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On the critical behaviour of hermitean f-matrix models in the double scaling limit with $f \geq 3$

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Abstract

An algorithm for the isolation of any singularity of f-matrix models in the double scaling limit is presented. In particular it is proved by construction that only those universality classes exist that are known from 2-matrix models.

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1 Introduction

We investigate the critical behaviour of hermitean matrix models in the double scaling limit $N \to \infty$ and $g \to g_c$, where $N \times N$ is the size of the matrices, g is a coupling parameter, and a finite number f of such random matrices is coupled to a chain. This scaling behaviour describes two-dimensional quantum gravity coupled to the matter fields of rational conformal field theories [1]-[3]. For $f = 1$ and $f = 2$ the existing analysis of double scaling behaviour is complete [4, 5]. For $f = 3$ only a small number of examples are known [6], leading to just three universality classes. We are going to present an algorithm which allows the systematic construction of all double scaling limits for $f \geq 3$. For an infinite subclass of $f = 3$ models this algorithm employs only linear algebra and all critical coefficients are rational. For all other cases of $f = 3$ and in particular for $f \geq 4$ we need to solve zeros of polynomials and the critical coefficients turn into algebraic numbers. We emphasize that matrix models are deeply related with Toda hierarchies [7] but that this relationship has not been fruitful for the elucidation of the critical behaviour.

In order to fix the notations we briefly describe hermitean matrix models. The action is

$$S(M^{(1)}, M^{(2)}, ..., M^{(f)}) = Tr\Big\{ \sum_{\alpha=1}^{f} V_\alpha(M^{(\alpha)}) - \sum_{\alpha=1}^{f-1} c_\alpha M^{(\alpha)} M^{(\alpha+1)} \Big\} \tag{1.1}$$

where each potential V_α is a polynomial of degree l_α (only $l_\alpha \geq 3$ is of interest).

$$V_\alpha(t) = \sum_{k=1}^{l_\alpha} g_k^{(\alpha)} \frac{t^k}{k} \tag{1.2}$$

Throughout the paper we assume that in relations between l_1 and l_3 w. l. o. g. $l_1 > l_3$, if not stated otherwise. Stability will be completely neglected in this work. The partition function is

$$Z = \int \sum_{\alpha=1}^{f} dM^{(\alpha)} e^{-S} \tag{1.3}$$

$$dM^{(\alpha)} = \prod_{\substack{i \leq j \\ k < l}} d(Re\, M_{ij}^{(\alpha)}) d(Im\, M_{kl}^{(\alpha)}) \tag{1.4}$$

The method of orthogonal polynomials [8, 9] makes use of biorthonomal systems of polynomials

$$\left\{\Pi_m(\lambda),\ \tilde{\Pi}_m(\mu)\right\}_{m=0}^{\infty} \qquad \lambda,\mu\in\mathbb{R} \tag{1.5}$$

satisfying

$$\begin{aligned}&\int\prod_{\alpha=1}^{f}d\lambda^{(\alpha)}\Pi_m(\lambda^{(1)})\tilde{\Pi}_n(\lambda^{(f)})\\ &\times\exp\left\{-\sum_{\alpha=1}^{f}V_\alpha(\lambda^{(\alpha)})+\sum_{\alpha=1}^{f-1}c_\alpha\lambda^{(\alpha)}\lambda^{(\alpha+1)}\right\}\\ &=\delta_{mn}\end{aligned} \tag{1.6}$$

Differentiation and multiplication matrices are introduced by

$$\begin{aligned}\Pi'_m &= \sum_n (A_1)_{mn}\Pi_n\\ \tilde{\Pi}'_m &= \sum_n (A_f)_{nm}\tilde{\Pi}_n\end{aligned} \tag{1.7}$$

$$\begin{aligned}\lambda\Pi_m(\lambda) &= \sum_n (B_1)_{mn}\Pi_n(\lambda)\\ \mu\tilde{\Pi}_m(\mu) &= \sum_n (B_f)_{nm}\tilde{\Pi}_n(\mu)\end{aligned} \tag{1.8}$$

With the help of auxiliary matrices

$$B_2, B_3, \ldots B_{f-2}$$

we can derive Dyson-Schwinger equations

$$\begin{aligned}A_1+c_1B_2&=V_1'(B_1)\\ c_{\alpha-1}B_{\alpha-1}+c_\alpha B_{\alpha+1}&=V_\alpha'(B_\alpha)\\ 2\le\alpha&\le f-1\\ A_f+c_{f-1}B_{f-1}&=V_f'(B_f)\end{aligned} \tag{1.9}$$

These matrices have support at

$$\begin{aligned}&(A_1)_{mn}=0 \qquad \text{except for} \quad -\prod_{\alpha=1}^{f}(l_\alpha-1)\le n-m\le -1\\ &(A_f)_{mn}=0 \qquad \text{except for} \quad 1\le n-m\le\prod_{\alpha=1}^{f}(l_\alpha-1)\end{aligned} \tag{1.10}$$

and

$$(B_\alpha)_{mn} = 0 \quad \text{except for} \quad -\prod_{\beta>\alpha}(l_\beta - 1) \le n - m \le \prod_{\beta<\alpha}(l_\beta - 1) \tag{1.11}$$

From (1.7), (1.8) follows

$$[B_1, A_1] = 1 \tag{1.12}$$

and the Dyson-Schwinger equations imply

$$\begin{aligned} [B_1, A_1] = c_1[B_2, B_1] = c_2[B_3, B_2] = ... \\ = c_{f-1}[B_f, B_{f-1}] = [A_f, B_f] \end{aligned} \tag{1.13}$$

One can easily scale all the $\{c_\alpha\}$ to one in (1.9), (1.13).

For convenience we present a summary of our results at the end of this work. However, we can anticipate that the universality classes found are those and only those $[p, q]$ of the two-matrix model: p and q are either coprime or coprime after a division by a factor r different from p and q [5]. Then p and q are the orders of a pair of differential operators of a generalized Korteweg–de Vries hierarchy.

2 Solving the Dyson-Schwinger equations

The solutions of the Dyson-Schwinger equations are obtained in two steps perturbatively. One observes that any singularity obtained in the double scaling limit is already determined by the solution of the Dyson-Schwinger equations at leading perturbative order. At this order we assume

$$(B_\alpha)_{n,n+m} \underset{\text{l.o.}}{\longrightarrow} \rho_m^{(\alpha)} \tag{2.1}$$

so that for each B_α we obtain a generating function

$$r^{(\alpha)}(z) = \sum_m \rho_m^{(\alpha)} z^m \tag{2.2}$$

with z a complex variable (see (2.17)). The $\rho_m^{(\alpha)}$ are submitted to the constraints (1.11) so that $r^{(\alpha)}(z)$ is rational with poles only at $z = 0$ and $z = \infty$. These generating functions are inserted into the "internal" Schwinger-Dyson equations ($f \ge 3$)

$$B_{\alpha-1} + B_{\alpha+1} = V'_\alpha(B_\alpha), \; 2 \le \alpha \le f - 1 \tag{2.3}$$

Each $r^{(\alpha)}(z)$ is required to exhibit a zero of order λ_α at $z = 1$. The ansatz that leads to a set of zeros $\{\lambda_\alpha\}$ is called a "maximal critical point" if it fixes all unknowns in the Schwinger-Dyson equations at leading order (no parameters are left over). We determine only maximal critical points. The $\{\rho_m^{(\alpha)}\}$ and the critical coupling constants (1.2)

$$\{g_k^{(\alpha)}\},\ 2 \leq \alpha \leq f-1 \tag{2.4}$$

can then be calculated. In the subsequent sections 3 (for $f = 3$) and 4 (for $f \geq 4$) this program is performed. The $\{\rho_m^{(\alpha)}\}$ are either rational or algebraic irrational. In the case $f = 3$ there is a whole subclass of solutions where this program can be executed analytically resulting in only rational solutions.

