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Perturbation theory for Wannier resonance states affected by ac field

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Abstract. The paper studies the effect of a weak ac field on the Wannier states, which are known to be the metastable states of a quantum particle in a periodic potential subject to a static field. Provided that the photon energy exactly matches the spacing of the Wannier-Stark ladder the system complex quasienergy spectrum is obtained. It is shown, in particular, that a weak ac field can increase the lifetime of the Wannier states by several orders of magnitude. The analytical results of a perturbation theoretical analysis are compared against the exact numerical calculation of the system spectrum.

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1 Introduction

The term Wannier states is currently used in the literature to denote the metastable states (resonances) of a Bloch particle in a homogeneous field (a model of crystal a electron subject to a static electric force):

$$\widehat{H}_W = \widehat{H}_0 + Fx \; , \tag{1}$$

$$\widehat{H}_0 = \hat{p}^2/2 + V(x) \; , \quad V(x+2\pi) = V(x) \; , \tag{2}$$

(To be concrete, we choose $V(x) = \cos x$ in what follows. Then the only parameter of the system (2) is the dimensionless Planck constant in the momentum operator.) These resonances form a set of equally spaced levels $E_{\alpha,l} = E_\alpha + 2\pi Fl$ known as the Wannier-Stark ladder of resonances. Thus, in contrast with the band spectrum $E_\alpha(k)$ of the Bloch Hamiltonian (2), the spectrum of the system (1) is discrete [1].

An interesting interplay between the band and discrete spectrum appears if there is a weak resonant ($\hbar\omega = 2\pi F$) ac field in addition to the dc field:

$$\widehat{H} = \widehat{H}_W + F_\omega x \cos(\omega t) \; , \tag{3}$$

In this case the system (quasi-)energy spectrum again has a band structure, where the band width is proportional to the amplitude F_ω of the ac field [2]. We note, however, that this fundamental result was obtained on the basis of the tight-binding model and its validity for the initial system (3) has not been proved. The aim of the present paper is to study the effect of the resonant periodic driving on the spectrum of the Wannier states beyond the tight-binding approximation.

It should be noted from the very beginning that the properties of the system (3) crucially depend on the system parameters. At least two limiting cases can be distinguished. These are the case of a small (scaled) Planck constant and a large amplitude of the driving force, which we refer to as the semiclassical region, and the opposite case of relatively large $\hbar$ ($\hbar > 1$) and a weak driving force, which we refer to as the deep quantum region. The semiclassical region was studied in some detail in the papers [3–5]. In this paper we are concerned with the deep quantum region. There are two special reasons for our interest in this case. First, as mentioned above, there is a body of theoretical results obtained for the tight-binding model, which is known to be a reasonable approximation of the initial system exactly in the deep quantum region. Second, this region is easily accessible by the experiment with cold neutral atoms in an accelerated standing wave [6], which suggest an almost perfect laboratory realization of the one-dimensional system (1). Anticipating further experimental study of the system (3), it is important to know what is missed in the tight-binding model and which are the exact conditions for its validity.

The structure of the paper is the following. Section 2 is devoted to the Wannier-Stark resonances (no ac field). Although the existence of these resonances is obvious from the physical point of view, the mathematical formalization of this intuitive result was a subtle problem for a long time. Recently we have shown that the Wannier resonances can be rigorously introduced as the complex poles of some effective scattering matrix [4]. This "scattering matrix" approach sets the basis for our further analysis and we briefly recall it in Sec. 2. The known results for the effect of an ac field obtained on the basis of the tight-binding model are presented in Sec. 3. The original part of the paper are sections 4-7. In Sec. 4 we develop a first-order perturbation theory for the Wannier states affected by a weak time-periodic force. This theory predicts the band structure $E_\alpha(k) \approx E_\alpha + \Delta E_\alpha \cos(2\pi k)$ for the system quasienergy spectrum, which is in qualitative agreement with the tight-binding model. However, the band width ΔE_α has a different dependence on the system parameters and, as shown in Sec. 5, the result of the tight-binding model can be reproduced only in the limit $F \to 0$, $F_\omega \to 0$. In the second part of the paper (Secs. 6-7) we study the influence of an ac field on the decay rate of the Wannier states – a problem which cannot be studied by using the tight-binding model in principle. The results of a numerical analysis of this problem reported in Sec. 6 indicate a drastic change in the decay rate. In particular, it is found that a weak periodic driving can increase the lifetime of the Wannier states by several orders of magnitude. An explanation for this surprising result is given in Sec. 7, where we approach the problem analytically by introducing a simple two-state model.

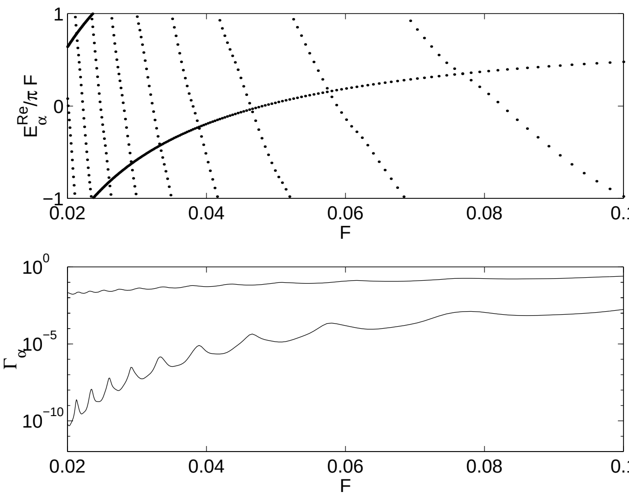

Fig. 1. Positions (a) and widths (b) of the first two Wannier resonances as a function of the static field amplitude. The scaled Planck constant $\hbar = 2$.

