AN INTRODUCTION TO THE NONLINEAR BOLTZMANN-VLASOV EQUATION

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An Introduction to the Nonlinear Boltzmann-Vlasov Equation

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1. The derivation of the modified Vlasov equation

The Vlasov equation in its simplest form describes the behaviour of a gas consisting of a large number of identical particles moving according to the laws of classical mechanics and interacting by a potential, which is proportional to the mass (or charge respectively) of the particles and which is - that is the main point - only weakly singular.

"Weakly singular" means precisely: The force, exerted by a particle x on a particle at y, both of mass m, has the form $m^2 G(x,y)$, where G might become singular for x=y only in a way, that at

$$\int_{\mathbb{R}^3} G(x,y)\varphi(y)\,dy$$

exists for all $x \in \mathbb{R}^3$, $\varphi \in L^1(\mathbb{R}^3)$.

The most typical examples are

$$(1) \quad G(x,y) = -\gamma \nabla_y \frac{1}{\|x-y\|} = -\gamma \frac{x-y}{\|x-y\|^3}$$

with $\gamma > 0$, if $m^2 G$ is the gravitational force and with $\gamma < 0$, if $m^2 G$ is the Coulomb force.

The Vlasov equation has the form

$$(2) \quad \frac{\partial f}{\partial t} + <v,\nabla_x f> + <\int_{\mathbb{R}^6} G(x,y)f(t,y,v)\,dy\,dv, \nabla_v f> = 0$$

and is assumed to describe for G given in (1) the behaviour of a stellar gas, if $\gamma > 0$ or the behaviour of an electron gas, if $\gamma < 0$. For the stellardynamic case (2) was already considered by Jeans in 1915, while Vlasov introduced that equation for the plasmaphysical case in 1938.

Our aim is to investigate, in which sense the Vlasov equation describes the behaviour of a system of N identical particles, which move under the influence of their mutual interaction forces. If we normalize the total mass (or charge) to 1, each particle has

mass (charge) $\frac{1}{N}$ and the state is described just by giving the positions and velocities of the particles, i.e. by

$$P_i := (x_i, v_i) \in \mathbb{R}^6, \quad i=1,\ldots,N.$$

The motion is then governed by Newton's law

$$(3) \quad \begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= \frac{1}{N} \sum_{j=1}^{N} G(x_i, x_j). \end{aligned}$$

So, given the state $\overset{o}{\omega}_N = \{\overset{o}{P}_1,\ldots,\overset{o}{P}_N\}$ at time 0, we get the state $\omega_N(t) := \{P_1(t),\ldots,P_N(t)\}$ at time t by solving the initial value problem for (3) with $P_i(0) = \overset{o}{P}_i$, $i=1,\ldots,N$. Such a solution might exist only for finite time - for $m^2 G$ being the gravitational force (3) represents the N-body-problem of celestial mechanics and has therefore only local solutions. Even more, the time interval of existence certainly depends on N and its length might tend to 0 if N tends to infinity.

In order to get rid of these difficulties we assume here, that G is not singular, but bounded and continuous. That can be achieved by a smooth cut-off of the potential or another mollifying procedure, for example by substituting the original G (which is weakly singular) by

$$G_\delta(x,y) := \int_{\mathbb{R}^3} G(x,z) \omega_\delta(z-y)\, dz$$

with the "mollifier" ω_δ defined by

$$\omega_\delta(x) := \begin{cases} 0 & \text{for } \|x\| \geq \delta \\ c_\delta \exp(-1+\frac{\|x\|^2}{\delta^2})^{-1} & \text{for } \|x\| < \delta \end{cases}$$

where c_δ is such that $\int_{\mathbb{R}^3} \omega_\delta\, dx = 1$.

Or by substituting the special $G(x,y) = \frac{x-y}{\|x-y\|^3}$ by

$$G_\delta(x,y) = \frac{x-y}{(\|x-y\|^2 + \delta)^{3/2}} \ .$$

G_δ is certainly bounded and continuous; we will write again G for G_δ in that lecture. Equation (2) with G_δ instead of G is called the "modified Vlasov equation". For bounded and continuous G the initial value problem for (3) has a global solution; $\omega_N(t)$ is defined for all $t \geq 0$.

We want to compare now $\omega_N(t)$ with a function $f(t,\cdot) : \mathbb{R}^6 \longrightarrow \mathbb{R}$, which is a solution of the Vlasov equation.

Let us begin with t=0: We have to compare $\omega_N(0) = \overset{o}{\omega}_N$ with $f(0,\cdot) = \overset{o}{f}$, where $\overset{o}{f}$ is the initial value for the Vlasov equation (2). In order to be able to compare $\overset{o}{\omega}_N = \{\overset{o}{P}_1,\ldots,\overset{o}{P}_N\}$ and $\overset{o}{f}$, we have to interprete both as mathematical objects of the same kind. Both describe mass (or charge) distributions. Mass distributions can be described by measures. Therefore interprete both - $\overset{o}{\omega}_N$ and $\overset{o}{f}$ as Borel measures.

(a) $\overset{o}{\omega}_N$ is substituted by the discrete measure

$$\delta_{\overset{o}{\omega}_N} := \frac{1}{N} \sum_{j=1}^{N} \delta_{\overset{o}{P}_j} \quad ,$$

where δ_P is the usual Dirac measure concentrated on P. $\delta_{\overset{o}{\omega}_N}$ has total mass 1; therefore it **is** called a probability measure.

(b) $\overset{o}{f}$ is supposed to be a positive function on $\mathbb{R}^6$ with

$$\int_{\mathbb{R}^6} \overset{o}{f} \, dx \, dv = 1.$$

Therefore $\overset{o}{f}$ might be interpreted as the density (with respect to the normal 6-dimensional Lebesgue measure) of a probability measure $\overset{o}{\mu}$:

$$\overset{o}{\mu}(M) := \int_M \overset{o}{f} \, dx dv \quad \text{for every Borel set M.}$$

So instead of using $\overset{o}{\omega}_N$ to describe the state of the N-particle system and $\overset{o}{f}$ to describe the initial condition for (2), we use the probability measures $\delta_{\overset{o}{\omega}_N}$ and $\overset{o}{\mu}$. We want to mention, that in spite of having probability measures there is no stochastic element in the description - we **only have** normalized the total mass.

We write M for the set of all probability measures on $\mathbb{R}^6$. The best method to compare $\delta_{\overset{o}{\omega}_N}$ and $\overset{o}{\mu}$ will be provided by a metric in M.

So we have to look for a metric in M. But there is one condition such a metric should fulfill. There is a natural topology in M, the weak[x]-topology, which I describe by the corresponding weak convergence in M:

$(\mu_n)_{N \in \mathbb{N}}$ converges weakly to μ, $\mu_n \longrightarrow \mu$, if and only if

$$\lim_{n \to \infty} \int_{\mathbb{R}^6} \varphi \, d\mu_n = \int_{\mathbb{R}^6} \varphi \, d\mu \quad \text{for all bounded continuous } \varphi.$$

Now it is important, that a metric ρ generates the weak convergence:

$$\lim_{n \to \infty} \rho(\mu_n, \mu) = 0 \quad \text{if and only if } \mu_n \longrightarrow \mu.$$

We have several choices for ρ, but not every choice is convenient for our purposes.

(I)   There is the so-called "bounded Lipschitz distance" ρ_1 defined by

(4) $\rho(\mu, \nu) := \sup\limits_{\varphi \in D} \left| \int \varphi \, d\mu - \int \varphi \, d\nu \right|$

where $D = \{ \varphi \mid \mathbb{R}^6 \longrightarrow [0,1], \; |\varphi(P) - \varphi(Q)| \leq \|P - Q\|$

for all $P, Q \subset \mathbb{R}^6 \}$.

