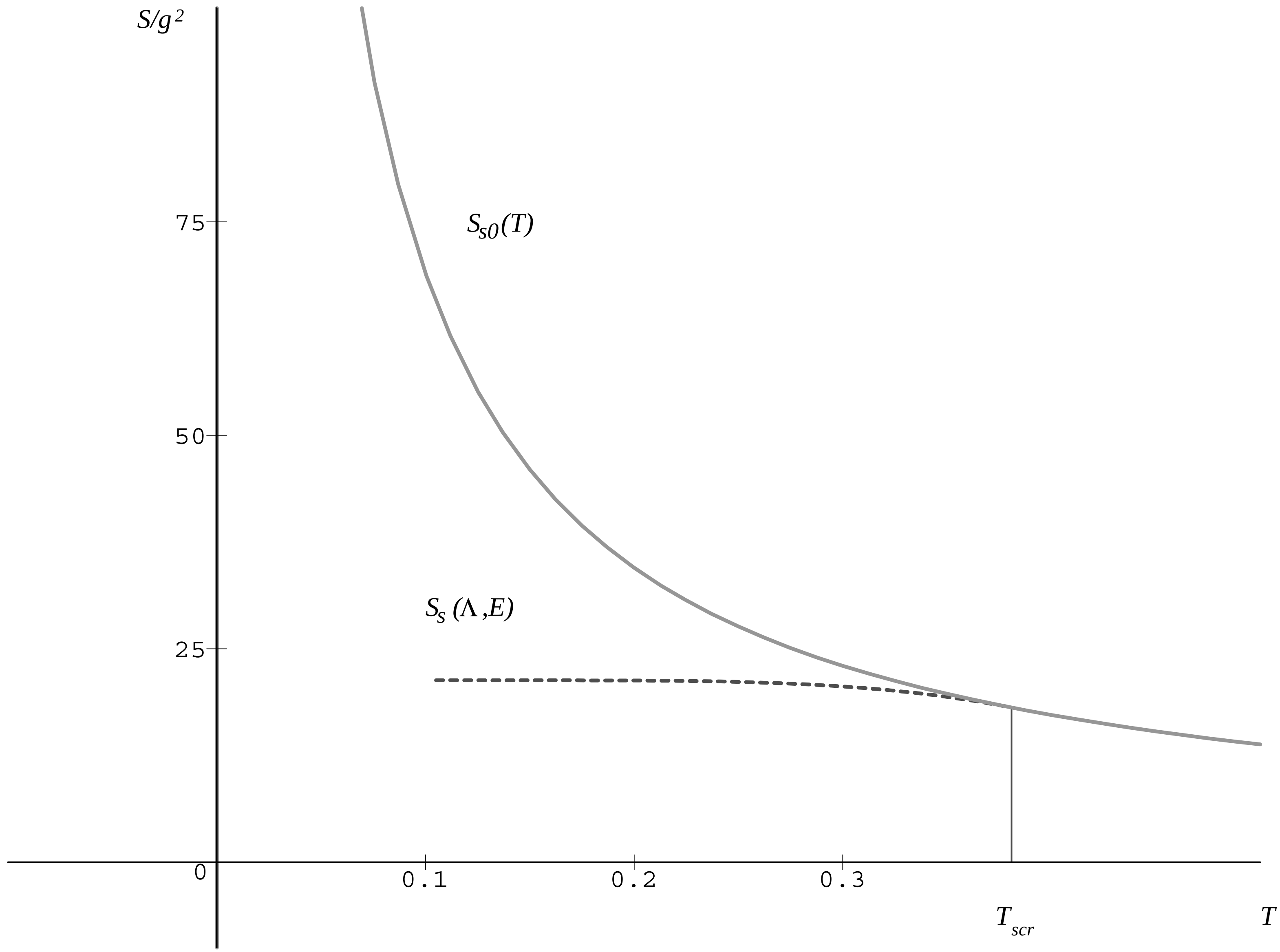

Fig. 4. b)