2 Wannier states

We recall some of the results of our previous papers [4, 5], where the Wannier resonances are defined as the poles of an effective scattering matrix constructed on the basis of the system evolution operator. Using the momentum representation for the Wannier-Bloch states

$$\psi_{\alpha,k}(x) = e^{ikx} \sum_{n=-\infty}^{\infty} c_n^{(\alpha,k)} \langle x|n\rangle \ , \quad \langle x|n\rangle = (2\pi)^{-1/2} e^{inx} \ , \tag{4}$$

the equation for the poles E_α of the scattering matrix has the form

$$U_W^{(k)} \mathbf{c}^{(\alpha,k)} = e^{-\frac{i}{\hbar} E_\alpha T_B} \mathbf{c}^{(\alpha,k)} \ , \tag{5}$$

$$\lim_{n\to\infty} |c_n^{(\alpha,k)}| = 0 \ , \quad F > 0 \ . \tag{6}$$

In Eq. (5) $U_W^{(k)}$ is the k-specific matrix of the system evolution operator $\widehat{U}_W$ over the Bloch period $T_B = \hbar/F$,

$$\left(U_W^{(k)}\right)_{n',n} = \langle n'| \exp(-ikx) \widehat{U}_W \exp(ikx) |n\rangle \ , \tag{7}$$

$$\widehat{U}_W = e^{-ix} \widehat{\exp}\left\{-\frac{i}{\hbar}\int_0^{T_B} \left[\frac{(\hat{p}-Ft)^2}{2} + \cos x\right] \mathrm{d}t\right\} \ , \tag{8}$$

and Eq. (6) is the resonance-like boundary condition (zero amplitude of the incoming wave) which ensures that the spectrum E_α is discrete (and complex). The matrices $U_W^{(k)}$ are unitarily equivalent to each other, and therefore the complex bands of the metastable Wannier-Bloch states are degenerate, i.e., the energies E_α do not depend on the quasimomentum k [7]. For the purpose of future use we also display the equation for the continuous evolution of the Wannier-Bloch states

$$\psi_{\alpha,k}(x,t) = e^{-\frac{i}{\hbar}E_\alpha t} \psi_{\alpha,k+Ft/\hbar}(x) \ , \tag{9}$$

where, by definition, $\psi_{\alpha,k+1}(x) = \psi_{\alpha,k}(x)$.

As an example, Fig. 1 shows the positions of the first two Wannier-Bloch bands (note, that $\mathrm{Re}[E_\alpha] = E_\alpha^{Re}$ is defined modulo $2\pi F$) and their decay rates $\Gamma_\alpha/\hbar = 2\mathrm{Im}\,[E_\alpha]/\hbar$ as a function of F for $\hbar = 2$. (We shall use this value of the scaled Planck constant in all our numerical illustrations.) It is seen that, in agreement with Landau-Zener theory [8], the decay rate decreases in average as an exponent of $1/F$ for $F \to 0$. The fluctuations of the decay rate are due to the band crossings and were studied in detail in Ref. [9].

To avoid a misunderstanding it should be noted that we distinguish Wannier-Bloch and Wannier-Stark states. The latter are related to the Wannier-Bloch states by the Fourier transformation

$$\Psi_{\alpha,l}(x) = \int_{-1/2}^{1/2} \mathrm{d}k e^{i2\pi lk} \psi_{\alpha,k}(x) \tag{10}$$

and (unlike the Wannier-Bloch states) are essentially localized within one l-th potential well. It is easy to prove the following relation between the expansion coefficient $c_n^{(\alpha,k)}$ of the Wannier-Bloch state and the Fourier image of the Wannier-Stark state:

$$c_n^{(\alpha,k)} = \bar{\Psi}_{\alpha,0}(n+k) \ , \quad \bar{\Psi}_{\alpha,0}(k) = \int_{-\infty}^{\infty} \mathrm{d}x e^{ikx} \Psi_{\alpha,0}(x) \ . \tag{11}$$

[Here we set $l = 0$. This does not cause a loss of generality, since the functions (10) possess the translational property $\Psi_{\alpha,l+1}(x) = \Psi_{\alpha,l}(x - 2\pi)$.] The asymptotic behavior of $\bar{\Psi}_{\alpha,0}(k)$ for large negative k coincides (up to a phase shift) with the asymptotic Fourier-image of the Airy function

$$\bar{\Psi}_{\alpha,0}(k) \sim \exp\left(i\frac{\hbar^2 k^3}{6F} - i\frac{E_\alpha k}{F}\right) \ . \tag{12}$$

Because E_α is complex, the function (12) diverges exponentially for $k \to -\infty$. The divergence of $\bar{\Psi}_{\alpha,0}(k)$ [and, according to Eq. (11), the expansion coefficients $c_n^{(\alpha,k)}$] brings about the problem of normalization of the metastable Wannier states. The common approach normalizing a metastable state is by complex scaling of the coordinate [12]. For the problem considered an appropriate scaling is

$$x \to x - i\delta \; , \quad \delta > \Gamma_\alpha/2F \; . \tag{13}$$

The complex translation of the coordinate is equivalent to the multiplication of the Fourier-image by factor $\exp(\delta k)$. Then the Wannier states $\Psi_{\alpha,l}(x-i\delta)$ are square integrable functions.

3 Tight-binding model

In this section we study the effect of a periodic perturbation by using the tight-binding model, which is known to be a reasonable approximation of the Hamiltonian (1) in the case of small tunneling rate (i.e., small resonance width $\Gamma_\alpha \ll 1$). The analysis mainly follows the paper [10].

