To appear in:
Physica E, in press

About universality of lifetime statistics in quantum chaotic scattering

M. Glück, A. R. Kolovsky[*] and H. J. Korsch

Fachbereich Physik, Universität Kaiserslautern, D-67653 Kaiserslautern, Germany

October 19, 2000

Abstract. The statistics of the resonance widths and the behavior of the survival probability is studied in a particular model of quantum chaotic scattering (a particle in a periodic potential subject to static and time-periodic forces) introduced earlier in Ref. [5,6]. The coarse-grained distribution of the resonance widths is shown to be in good agreement with the prediction of Random Matrix Theory (RMT). The behavior of the survival probability shows, however, some deviation from RMT.

PACS. : 03.65.†, 05.45.+b, 73.20Dx
Keywords: Chaotic scattering; Quantum chaos; Statistics of resonances.

1. The abstract Random Matrix Theory (RMT) is known as a powerful tool for analyzing complex quantum systems. During the last two decades the predictions of the *hermitian* RMT (which is supposed to describe the properties of a closed system) were checked for a large number of physical models and an understanding of the conditions of applicability of RMT was reached. The situation is different, however, for *nonhermitian* RMT, which is aimed to describe the spectral properties of open systems. Here we have quite a few physical models which can serve a suitable test for a nonhermitian RMT. Up to our knowledge the following systems are mainly under discussion: chaotic 2D billiards with attached leads [1]; the kicked rotor with an absorbing boundary condition [2]; simplified models of a dissociating molecule [3]; scattering on graphs [4]; and the Bloch particle affected by static and time periodic forces (1D model of a crystal electron in dc and ac fields) [5–7]. The latter model has a number of nice features, which distinguishes it among the other systems. First, it is simple for numerical analysis. Second, it always realizes the so-called case of perfect coupling (where the predictions of the hermitian and nonhermitian RMT are most different). Third, as a physical model it can be and has been studied under laboratory conditions [8]. This present paper continues our study of the Bloch particle in ac and dc fields in relation to RMT. In particular we discuss here the decay of the probability in the system (probability leakage), which is the simplest quantity measured in the laboratory experiment.

2. We briefly recall some of the results of our preceding papers [5–7]. After an appropriate rescaling the Hamiltonian of the system of interest can be presented in the dimensionless form

$$\widehat{H} = \frac{\hat{p}^2}{2} + \cos x + Fx + F_\omega x \cos(\omega t) \; . \qquad (1)$$

The parameters of the system (1) are the amplitude of the static force F, the amplitude F_ω and the frequency ω of the time-periodic force, and the scaled Planck constant $\hbar$ which enters the momentum operator $\hat{p} = -i\hbar \mathrm{d}/\mathrm{d}x$. Another, unitary equivalent, form of the Hamiltonian (1) reads as

$$\widehat{H} = \frac{\hat{p}^2}{2} + \cos[x - \epsilon\cos(\omega t)] + Fx \; , \; \epsilon = \frac{F_\omega}{\omega^2} \; . \qquad (2)$$

When some condition (based on Chirikov's overlap criterion) on ϵ is satisfied, the classical dynamics of the system (2) is an example of chaotic scattering. In fact, one of the main characteristics of the classical chaotic scattering is the delay or dwell time

$$\tau = \lim_{p_0\to\infty} [\tau(p_0 \to -p_0) - 2p_0/F] \; . \qquad (3)$$

(Here $\tau(p_0 \to -p_0)$ is the time taken by the particle to change its initial momentum p_0 to the opposite one.) Figure 1 shows the delay time (3) as a function of the initial coordinate x. The fractal behavior is typical for chaotic scattering.

We proceed with the quantum case. It was shown in Ref. [6,7] that the dynamics and spectral properties of the system (2) depend crucially on the condition of commensurability between the so-called Bloch period $T_B = \hbar/F$ and the period $T_\omega = 2\pi/\omega$ of the driving force,