(II)   Dobrushin and Braun-Hepp used the Wasserstein-metric $\tilde{\rho}$, defined as follows:

Consider the class Σ of measures σ on $\mathbb{R}^{12}$, such that $\sigma(M \times \mathbb{R}^6) = \mu(M)$, $\sigma(\mathbb{R}^6 \times N) = \nu(N)$. Then

(5) $\tilde{\rho}(\mu, \nu) = \inf\limits_{\sigma \in \Sigma} \int_{\mathbb{R}^{12}} \min \{ \|P - Q\|, 1 \} \, d\sigma(P, Q)$.

(III)   The most simple possible choice is the "discrepancy", defined by

(6) $D(\mu, \nu) := \sup\limits_{R \in \mathcal{R}} |\mu(R) - \nu(R)|$

where $\mathcal{R}$ is the class of all subsets of R_6 of the form $\{P \in \mathbb{R}^6 \mid P \leq Q\}$ with arbitrary but given Q and the usual semiorder in $\mathbb{R}^6$.

But while ρ_1, ρ_2 generate the weak convergence, this is not true for D.

Only for absolutely continuous measure μ is it true, that

$$\mu_n \longrightarrow \mu \Longleftrightarrow D(\mu_n, \mu) \longrightarrow 0.$$

Therefore we have to be careful in using the discrepancy and we prefer ρ_1 to work with.

Now suppose we have a sequence of N-particle systems $\overset{o}{\omega}_N$ with N increasing and an initial distribution $\overset{o}{\mu}$, so that

$$\delta_{\overset{o}{\omega}_N} \to \overset{o}{\mu} \quad \text{with } N \to \infty, \text{ or, which is the same}$$

$$\lim_{N \to \infty} \rho_1(\delta_{\overset{o}{\omega}_N}, \overset{o}{\mu}) = 0.$$

We solve, for each N, the initial value problem for (3) with

$$P_i(0) = \overset{oN}{P_i}, \quad \text{where } \overset{o}{\omega}_N = \{\overset{oN}{P_1}, \ldots, \overset{oN}{P_N}\}$$

and get $\omega_N(t) = \{P_1^N(t), \ldots, P_N^N(t)\}$ or correspondingly $\delta_{\omega_N(t)}$.

On the other hand, let us assume, that $f(t, \cdot)$ is a solution of the initial value problem for (2) with $f(0, \cdot) = \overset{o}{f}$, where $\overset{o}{f}$ is the density of $\overset{o}{\mu}$. (We return to the question of existence and uniqueness of the solution of that problem later.)

It follows immediately from (2), that

$$\int_{\mathbb{R}^6} f(t, P) \, dP = \int_{\mathbb{R}^6} \overset{o}{f}(P) \, dP = 1$$

and $f(t, P) \geq 0$ for all $t \geq 0$. So $f(t, \cdot)$ is again the density of a probability measure μ_t.

We are now able to formulate the notion "derivation of the Vlasov equation".

Definition 1: We call the Vlasov equation (2) "strictly derivable in $[0, T]$", if for every solution μ_t of the initial value problem for (2) with initial condition $\overset{o}{\mu}$

$$\lim_{N \to \infty} \rho_1(\delta_{\omega_N(t)}, \mu_t) = 0 \quad \text{holds, if}$$

$$\lim_{N \to \infty} \rho_1(\delta_{\overset{o}{\omega}_N}, \overset{o}{\mu}) = 0.$$

Remark: It is clear that the definition depends on what we call a solution. If we allow a solution to be a weak solution, the definition is even stronger. On the other hand, the strong derivability

of the Vlasov equation with respect to a certain class of solutions gives you also a hint what kind of solutions are physically meaningful.

Now the question is: What conditions should be fullfilled by G, so that (2) is strictly derivable? The answer is

Theorem 1: If G is bounded, globally Lipschitz continuous, then (2) is strongly derivable (with respect to a class of very weak "measure solutions", which will be defined in the following sketch of the proof).

Sketch of the proof : We put things in a more general setting. Let

$$\mu_\bullet : t \longrightarrow \mu_t \in M, \quad t \in [0,T]$$

be a weakly continuous measure valued function (weakly continuous means that $t \longrightarrow \int_{\mathbb{R}^6} \varphi \, d\mu_t$ is continuous for any bounded, continuous φ).

We denote the set of these functions by C_M. Let us assume, that for any $\mu_\bullet \in C_M$ there is defined a six-dimensional time depending vectorfield

$$(t,P) \longrightarrow V[\mu_\bullet](t,P) \in \mathbb{R}^6, \quad t \in [0,T], \quad P \in \mathbb{R}^6 .$$

In case of (2), this vectorfield is given by

$$(7) \qquad V[\mu_\bullet](t,x,v) = \begin{pmatrix} v \\ \int_{\mathbb{R}^6} G(x,y) \, d\mu_t(y,v) \end{pmatrix}$$

(remember, that $\int_{\mathbb{R}^6} G(x,y) \, d\mu_t(y,v) = \int_{\mathbb{R}^6} G(x,y) f(t,y,v) \, dy \, dv$.

$V[\mu_\bullet]$ has to satisfy two conditions, the first of which reads

(I) $V[\mu_\bullet](t,P)$ should be continuous in t and globally Lipschitz continuous in P (with Lipschitz constant L).

This is true for V given by (7), since $\mu_\bullet$ is weakly continuous, $G(x,\cdot)$ is continuous for each x and $G(\cdot,y)$ is globally Lipschitz continuous.

If (I) is true, the initial value problem for the characteristic equations

$$(8) \quad \dot{P} = V[\mu_\bullet](t,P), \quad P(s) = \overset{o}{P}$$

has a unique globally existing solution, which we denote by

$$P(t) = T_{t,s}[\mu_\bullet]\overset{o}{P} .$$

$T_{t,s}$ is bijective and $T_{s,s} = id$, $T_{t,s}^{-1} = T_{s,t}$.

Now we consider the "fix point equation"

$$(9) \quad \mu_t = \overset{o}{\mu} \circ T_{0,t}[\mu_\bullet] , \quad t \in [0,T]$$

which is to interpret as

$$\mu_t(M) = \overset{o}{\mu}(T_{0,t}[\mu_\bullet]M) \quad \text{for every Borel set } M \subset \mathbb{R}^6 .$$

(9) is a measure theoretic formulation of (2), when V is given by (7) as the following lemma shows:

Lemma: If $\overset{o}{\mu}$ is absolutely continuous with density $\overset{o}{f}$, then any solution μ_t of (9) is absolutely continuous with density $f(t,\cdot)$, which is a weak solution (in the sense defined by Lax for conservation laws) of the initial value problem

$$(10) \quad \frac{\partial f}{\partial t} + \mathrm{div}_p(f \cdot V) = 0$$
$$f(0,\cdot) = \overset{o}{f}$$

Remarks:

1. If V is given by (7), equation (10) is the Vlasov equation (2).

2. A weak solution of (10) is defined to be a function f with

   (i) $f(t,\cdot)$ is weakly continuous in $[0,T]$
   (ii) for all test functions $w \in C^1([0,T] \times \mathbb{R}^6)$ with
         supp $w \subset [0,T) \times K$, K compact in $\mathbb{R}^6$,

the equation

$$(11) \quad \int_0^T \int_{\mathbb{R}^6} f[\frac{\partial w}{\partial t} + \langle V, \mathrm{grad}_p w \rangle] dP dt + \int_{\mathbb{R}^6} w(0,\cdot)\overset{o}{f} dP = 0$$

holds. f is a classical solution of (10), if $\overset{o}{f}$ and V is sufficiently smooth.