In the notation used the tight-binding counterpart of the Hamiltonian (2) has the form

$$H_0 = \frac{\Delta_0}{2} \sum_l \left(|l\rangle\langle l+1| + |l+1\rangle\langle l| \right) \; , \tag{14}$$

In Eq. (14) $|l\rangle$ is the set of the localized states associated with the l-th cell of the periodic potential. The eigenfunction of the system (14) are Bloch-like functions

$$|\phi_k\rangle = \sum_l e^{i2\pi lk} |l\rangle \tag{15}$$

corresponding to the energies $E(k) = \Delta_0 \cos(2\pi k)$. The influence of the static force is mimicked by the term $F \sum_l |l\rangle 2\pi l \langle l|$. Then, as it is easy to show [see Eq. (19) below], the tight-binding counterpart of the Wannier-Bloch state (4) is

$$|\psi_k\rangle = \exp\left[i \frac{\Delta_0}{2\pi F} \sin(2\pi k) \right] \sum_l e^{i2\pi lk} |l\rangle \; . \tag{16}$$

The functions (16) are the eigenfunction of the system evolution operator over the Bloch period and they form the degenerate Wannier-Bloch band with the energy $E = 0$. The continuous time evolution of the function (16) is given by Eq. (9) and corresponds (since $E = 0$) to the substitution $k \to k + Ft/\hbar$ in Eq. (16). At last, the Wannier-Stark states are given by the transformation (10) and can be explicitly written in terms of Bessel functions:

$$|\Psi_l\rangle = \sum_m J_{m-l} \left(\frac{\Delta_0}{2\pi F} \right) |l\rangle \; . \tag{17}$$

Now we discuss the effect of the periodic perturbation $F_\omega \cos(\omega t) \sum_l |l\rangle 2\pi l \langle l|$. In this case the notion of energy should be substituted by the notion of quasienergy. The key point is, however, that the system already has an intrinsic time-period $T_B = \hbar/F$. Thus the notion of quasienergy can be introduced only in the case of commensurate periods [11]

$$qT_B = rT_\omega \; , \quad T_\omega = 2\pi/\omega \; , \tag{18}$$

(r, q are coprime integers). In what follows we restrict ourselves to the simplest case $T_\omega = T_B$ ($\hbar\omega = 2\pi F$). Using the general solution of the Schrödinger equation for the perturbed tight-binding model

$$|\psi(t)\rangle = \sum_l \exp\left[i2\pi lk(t) + i\frac{\Delta_0}{\hbar} \int_0^t \mathrm{d}t' \cos(2\pi k(t')) \right] |l\rangle \; ,$$
$$k(t) = k + Ft/\hbar - (F_\omega/\omega)\sin(\omega t) \; , \tag{19}$$

one obtains the following expression for the quasienergy spectrum

$$E(k) = \Delta_0 J_1 \left(\frac{2\pi F_\omega}{\hbar\omega} \right) \cos(2\pi k) \; . \tag{20}$$

Equation (20) shows that the resonant periodic driving removes the degeneracy of the Wannier-Bloch band and it gains a finite width proportional (for $2\pi F_\omega/\hbar\omega \ll 1$) to the amplitude of the driving force.

4 First order correction

As stated in Sec. 2, one obtains the spectrum of the system (1) by solving the eigenvalue problem

$$\widehat{U}_W \psi_{\alpha,k}(x) = e^{-\frac{i}{\hbar} E_\alpha T_B} \psi_{\alpha,k}(x) \; , \tag{21}$$

where $\widehat{U}_W = \widehat{U}_W(T_B, 0)$ is the evolution operator (8). We recall that the spectrum E_α depends on the type of the boundary condition – a real continuous spectrum corresponds to a hermitian boundary condition and a nonhermitian boundary condition leads to a complex discrete spectrum. Here we are interested in the case of the nonhermitian boundary condition (6). However, the type of the boundary condition will be actually irrelevant in all intermediate equations of this section. The difference appears only in the final equation, where one should substitute the proper eigenfunctions depending on the boundary condition.

Now we find the correction to the spectrum E_α due to the resonant ($\hbar\omega = 2\pi F$) time-periodic perturbation. Using the Kramers-Henneberger transformation we reduce the Hamiltonian (3) to the form

$$\widehat{H} = \frac{\widehat{p}^2}{2} + \cos[x - \epsilon\cos(\omega t)] + Fx \; , \quad \epsilon = \frac{F_\omega}{\omega^2} \; , \tag{22}$$

which we approximate for $\epsilon \ll 1$ as

$$\widehat{H} = \widehat{H}_W + \epsilon \sin x \cos(\omega t) \; . \tag{23}$$

(We would like to stress that the perturbation parameter is $\epsilon = F_\omega/\omega^2$ and not $z = 2\pi F_\omega/\hbar\omega$ as it could be expected on the basis of the tight-binding model.) Calculating the effect of the perturbation by using the interaction representation of the Schrödinger equation, the time evolution operator takes the form

$$\widehat{U} = \widehat{U}_W \, \widehat{U}_A \, , \tag{24}$$

where the operator $\widehat{U}_A$ reads

$$\widehat{U}_A = \widehat{\exp}\left\{ -\frac{i\epsilon}{\hbar}\int_0^{T_B} \widehat{U}_W^{+}(t,0)\cos(\omega t)\sin x\, \widehat{U}_W(t,0)\mathrm{d}t \right\} \tag{25}$$

with the continuous-time system evolution operator [compare with Eq. (8)]

$$\widehat{U}_W(t,0) = e^{-iFxt/\hbar}\,\widehat{\exp}\left\{ -\frac{i}{\hbar}\int_0^t \left[\frac{(\hat{p}-Ft)^2}{2} + \cos x\right]\mathrm{d}t \right\} . \tag{26}$$

According to the usual perturbation theory, the first order correction is given by the diagonal elements of the operator (24), which directly yields

$$\exp[-i\varDelta E_\alpha T_B/\hbar] = \langle \psi^*_{\alpha,k}(x)\, \widehat{U}_A\, \psi_{\alpha,k}(x)\rangle \, . \tag{27}$$

[In Eq. (27) and below the angle brackets denote an integral over x, i.e., $\langle\ldots\rangle = \lim_{L\to\infty}\frac{1}{L}\int_{-L/2}^{L/2}\ldots\mathrm{d}x$.] Expanding the operator exponent (25) in a series over ϵ and keeping only the linear term, we obtain

$$\varDelta E_\alpha(k)/F = \epsilon\,\langle \psi^*_{\alpha,k}(x)\widehat{A}\psi_{\alpha,k}(x)\rangle \, , \tag{28}$$

where $\widehat{A}$ is the hermitian operator

$$\widehat{A} = \frac{1}{\hbar}\int_0^{T_B}\cos(\omega t)\,\widehat{U}_W^{+}(t,0)\sin x\,\widehat{U}_W(t,0)\mathrm{d}t \, . \tag{29}$$