In the second step we consider the "external" Schwinger-Dyson equations

$$\begin{aligned} A_1 + B_2 &= V_1'(B_1) \\ A_f + B_{f-1} &= V_f'(B_f) \end{aligned} \tag{2.5}$$

They are (at this perturbative order) only used to fix the remaining critical coupling constants

$$\{g_k^{(\alpha)}\}, \quad \alpha = 1 \quad \text{or} \quad \alpha = f \tag{2.6}$$

This is done as follows. Denote the restriction of $r^{(\alpha)}(z)$ to its holomorphic part at zero (infinity) by

$$r^{(\alpha)}(z)_{\geq}, \quad (r^{(\alpha)}(z)_{\leq})$$

respectively. Then using (1.10) we can reformulate (2.5) as

$$\begin{aligned} r^{(2)}(z)_{\geq} &= V_1'(r^{(1)}(z))_{\geq} \\ r^{(f-1)}(z)_{\leq} &= V_f'(r^{(f)}(z))_{\leq} \end{aligned} \tag{2.7}$$

Now from (1.11)

$$\begin{aligned} r^{(1)}(z) &= \rho_1^{(1)} z + r^{(1)}(z)_{\leq} \\ r^{(f)}(z) &= \rho_{-1}^{(f)} z^{-1} + r^{(f)}(z)_{\geq} \\ & \quad (\rho_1^{(1)} \neq 0,\ \rho_{-1}^{(f)} \neq 0). \end{aligned} \tag{2.8}$$

We see that we can introduce new variables

$$\begin{aligned} \tilde{r}_1 &= r^{(1)}(z) \\ \tilde{r}_f^{-1} &= r^{(f)}(z) \end{aligned} \tag{2.9}$$

by holomorphic maps in a neighborhood of $z=\infty$ ($z=0$), respectively, consequently

$$
\begin{aligned}
r^{(1)}(z)^k_{\geq} &= \tilde{r}_1^k + O(\frac{1}{\tilde{r}_1}) \quad (\tilde{r}_1 \to \infty) \\
r^{(f)}(z)^k_{\leq} &= \tilde{r}_f^{-k} + O(\tilde{r}_f) \quad (\tilde{r}_f \to 0)
\end{aligned} \tag{2.10}
$$

and by a series expansion we get

$$
\begin{aligned}
z^n &= \sum_{k=-\infty}^{n} a_{nk}^{(1)}(\tilde{r}_1)^k, \quad n \geq 0 \\
z^n &= \sum_{k=n}^{+\infty} a_{nk}^{(f)}(\tilde{r}_f)^k, \quad n \leq 0
\end{aligned} \tag{2.11}
$$

The $\{a_{nk}^{(\alpha)}\}$ are rational functions of the $\{\rho_m^{(\alpha)}\}$ and can be calculated recursively. Then the desired critical coupling constants result from

$$
\begin{aligned}
g_{k+1}^{(1)} &= \sum_{m=0}^{l_1-1} \rho_m^{(2)} a_{mk}^{(1)} \\
g_{k+1}^{(f)} &= \sum_{m=-(l_f-1)}^{0} \rho_m^{(f-1)} a_{mk}^{(f)}
\end{aligned} \tag{2.12}
$$

Having performed the leading order solution, the perturbative expansion proceeds as in the case $f=2$. We keep the critical coupling constants fixed, multiply the whole action with

$$\frac{N}{g}$$

and tune

$$N \to \infty, \quad g \to g_c \tag{2.13}$$

as follows.

The matrix labels n, m become continuous in this "double scaling" limit

$$\frac{n}{N} = \xi, \quad 0 \leq \xi \leq 1 \tag{2.14}$$

We replace the label N by the "string coupling constant" a :

$$\frac{1}{N} = a^{2-\gamma}, \quad \gamma < 0 \tag{2.15}$$

so that $a \to 0$ for $N \to \infty$. Moreover

$$\xi = \frac{g_c}{g}(1 - a^2 x) \tag{2.16}$$

The variable z (2.2) is dual in the Fourier series sense to the discrete matrix label m (see (2.1)). Now we set

$$z = e^{i\varphi} \tag{2.17}$$

$$\varphi = a^{-\gamma} p \tag{2.18}$$

so that due to (2.16), (2.17)

$$p = i \frac{d}{dx} \tag{2.19}$$

is the quantum mechanical momentum operator corresponding to x. The perturbative expansion is in powers of

$$a^{-\gamma}$$

and γ is defined at the end from the Korteweg-de Vries hierarchy operators by

$$\gamma = \frac{-2}{p+q-1} \tag{2.20}$$

In this limit the matrices $\{B_\alpha\}_{\alpha=1}^{f}$ become differential operators $\{R_\alpha\}_{\alpha=1}^{f}$

$$B_\alpha \to a^{-\gamma\lambda_\alpha} R_\alpha \quad \alpha \in \{1, \ldots, f\} \tag{2.21}$$

where the order of R_α is λ_α. While the f-matrix model belongs to a class denoted

$$(l_1, l_2, \ldots, l_f)$$

we use square brackets

$$[\lambda_1, \lambda_2, \ldots, \lambda_f]$$

to denote the critical points.

3 The three-matrix model

For the three-matrix model we proceed as follows. We set

$$\begin{aligned} (B_1)_{n,n+k} &= r_k^{(1)}(n) \\ (B_2)_{n,n+k} &= r_k^{(2)}(n) \\ (B_3)_{n,n+k} &= r_k^{(3)}(n) \end{aligned} \tag{3.1}$$

and make the ansatz

$$\begin{aligned} r_k^{(1)}(n-\frac{1}{2}) &= \rho_k^{(1)} + a^{-2\gamma} u_k^{(1)}(x) \\ r_k^{(2)}(n-\frac{1}{2}) &= \rho_k^{(2)} + a^{-2\gamma} u_k^{(2)}(x) \\ r_k^{(3)}(n-\frac{1}{2}) &= \rho_k^{(3)} + a^{-2\gamma} u_k^{(3)}(x) \end{aligned} \tag{3.2}$$

By inserting this ansatz in the Dyson-Schwinger equations we get a perturbed system in powers of $a^{-\gamma}$.

To 0-th order one has equations between the $\rho_k^{(1)}, \rho_k^{(2)}, \rho_k^{(3)}$ and the coupling constants $g_k^{(1)}, g_k^{(2)}, g_k^{(3)}$. First we will concentrate on this system of equations.

In order to solve them we make a more precise ansatz for the $\rho_k^{(1)}$, $\rho_k^{(2)}$ and $\rho_k^{(3)}$. We introduce the generating functions

$$\begin{aligned} r^{(1)}(z) &= \sum_{m=-(l_2-1)(l_3-1)}^{1} \rho_m^{(1)} z^m \\ r^{(2)}(z) &= \sum_{m=-(l_3-1)}^{(l_1-1)} \rho_m^{(2)} z^m \\ r^{(3)}(z) &= \sum_{m=-1}^{(l_2-1)(l_1-1)} \rho_m^{(3)} z^m \end{aligned} \tag{3.3}$$

and assume they are of the form

$$\begin{aligned} r^{(1)}(z) &= \frac{(z-1)^{\lambda_1} P_{(l_2-1)(l_3-1)+1-\lambda_1}(z)}{z^{(l_2-1)(l_3-1)}} \\ r^{(2)}(z) &= \frac{(z-1)^{\lambda_2} P_{(l_1+l_3-2)-\lambda_2}(z)}{z^{(l_3-1)}} \\ r^{(3)}(z) &= \frac{(z-1)^{\lambda_3} P_{(l_1-1)(l_2-1)+1-\lambda_3}(z)}{z} \end{aligned} \tag{3.4}$$

where the P_n are polynomials of degree n.