Remarks on the proof of the Lemma: The main point is to prove that μ_t is absolutely continuous, if $\overset{o}{\mu}$ is. But that follows by a rather unknown statement of Rademacher from 1916, that $\mu \circ T^{-1}$ is absolutely continuous together with μ, if T is a bijective measurable mapping and T^{-1} is locally Lipschitz continuous. So putting $T = T_{o,t}[\mu.]$, $\mu = \overset{o}{\mu}$, the absolute continuity of μ_t follows. The rest is essentially done by direct calculation.

The lemma shows, that (9) is a reasonable generalization of (2). But (10) includes also the N-body problem given by (3):

If we put $\mu_t = \delta_{\omega_N}(t)$, then $\int_{\mathbb{R}^6} G(x,y) d\delta_{\omega_N}(t) = \frac{1}{N} \sum_{j=1}^{N} G(x, x_j(t))$

and $P_i(t)$ is given as the solution of

$$P = V[\delta_{\omega_N}(\cdot)](t,P), \quad P(0) = \overset{o}{P}_i, \quad i=1,\ldots,N$$

where V has the form (7). It follows by (8), that

$$(12) \qquad \dot{P}_i(t) = T_{t,o}[\delta_{\omega_N}(\cdot)]\overset{o}{P}_i .$$

Therefore a solution of the N-body problem (3) is a solution of equation (12) and (12) now is equivalent to

$$(13) \qquad \delta_{\omega_N}(t) = \delta_{\overset{o}{\omega_N}} \circ T_{o,t}[\delta_{\omega_N}(\cdot)]$$

which is just (9) with $\overset{o}{\mu} = \delta_{\overset{o}{\omega_N}}$. To verify the last equivalence of (12) and (13), just calculate for an arbitrary Borel set M

$$\delta_{P_i(t)}(M) = \delta_{T_{t,o}[\delta_{\omega_N}(\cdot)]P_i}(M)$$

$$= \begin{cases} 1 & \text{if } T_{t,o}[\delta_{\omega_N}(\cdot)]\overset{o}{P}_i \in M \\ 0 & \text{else} \end{cases} = \begin{cases} 1 & \text{if } \overset{o}{P}_i \in T_{o,t}[\delta_{\omega_N}(\cdot)]M \\ 0 & \text{else} \end{cases}$$

$$= \delta_{\overset{o}{P}_i}(T_{o,t}[\cdot]M) = \delta_{\overset{o}{P}_i} \circ T_{o,t}[\delta_{\omega_N}(\cdot)](M) .$$

Summing up all i and dividing by N gives (13).

Conclusion: Both, the discrete N-body problem as well as the initial value problem for the Vlasov equation are special cases of equation (9), which correspond to different initial distributions $\overset{o}{\mu}$ and $\delta_{\overset{o}{\omega}_N}$ respectively. Our aim is to compare the solutions of these two special cases: We have to prove, that the solution of (9) is continuously depending on the initial data $\overset{o}{\mu}$, i.e. that

$$\delta_{\omega_N}(t) \longrightarrow \mu_t \quad \text{if} \quad \delta_{\overset{o}{\omega}_N} \longrightarrow \overset{o}{\mu} \quad \text{for } N \to \infty .$$

Now we need the second condition on V.

(II) The mapping $V : \mu_{\bullet} \longrightarrow V[\mu_{\bullet}]$ for $\mu_{\bullet} \in C_M$ is Lipschitz continuous in the following sense:

$$(14) \qquad \int_{\mathbb{R}^6} \| V[\mu_{\bullet}](t,P) - V[\nu_{\bullet}](t,P) \| \, d\mu_t(P) \leq K \, \rho_1(\mu_t, \nu_t)$$

for fixed K and all $\mu_{\bullet}, \nu_{\bullet} \in C_M$.

That condition (II) is fulfilled by V given in (7):

$$\| V[\mu_{\bullet}](t,P) - V[\nu_{\bullet}](t,P) \| = \| \int_{\mathbb{R}^6} G(x,y)(d\mu_t(y,v) - d\nu_t(y,v)) \|$$

$$= 2L \cdot B \| \int_{\mathbb{R}^6} \frac{G(x,y)+B}{2LB} (d\mu_t(y,v) - d\nu_t(y,v)) \| ,$$

where B is the bound for $\| G(x,y) \|$ and L the Lipschitz constant of G (and V) (remember, that $\int_{\mathbb{R}^6} (d\mu_t - d\nu_t) = 0$, since μ_t and ν_t are probability measures).

Now $\frac{G(x,\cdot)+B}{2LB}$ is for each x in class D used in definition (4) if $L \geq 1$; therefore we get

$$\| V[\mu_{\bullet}](t,P) - V[\nu_{\bullet}](t,P) \| \leq 2LB \, \rho_1(\mu_t, \nu_t).$$

Since the right hand side is independent of P and $\overset{o}{\mu}$ is again a probability measure, we get (14) with $K = 2LB$.

The rest of the proof of theorem 1 is as follows, where we write $\overset{o}{\nu}$ for $\delta_{\overset{o}{\omega}_N}$, ν_t for $\delta_{\omega_N}(t)$:

$$\rho_1(\mu_t, \nu_t) = \rho_1(\overset{o}{\mu} \circ T_{o,t}[\mu_\cdot], \overset{o}{\nu} \circ T_{o,t}[\nu_\cdot])$$

$$\leq \rho_1(\overset{o}{\mu} \circ T_{o,t}[\mu_\cdot], \overset{o}{\mu} \circ T_{o,t}[\nu_\cdot]) + \rho_1(\overset{o}{\mu} \circ T_{o,t}[\nu_\cdot], \overset{o}{\nu} \circ T_{o,t}[\nu_\cdot])$$

The first term on the right hand side might be estimated by using definition (4).

$$\rho_1(\overset{o}{\mu} \circ T_{o,t}[\mu_\cdot], \overset{o}{\mu} \circ T_{o,t}[\nu_\cdot]) = \sup_{\varphi \in D} \left| \int_{\mathbb{R}^6} \varphi d(\overset{o}{\mu} \circ T_{o,t}[\mu_\cdot]) - \varphi d(\overset{o}{\mu} \circ T_{o,t}[\nu_\cdot]) \right|$$

$$= \sup_{\varphi \in D} \left| \int_{\mathbb{R}^6} (\varphi \circ T_{t,o}[\mu_\cdot] - \varphi \circ T_{t,o}[\nu_\cdot]) d\mu_o \right|$$

(since $\varphi \in D$)

$$\leq \sup \int_{\mathbb{R}^6} \| T_{t,o}[\mu_\cdot] - T_{t,o}[\nu_\cdot] \| d\mu_o =: \lambda(t)$$

Now $T_{t,o}[\mu_\cdot]P$, $T_{t,o}[\nu_\cdot]P$ are solutions of the initial value problems

$$\dot{P} = V[\mu_\cdot](t,P) \quad \text{and} \quad \dot{P} = V[\nu_\cdot](t,P), \quad P(0) = P \quad \text{respectively.}$$