Substituting Eq. (29) into Eq. (28) and taking into account that $\widehat{U}_W(t,0)\psi_{\alpha,k}(x) = \exp(-iE_\alpha t/\hbar)\psi_{\alpha,k+Ft/\hbar}(x)$ we obtain

$$\varDelta E_\alpha(k) = \epsilon\frac{1}{T_B}\int_0^{T_B}\mathrm{d}t\cos(\omega t)\langle \sin x|\psi_{\alpha,k+\frac{Ft}{\hbar}}(x)|^2\rangle \, . \tag{30}$$

Finally, using the the fact that $|\psi_{\alpha,k}(x)|$ is periodic both in x and k and an even function of k, we come to the result

$$\varDelta E_\alpha(k) = \epsilon I_\alpha \cos(2\pi k) \, , \tag{31}$$

where

$$I_\alpha = \int_{-1/2}^{1/2}\mathrm{d}k\cos(2\pi k)\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}x\sin x|\psi_{\alpha,k}(x)|^2 \tag{32}$$

is the amplitude of the first Fourier harmonic of the Bloch oscillations. It follows from Eq. (31) that a weak periodic driving removes the degeneracy of the Wannier-Bloch bands which gain a finite width $\varDelta E_\alpha = 2\epsilon I_\alpha$. We draw attention to the fact that the width of the quasienergy Wannier-Bloch band depends on F both through the perturbation parameter $\epsilon = F_\omega/\omega^2 = F_\omega(\hbar/2\pi F)^2$ and the integral (32). We study this dependence in the next section.

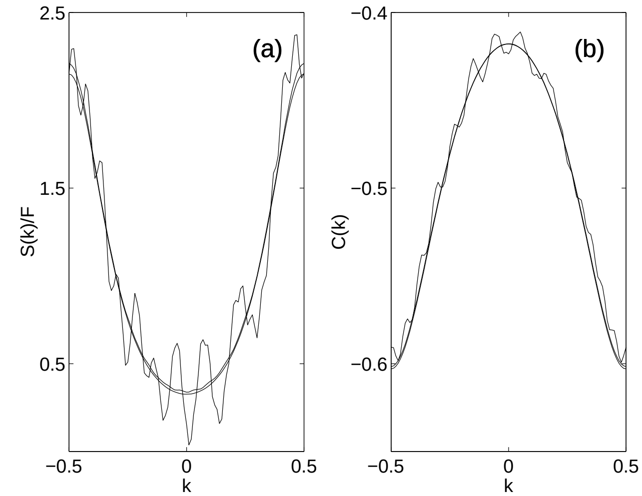

Fig. 2. (a) The integral $S(k) = (2\pi)^{-1}\int_{-\pi}^{\pi}\mathrm{d}x\sin x|\psi_{0,k}(x)|^2$ for $F = 0.04$, $F = 0.02$, and $F = 0.01$. The figure illustrates the convergence of $S(k)/F$ for $F\to 0$. (b) The integral $C(k) = \int_{-\pi}^{\pi}\mathrm{d}x\cos x|\psi_{0,k}(x)|^2$ for the same values of F, which illustrates the convergence of $C(k)$ to the potential energy of the ground Bloch band (depicted by dashed line).

5 Width of the quasienergy band

In this section we show that in the limit $F\to 0$ Eq. (31) reduces to the result of the tight-binding model. In fact, for $F \ll 1$ one can neglect the decay of the Wannier-Bloch states and, after taking into account the shift in the positions for minima of the potential wells, they can be well approximated by the Bloch functions [7]

$$|\psi_{\alpha,k}(x)| \approx |\phi_{\alpha,k}(x+F)| \, . \tag{33}$$

Expanding (33) in F we obtain an estimate for the integral over x in Eq. (32)

$$\begin{aligned} S(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}x\sin x|\psi_{\alpha,k}(x)|^2 \\ &\approx \frac{F}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}x\sin x\frac{\mathrm{d}}{\mathrm{d}x}|\phi_{\alpha,k}(x)|^2 \\ &= -\frac{F}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}x\cos x|\phi_{\alpha,k}(x)|^2 = -FC(k) \, , \end{aligned} \tag{34}$$

where $C(k)$ is the mean potential energy of the Bloch states. To illustrate the validity of the approximation (33) we compare in Fig. 2(b) the mean potential energy $C(k)$ of the ground Bloch state with that of Wannier-Bloch states for $F = 0.04$, 0.02, and 0.01. A good convergence is noticed. The left panel in Fig. 2 shows the function $S(k)$ calculated for the same values of static field amplitude. It is seen that Eq. (34) captures correctly the functional dependence of the function $S(k)$ on both F and k. Bearing in mind that the amplitude of the potential energy variation is proportional to the width $\varDelta_\alpha$ of the Bloch band,

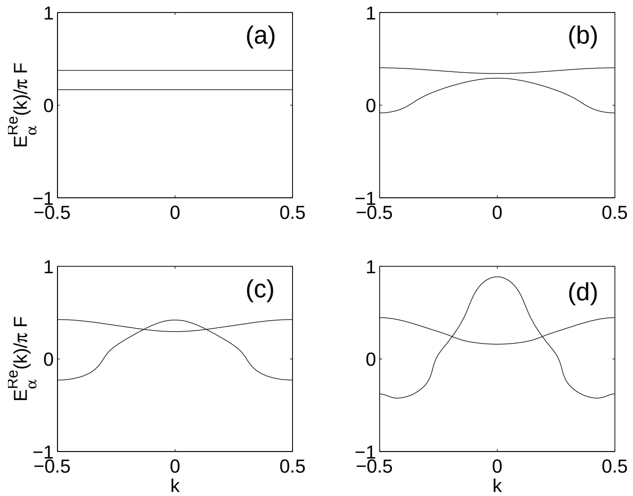

Fig. 3. The real part of the first two (quasi-)energy Wannier-Bloch bands for $\epsilon = 0$ (a), $\epsilon = 0.2$ (b), $\epsilon = 0.4$ (c), and $\epsilon = 1$ (d). The value of the static force $F = 0.08$, $\hbar = 2$.