$$\frac{T_B}{T_\omega} = \frac{r}{q} \; . \qquad (4)$$

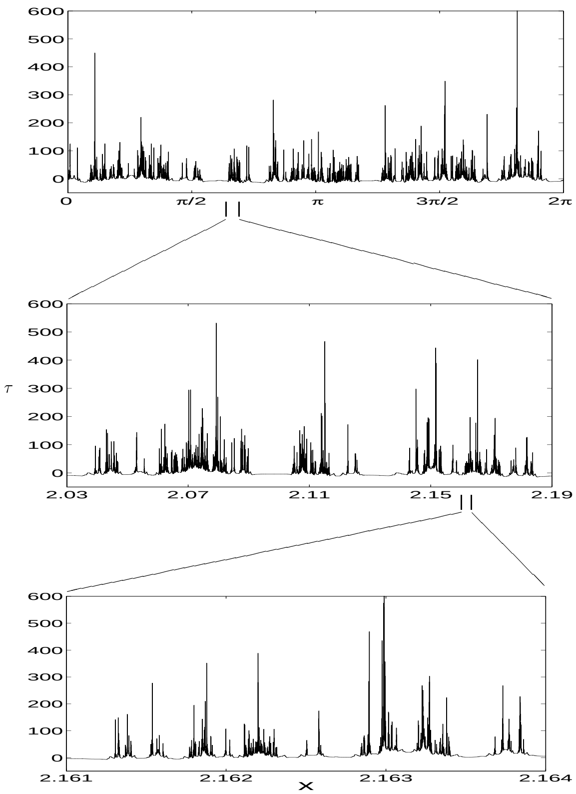

Fig. 1. Fractal structure of the classical delay time (3) as a function of the initial coordinate. The system parameters are $F=0.3$, $\epsilon=1.5$, and $\omega=10/6$.

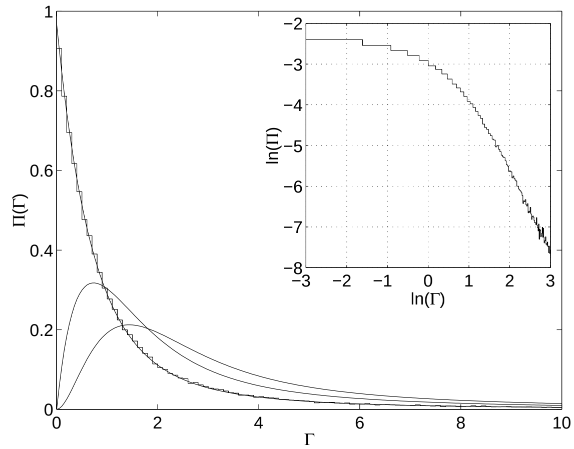

Fig. 2. Distribution of the scaled resonance width $\mathit{\Gamma}_s$ of the nonunitary random matrix U_{ran}. Parameters are $N=41$ and $M=1$. The statistical ensemble involves 5000 matrices. The smooth curves are the distribution (8) for $M=1,2,3$, respectively.

Providing the condition (4) is satisfied, the complex quasienergy spectrum of the system (the resonances) is given by the eigenvalues of the following nonunitary matrix

$$U_{sys}=\begin{pmatrix} 0_{M\times N} & 0_{M\times M} \\ W_{N\times N} & 0_{N\times M}\end{pmatrix} . \qquad (5)$$

In Eq. (5) $W_{N\times N}$ is the unitary matrix with the coefficients

$$W_{n',n}=\langle n'|\exp(-ikx)\widehat{W}\exp(ikx)|n\rangle , \qquad (6)$$

$$\widehat{W}=\widehat{\exp}\left\{-\frac{i}{\hbar}\int_0^T\left[\frac{(\hat{p}-Ft+\hbar k)^2}{2}+\widehat{V}\right]\mathrm{d}t\right\} ,$$

$$\widehat{V}=\cos[x-\epsilon\cos(\omega t)] ,$$

and $0_{M\times N}$, $0_{M\times M}$, $0_{N\times M}$ are blocks of zeros. The nonunitary matrix (5) can be thought of as the truncated [to the size $(N+M)\times(N+M)$] unitary matrix of the system evolution operator over the common period $T=qT_B=rT_\omega$. Then the parameter M, which plays the role of the number of channels, is identical with the integer q in condition (4). The additional parameter N measures the number of states supported by the chaotic component of the classical phase space. (Unimportant for our present aim is the quasimomentum k which can take any value in the interval $-\pi/q\le k<\pi/q$.)