Inserting such an ansatz in the Dyson-Schwinger equations of order zero one obtains the values for the critical coupling constants $g_k^{(1)}, g_k^{(2)}, g_k^{(3)}$ and can fix the coefficients of the polynomials P_n.

In the case of the three-matrix model, in contrast to the two-matrix model (compare [6]), more than one maximal critical point can be found for each model. For example the (4,3,3) three-matrix model has three different maximal critical points. They will be discussed in detail in Appendix B.

Among the critical points exist in general two different types which will be called type I and type II. Critical points of type I are given if $\lambda_1 = \lambda_3$. Then we have $R_1 = -R_3$ and the commutators $[R_2, R_1]$ and $[R_3, R_2]$ are obviously identical and therefore compatible with (1.13). For the critical points of type II we have $\lambda_1 \neq \lambda_3$. If $\lambda_1 < \lambda_3$ then to leading order R_1 and R_2 are identical or R_1 is a power of R_2 up to a multiplicative constant and their commutator is zero. But in higher orders the commutator of R_1 and R_2 is the same as the one between R_2 and R_3 and again $[R_2, R_1]$ and $[R_3, R_2]$ are compatible. Of course, in this case the rôle of R_1 and R_3 can be exchanged.

The appearance of the two types can be understood from examining the "internal" Dyson-Schwinger equation (2.3). We may choose the ansatz such that to leading order B_1 and B_3 compensate each other (type I) and $V_2'(B_2)$ only contributes in higher orders. Or to leading order B_1 and $V_2'(B_2)$ compensate and B_3 only is important for higher order contributions (type II).

In particular with the ansatz

$$r^{(2)}(z) = \frac{(z-1)^{(l_1+l_3-2)}}{z^{(l_3-1)}} \tag{3.5}$$

which yields maximal order for the differential operator R_2, and for which the Dyson-Schwinger equations can be solved by linear algebra only (see Appendix A), we obtain several interesting cases. If $(l_1 - 1)$ and $(l_3 - 1)$ have no common divisor then we found a critical point of type $[l_2 + 1,\, l_1 + l_3 - 2,\, l_2 + 1]$. In the other cases, in which $n(l_1 - 1) = m(l_3 - 1),\ n, m \in \mathbb{N}\backslash\{0\}$ holds, we got critical points of higher order than $[l_2 + 1,\, l_1 + l_3 - 2,\, l_2 + 1]$ except in the cases where $(l_1 - 1) = m(l_3 - 1), m \in \mathbb{N}\backslash\{0, 1\}$ and $l_2 > m$. In these cases the construction of the maximal critical point fails which is shown in detail in Appendix A.

In the following table we have listed some three-matrix models and their maximal critical points. Rational solutions are marked with the abbreviation "rat". Moreover we give as last entries in this table the universality class $[p, q]$ of these critical points which were defined first for two-matrix models.

(l_1, l_2, l_3)	$[\lambda_1, \lambda_2, \lambda_3]$ first type	$[\lambda_1, \lambda_2, \lambda_3]$ second type	$[p, q]$
4,3,3	4,5,4 rat.		5,4
		4,4,5	5,4
		3,3,6	7,3
4,4,3	5,5,5 rat.		6,5
	6,3,6		7,3
		4,4,6	6,4
		3,3,7	7,3
4,5,3	6,5,6 rat.		6,5
		4,4,7	7,4
		3,3,8	8,3
4,6,3	7,5,7, rat.		7,5
		4,4,8	9,4
		3,3,9	10,3
4,7,3	8,5,8 rat.		8,5
		6,3,9	10,3
		4,4,9	9,4
		3,3,10	10,3

Table 1: Examples of singularities and their universality classes of three-matrix models with $(l_1, l_2, l_3) = (4, n, 3)$

4 Four-matrix models and beyond

If the model involves four or more matrices we must always start from an ansatz with free parameters such as (four matrices)

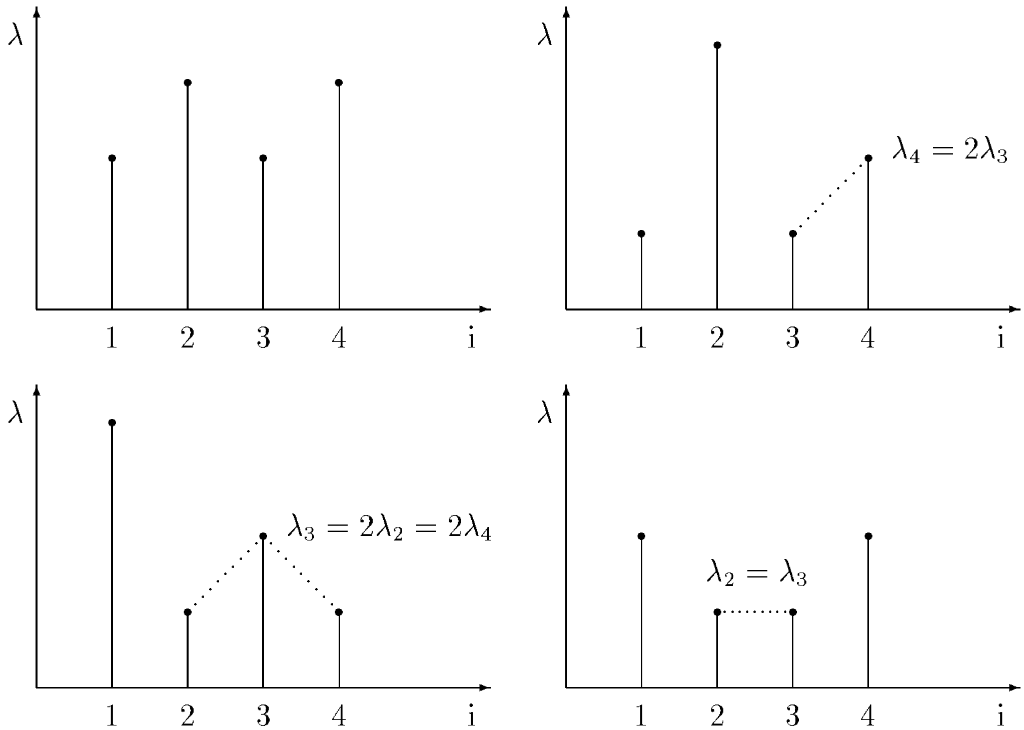
$$r^{(2)}(z) = \frac{(z-1)^{\lambda_2}}{z^{(l_3-1)(l_4-1)}} \sum_{n=0}^{m_2} a_m^{(2)} z^m \tag{4.1}$$

$$m_2 = (l_3 - 1)(l_4 - 1) + (l_1 - 1) - \lambda_2 \tag{4.2}$$

The two "internal" Schwinger-Dyson equations

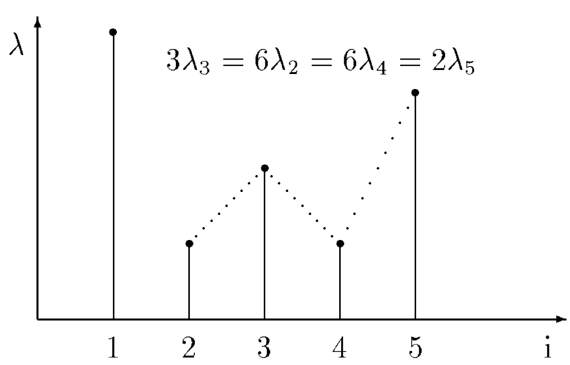
$$\begin{aligned} B_1 + B_3 &= V_2'(B_2) \\ B_2 + B_4 &= V_3'(B_3) \end{aligned} \tag{4.3}$$

admit two compensation mechanisms of type I or one of type I and the other of type II as illustrated in the four graphs:



Without doubt the universality classes are always $[p, q]$, this conclusion is drawn solely from the possible compensation types.