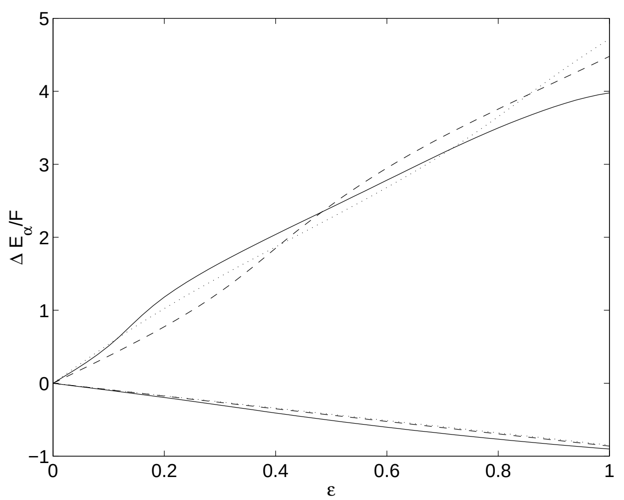

Fig. 4. The width of the Wannier-Bloch bands as a function of the perturbation parameter ϵ for $F = 0.08$ (solid line), $F = 0.04$ (dashed line), and $F = 0.02$ (dotted line).

we finally obtain

$$\Delta E_\alpha(k) \sim \epsilon \Delta_\alpha F \cos(2\pi k) \ . \tag{35}$$

(We keep the sign of Δ_α, which is negative for the ground Bloch band, positive for first excited, and so on.) The dispersion relation (35) coincides with the dispersion relation (20) in the case $F_\omega/\hbar\omega \ll 1$. We note, however, that the validity condition of Eq. (35) is $\epsilon \ll 1$ and $F \ll 1$, which is not the same but stronger than the condition $\epsilon F \sim F_\omega/\hbar\omega \ll 1$.

To check the analytical result (35) we calculated numerically the complex spectrum of the system. As an example, Fig. 3 shows the real part of the first two quasienergy Wannier-Bloch bands for $F = 0.08$ and different values of the perturbation parameter ϵ. The width of the bands as a function of ϵ are given by the solid lines in Fig. 4. (In addition, the dashed and dotted lines in Fig. 4 show the width of the bands for $F = 0.04$ and $F = 0.02$.) It is seen from the figures that, in agreement with Eq. (35), the dispersion relation for the ground band is well approximated by a cosine function and the band width grows (approximately) linear with ϵ. For the first excited band, which is quite unstable, the approximation (33) is not valid and one observes a deviation from the dependence (35).

6 Correction to the imaginary part

Because the Wannier-Bloch states are metastable states, any perturbation should also influence the imaginary part of the energy. In our numerical study of the problem we have found a dramatic change in the imaginary part of the energy (which defines the decay rate of the states) due to a weak periodic driving. Some of the numerical results are presented below.

Figure 5 shows the decay rate of the ground state as a function of the quasimomentum k for the parameters of Fig. 3. It is seen that even for the small value of $\epsilon = 0.2$ there is an essential deviation from “cosine dependence”, which could be naively expected basing on the results of the previous section. By further increasing ϵ the deviation from the cosine dependence increases and leads to a more complicated structure of $\Gamma_0(k)$ [see Fig. 5(c,d)]. By comparing Fig. 5 with Fig. 3 one concludes that this structure is related to the crossing of the ground Wannier band with the first excited band.

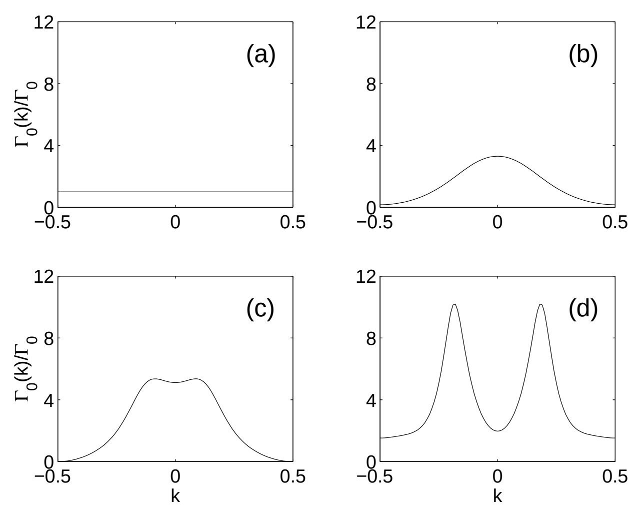

Fig. 5. The decay rate of the ground Wannier-Bloch states as a function of the quasimomentum k for the case of Fig. 3. In this and the following figures we normalize the decay rate $\Gamma_\alpha(k)/\hbar$ against the decay rate $\Gamma_\alpha/\hbar$ at $\epsilon = 0$.

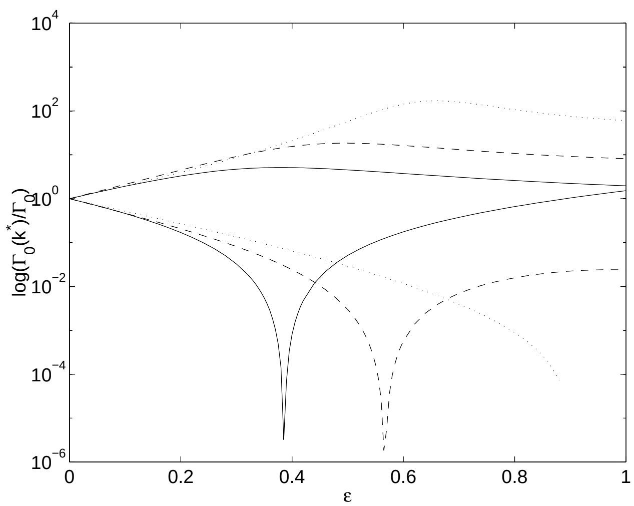

Fig. 6. The decay rate of the ground Wannier-Bloch band at $k^* = 0$ (upper family of curves) and $k^* = \pm 1/2$ (lower curves) for the parameters of Fig. 4.