Using that and condition (II) for V we get an estimate of $\lambda(t)$:

$$\lambda(t) = \int_{\mathbb{R}^6} \| T_{t,o}[\mu_\cdot]P - T_{t,o}[\nu_\cdot]P \| d\mu_o$$

$$= \int_{\mathbb{R}^6} \left\| \int_o^t V[\mu_\cdot](\tau, T_{\tau,o}[\mu_\cdot]P) d\tau - \int_o^t V[\nu_\cdot](\tau, T_{\tau,o}[\nu_\cdot]P) d\tau \right\| d\mu_o$$

$$\leq \int_{\mathbb{R}^6} \left\| \int_o^t [V[\mu_\cdot](\tau, T_{\tau,o}[\mu_\cdot]P) - V[\nu_\cdot](\tau, T_{\tau,o}[\mu_\cdot]P)] d\tau \right\| d\mu_o +$$

$$+ \int_{\mathbb{R}^6} \left\| \int_o^t [V[\nu_\cdot](\tau, T_{\tau,o}[\mu_\cdot]P) - V[\nu_\cdot](\tau, T_{\tau,o}[\nu_\cdot]P)] d\tau \right\| d\mu_o$$

$$\leq \int_{\mathbb{R}^6} (\int_o^t \| \ldots \| d\tau) d\mu_o + \int_{\mathbb{R}^6} (\int_o^t \| \ldots \| d\tau) d\mu_o = \text{(Fubini)}$$

$$= \int_o^t (\int_{\mathbb{R}^6} \| V[\mu_\cdot](\tau, T_{\tau,o}[\mu_\cdot]P) - V[\nu_\cdot](\tau, T_{\tau,o}[\mu_\cdot]P) \| d\mu_o(P) d\tau +$$

$$+ \int_{\mathbb{R}^6} (\int_o^t \| V[\nu_\cdot](\tau, T_{\tau,o}[\mu_\cdot]P) - V[\nu_\cdot](\tau, T_{\tau,o}[\nu_\cdot]P) \| d\tau d\mu_o$$

$$\int_o^t (\int_{\mathbb{R}^6} \| V[\mu_\cdot](\tau,Q) - V[\nu_\cdot](\tau,Q) \| d\mu_\tau(Q)) d\tau +$$

$$+ L \int_o^t \int_{\mathbb{R}^6} \| T_{\tau,o}[\mu_\cdot]P - T_{\tau,o}[\nu_\cdot]P \| d\mu_o(P) d\tau \quad \text{(II)}$$

$$\leq K \int_0^t \rho_1(\mu_\tau, \nu_\tau) d\tau + L \int_0^t \lambda(\tau) d\tau \quad .$$

So we have

$$\lambda(t) \leq K \int_0^t \rho_1(\mu_\tau, \nu_\tau) d\tau + L \int_0^t \lambda(\tau) d\tau \quad \text{and}$$

the wellknown lemma of Gronwall provides

$$\lambda(t) \leq K e^{Lt} \int_0^t \rho_1(\mu_\tau, \nu_\tau) e^{-L\tau} d\tau$$

The second term in the estimation for $\rho_1(\mu_t, \nu_t)$ is more simple:

$$\rho_1(\overset{o}{\mu} \circ T_{o,t}[\nu_\cdot], \overset{o}{\nu} \circ T_{o,t}[\nu_\cdot]) = \sup_{\varphi \in D} | \int \varphi \circ T_{o,t}[\nu_\cdot] (d\overset{o}{\mu} - d\overset{o}{\nu}) |$$

Since $T_{o,t}[\nu_\cdot]$ is Lipschitz continuous with constant e^{Lt},
$e^{-Lt}(\varphi \circ T_{o,t}[\nu_\cdot])$ is in D and we therefore get as a bound for the second term $e^{Lt} \rho_1(\mu_o, \nu_o)$.

Putting things together we have

$$\rho_1(\mu_t, \nu_t) \leq e^{Lt} \rho_1(\mu_o, \nu_o) + K e^{Lt} \int_0^t \rho_1(\mu_\tau, \nu_\tau) e^{-L\tau} d\tau \quad .$$

Applying Gronwalls lemma again, we get

$$\rho_1(\mu_t, \nu_t) \leq e^{(K+L)t} \rho_1(\overset{o}{\mu}, \overset{o}{\nu}) \quad .$$

So, if $\overset{o}{\nu} = \delta_{\overset{o}{\omega}_N}$ tends to $\overset{o}{\mu}$, $\nu_t = \delta_{\omega_N(t)}$ tends to μ_t, which is the statement of the theorem.

Remarks:

1. One recognizes, that the solution of (9) depends even in a Lipschitz continuous way with Lipschitz constant $e^{(K+L)t}$ on the initial data $\overset{o}{\mu}$. But one also can realize that the boundedness and the Lipschitz continuity of G is essential: Since K = 2LB, both properties expressed by B and L are used.

2. Theorem 1 covers by far more than the special situation given by the Vlasov equation. Especially V may depend in a rather general way on μ_o, whereas in the Vlasov case it depends

linearly on $f(t, \cdot)$.

3. The crucial assumption is condition (II). There the integration with respect to μ_t may be substituted by the integration with respect to $\delta_{\omega_N(t)}$.

2. Existence and uniqueness for the modified Boltzmann equation

So far, we have assumed the existence of at least a measure theoretic solution of the modified Vlasov equation (2). It is now easy to establish existence and uniqueness in using the theory we have developed in lecture 1.

Theorem 2: If G is bounded and globally Lipschitz, then for any $\overset{o}{f} \in L^1$ the modified Vlasov equation (2) has a global unique solution, which is a weak solution in the sense defined in (11). The solution is classical, if $\overset{o}{f}$ and G are in addition continuously differentiable.

Again we prove a more general statement: If V fullfills conditions (I) and (II), given in lecture 1, then the measure theoretic equation (9) has a unique weakly continuous solution $\mu_{\bullet}$ in any interval $[0,T]$.

Theorem 2 follows from that statement, since V given by (7) fullfills conditions (I) and (II) and, due to the lemma, a solution of (9) with absolutely continuous $\overset{o}{\mu}$ gives a weak solution of (2).

Proof: In M we defined the metric ρ_1; it induces a metric in C_M, the set of all weakly continuous functions $t \longrightarrow \mu_t \in M$, $t \in [0,T]$, where T is arbitrary, but fixed:

$$d_\alpha(\mu_{\bullet}, \nu_{\bullet}) := \sup_{t \in [0,T]} \rho_1(\mu_t, \nu_t) e^{-\alpha t}$$

Here $\alpha > 0$ may be chosen freely. Since (M, ρ_1) is a complete metric space (Kellerer), the same is true for (C_M, d_α).

We want to solve the equation (9)

$$\mu_t = \mu_o \circ T_{o,t}[\mu_{\bullet}]$$

To this end, we introduce the operator

$$(15) \quad A : C_M \longrightarrow C_M \text{ with } (A\mu_{\bullet})(t) := \mu_o \circ T_{o,t}[\mu_{\bullet}], \quad t \in [0,T] .$$

Here we have to prove, that $A\mu_{\bullet} \in C_M$, i.e. that

$$t \longrightarrow A\mu_{\bullet}(t), \quad t \in [0,T]$$

is weakly continuous. But this is an immediate consequence of a

well known theorem, which says, that $\mu_o \circ T_{o,t}[\mu_{\textbf{.}}]$ is weakly continuous, if

$$\mu_o(\{P = \mathbb{R}^6 \mid T_{o,\textbf{.}} \text{ is continuous at } (t,P)\}) = 1.$$

Since $T_{o,\textbf{.}}$, considered as a function of (t,P) is everywhere continuous (it is a solution of an ordinary differential equation system), $\mu_o \circ T_{o,t}[\mu_{\textbf{.}}]$ is weakly continuous for each μ_o.

Now we simply have to show, that A is for suitable α a contractive mapping in (C_M, d_α).

First we get for arbitrary $\mu_{\textbf{.}}, \nu_{\textbf{.}} \in C_M$

$$\rho_1(A\mu_{\textbf{.}}(t), A\nu_{\textbf{.}}(t)) = \rho_1(\overset{0}{\mu} \circ T_{o,t}[\mu_{\textbf{.}}], \overset{0}{\mu} \circ T_{o,t}[\nu_{\textbf{.}}])$$

$$\leq Ke^{Lt} \int_0^t \rho_1(\mu_\tau, \nu_\tau) e^{-L\tau} d\tau$$

as we already saw in the proof of theorem 1.