The main conjecture made in Ref. [7] is that the spectral statistics of the system (2) [i.e., the eigenvalues statistics of the matrix U_{sys}] is the same as the statistics of the eigenvalues of a random matrix U_{ran} of the structure (5) but with the matrix $W_{N\times N}$ substituted by a member of the Circular Unitary Ensemble (CUE)

$$W_{N\times N}\to A_{N\times N} , \quad A_{N\times N}\in \mathrm{CUE} . \qquad (7)$$

In what follows we examine this conjecture in more detail.

3. First we discuss the spectral statistics of the random matrix U_{ran}. There is strong numerical evidence that the statistics of the eigenvalues $\exp(-i\mathcal{E})=\exp(-iE-\mathit{\Gamma}/2)$ of the matrix U_{ran} is given by the universal distribution derived in Ref. [9]. (An analytical proof of this result is still an open problem. [1]) In particular, the distribution of the scaled resonance width $\mathit{\Gamma}_s=\pi\mathit{\Gamma}/\mathit{\Delta}$ ($\mathit{\Delta}=2\pi/N$ is the mean level spacing) obeys the equation

$$\mathit{\Pi}_M(\mathit{\Gamma}_s)=\frac{(-1)^M\mathit{\Gamma}_s^{M-1}}{(M-1)!}\frac{d^M}{d\mathit{\Gamma}_s^M}\left(e^{-\mathit{\Gamma}_s}\frac{\sinh\mathit{\Gamma}_s}{\mathit{\Gamma}_s}\right) . \qquad (8)$$

As an example Fig. 2 (adopted from paper [7]) shows the histogram for the distribution of $\mathit{\Gamma}_s$ for $N=41$ and $M=1$. We note that $\mathit{\Pi}_M(\mathit{\Gamma}_s)\approx M/\mathit{\Gamma}_s^2$ for $\mathit{\Gamma}_s\gg 1$ and, thus, the notion of mean resonance width is not defined.

Let us discuss now the decay of the probability $P(t)$, which is given by the following equations

$$P(t)=|\mathbf{c}_t|^2 , \quad \mathbf{c}_{t+1}=U_{ran}\mathbf{c}_t , \quad |\mathbf{c}_0|=1 . \qquad (9)$$

It is obvious that the dynamics of $P(t)$ is determined by the spectrum of the system. Thus the behavior of $P(t)$

[1] After the paper was submitted we learned that this result has been proved in Ref. [13].

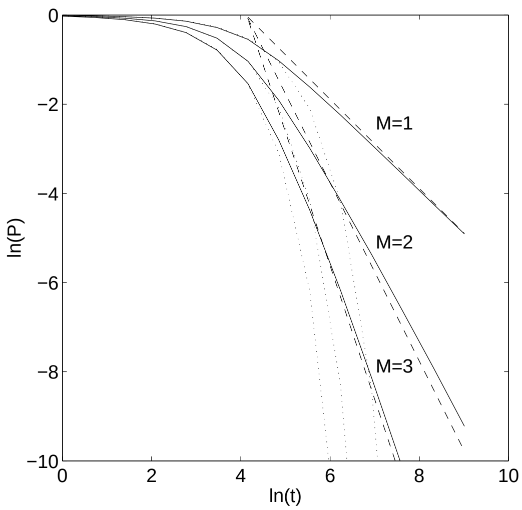

Fig. 3. Decay of the probability for the random matrix model (9) for $N = 121$ and $M = 1, 2, 3$. (Statistical ensemble involves 100 matrices). The dotted and dashed lines correspond to the asymptotic (10).

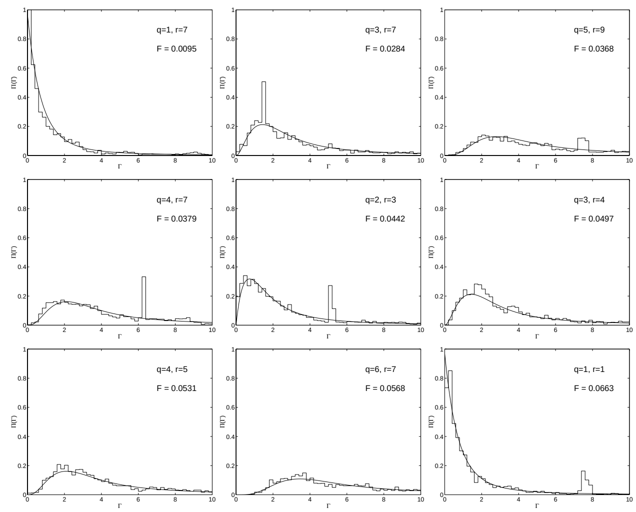

Fig. 4. Distribution of the scaled resonance width Γ_s of the system (2) for some values of the static force F satisfying the commensurability condition (4). The other system parameters are $\omega = 10/6$, $\epsilon = 1.5$, and $\hbar = 0.25$. The smooth curves are the distribution (8) for $M = q$.

suggests an additional test of the eigenvalue statistics [10]. The main advantage of studying $P(t)$ is that this quantity is more easily measured in the laboratory experiments [11].