It is also seen in Fig. 5 that the periodic perturbation can both increase and decrease the decay rate of the Wannier states. The regions of the enhanced and suppressed tunneling depend on the phase difference between the phase of the field and the phase of the Bloch oscillations. For the chosen Hamiltonian $\widehat{H} = \widehat{H}_W + F_\omega x \cos(\omega t)$ these are (at least for small ϵ) the middle and the edges of the Brillouin zone, respectively. (For $\widehat{H} = \widehat{H}_W - F_\omega x \cos(\omega t)$ the situation is reversed.) The ϵ-dependence of the decay rates $\Gamma_0(k = 0)$ and $\Gamma_0(k = \pm 1/2)$ is shown in Fig. 6 for $F = 0.08$ (solid line), $F = 0.04$ (dashed line), and $F = 0.02$ (dotted line). A highly nontrivial dependence is noticed. In particular, we would like to draw attention to the points of nonanalyticity, where the decay rate is suppressed by more than a factor of 10^5! This tremendous decrease of decay at the edges of the Brillouin zone has consequences in the global increase of the stability of the ground state. As an example, Fig. 7 shows the survival probability

$$P(t) = \int_{-1/2}^{1/2} \exp\left[\frac{\Gamma_0(k)t}{\hbar}\right] \mathrm{d}k$$

of the ground Wannier state for $F = 0.04$ and $\epsilon = 0$ (a), $\epsilon = 0.2$ (b), and $\epsilon = 0.54$ (c). It is seen that the decay rate slows down when ϵ approaches the nonanalytic point $\epsilon_{cr} \approx 0.55$. To conclude this section, we note that the periodic perturbation essentially affects only the ground, relatively stable, band. The lifetime of the unstable excited states remains practically unchanged.

7 Two-state model

The numerical results of the previous section shows a complicated behavior of the decay rate as the amplitude of the driving force is increased. Unfortunately, this observed ϵ-dependence of $\Gamma_\alpha(k)$ cannot be described in terms of the first-order perturbation theory used in Sec. 4 to find the correction to the real part of the energy. The reason for this can be understood by considering the complex energy in the polar coordinate

$$\lambda_\alpha = \exp\left(-i\frac{E_\alpha T_B}{\hbar}\right) = \exp\left(-\frac{i}{F}\left[E_\alpha^{Re} - i\frac{\Gamma_\alpha}{2}\right]\right) . \tag{36}$$

Then Eq. (28) gives the tangent correction to the energy and, thus, artificially increases $|\lambda_\alpha|$ or even moves λ_α out of the unit circle. Because of this we employ here a different method, which was used earlier in Ref. [9] to find the fluctuation of decay rate when the static field amplitude is varied. We describe this approach in the following using $\epsilon = 0$ for the moment.

The main idea behind our approach is that a channel for decaying of the α-th Wannier-Bloch state is its coupling to the next $(\alpha + 1)$-th state. Thus, being interested in the decay rate only of the ground state, we can approximate the system by a two-state model. In this model we assign zero width to the ground resonance, a finite width to the 1-st resonance, and explicitly introduce a coupling between these states. Then the ground Wannier resonance is given by the largest eigenvalue of the following 2×2 matrix

$$\mathbf{U}_W = \bar{\mathbf{U}}\mathbf{W} , \quad \bar{\mathbf{U}}_{\alpha,\beta} = \delta_{\alpha,\beta}\bar{\lambda}_\alpha , \tag{37}$$

$$\mathbf{W} = \exp\left\{-ia(F)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\} . \tag{38}$$

In Eqs. (36)–(38) the resonances positions E_0^{Re}, E_1^{Re} and the resonance width Γ_1 are assumed to be known (for example, from a semiclassical analysis) and $a(F) \sim \exp(-const/F) \ll 1$. Because of the coupling, the ground resonance now gains a finite width Γ_0 which depends on

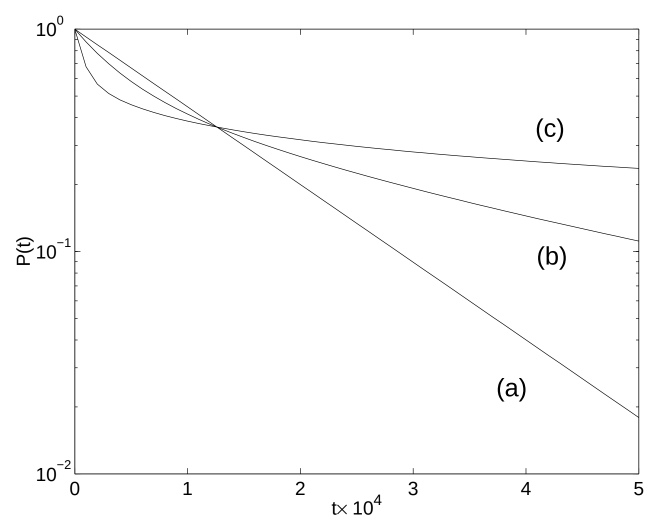

Fig. 7. The decay of the ground Wannier-Bloch band for different values of the perturbation parameter $\epsilon = 0$ (a), $\epsilon = 0.2$, and $\epsilon = 0.54 \approx \epsilon_{cr}$, $(F = 0.04,\ \hbar = 2)$.