Therefore
$$d_\alpha(A\mu_{\textbf{.}}, A\nu_{\textbf{.}}) \leq K \sup_{t \in [0,T]} e^{(L-\alpha)t} \int_0^t \rho_1(\mu_\tau, \nu_\tau) e^{-L\tau} d\tau$$

$$\leq K \sup_{t \in [0,T]} e^{(L-\alpha)t} \int_0^t d_\alpha(\mu_{\textbf{.}}, \nu_{\textbf{.}}) e^{-(L-\alpha)\tau} d\tau$$

$$= \frac{K}{\alpha - L} d_\alpha(\mu_{\textbf{.}}, \nu_{\textbf{.}}) \quad \text{for } \alpha > L$$

Choosing $\alpha = K + L + \delta$ for $\delta > 0$ yields

$$d_\alpha(A\mu_{\textbf{.}}, A\nu_{\textbf{.}}) \leq \frac{1}{1 + \frac{\delta}{K}} d_\alpha(\mu_{\textbf{.}}, \nu_{\textbf{.}})$$

so A is a contractive operator and from the Banach fixpoint theorem the statement of the theorem follows.

Remark: The proof of theorem 2 is of Picard-Lindelöf type. One may also make a proof of Peano type: For $\overset{0}{\mu} = \delta_{\omega_N}$ the solution $\delta_{\omega_N}(t)$ of (9) exists. One can show by using Prohorovs theorem on compactness in M, that $\{\delta_{\omega_N}(t), n \in \mathbb{N}\}$ is relatively compact in M for each t. Therefore for each t a sequence $(n_j(t))_{j \in \mathbb{N}}$ exists, such that $\delta_{\omega_{n_j}(t)}(t)$ converges to a measure μ_t. Now one has to show, that

n_j can be chosen independently of t, so that

$$\delta_{\omega_{n_j}(t)} \longrightarrow \mu_t \qquad \text{for all } t \in [0,T].$$

That μ_t gives a solution of (9). The advantage of that procedure is not very big; one can weaken a little bit condition (I). But we need that kind of arguments in order to prove the existence of a weak solution for the non-modified Vlasov equation.

Historical remarks: The first existence proof for the modified Vlasov equation was given by Batt in 1963, who also introduced the modification. He had stricter assumptions concerning $\overset{o}{f}$. The theorems 1 and 2 given here were proved by me in 1975. Braun and Hepp in 1977 as well as Dobrushin in 1978 were not aware of these results and proved some slightly weaker results using the metric ρ_2.

3. Existence and uniqueness for the unmodified Vlasov equation

The research concerning existence and uniqueness for the unmodified Vlasov equation proceeded in two different directions:

(a) One looks for weak solutions, gets existence for dimension 3, but uniqueness is not proved up till now. Existence theorems for weak solutions have been shown by Arseneev and Illner and myself.

(b) One looks for classical solutions, for which uniqueness is easy to show. But one has to make some assumption on the initial data, which are restricting. So for $\overset{o}{f}$ depending only on two space and two velocity variables or for f being in a certain way symmetric, one gets existence. These results are mainly due to E. Horst (1980), but Batt did the first step in that direction; also Ukai, Okabe and Wollmann showed the existence of classical solutions for the "two-dimensional" case.

Since all the proofs of these theorems are rather long and technical, I just try to give the results and some impressions of the ideas lying behind.

In both cases the idea is initially the same: In order to solve (2) with initial condition $f(0,\cdot) = \overset{o}{f}$ and G given (1) one first solves the Vlasov equation with a modified G_δ - we call that equation now (2_δ) - and the same initial value $\overset{o}{f}$; the solution of that problem, which for any $\overset{o}{f} \in L^1$ exists as theorem 2 shows, is denoted by f_δ. Letting δ tend to 0, one hopes to get a solution of (2). The difference of both cases lies in the kind of convergence one has to prove: Whereas in (a), where $\overset{o}{f}$ may be chosen arbitrarily in L^1, one only has to prove weak convergence of f_δ, in case (b), where $\overset{o}{f}$ has to be continuously differentiable one has to show uniform convergence.

But there is another thing in common for both cases: One has to show, that, if

$$E_o = \int_{\mathbb{R}^6} \|v\|^2 \overset{o}{f}(x,v)\,dxdv < \infty$$

then there exists a constant C, not depending on t and δ, so that
the kinetic energy of the gas fullfills the estimate

$$(16) \qquad E_\delta(t) := \int_{\mathbb{R}^6} \|v\|^2 f_\delta(t,x,v)\,dxdv \le C.$$

(16) is proved by showing that energy conservation holds:
Let $U_\delta(x,y)$ be the potential of G_δ, i.e.

$$G_\delta(x,y) = +\nabla_x U_\delta(x-y)$$

(for example $U_\delta(x-y) := -\gamma \cdot \int_{\mathbb{R}^3} \frac{1}{\|x-z\|} \omega_\delta(z-y)\,dz$ for one kind of

mollifying or $U_\delta(x-y) = \dfrac{-\gamma}{(\|x-y\|^2 + \delta)^{1/2}}$ for the other)

and define the potential energy of the solution f_δ by

$$(17) \qquad V_\delta(t) := \int_{\mathbb{R}^6} \left(\int_{\mathbb{R}^3} U_\delta(x-y) f_\delta(t,y,v)\,dydv \right) f_\delta(t,x,v)\,dxdv;$$

then

$$(18) \qquad E_\delta(t) + V_\delta(t) = E_0 + V_\delta(0)$$

holds for all $t \ge 0$.
(18) is rather easy to prove by using (9).
(16) follows immediately from (18) in the plasmaphysical case,
where $\gamma < 0$: $U_\delta(x-y)$ is then nonnegative, so this is true for $V_\delta(t)$
and

$$E_\delta(t) = E_0 + V_\delta(0) - V_\delta(t) \le E_0 + V_\delta(0),$$

where $V_\delta(0)$ may be bounded uniformly with respect to δ.

(16) is harder to prove in the stellardynamic case, where $\gamma > 0$.
But nevertheless it is true as was shown by Horst in using some
Sobolev inequalities.

Let us now present the existence theorems:

Theorem 3: If $\overset{0}{f} \in L^1$ has the following properties:

(i)   There exists a M such that $0 \leq \overset{0}{f}(P) \leq M$ a.e. in $\mathbb{R}^6$

(ii) $\int_{\mathbb{R}^6} \|P\|^2 \overset{0}{f}(P)\,dP < \infty$

(iii) $\overset{0}{\rho} : x \longrightarrow \int \overset{0}{f}(x,v)\,dv$ is essentially bounded.
then a weak solution $f(t,\cdot)$ of the initial value problem for (2) exists.