In the five-matrix-model case the conclusion is the same, but the arguments are more general. Namely, there are up to three types of compensations of type II possible (see the graph)



Of course, for a compensation one needs simply $\lambda_i = n\lambda_{i\pm1}$, $n \in \mathbb{N}$, the explicit cases given here (as $\lambda_4 = 2\lambda_3$) are just examples.

One can imagine that what was shown with these examples holds true for all f-matrix models with arbitrary f. Thus the universality classes are always of the two-matrix model type. The actual calculation proceeds by solving the first Schwinger-Dyson equation (4.3) with the ansatz (4.1) and the method of the three-matrix model. Then $\{\rho_m^{(1)}, \rho_m^{(3)}\}$ and $\{g_k^{(2)}\}$ result as polynomials in $\{a_m^{(2)}\}$. Next we have to adjust the $\{a_m^{(2)}\}$ and the $\{g_m^{(2)}\}$ so that

1.
$$\begin{aligned} r^{(4)}(z) &= V_3'(r^{(3)}(z)) - r^{(2)}(z) \\ \rho_m^{(4)} &= 0 \quad \text{for} - (l_3-1)(l_4-1) \leq m \leq -2 \end{aligned} \tag{4.4}$$

2. $r^{(4)}(z)$ has a maximal zero of degree λ_4 at $z=1$.

For simple four-matrix models this program can even be performed by hand, e.g. for $(l_1, l_2, l_3, l_4) = (3,3,3,3)$.

In this case we find a

$$[\lambda_1, \lambda_2, \lambda_3, \lambda_4] = [5,4,5,4] \tag{4.5}$$

with

$$\mu_2 = 2, \quad g_3^{(2)} = 1 \quad \text{(by normalization)} \tag{4.6}$$

$$\begin{aligned} \mu_3 = 1, \quad g_2^{(3)} &= \frac{3}{4\eta^3}(17\eta^2 - 24\eta - 5) \\ g_3^{(3)} &= \frac{1}{8\eta^2}(-\eta^2 - 6\eta + 15) \end{aligned} \tag{4.7}$$

where η is any of the three real solutions of

$$\eta^3 + 7\eta^2 - 15\eta - 5 = 0 \tag{4.8}$$

Examples how the different compensations, that we presented in this section actually occur, can be seen in appendix B.

5 Summary

An algorithm has been described by which all critical points of f-matrix models with polynomial potentials in the double scaling limit $N \to \infty$ can be derived. These critical points form universality classes $[p,q]$ where p and q are the orders of differential operators of a generalized KdV hierarchy and satisfy only

$$p \geq q, \quad p/q \notin \mathbb{N}.$$

We have given a new constructive argument to move this.

The algorithm treats the Dyson-Schwinger equations, derived from the orthogonal polynomial approach, perturbatively by expansion in power series of a rational power of $\frac{1}{N}$. For $f \geq 3$ these Dyson-Schwinger equations split into $(f-2)$ "internal" and (two) "external" equations. A leading order analysis of the internal equations fixes the critical point. The external equations determine only the two external critical potentials. The fact that only two differential operators result from f matrices $\{B_i\}_{i=1}^{f}$ in the double scaling limit is the result of $f-2$ compensations. We distinguish between two different compensation mechanisms (type I and type II) that act on any triplet B_i, B_{i+1}, B_{i+2}. The differential operators of order p and q arise after the compensations and at higer perturbative order. Thus we must be able to push the perturbative order to any desired value. This fact was responsible already for the derivation of all critical points of the two-matrix models in [6] (including some not seen before) and is even more decisive for the analysis of multi-matrix models. As usual the perturbative order of the commutator

$$[B_1, B_2]$$

is $p+q-1$ and gives the string susceptibility exponent γ by

$$\gamma = \frac{-2}{p+q-1}.$$

In the case of the three-matrix models more detailed results have been obtained. We determined the maximal (i.e. parameter free) critical points and found classes of "rational" and "algebraic" critical points, referring to the values of their critical coupling constants. The rational class derived from (3.5) is absent however, if

$$\frac{l_1-1}{l_3-1} \in \mathbb{N} \quad \text{or} \quad \frac{l_3-1}{l_1-1} \in \mathbb{N}$$

where l_1, l_3 denote the degree of the external critical potentials.

A A linear algorithm

We make the ansatz

$$r^{(2)}(z) = \frac{(z-1)^{l_1+l_3-2}}{z^{l_3-1}} \tag{A.1}$$

$$\rho_m^{(2)} = (-1)^{l_1-m+1}\binom{l_1+l_3-2}{m+l_3-1} \tag{A.2}$$

We have the "internal" Schwinger-Dyson equation

$$r^{(1)}(z) + r^{(3)}(z) = V_2'(r^{(2)}(z)) \tag{A.3}$$

Then (A.3) amounts to

$$\sum_{k=1}^{l_2-1} g_k^{(2)} A_{m,k-1} = \begin{cases} \rho_m^{(2)}, \ -(l_2-1)(l_3-1) \le m \le -2 \\ \rho_m^{(1)} + \rho_m^{(3)}, \ -1 \le m \le +1 \\ \rho_m^{(3)}, \ 2 \le m \le (l_1-1)(l_2-1) \end{cases} \tag{A.4}$$

with

$$A_{mk} = (-1)^{k(l_1-1)-m}\binom{k(l_1+l_3-2)}{k(l_3-1)+m} \tag{A.5}$$

In order to produce zeros of order λ_1, λ_3 at $z=1$, the coefficients $\rho_m^{(1)}, \rho_m^{(3)}$ must fulfil

$$\begin{aligned} \sum_{m=1}^{-(l_2-1)(l_3-1)} \binom{m}{n}\rho_m^{(1)} &= 0, \quad 0 \le n \le \lambda_1 - 1 \\ \sum_{m=-1}^{(l_1-1)(l_2-1)} \binom{m}{n}\rho_m^{(3)} &= 0, \quad 0 \le n \le \lambda_3 - 1 \end{aligned} \tag{A.6}$$

where we want to maximize λ_1, λ_3.