the relative position of the resonances within the fundamental energy interval. As shown in Ref. [9], this model reproduces the numerically observed dependence $\Gamma_0 = \Gamma_0(F)$ with high accuracy. Moreover, the model allows us to obtain an analytical expression for the decay rate of the ground Wannier state. In fact, in the considered case of a 2×2 matrix $\mathbf{U}_W$ one easily solves the eigenvalue problem, which gives

$$\lambda_0 = \cos a\left(\frac{\bar\lambda_0+\bar\lambda_1}{2}\right) + \left[\cos^2 a\left(\frac{\bar\lambda_0+\bar\lambda_1}{2}\right)^2 - \bar\lambda_0\bar\lambda\right]^{1/2} . \tag{39}$$

In Eq. (39) we use the short notation (36) and the bar denotes the "zero" approximation for the resonances. Because $a \ll 1$ we can simplify Eq. (39) to

$$\lambda_0 = \bar\lambda_0\left[1-\frac{\epsilon^2}{2}\left(\frac{\bar\lambda_0+\bar\lambda_1}{\bar\lambda_0-\bar\lambda_1}\right)\right] . \tag{40}$$

Then, bearing in mind that $\Gamma_\alpha = -2F\,\mathrm{Re}[\ln\lambda_\alpha]$, we obtain for the resonance width Γ_0 the equation

$$\frac{\Gamma_0}{F} = a^2\mathrm{Re}\left[\frac{\bar\lambda_0+\bar\lambda_1}{\bar\lambda_0-\bar\lambda_1}\right] . \tag{41}$$

In Eq. (41), the function $a(F) \sim \exp(-const/F)$ [with $const$ being an adjusted parameter] gives "an average" decrease of the decay rate $\Gamma_0 \sim F\exp(-2const/F)$ as $F\to 0$, and the terms in the square brackets is responsible for the fluctuation of the decay rate due to level crossings (see Fig. 1).

Now we shall adopt this model to analyze the currently considered case of a time-periodic perturbation ($\epsilon \neq 0$). Our starting point is the equation

$$\widehat U = \widehat U_W\,\widehat U_A \approx \widehat U_W(1-i\epsilon\widehat A) \approx \widehat U_W\exp(-i\epsilon\widehat A) , \tag{42}$$

where the hermitian operator $\widehat A$ is given by Eq. (29). We note that here we use a representation which preserves unitarity of the evolution operator and, thus, ensures $|\lambda_\alpha| \le 1$. (It should be remembered, however, that Eq. (42) is valid only for $\epsilon \ll 1$.) It follows from Eq. (42) that to include the effect of the ac field in the model (37), the matrix $\mathbf{U}_W$ should be multiplied by the operator exponent

$$\mathbf{U}^{(k)} = \mathbf{U}_W\exp\left[-i\epsilon\cos(2\pi k)\mathbf{A}\right] . \tag{43}$$

For the moment, $\mathbf{A}$ is an arbitrary symmetric 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} c_0 & b\\ b & c_1\end{pmatrix} , \tag{44}$$

and the prefactor $\cos(2\pi k)$ mimics the cosine dependence of the operator $\widehat A$ on the quasimomentum. Then, using the approximate relation

$$\exp\left[-i\epsilon\begin{pmatrix} c_0 & b\\ b & c_1\end{pmatrix}\right] \approx \exp\left[-i\epsilon\begin{pmatrix} 0 & b\\ b & 0\end{pmatrix}\right]\exp\left[-i\epsilon\begin{pmatrix} c_0 & 0\\ 0 & c_1\end{pmatrix}\right] ,\ \epsilon \ll 1 , \tag{45}$$

we can present Eq. (43) in the form

$$\mathbf{U}^{(k)} = \bar{\mathbf{U}}\mathbf{W} ,\quad \bar{\mathbf{U}}_{\alpha,\beta} = \delta_{\alpha,\beta}\bar\lambda_\alpha , \tag{46}$$

where

$$\bar\lambda_\alpha = \exp\left(-\frac{i}{F}\left[E_\alpha^{Re} + c_\alpha\epsilon\cos(2\pi k) - i\frac{\Gamma_\alpha}{2}\right]\right) , \tag{47}$$

$\Gamma_0 = 0$, and

$$\mathbf{W}^{(k)} = \exp\left\{-i[a+b\epsilon\cos(2\pi k)]\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\right\} . \tag{48}$$

Equations (46)–(48) have the same structure as Eqs. (36)–(38) and, therefore, the solution (41) can be directly used, namely

$$\frac{\Gamma_0(k)}{F} = [a+b\epsilon\cos(2\pi k)]^2\,\mathrm{Re}\left[\frac{\bar\lambda_0+\bar\lambda_1}{\bar\lambda_0-\bar\lambda_1}\right] . \tag{49}$$

The roles played by the two factors on the right hand side of Eq. (49) are similar to those in Eq. (41) – the first factor gives a smooth "average" dependence of the decay rate (now on the quasimomentum k), and the term in the square brackets takes into account the effect of level (band) crossings.

Equation (49) contains a number of still undefined parameters. Among them, E_0^{Re}, E_1^{Re}, Γ_1, and a refer to the unperturbed case $\epsilon = 0$, and the coefficients c_0, c_1 are easy to identify with the integrals (32). Thus we are left with only one adjustable parameter b. Based on the numerical result (see Fig. 5) we conclude that b is positive. Moreover, one can see that Eq. (49) correctly captures the decrease of the decay rate at the edges of the Brillouin zone leading to a non-analytical behavior of $\Gamma_0(k=\pm 1/2)$ at $\epsilon = \epsilon_{cr}$. Using the numerically known value ϵ_{cr} we estimate the parameter b as $b = a/\epsilon_{cr}$.