Sketch of the proof:

1. step: Let $\mu_{\bullet}^{\delta} : t \longrightarrow \mu_t^{\delta}$, $t \in [0,T]$ the solution of (9) with molli-fied G_{δ}. Then one shows, that the set

$$\{\mu_t^{\delta} \mid t \in [0,T], \ \delta > 0\}$$

is uniformly tight in M; that means: to every $\varepsilon > 0$ there exists a $R > 0$, such that

$$\mu_t^{\delta}(\mathbb{R}^6 \setminus K_R) < \varepsilon \quad \text{for all } t \in [0,T], \ \delta > 0;$$

here $K_R := \{P \in \mathbb{R}^6 \mid \|P\| \leq R\}$.
This is shown by considering

$$h_{\delta}(t) := \int_{\mathbb{R}^6} \|x\|^2 f_{\delta}(t,x,v)\,dx\,dv;$$

using, that f_{δ} is a solution of (9) and using (16) one gets

$$0 \leq h_{\delta}(t) \leq C' \quad \text{independent on } \delta > 0 \text{ and } t \in [0,T].$$

Now, for any $R > 0$

$$R^2 \mu_t^{\delta}(\mathbb{R}^6 \setminus K_R) = R^2 \int_{\mathbb{R}^6 \setminus K_R} f_{\delta}(t,P)\,dP \leq \int_{\mathbb{R}^6 \setminus K_R} \|P\|^2 f_{\delta}(t,P)\,dP$$

$$\leq \int_{\mathbb{R}^6} \|P\|^2 f_{\delta}(t,P)\,dP = h_{\delta}(t) + E_{\delta}(t) \leq C + C'$$

again using (16); therefore $\mu_t^{\delta}(\mathbb{R}^6 \setminus K_R)$ can be made smaller than an arbitrary $\varepsilon > 0$ by choosing R large enough.

Now Prohorovs theorem shows, that for any countable dense subset $T' \in [0,T]$ there is a monotone sequence $(\delta_n)_{n \in \mathbb{N}}$, $\delta_n \searrow 0$, such that

$$\mu_t^n := \mu_t^{\delta_n} \longrightarrow \mu_t \in M \quad \text{for } n \to \infty, \ t \in T'.$$

We have next to show, that the convergence holds also for $t \notin T'$.

2. step: We show, that for any continuously differentiable φ with compact support, the set of functions

$$t \longrightarrow \phi_n(t) := \int_{\mathbb{R}^6} \varphi \, d\mu_t^n, \quad t \in [0,T]$$

is equicontinuous. This is a technical proof, using mainly the boundedness of $\overset{o}{f}$ and therefore also of f_δ.

Now, since ϕ_N converges pointwise on T' and forms an equicontinuous set, it converges uniformly on $[0,T]$. The limit has to be

$$\lim_{n \to o} \phi_n(t) = \int_{\mathbb{R}^6} \varphi \, d\mu_t$$

with $\mu_t \in M$ defined for all $t \in [0,T]$. It follows immediately, that

$$\mu_t^n \longrightarrow \mu_t \quad \text{for } t \in [0,T]$$

and that $\mu_{\cdot} : t \longrightarrow \mu_t$ is weakly continuous.

It is also easy to show, that μ_t is absolutely continuous with a density $f(t,\cdot)$ essentially bounded by M.

3. step: What stays is a long but straightforward proof, that $f(t,\cdot)$, constructed in step 2 is a weak solution of the initial value problem for (2). We will not go into details.

Remark: The existence proof sketched here is not constructive, since it uses a compactness argument and we are not able to show, that for any sequence $(\delta_n)_{n \in \mathbb{N}}$, $\delta_n \searrow 0$ the measures $\mu_t^{\delta_n}$ converge to μ_t. This has two consequences:

(i) We are not able to prove uniqueness - we will even not make any conjecture on that problem.

(ii) Since it is not possible to show, that $\rho_1(\mu_t^\delta, \mu_t)$ is small for small δ, we cannot prove, that the unmodified Vlasov equation is strictly derivable - even for the plasmaphysical case, where the discrete problem has a global solution. It might be true, that the equation is only stochastically derivable - a notion, which we will not define here, since it is more connected with the Boltzmann equation.

We now turn to the question of existence of classical solutions, i.e. to Horst's work. Here we will always assume, that $\overset{o}{f}$ is continuously differentiable and has compact support.

As already mentioned there is up till now no existence theorem for the full 3-dimensional problem. Therefore we have to consider the following lower dimensional cases:

(A) The function f in equation (2) depends only on 2 space and 2 velocity variables. Then equation (1) has to be changed into

$$G(x,y) = \gamma \nabla_y \ln \|x-y\| = -\gamma \frac{x-y}{\|x-y\|^2}$$

(B) Let H be the group of all orthogonal transformations S of $\mathbb{R}^3$ with det $S = 1$. A function $g : (x,y) \longrightarrow \mathbb{R}$, $(x,v) \in \mathbb{R}^6$ is called to be spherically symmetric if $g(Sx,Sv) = g(x,v)$ for all $S \in H$, $(x,v) \in \mathbb{R}^6$.

Now it is easy to show, that if $\overset{o}{f}$ is spherically symmetric, a solution of the corresponding initial value problem for (2) is also spherically symmetric. So the question occurs, whether such a solution for a spherically symmetric initial condition exists.

(C) Let Z be the group of orthogonal transformations Z_δ of $\mathbb{R}^3$, which are represented by matrices of the form

$$Z_\delta := \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad 0 \le \theta \le 2\pi$$

A function g is called rotationally symmetric, if $g(Z_\delta x, Z_\delta) = g(x,v)$ for all $(x,v) \in \mathbb{R}^6$, $Z_\delta \in Z$. Again for $\overset{o}{f}$ rotationally symmetric the corresponding solution $f(t,\cdot)$ of (2) is also rotationally symmetric and one may ask for the existence of such solutions. This problem may be considered as a 5-dimensional version of the original 6-dimensional problem. The special part of the z-axis given by the form of Z_δ is not essential.

Theorem 4: In each case (A), (B) or (C) there exists a global classical solution, which is unique.

Sketch of the proof: One again starts with the solutions f_δ of the modified problem. The main tool is the following lemma, which was in a similar form first proved by Batt.

Lemma: There exists a unique classical solution of the initial value problem for (2) in $[0,T]$, if and only if

$$(19) \qquad \sup\{ \int_{\mathbb{R}^3} f_\delta(t,x,v)dv \mid t \in [0,T], \ x \in \mathbb{R}^3, \delta > 0\} < \infty$$

The proof of that lemma is rather long but purely technical. One gets an estimate of the form

$$\| T_{t,\tau}[\mu_\bullet^\delta]P - T_{t,\tau}[\mu_\bullet^{\tilde\delta}]P \| \le C |\delta^{\frac{1}{4}} - \tilde\delta^{\frac{1}{4}}| \ ,$$

C depending only on the length T of the time interval.
Therefore $T_{t,\tau}[\mu_\bullet^\delta]P$ converges uniformly with respect to $t, \tau \in [0,T]$, $P \in \mathbb{R}^6$ for δ tending to zero to a mapping $T_{t,\tau}(P)$.
Defining $f(t,P) := \overset{o}{f}(T_{0,t}P)$ one gets a solution of (2).
Since we have now full convergence for δ tending to zero (in contrast to the situation with weak solutions, where we only know the existence of a sequence $(\delta_j)_{j \in \mathbb{N}}$ for which $\mu_\bullet^{\delta_j}$ and $T_{t,0}[\mu_\bullet^{\delta_j}]$ converges), uniqueness also follows immediately.

For the rest one needs again (16), i.e. the fact, that the kinetic energy of f_δ is bounded uniformly with respect to δ. Applying Hölder's inequality, one gets for

$$\rho_\delta(t,x) = \int_{\mathbb{R}^3} f_\delta(t,x,v)dv$$

that $\qquad \| \rho_\delta(t,\cdot) \|_{\frac{5}{3}} \le K \| \overset{o}{f} \|_\infty^{\frac{2}{5}} \cdot E_\delta^{\frac{3}{5}} \le C_1 \ .$

But in order to use the lemma, we need a uniform bound for $\| \rho_\delta \|_\omega$.
Therefore one proceeds as follows: We estimate

$$(20) \qquad K_\delta(t,x) := \int_{\mathbb{R}^3} G_\delta(x-y) f_\delta(t,y,v)dydv = \int_{\mathbb{R}^3} G_\delta(x-y)\rho_\delta(t,y)dy$$

by Sobolev's inequality to get

(21) $\qquad \| X_\delta(t,\cdot) \|_\infty \leq C_2 \| \rho_\delta \|_\infty^{\frac{4}{9}}$

But if we are aware of the symmetry of f_δ, for example the spherical symmetry, one gets better estimates:

(22) $\qquad | K_\delta(t,x) | \leq C_3 \dfrac{1}{\sqrt{x_1^2+x_2^2}} \| \rho_\delta \|_\infty^{\frac{1}{6}}$.