With the shorthands

$$R_n^{(1,3)} = \sum_{m=-1}^{+1}\binom{m}{n}\rho_m^{(1,3)} \tag{A.7}$$

the m-summations of (A.2) can be done analytically to yield

$$\begin{aligned} \sum_{k=0}^{l_2-1} g_{k+1}^{(2)}\Phi_{nk} &= R_n^{(3)} \\ \sum_{k=0}^{l_2-1} g_{k+1}^{(2)}\Psi_{nk} &= R_n^{(1)} \end{aligned} \tag{A.8}$$

with

$$\begin{aligned}\Phi_{n,k} &= (-1)^{K_1}\binom{K_1+K_3}{K_3}\delta_{n0} \\ &+(-1)^{K_1+1}\binom{K_1+K_3}{K_3+1}(\delta_{n0}+\delta_{n1}) \\ &+(-1)^{K_1+1+n}\binom{K_1+K_3-n-1}{K_3-1} \end{aligned} \tag{A.9}$$

$$\begin{aligned}\Psi_{n,k} &= (-1)^{K_1+n+1}\binom{K_1+K_3}{K_3-1} \\ &+(-1)^{K_1+n}\binom{K_1+K_3-n-1}{K_3-1} \end{aligned} \tag{A.10}$$

and

$$\begin{aligned} K_1 &= k(l_1-1) \\ K_3 &= k(l_3-1). \end{aligned} \tag{A.11}$$

We have ($\alpha \in \{1,3\}$)

$$R_n^{(\alpha)} = (-1)^n R_2^{(\alpha)}, \quad n \geq 2 \tag{A.12}$$

and

$$\begin{aligned}(-1)^n(\Phi_{n,k}+\Phi_{n-1,k}) &= (-1)^{n+1}(\Psi_{n,k}+\Psi_{n-1,k}) \\ &= (-1)^{K_1}\binom{K_1+K_3-n-1}{K_3-2}\end{aligned} \tag{A.13}$$

The critical coupling constants $\{g_k^{(2)}\}$ are then determined from homogeneous equations

$$\sum_{k=0}^{l_2-1}(\Phi_{n,k}+\Phi_{n-1,k})g_{k+1}^{(2)} = 0, \quad n \geq 3 \tag{A.14}$$

with (A.13) inserted. All $\rho_m^{(1)}, \rho_m^{(3)}$ can then be calculated from (A.4) except say

$$\rho_{+1}^{(3)}, \rho_0^{(3)}, \rho_{-1}^{(3)}$$

But these follow from the first equations (A.8) for $n \in \{0,1,2\}$ since $R_0^{(3)}, R_1^{(3)}, R_2^{(3)}$ determine these parameters uniquely.

Now we consider the case $(l_1 - 1) = m(l_3 - 1)$, $m \in \mathbb{N}\backslash\{0,1\}$. Then we have $K_1 = mK_3$ and the binomial coefficients in (A.13) have the form

$$(-1)^n(\Phi_{n,k} + \Phi_{n-1,k}) = (-1)^{mK_3} \binom{(m+1)K_3 - n - 1}{K_3 - 2} \tag{A.15}$$

and especially $\Phi_{2,k}$ can be expressed in terms of $(\Phi_{n,k} + \Phi_{n-1,k})$, $n \geq 3$:

$$\sum_{n=3}^{m+1} \alpha_n \binom{(m+1)K_3 - n - 1}{K_3 - 2} = \binom{(m+1)K_3 - 3}{K_3 - 1} \tag{A.16}$$

where $\alpha_3 = \alpha_4 = \ldots = \alpha_m = 1$ and $\alpha_{m+1} = m + 1$. From this we see that

$$R_2^{(3)} = \sum_{k=0}^{l_2-1} g_{k+1}^{(2)} \Phi_{2,k} \tag{A.17}$$

can be written as

$$R_2^{(3)} = \sum_{i=3}^{m+1} \alpha_i (-1)^{i+1} \sum_{k=0}^{l_2-1} g_{k+1}^{(2)} (\Phi_{i,k} + \Phi_{i-1,k}). \tag{A.18}$$

Because of (A.14) the r.h.s. of (A.18) is zero and $R_2^{(3)}$ is zero, too. This means that $\rho_{-1}^{(3)}$ is zero (see (A.7)) which is not allowed because then the "external" Dyson-Schwinger equations can not be solved.

B Maximal critical points for the (4,3,3) model

To illustrate the fact, that for the f-matrix models with $f \geq 3$ one can have more than one maximal critical point, we discuss the three maximal critical points ([4,5,4], [3,3,6], [4,4,5]) of the (4,3,3) three-matrix model.

The critical point [4,5,4]:
The functions $r^{(i)}(z)$, $i = 1, 2, 3$ are

$$r_1(z) = \frac{1}{15}\frac{1}{z^4} - \frac{2}{3}\frac{1}{z^3} + \frac{2}{z^2} - \frac{8}{3}\frac{1}{z} + \frac{5}{3} - \frac{2}{5}z \tag{B.1}$$

$$r_2(z) = -\frac{1}{z^2} + \frac{5}{z} - 10 + 10z - 5z^2 + z^3 \tag{B.2}$$

$$r_3(z) = -\frac{1}{3}\frac{1}{z} + \frac{7}{3} - \frac{32}{5}z + 9z^2 - 7z^3 + 3z^4 - \frac{2}{3}z^5 + \frac{1}{15}z^6 \tag{B.3}$$

and we have the critical potentials

$$V_1(x) = \frac{3355}{216}x - \frac{25}{48}x^2 + \frac{125}{8}x^3 - \frac{125}{32}x^4 \tag{B.4}$$

$$V_2(x) = \frac{1}{2}x^2 + \frac{1}{45}x^3 \tag{B.5}$$

$$V_3(x) = \frac{72}{5}x + \frac{27}{2}x^2 - 3x^3. \tag{B.6}$$

In this case the operators R_1 and $-R_3$ are equal and therefore the two commutators $[R_2, R_1]$, $[R_3, R_2]$ are obviously equal and this critical point leads to the universality class [5,4].

The critical point [3,3,6]:
The functions $r^{(i)}(z)$, $i = 1, 2, 3$ are

$$r_1(z) = -\frac{1}{2}\frac{\alpha}{z^4} + \frac{13}{6}\frac{\alpha}{z^3} - \frac{1}{10}\frac{\alpha}{z^2} - \frac{77}{10}\frac{\alpha}{z} + \frac{143}{15}\alpha - \frac{17}{5}\alpha z \tag{B.7}$$

$$r_2(z) = \frac{12}{5}\frac{\alpha}{z^2} - \frac{26}{5}\frac{\alpha}{z} + \frac{11}{30}\alpha + \frac{61}{10}\alpha z - \frac{9}{2}\alpha z^2 + \frac{5}{6}\alpha z^3 \tag{B.8}$$

$$\begin{aligned} r_3(z) = {} & \frac{125}{432}\frac{\alpha}{z} - \frac{18625}{10368}\alpha + \frac{8125}{1728}\alpha z - \frac{23125}{3456}\alpha z^2 + \frac{14375}{2592}\alpha z^3 \\ & - \frac{9125}{3456}\alpha z^4 + \frac{125}{192}\alpha z^5 - \frac{625}{10368}\alpha z^6 \end{aligned} \tag{B.9}$$

and we have the critical potentials

$$\begin{aligned}
V_1(x) &= \frac{1948250}{397953}\alpha x + \frac{267493}{353736}x^2 + \frac{1}{3}(\frac{6400}{14739} + \frac{6400}{14739}\alpha)x^3 \\
&+\frac{1}{4}(-\frac{1250}{14739}\alpha - \frac{625}{4913})x^4 \qquad (B.10) \\
V_2(x) &= \frac{1}{2}x^2 + \frac{1}{3}(-\frac{25}{144}\alpha - \frac{25}{144})x^3 \qquad (B.11) \\
V_3(x) &= -\frac{1393}{80}\alpha x + \frac{26568}{625}x^2 + \frac{1}{3}(\frac{4478976}{78125}\alpha + \frac{4478976}{78125})x^3, \qquad (B.12)
\end{aligned}$$

with α being a solution of

$$2\alpha^2 + 2\alpha - 1 = 0. \qquad (B.13)$$

This is one of the nonrational cases. If one proceeds to higher order one can get higher order algebraic equations for more than one variable in order to establish a certain critical point. This means that calculations become very complicated or even impossible. The operators in this case can be written as

$$\begin{aligned}
R_1 &= R_{1,3} + a^{-\gamma}R_{1,4} + a^{-2\gamma}R_{1,5} + a^{-3\gamma}R_{1,6} + a^{-4\gamma}R_{1,7} \qquad (B.14) \\
R_2 &= R_{2,3} + a^{-\gamma}R_{2,4} + a^{-2\gamma}R_{2,5} + a^{-3\gamma}R_{2,6} + a^{-4\gamma}R_{2,7} \qquad (B.15) \\
R_3 &= R_{3,6} + a^{-\gamma}R_{3,7} \qquad (B.16)
\end{aligned}$$

where the second index denotes the highest order of the differential operators. Because of $R_{1,3}^2 = R_{2,3}^2 = R_{3,6}$ both commutators vanish to leading order. But in higher orders one finds

$$\begin{aligned}
[R_2, R_1] &= -a^{-4\gamma}[(-\tfrac{25}{144}\alpha - \tfrac{169}{144})\{R_{2,3}, R_{2,4}\} + R_{1,7} - R_{2,7}, R_{2,3}] \qquad (B.17) \\
[R_3, R_2] &= a^{-\gamma}[\{R_{2,3}, R_{2,4}\} + R_{3,7}, R_{2,3}]. \qquad (B.18)
\end{aligned}$$