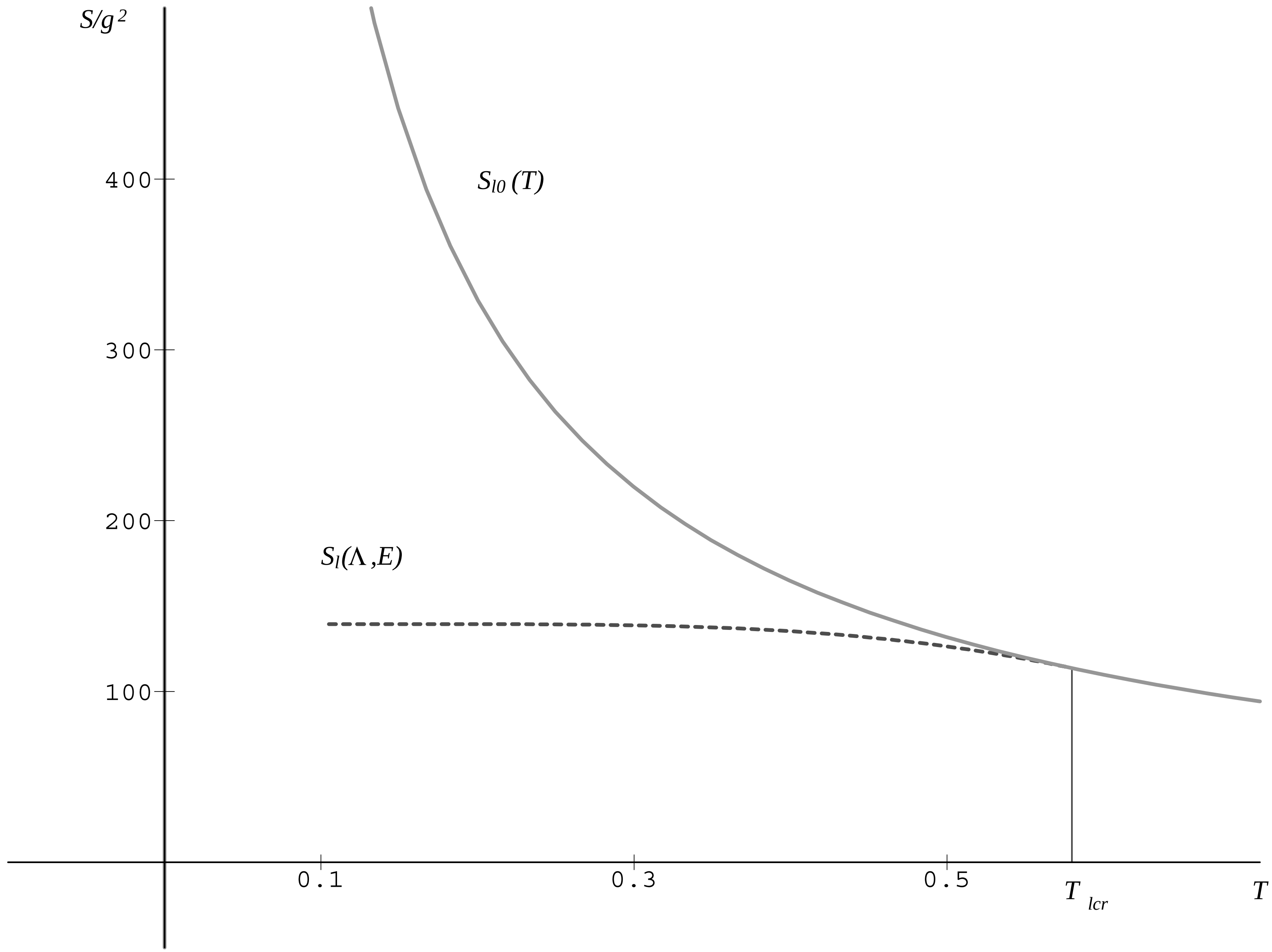

Fig. 5