The problem of probability decay was considered (although within a different RMT model) in paper [12]. It was proved there that the function $P(t)$ has an exponential short-term and an algebraic long-term asymptotic. Adopting these results to the present model (9) we obtain

$$P(t) = \begin{cases} \exp\left(-\frac{M}{N}t\right) \, , & t \ll t^* \\ \left(\frac{2t}{N}\right)^{-M} \, , & t \gg t^* \end{cases} \, , \qquad (10)$$

where $t^* \sim N/2 = \pi/\Delta$ is of order of the Heisenberg time. The results of a numerical simulation of the dynamics of $P(t)$ depicted in Fig. 3 well support the analytical expression (10).

It is interesting to note that Eq. (10) can be obtained by using rather simple arguments. In fact, expanding the initial vector $\mathbf{c}_0$ over the set of eigenvectors of the matrix U_{ran} and following Ref. [12] using the diagonal approximation we have

$$P(t) = \int_0^\infty \Pi_M(\Gamma_s) \exp(-\Gamma t) \mathrm{d}\Gamma_s \, . \qquad (11)$$

For the long-term asymptotic only the narrow resonances are of importance. Then, substituting $\Pi_M(\Gamma_s)$ by its asymptotic expression $\Pi_M(\Gamma_s) \sim \Gamma_s^{M-1}$, $\Gamma_s \ll 1$ [see Eq. (8)], we obtain $P(t) \sim (2t/N)^{-M}$. This power law decay takes place only for a "coherent" evolution of the initial state. In contrast, the short-term asymptotic of $P(t)$ coincides with an "incoherent" evolution, which would take place if one used uncorrelated matrices U_{ran} in Eq. (9) at each time step. Then, as follows from the structure of the matrix U_{ran}, at each time step the state vector decreases its norm by the factor $N/(N + M)$ and

$$P(t) = \left(\frac{N}{N+M}\right)^t \approx \exp\left(-\frac{M}{N}t\right) \, . \qquad (12)$$

4. We proceed with the statistics of the resonances for the physical model (2). Numerically we find them as the eigenvalues of the matrix (5), where the parameter M is identical with the denominator q in the condition of commensurability (4).

Figure 4 compares the distribution of the scaled resonance widths $\Gamma_s = \pi\Gamma/\Delta$ in the system (2) against the prediction of RMT given by Eq. (8). It seen that the global features of the distribution $\Pi(\Gamma_s)$ fit well to the result of RMT. (The peak-like peculiarities of $\Pi(\Gamma_s)$ are due to the resonances associated with the stability islands of the classical phase space. In principle these resonances should be removed from the analyzed data.)

We would like to note that to satisfy the condition (4) we adjusted the amplitude of the static force F (the other system parameters are kept fixed). By changing F we actually change the classical properties of the system [in particular, the distribution of the classical delay time (3)]. Nevertheless, the quantum distribution $\Pi(\Gamma_s)$ remains practically unchanged (see cases $F = 0.0095$ and $F = 0.0663$, for example) and is defined exclusively by the channel number q. This fact clearly demonstrates the applicability of RMT for the system under study.