As an example, Fig. 8 compares the function $\Gamma_0(k)$ calculated on the basis of Eq. (49) (solid line) with the exact result (dotted line) for $F = 0.04$ and $\epsilon = 0.2$ (a), $\epsilon = 0.5$ (b). The procedure of adjusting the free parameters was organized as follows. The values $E_0^{Re} = -0.6055\pi F$, $E_1^{Re} = -1.7202\pi F$, and $\Gamma_1 = 1.3098F$ were taken from the data of Fig. 1. Then we adjusted the parameter a to $a = 0.0096$ to get the correct value $\Gamma_0 = 8.0449\cdot 10^{-5}F$. The coefficients $c_0 = -0.43$, $c_1 = 2.3$ were calculated by using Eq. (32). Alternatively, these values can be extracted from the numerical data of Fig. 4. The value $\epsilon_{cr} = 0.55$ is taken from Fig. 6, which gives $b = 0.0171$. It is seen in Fig. 8 that there is a good qualitative agreement with the exact result even for a large $\epsilon = 0.5$. For a small ϵ the agreement becomes quantitative. In this case of a very small ϵ one can neglect the ϵ-,k-dependence of the second factor in Eq. (49) and it takes the form

$$\frac{\Gamma_0(k)}{\Gamma_0} = \left[1+\frac{\epsilon}{\epsilon_{cr}}\cos(2\pi k)\right]^2 ,\quad \epsilon \ll \epsilon_{cr} . \tag{50}$$

This formula actually gives the correction to the imaginary part of the energy up to the second order in ϵ.

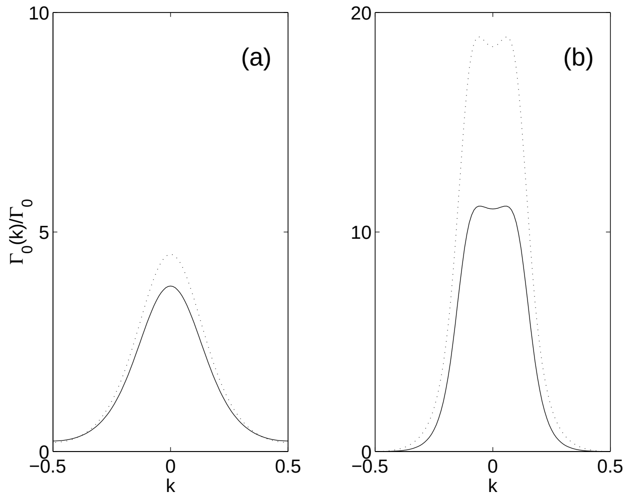

Fig. 8. Comparison of the ground state decay rate calculated on the basis of two-state model against the exact result for $\epsilon = 0.2$ (a) and $\epsilon = 0.5$ ($F = 0.04$, $\hbar = 2$).

8 Conclusion

We have studied the effect of a weak resonant ac field on the spectrum of the Wannier states. In the paper we restrict ourselves to the case where the Bloch period of the particle in dc field exactly coincides with the driving period. (In principle, the analysis can be extended to the case of arbitrary rational ratio between the periods.) Below we summarize the main results obtained.

To relate the dimensionless Hamiltonian (3) to some physical system we shall consider it in context of the experiment with cold atoms in an accelerated standing wave. Then the system Hamiltonian has the form [6]

$$\widehat{H} = \frac{\hat{p}^2}{2M} + V_0 \cos\left(2k_L\left[x - \frac{F_\omega}{M\omega^2}\cos(\omega t)\right]\right) + Fx\ . \quad (51)$$

In the Hamiltonian (51), M is an atomic mass, V_0 the amplitude of the optical potential (which is proportional to the laser intensity), k_L the wave number of the laser field, $F = Ma$ the inertial force due to the acceleration, and F_ω the amplitude of the periodic driving force with frequency ω. The condition $T_B = T_\omega$ implies

$$\frac{\hbar}{dF} = \frac{2\pi}{\omega}\ , \quad d = \frac{\pi}{k_L}\ . \quad (52)$$

It is easy to show that the Hamiltonian (51) can be presented in the dimensionless form (22) where the values of the scaled Planck constant $\hbar'$ is given by

$$\hbar' = \left(\frac{\hbar\omega_{rec}}{V_0}\right)^{1/2}\ , \quad \omega_{rec} = \frac{\hbar k_L^2}{2M}\ . \quad (53)$$

The deep quantum region of the system parameters we were interested in corresponds to $\hbar' > 1$.

In this paper we treated the driving force as a perturbation where the perturbation parameter was

$$\epsilon = \frac{2F_\omega k_L}{M\omega^2} = \frac{F_\omega d}{\hbar\omega}\frac{\omega_{rec}}{\pi\omega}\ . \quad (54)$$

It was shown that a perturbation ($\epsilon < 1$) removed the degeneracy of the Wannier-Bloch bands and the system quasienergy spectrum obeyed the equation

$$E_\alpha^{Re}(k) = \hbar\omega\left[E_\alpha^{Re} + \epsilon I_\alpha \cos(2\pi k)\right]_{\mathrm{mod}:\hbar\omega}\ . \quad (55)$$

In Eq. (55), E_α^{Re} is the position of the α-th Wannier resonance and I_α is a constant which depends primarily on the static field amplitude. In the limit of small F the coefficient I_α is proportional to F and Eq. (55) reproduces the result obtained earlier on the basis of the tight-binding model.

The periodic driving also changes the decay rate of the Wannier-Bloch states, which is defined by the imaginary part of the quasienergy. In the case of only one (ground) narrow resonance, the second order correction to the resonance width is given by the expression

$$\Gamma_0(k) = \Gamma_0\left[1 + \frac{\epsilon}{\epsilon_{cr}}\cos(2\pi k)\right]^2\ , \quad (56)$$

where Γ_0 is the width of the unperturbed Wannier resonance and ϵ_{cr} is some characteristic value of the perturbation. Equation (56) predicts a singularity in the lifetime of the Wannier states as ϵ approaches ϵ_{cr}. In fact, it is confirmed numerically that at $\epsilon = \epsilon_{cr}$ the periodic driving can increase the lifetime of the ground resonance by several orders of magnitude. This coherent suppression of Landau-Zener tunneling might find an application both in the field of solid state physics and atomic optics.

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