There the symmetry comes in! Now recall, that $T_{t,o}[\mu_\bullet^\delta]P$ is a solution of the system

$$\dot{x} = v$$
$$\dot{v} = K_\delta(t,x).$$

Therefore estimates for E_δ give estimates for $T_{t,o}[\mu_\bullet^\delta]$. Considering only the last three components of $T_{t,o}[\mu_\bullet^\delta]$, which correspond to the velocities and estimating

$$G_j^\delta := \sup\{ | (T_{t,o}[\mu_\bullet^\delta]P)_{3+j} - P_{3+j} | \ | P \in \mathbb{R}^6, t \in [0,T]\}$$

one gets, that if $| K_\delta(t,x) | \leq g(x_j)$ for some $j \in \{1,2,3\}$ and $g \in L^p$, then

$$G_j^\delta \leq C_4 \| g \|_p^{\frac{p}{p+1}} ,$$

C_4 only depending on p and T.
Now using (21) for G_3^δ gives

$$G_3^\delta \leq C_5 \| \rho_\delta \|_\infty^{\frac{4}{9}} .$$

Using (22) for G_1^δ and G_2^δ gives

$$G_1^\delta, G_2^\delta \leq C_6 \| \rho_\delta \|_\infty^{\frac{11}{45}} .$$

The exponents look strange but are important. Since

$$\rho_\delta(t,x) = \int_{\mathbb{R}^3} \overset{o}{f}(T_{o,t}[\mu_\bullet^\delta]P) dv$$

these estimates for G_j^δ can be used again to give estimates for ρ_δ. One gets

$$\| \rho_\delta \|_\infty \leq \alpha (G_1^\delta + \beta)(G_2^\delta + \beta)(G_3^\delta + \beta)$$

$(\alpha, \beta$ independent of $\delta)$ and therefore

(23) $\|\rho_\delta\|_\infty \leq O(\|\rho_\delta\|_\infty^{\frac{14}{15}})$ uniformly in δ.

Note, that $\frac{14}{15} = \frac{4}{9} + \frac{11}{45} + \frac{11}{45}$ - and note that without symmetry we would have instead of $\frac{14}{15}$ a number larger than one: $3 \cdot \frac{4}{9} = \frac{4}{3}$.

You see, that (23) cannot be true, if $\|\rho_\delta\|_\infty$ is not bounded. This proves (19) and therefore theorem 4.

Remark: I gave some of the most important estimates in Horst's work in order to show you how tough the stuff is. The methods end up with an estimate of the type

$$\|\rho_\delta\|_\infty = O(\|\rho_\delta\|_\infty^k)$$

and if $k < 1$, then we are through. In the twodimensional case we have just $k = 2 \cdot \frac{4}{9} < 1$ without symmetries. But for three dimensions this method fails.
It is hard to guess whether there is a unique classical solution of (2) in 3 dimensions. The situation is similar to the situation in the field of Navier-Stokes equations:

In 3 dimensions one has existence but not uniqueness for weak solutions, uniqueness but not existence for classical solutions; in lower dimensions existence and uniqueness of classical solutions.

What about higher dimensions? Nothing is known for Navier-Stokes and nothing is known for Vlasov in the plasmaphysical case. But - here is the only essential difference between $\gamma = +1$ and $\gamma = -1$ - Horst has shown, that there is no global solution in 4 dimensions for the stellardynamic case. Since that result is at least mathematically surprising and since the proof is simple, I state it as

Theorem 5: In 4 dimensions for $\gamma > 0$ there are initial values $\overset{o}{f}$ (positive, continuously differentiable with compact support), such that the corresponding initial value problem for (L) has no global solution.

Remark: 4 dimensions mean 4 space and 4 velocity variables. G has to be then

$$G(x,y) = -\gamma \frac{x-y}{\|x-y\|^4}, \quad x,y \in \mathbb{R}^4.$$

Proof: We consider the moment of inertia for a solution of (2) (for times t, when it exists).

$$(24) \quad mi(t) := \int \|x\|^2 f(t,x,v) \, dx \, dv$$

With V defined like in (17) and with f instead of f_δ and

$$U(x-y) = \frac{\gamma}{2} \frac{1}{\|x-y\|^2}$$

instead of U_δ (V is the potential energy) one gets by straight-forward calculation, that mi is twice continuously differentiable and

$$mi''(t) = 2(E(t) + V(t)) = 2(E(O) + V(O))$$

(here the 3 dimension comes in; in 3 dimensions it would just be $mi''(t) = 2E(t) + V(t)$ and we could not use the energy conservation).

Now choose a $\overset{o}{f}$ such that $E(O) + V(O) < O$. This is possible only in the stellardynamic case, where $\gamma > O$, since $E(O)$ is positive but linear in $\overset{o}{f}$ and $V(O)$ is negative but quadratic in $\overset{o}{f}$. Therefore $mi''(t)$ is a negative constant. mi itself is positive, since $\overset{o}{f}$ and therefore f is positive. This is possible only in certain finite time intervals. So the solution can only exist in a finite time interval (which depends only on $\overset{o}{f}$). That was to be shown.

4. The plasmaphysical case with selfconsistent magnetic field

If the particles considered are electrons, they do not only interact
by means of the electric field but also by the magnetic field gene-
rated by themselves. Therefore instead of looking only at the elec-
tric field

$$E(t,x) = \int_{\mathbb{R}^3} G(x,y)f(t,y,v)\,dy\,dv = \int_{\mathbb{R}^3} G(x,y)\rho(t,y)\,dy$$

in (2), which is a solution of $\operatorname{div} E = -4\pi\gamma\rho$, we have to take into
account the full Maxwell equations. In order to be in accordance
with a familiar form of these equations, we slightly change the
notation.

Instead of (1), (2) the system of equations we have to consider now
is the following system for the function f, the electric field E and
the magnetic field B:

$$(2') \quad \frac{\partial f}{\partial t} + \langle v, \operatorname{grad}_x f \rangle - \langle E[f] + \frac{1}{c} v \times B[f], \operatorname{grad}_v f \rangle = 0$$

$$(1a) \quad \operatorname{div} E = 4\pi(n_e + n_i)$$

$$(1b) \quad \operatorname{div} B = 0$$

$$(1c) \quad \operatorname{rot} E = -\frac{1}{c}\frac{\partial B}{\partial t}$$

$$(1d) \quad \operatorname{rot} B = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c} j$$

where

$$(25) \quad n_e(t,x) := -\int_{\mathbb{R}^3} f(t,x,v)\,dv, \quad j(t,x) := -\int_{\mathbb{R}^3} vf(t,x,v)\,dv$$

and $n_i = n_i(x)$ is a given spatial density of a fixed ion background.

Remarks:

(a) If $B = 0$ and $n_i = 0$, then (2') is essentially (2), provided we
    add to (1a) the boundary condition that E has to vanish at infi-
    nity. Then (1a) has the solution

$$E(t,x) = \int G(x,y)n_e(t,y)\,dy$$

with $\gamma = -1$.

(b) In (2') we write $E[f]$ and $B[f]$ in order to make clear that the
    fields depend on f: n_e and j are moments of f, therefore a solu-

tion of the system (1a) - (1d) depends on f.