We come to the problem of the probability decay. In our numerical analysis of the system we calculated the dynamics of probability $P(t)$ by two different methods. The first method utilizes Eq. (9) where the random matrix U_{ran} is substituted by the matrix (5). The second

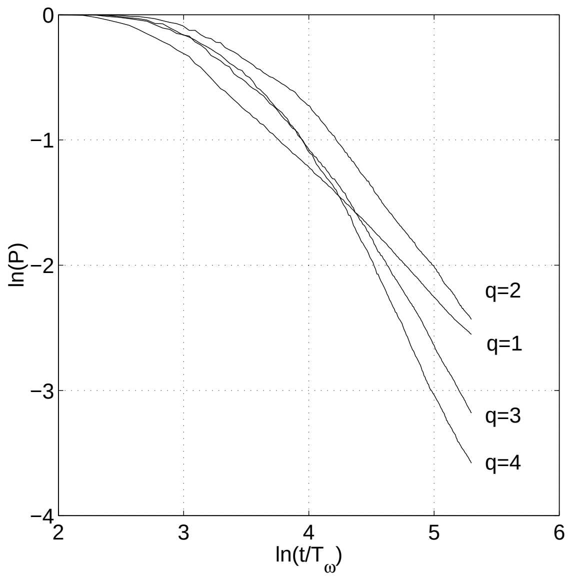

Fig. 5. Decay of the probability of the system (2) for the values of F corresponding to $r/q = 1$, 3/2, 4/3, and 5/4 in Eq. (4). The other parameters are the same as in Fig. 4.

method is the direct numerical simulation of the wave-packet dynamics of the system (2). The latter method has the advantage that it allows to study the incommensurate case but it is essentially more time consuming. In the commensurate case $T_B/T_\omega = r/q$ (with relatively small r and q) both methods give the same result.

Figure 5 shows the behavior of $P(t)$ on a double logarithmic scale for $r/q = 1$, 3/2, 4/3, and 5/4. It is seen that the survival probability follows asymptotically a power law $P(t) \sim t^{-\alpha}$. However, the value of α ($\alpha \approx 1$, 4/3, 5/3, 2, respectively) differs from that predicted by Eq. (10). This means that for very small resonance width (not resolved on the scale of Fig. 4) the actual distribution $\Pi(\Gamma_s)$ deviates from the distribution (8). For the moment we have no explanation for this deviation from RMT.

It is also worth to note that the algebraic decay discussed is actually a transient phenomenon for physical systems. The point is that RMT deals with an infinite ensemble while in practice it is always finite (and often consists of a single representative). For a finite ensemble most narrow resonance exists and, thus, a very far asymptotic is again an exponential decay with the increment given by the width of this most narrow resonance.

References

[*] Also at L. V. Kirensky Institute of Physics, 660036 Krasnoyarsk, Russia.

1. J. Burgdörfer and H. Ishio, unpublished.
2. F. Borgonovi, I. Guarneri, and D. L. Shepelyansky, Phys. Rev. A **43**, 4517 (1991); G. Casati, G. Maspero, and D. L. Shepelyansky, Phys. Rev. E **56**, R6233 (1997).
3. Th. Zimmermann, H. Köppel, and L. S. Cederbaum, J. Phys. **91**, 3934 (1989); R. Schinke, H.-M. Keller, M. Stumpf, and A. J. Dobbyn, J. Phys. B: Mol. Opt. Phys. **28**, 3081 (1995); V. A. Mandelshtam and H. S. Taylor, J. Chem. Soc., Faraday Tras. 1 **93**, 847 (1997).
4. S. Kottos and U. Smilanski, preprint (chao-dyn/9906008).
5. M. Glück, A. R. Kolovsky, and H. J. Korsch, Phys. Lett. A **249**, 483 (1998).
6. M. Glück, A. R. Kolovsky, and H. J. Korsch, Phys. Rev. Lett. **82**, 1534 (1999).
7. M. Glück, A. R. Kolovsky, and H. J. Korsch, Phys. Rev. E **60**, 247 (1999).
8. F. L. Moore, J. C. Robinson, C. F. Bharucha, Bala Sundaram, and M. Raizen, Phys. Rev. Lett. **75**, 4598 (1995).
9. Y. V. Fyodorov and H. J. Sommers, J. Math. Phys. **38**, 1918 (1997).
10. For a closed system the relation between time behavior and spectrum statistics was discussed in Y. Alhassid and N. Whelan, Phys. Rev. Lett. **70**, 572 (1993); Y. Alhassid and R. D. Levine, Phys. Rev. A **46**, 4650 (1992).
11. M. Glück, A. R. Kolovsky, and H. J. Korsch, preprint (nlin.CD/0002013).
12. D. V. Savin and V. V. Sokolov, Phys. Rev. E **56**, R4911 (1997).
13. K. Zyczkowski and H.-J. Sommers J.Phys.A: Math.Gen. **33** 2045 (2000).