During the calculation of these commutators we have used the Dyson-Schwinger equations which provide us with equalities such as $R_{1,4} = R_{2,4}$, $R_{1,5} = R_{2,5}$, etc.. The two commutators (B.17, B.18) are then commutators of an operator of order three with one of order seven and they are equal to the operators of the (7,3) two-matrix model, as expected. This critical point then gives a solution that belongs to the universality class [7,3].

The critical point [4,4,5]:

The functions $r^{(i)}(z)$, $i=1,2,3$ are

$$\begin{aligned} r_1(z) &= \frac{\frac{1}{5}+\frac{3}{5}\beta-\frac{2}{5}\beta^2}{z^4}+\frac{2\beta^2-4\beta}{z^3}+\frac{-4\beta^2-2+10\beta}{z^2} \\ &\quad +\frac{4\beta^2+4-12\beta}{z}-(2\beta^2-3+7\beta)+(\frac{2}{5}\beta^2+\frac{4}{5}-\frac{8}{5}\beta)z \qquad \text{(B.19)} \\ r_2(z) &= -\frac{\beta}{z^2}+\frac{4\beta+1}{z}-6\beta-4+(4\beta+6)z+(-4-\beta)z^2+z^3 \qquad \text{(B.20)} \\ r_3(z) &= \frac{-11+4\beta^2}{z}+69+\beta-26\beta^2+(-\frac{926}{5}-\frac{28}{5}\beta+\frac{362}{5}\beta^2)z \\ &\quad +(276+13\beta-112\beta^2)z^2+(-247-16\beta+104\beta^2)z^3 \\ &\quad +(11\beta-58\beta^2+133)z^4+(-4\beta+18\beta^2-40)z^5 \\ &\quad +(\frac{3}{5}\beta-\frac{12}{5}\beta^2+\frac{26}{5})z^6 \qquad \text{(B.21)} \end{aligned}$$

and we have the critical potentials

$$\begin{aligned} V_1(x) &= \frac{11937}{19652}\beta^2x+\frac{294725}{39304}x+\frac{263361}{39304}\beta x-\frac{1}{2}(\frac{8013}{19652}\beta^2-\frac{157853}{39304}+\frac{1221}{19652}\beta)x^2 \\ &\quad -\frac{1}{3}(\frac{22959}{39304}\beta^2-\frac{154635}{19652}-\frac{161649}{39304}\beta)x^3+\frac{1}{4}(\frac{7933}{9826}-\frac{27915}{39304}\beta-\frac{4254}{4913}\beta^2)x^4 \qquad \text{(B.22)} \\ V_2(x) &= \frac{1}{2}x^2+\frac{1}{3}(\frac{3}{5}\beta-\frac{12}{5}\beta^2+\frac{26}{5})x^3 \qquad \text{(B.23)} \\ V_3(x) &= \frac{115848}{222605}\beta^2x+\frac{378696}{222605}x+\frac{2134863}{222605}\beta x+\frac{1}{2}(\frac{21225}{44521}+\frac{219258}{44521}\beta+\frac{62334}{44521}\beta^2)x^2 \\ &\quad +\frac{1}{3}(-\frac{34521}{44521}\beta-\frac{17040}{44521}\beta^2-\frac{4568}{44521})x^3 \qquad \text{(B.24)} \end{aligned}$$

with β being a solution of

$$3\beta^3-7\beta-1=0. \qquad \text{(B.25)}$$

The operators in this case are

$$\begin{aligned} R_1 &= R_{1,4}+a^{-\gamma}R_{1,5} \qquad \text{(B.26)} \\ R_2 &= R_{2,4}+a^{-\gamma}R_{2,5} \qquad \text{(B.27)} \\ R_3 &= R_{3,5} \qquad \text{(B.28)} \end{aligned}$$

The leading order of the first commutator vanishes again ($R_{1,4}=R_{2,4}$) but the second commmutator does not:

$$\begin{aligned} [R_2,R_1] &= -a^{-\gamma}[R_{1,5}-R_{2,5},R_{2,4}] \qquad \text{(B.29)} \\ [R_3,R_2] &= [R_{3,5},R_{2,4}]. \qquad \text{(B.30)} \end{aligned}$$

Then we have $-R_{1,5}+R_{2,5}=R_{3,5}$ as operators of order five and find the universality class [5,4].

C Examples for the three-matrix model

In this appendix we give the potentials that belong to rational critical points of the type that is discussed in Appendix A. This means that λ_2 assumes its maximal possible value (l_1+l_3-2). Models of the class $(5, x, 3)$ were omitted because they belong to the cases where the construction based on the ansatz (3.5) fails.

model	critical point	critical potentials
3, 2, 3	[3, 4, 3]	$V_1 = -3\,x - x^2 + \frac{1}{3}\,x^3$ $V_2 = \frac{1}{2}\,x^2$ $V_3 = V_1$
3, 3, 3	[4, 4, 4]	$V_1 = -\frac{111}{16}\,x + \frac{27}{8}\,x^2 + \frac{3}{4}\,x^3$ $V_2 = \frac{1}{2}\,x^2 - \frac{1}{18}\,x^3$ $V_3 = V_1$
3, 4, 3	[5, 4, 5]	$V_1 = -\frac{1305}{112}\,x + \frac{75}{16}\,x^2 + \frac{25}{48}\,x^3$ $V_2 = \frac{1}{3}\,x^3 - \frac{3}{140}\,x^4$ $V_3 = V_1$
3, 5, 3	[7, 4, 7]	$V_1 = -\frac{51485}{3072}\,x - \frac{6125}{512}\,x^2 + \frac{1225}{768}\,x^3$ $V_2 = \frac{1}{3}\,x^3 - \frac{1}{10}\,x^4 + \frac{1}{210}\,x^5$ $V_3 = V_1$
3, 6, 3	[8, 4, 8]	$V_1 = -\frac{957285}{45056}\,x + \frac{9261}{2048}\,x^2 + \frac{147}{1024}\,x^3$ $V_2 = \frac{1}{3}\,x^3 - \frac{3}{4}\,x^4 + \frac{9}{70}\,x^5 - \frac{1}{198}\,x^6$ $V_3 = V_1$
4, 2, 4	[3, 6, 3]	$V_1 = \frac{388}{27}\,x - \frac{11}{18}\,x^2 + \frac{4}{27}\,x^3 + \frac{1}{108}\,x^4$ $V_2 = \frac{1}{2}\,x^2$ $V_3 = V_1$
4, 3, 4	[5, 6, 5]	$V_1 = \frac{6595}{189}\,x - \frac{125}{126}\,x^2 + \frac{250}{27}\,x^3 - \frac{125}{108}\,x^4$ $V_2 = \frac{1}{2}\,x^2 + \frac{1}{105}\,x^3$ $V_3 = V_1$
4, 4, 4	[6, 6, 6]	$V_1 = \frac{311127320}{5845851}\,x + \frac{287800}{59049}\,x^2 + \frac{160000}{59049}\,x^3 + \frac{2000}{19683}\,x^4$ $V_2 = \frac{1}{2}\,x^2 + \frac{2}{15}\,x^3 + \frac{1}{528}\,x^4$ $V_3 = V_1$
4, 5, 4	[7, 6, 7]	$V_1 = \frac{508518850}{6908733}\,x - \frac{11502260}{767637}\,x^2 + \frac{266200}{59049}\,x^3 - \frac{2662}{19683}\,x^4$ $V_2 = \frac{1}{3}\,x^3 + \frac{5}{264}\,x^4 + \frac{1}{4862}\,x^5$ $V_3 = V_1$

4, 6, 4	[9, 6, 9]	$V_1 = \frac{300701859304}{3171108447}\,x + \frac{3004141294}{62178597}\,x^2 + \frac{204526784}{14348907}\,x^3 + \frac{7304528}{14348907}\,x^4$ $V_2 = \frac{1}{3}\,x^3 + \frac{1}{16}\,x^4 + \frac{3}{1430}\,x^5 + \frac{1}{54264}\,x^6$ $V_3 = V_1$
5, 2, 5	[3, 8, 3]	$V_1 = -\frac{24567}{400}\,x - \frac{103}{200}\,x^2 - \frac{3}{1000}\,x^3 - \frac{3}{2000}\,x^4 + \frac{1}{50000}\,x^5$ $V_2 = \frac{1}{2}\,x^2$ $V_3 = V_1$
5, 3, 5	[5, 8, 5]	$V_1 = -\frac{109122293}{720896}\,x - \frac{5831}{16384}\,x^2 + \frac{232211}{24576}\,x^3 + \frac{21609}{8192}\,x^4 + \frac{2401}{20480}\,x^5$ $V_2 = \frac{1}{2}\,x^2 - \frac{1}{462}\,x^3$ $V_3 = V_1$
5, 4, 5	[7, 8, 7]	$V_1 = -\frac{5926366827535}{25518145536}\,x - \frac{500520251}{327155712}\,x^2 + \frac{50463788221}{981467136}\,x^3$ $-\frac{175765205}{16777216}\,x^4 + \frac{35153041}{83886080}\,x^5$ $V_2 = \frac{1}{2}\,x^2 - \frac{1}{63}\,x^3 + \frac{1}{19448}\,x^4$ $V_3 = V_1$
5, 5, 5	[8, 8, 8]	$V_1 = -\frac{318301318564336604805}{1020323700831944704}\,x + \frac{195795357608389}{30374008717312}\,x^2$ $+\frac{45439661273853}{7593502179328}\,x^3 + \frac{3691069305}{17179869184}\,x^4 + \frac{35153041}{21474836480}\,x^5$ $V_2 = \frac{1}{2}\,x^2 - \frac{3}{14}\,x^3 + \frac{3}{1144}\,x^4 - \frac{1}{148580}\,x^5$ $V_3 = V_1$
5, 6, 5	[9, 8, 9]	$V_1 = -\frac{489711657601739357169}{1235128690480775168}\,x + \frac{2079610272730221}{60748017434624}\,x^2$ $+\frac{1697130639464183}{91122026151936}\,x^3 + \frac{93493458289}{137438953472}\,x^4 + \frac{1908029761}{343597383680}\,x^5$ $V_2 = \frac{1}{3}\,x^3 - \frac{7}{572}\,x^4 + \frac{7}{74290}\,x^5 - \frac{7}{35357670}\,x^6$ $V_3 = V_1$
4, 2, 3	[3, 5, 3]	$V_1 = 8\,x - x^2 + \frac{4}{3}\,x^3 + \frac{1}{4}\,x^4$ $V_2 = \frac{1}{2}\,x^2$ $V_3 = \frac{17}{4}\,x - \frac{1}{2}\,x^2 - \frac{1}{12}\,x^3$
4, 3, 3	[4, 5, 4]	$V_1 = \frac{3355}{216}\,x - \frac{25}{48}\,x^2 + \frac{125}{8}\,x^3 - \frac{125}{32}\,x^4$ $V_2 = \frac{1}{2}\,x^2 + \frac{1}{45}\,x^3$ $V_3 = \frac{72}{5}\,x + \frac{27}{2}\,x^2 - 3\,x^3$
4, 4, 3	[5, 5, 5]	$V_1 = \frac{223600}{9261}\,x + \frac{1940}{343}\,x^2 + \frac{8000}{1029}\,x^3 + \frac{250}{343}\,x^4$ $V_2 = \frac{1}{2}\,x^2 + \frac{4}{15}\,x^3 + \frac{1}{120}\,x^4$ $V_3 = \frac{29}{4}\,x - 25\,x^2 - \frac{25}{12}\,x^3$
4, 5, 3	[6, 5, 6]	$V_1 = \frac{4014340}{120393}\,x - \frac{154396}{4459}\,x^2 + \frac{53240}{1029}\,x^3 - \frac{2662}{343}\,x^4$ $V_2 = \frac{1}{3}\,x^3 + \frac{1}{24}\,x^4 + \frac{1}{1001}\,x^5$ $V_3 = -56\,x + \frac{245}{2}\,x^2 - \frac{49}{3}\,x^3$

4, 6, 3	[7, 5, 7]	$V_1 = \frac{43138744}{1008423}\,x + \frac{1969880}{19773}\,x^2 + \frac{2725888}{19773}\,x^3 + \frac{1362944}{59319}\,x^4$ $V_2 = \frac{1}{3}\,x^3 + \frac{9}{64}\,x^4 + \frac{3}{286}\,x^5 + \frac{1}{4896}\,x^6$ $V_3 = -\frac{927}{4}\,x - 324\,x^2 - 48\,x^3$
5, 2, 4	[3, 7, 3]	$V_1 = -\frac{8331}{256}\,x - \frac{21}{32}\,x^2 + \frac{3}{128}\,x^3 - \frac{3}{128}\,x^4 + \frac{1}{1280}\,x^5$ $V_2 = \frac{1}{2}\,x^2$ $V_3 = -\frac{626}{27}\,x - \frac{11}{24}\,x^2 - \frac{1}{27}\,x^3 + \frac{1}{864}\,x^4$
5, 3, 4	[4, 7, 4]	$V_1 = -\frac{694729}{10000}\,x - \frac{1029}{5000}\,x^2 + \frac{28126}{1875}\,x^3 + \frac{21609}{2500}\,x^4 + \frac{2401}{3125}\,x^5$ $V_2 = \frac{1}{2}\,x^2 - \frac{1}{210}\,x^3$ $V_3 = -\frac{26045}{378}\,x - \frac{1375}{126}\,x^2 - \frac{125}{9}\,x^3 - \frac{125}{108}\,x^4$
5, 4, 4	[5, 7, 5]	$V_1 = -\frac{239105243531}{2190240000}\,x - \frac{4369673}{780000}\,x^2 + \frac{10962726929}{63180000}\,x^3$ $-\frac{35153041}{405000}\,x^4 + \frac{35153041}{4050000}\,x^5$ $V_2 = \frac{1}{2}\,x^2 - \frac{2}{63}\,x^3 + \frac{1}{4576}\,x^4$ $V_3 = -\frac{12680}{77}\,x + \frac{2144}{7}\,x^2 - \frac{512}{3}\,x^3 + 16\,x^4$
5, 5, 4	[6, 7, 6]	$V_1 = -\frac{140287221613}{940970706}\,x + \frac{665951209}{8739666}\,x^2 + \frac{80349808}{336141}\,x^3$ $+\frac{175765205}{3084588}\,x^4 + \frac{35153041}{11567205}\,x^5$ $V_2 = \frac{1}{2}\,x^2 - \frac{3}{14}\,x^3 + \frac{3}{572}\,x^4 - \frac{1}{35530}\,x^5$ $V_3 = -\frac{326419}{378}\,x - \frac{14641}{14}\,x^2 - \frac{6655}{27}\,x^3 - \frac{1331}{108}\,x^4$
5, 6, 4	[7, 7, 7]	$V_1 = -\frac{2615115300936941}{14045490038528}\,x - \frac{7713352643381}{85875958116}\,x^2 + \frac{98261624661739}{2748030659712}\,x^3$ $-\frac{4581179456161}{2748030659712}\,x^4 + \frac{4581179456161}{247322759374080}\,x^5$ $V_2 = \frac{1}{2}\,x^2 - \frac{16}{7}\,x^3 + \frac{45}{286}\,x^4 - \frac{4}{1615}\,x^5 + \frac{1}{91770}\,x^6$ $V_3 = -\frac{1220198}{297}\,x + \frac{62209}{72}\,x^2 - \frac{343}{9}\,x^3 + \frac{343}{864}\,x^4$
6, 2, 3	[3, 7, 3]	$V_1 = 24\,x - x^2 + 3\,x^4 + \frac{8}{5}\,x^5 + \frac{1}{6}\,x^6$ $V_2 = \frac{1}{2}\,x^2$ $V_3 = \frac{29}{4}\,x - \frac{1}{4}\,x^2 - \frac{1}{48}\,x^3$
6, 3, 3	[4, 7, 4]	$V_1 = \frac{4217031}{100000}\,x - \frac{729}{1600}\,x^2 + \frac{40743}{400}\,x^3 - \frac{308367}{320}\,x^4$ $+\frac{177147}{160}\,x^5 - \frac{19683}{64}\,x^6$ $V_2 = \frac{1}{2}\,x^2 + \frac{1}{135}\,x^3$ $V_3 = -\frac{423}{5}\,x + \frac{27}{2}\,x^2 - 3\,x^3$
6, 4, 3	[5, 7, 5]	$V_1 = \frac{1157614349283}{18564650000}\,x - \frac{6163075107}{11881376000}\,x^2 + \frac{20496602637}{14851720}\,x^3$ $+\frac{4321777591467}{475255040}\,x^4 + \frac{80387359983}{7425860}\,x^5 + \frac{80387359983}{23762752}\,x^6$ $V_2 = \frac{1}{2}\,x^2 + \frac{4}{81}\,x^3 + \frac{1}{1680}\,x^4$ $V_3 = -\frac{44505}{28}\,x - 675\,x^2 - \frac{675}{4}\,x^3$

6, 5, 3	[6, 7, 6]	$V_1 = \frac{34173103071667}{408422300000}\,x - \frac{18059440511}{32673784000}\,x^2 + \frac{3311338930137}{1633689200}\,x^3$ $-\frac{684760877857}{118813760}\,x^4 + \frac{41615795893}{11881376}\,x^5 - \frac{41615795893}{71288256}\,x^6$ $V_2 = \frac{1}{2}\,x^2 + \frac{1}{3}\,x^3 + \frac{1}{70}\,x^4 + \frac{1}{7315}\,x^5$ $V_3 = -10695\,x + \frac{6125}{2}\,x^2 - \frac{1225}{3}\,x^3$
6, 6, 3	[7, 7, 7]	$V_1 = \frac{788322756540293184}{7468028442184375}\,x + \frac{3521389972070481}{298721137687375}\,x^2 + \frac{865581376911776}{2757425886345}\,x^3$ $+\frac{1969585683008454}{11948845507495}\,x^4 + \frac{242703321647976}{11948845507495}\,x^5 + \frac{3370879467333}{4779538202998}\,x^6$ $V_2 = \frac{1}{2}\,x^2 + \frac{32}{9}\,x^3 + \frac{3}{7}\,x^4 + \frac{8}{665}\,x^5 + \frac{1}{10530}\,x^6$ $V_3 = -\frac{180675}{4}\,x - \frac{11907}{4}\,x^2 - \frac{1323}{16}\,x^3$
6, 2, 4	[3, 8, 3]	$V_1 = \frac{190536}{3125}\,x - \frac{851}{1250}\,x^2 - \frac{16}{1875}\,x^3 + \frac{1}{50}\,x^4$ $+\frac{8}{3125}\,x^5 + \frac{1}{18750}\,x^6$ $V_2 = \frac{1}{2}\,x^2$ $V_3 = \frac{174}{5}\,x - \frac{3}{8}\,x^2 + \frac{1}{75}\,x^3 + \frac{1}{4000}\,x^4$
6, 3, 4	[4, 8, 4]	$V_1 = \frac{775352871324}{6355146875}\,x - \frac{3256929}{20336470}\,x^2 + \frac{41065704}{4621925}\,x^3 - \frac{52337097}{1848770}\,x^4$ $+\frac{708588}{84035}\,x^5 - \frac{19683}{33614}\,x^6$ $V_2 = \frac{1}{2}\,x^2 + \frac{4}{1485}\,x^3$ $V_3 = \frac{5825}{27}\,x - \frac{4325}{24}\,x^2 + \frac{125}{2}\,x^3 - \frac{125}{32}\,x^4$
6, 4, 4	[5, 8, 5]	$V_1 = \frac{45952369797}{244268750}\,x - \frac{986577}{1645600}\,x^2 + \frac{2048274}{4675}\,x^3 + \frac{859977}{880}\,x^4$ $+\frac{1944}{5}\,x^5 + \frac{81}{2}\,x^6$ $V_2 = \frac{1}{2}\,x^2 + \frac{8}{495}\,x^3 + \frac{1}{15504}\,x^4$ $V_3 = \frac{169600}{27}\,x + 7550\,x^2 + \frac{8000}{3}\,x^3 + 250\,x^4$
6, 5, 4	[6, 8, 6]	$V_1 = \frac{2294017899932829}{8980595525000}\,x - \frac{30776961716517}{1436895284000}\,x^2 + \frac{1636161080121}{780921350}\,x^3$ $-\frac{320385186351}{113588560}\,x^4 + \frac{1203384114}{1419857}\,x^5 - \frac{200564019}{2839714}\,x^6$ $V_2 = \frac{1}{2}\,x^2 + \frac{4}{45}\,x^3 + \frac{1}{816}\,x^4 + \frac{6}{1562275}\,x^5$ $V_3 = \frac{8025008200}{120393}\,x - \frac{227667550}{4459}\,x^2 + \frac{13310000}{1029}\,x^3 - \frac{332750}{343}\,x^4$
6, 6, 4	[7, 8, 7]	$V_1 = \frac{15407551189581674648}{476632759170528125}\,x + \frac{23090824363335471}{131484899081525}\,x^2 + \frac{9611096332631616}{5716734742675}\,x^3$ $+\frac{928245980327382}{1143346948535}\,x^4 + \frac{112290178445568}{1143346948535}\,x^5 + \frac{779792905872}{228669389707}\,x^6$ $V_2 = \frac{1}{2}\,x^2 + \frac{16}{27}\,x^3 + \frac{3}{136}\,x^4 + \frac{24}{120175}\,x^5 + \frac{1}{1941840}\,x^6$ $V_3 = \frac{10068806}{27}\,x + \frac{2546203}{24}\,x^2 + 10648\,x^3 + \frac{1331}{4}\,x^4$

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