(c) We introduced the function n_i representing the spatial density
of a ion gas, while f is the μ-space density for the electrons.
We assume that the ions form a fixed background. Otherwise we
would have to consider two distribution functions f_e and f_i and
two equations of the form (2'), which differ only by a constant
and a sign in front of the third term. This would not cause
principal difficulties but increase the formal complexity of
the system.
We mention that the introduction of a n_i and even of an external
magnetic field, which does not depend on f, into equations
(1) - (2) of the preceding lectures would not have generated
essential troubles and could have been handled easily.

In order not to get lost in the complexity of the problem we are
looking for the simplest geometry in which the typical features of
the problem are present. If f would depend only on one space and one
velocity variable no magnetic field would occur. Therefore we look
for solutions which depend only on one space and on two velocity
variables:

$$f(t,\cdot) : (x_1,v_1,v_2) \longrightarrow f(t,x_1,v_1,v_2) \geqslant 0 \text{ for all } P := (x_1,v_1,v_2) \in \mathbb{R}^3.$$

For such a function n_e and j defined by (25) do not exist. Therefore
we change these definitions into

$$(25') \quad n_e(t,x_1) = - \int_{\mathbb{R}^2} f(t,x_1,v)dv, \quad j(t,x) = - \int_{\mathbb{R}^2} vf(t,x_1,v)dv$$

$$\text{with } v = (v_1,v_2).$$

Furthermore, we add some boundary conditions for E and B, which also
will depend only on x_1:

$$(26a) \quad \lim_{x_1 \to -\infty} E_1(t,x_1) = - \lim_{x_1 \to +\infty} E_1(t,x_1), \quad t \in [0,T]$$

$$(26b) \quad E_2(0,x_1) = \overset{0}{E}_2(x_1), \quad E_3(0,x_1) = 0 \quad \text{for all } x_1 \in \mathbb{R}$$

$$(26c) \quad B_1(0,x_1) = B_2(0,x_1) = 0, \quad B_3(0,x_1) = \overset{0}{B}_3(x_1) \quad \text{for all } x_1 \in \mathbb{R}.$$

It follows that $B_1(t,x_1) = B_2(t,x_1) = E_3(t,x_1) = 0$ everywhere and instead of (1a) - (1d) the simpler system

$$(1a') \qquad \frac{\partial E_1}{\partial x_1} = 4\pi(n_e + n_i)$$

$$(1c') \qquad \frac{\partial E_2}{\partial x_1} + \frac{1}{c}\frac{\partial B_3}{\partial t} = 0$$

$$(1d') \qquad \frac{\partial E_1}{\partial t} = -4\pi j_1, \quad \frac{\partial E_2}{\partial t} + c\frac{\partial B_3}{\partial x_1} = -4\pi j_2.$$

remains to be considered.

(2') becomes

$$(2'') \qquad \frac{\partial f}{\partial t} + v_1\frac{\partial f}{\partial x_1} - (E_1[f] + \frac{1}{c}v_2 B_3[f])\frac{\partial f}{\partial v_1} - (E_2[f] - \frac{1}{c}v_1 B_3[f])\frac{\partial f}{\partial v_2} = 0$$

The initial value problem for the system (2'), (1a'), (1c') and (1d') is unmodified in the sense of lecture 1 but lower dimensional. I suggest that it has a unique classical solution. But this is not yet proved. Only for a "mollified" problem existence and uniqueness is assured and I will sketch that result (see [25]).

Instead of defining n_e and j as in (25') we use a smoothed f_δ instead of f and define

$$(25'') \qquad n_e^\delta(t,x_1) = -\int_{\mathbb{R}^3} \omega_\delta(x_1-\xi)\,f(t,\xi,v)\,d\xi\,dv$$

$$j^\delta(t,x_1) = -\int_{\mathbb{R}^3} v\omega_\delta(x_1-\xi)\,f(t,\xi,y)\,d\xi\,dv.$$

Now our problem is completely stated by the equations (1a'), (1c'), (1d'), (2'') and (25''), where into the equations (1') n_e^δ, j^δ must be inserted instead of n_e, j. We have to add the initial and boundary conditions (26) for B, E and f.

We first solve the equations (1a'), (1c') and (1d') for given $f(t,\cdot)$. Let μ_t be again the measure with the density $f(t,\cdot)$. We have to be a little bit more careful, as μ_t has to be such that

$$j^\delta(t,x_1) = \int_{\mathbb{R}^3} v\dot\omega_\delta(x_1-\xi)\,d\mu_t(\xi,v)$$

exists. But the function $(\xi,v) \longrightarrow v\omega_\delta(x_1-\xi)$, x_1 fixed, is not bounded. That causes some trouble, in contrast to the situation of lecture 1.

We will therefore assume, that the supports of μ_t, $t \in [0,T]$ are uniformly bounded with respect to t; i.e. there exists a ball $\kappa_R = \{P \in \mathbb{R}^3 / \|P\| \leq R\}$ such that

$$\text{supp } \mu_t \in \kappa_R \quad \text{for } t \in [0,T].$$

Now (1a') with the boundary condition (26a) has the solution

$$E_1(t,x_1) = 2\pi \int_{-\infty}^{+\infty} \text{sign}(x_1-y)[n_e^\delta(t,y) + n_i(y)]dy$$

$$= 2\pi \int_{\mathbb{R}^3} (\int_{-\infty}^{+\infty} \text{sign}(x_1-y)\omega_\delta(y-z)\,dy)\,d\mu_t(z,v)$$

$$+ 2\pi \int_{-\infty}^{+\infty} \text{sign}(x_1-y)n_i(y)\,dy.$$

It is easy to check, that E_1 is continuous and bounded in $[0,T] \times \mathbb{R}$, is globally Lipschitz continuous with respect to x with a Lipschitz constant independent of t and $\mu_.$. Furthermore, by doing the same small calculations as in lecture 1 one gets

$$|E_1[\mu.](t,x) - E_1[v.](t,x)| \leq K_1\rho(\mu_t,v_t).$$

For arbitrary $\mu.$ the function E_1 given by (27) will in general not satisfy the first equation in (1d'). We forget that equation for a moment; it will turn out at the end that, if $\mu.$ is a solution of our problem, this equation is automatically satisfied.
Now we have to consider the equations

$$\frac{\partial E_2}{\partial x_1} + \frac{1}{c}\frac{\partial B_3}{\partial t} = 0, \quad \frac{\partial E_2}{\partial t} + c\frac{\partial B_3}{\partial x} = -4\pi j_2^\delta$$

with the initial condition $E_2(0,x_1) = E_2^0(x_1)$, $B_3(0,x_1) = B_3^0(x_1)$.

To solve them we look for a function $A: [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$, so that

$$(28) \qquad B_3 = \frac{\partial A}{\partial x_1}, \quad E_2 = -\frac{1}{c}\frac{\partial A}{\partial t}$$

is a solution. Inserting this "Ansatz" into the equations for E_2, B_3 we get

$$(29) \qquad \frac{\partial^2 A}{\partial x_1^2} - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\frac{4\pi}{c} j_2^\delta$$

i.e. the inhomogenous wave equation. Its solution is

$$A(t,x_1) = -2\pi \int_C^t \left(\int_{x_1-c(t-\tau)}^{x_1+c(t-\tau)} j_2^\delta(\tau,y)\, dy \right) d\tau - \frac{1}{2} \int_{x_1-ct}^{x_1+ct} \overset{o}{E_2}(y)\, dy$$

$$(30)$$

$$+ \frac{1}{2}\left(\int_{-\infty}^{x_1+ct} \overset{o}{B_3}(y)\, dy + \int_{-\infty}^{x_1-ct} \overset{o}{B_3}(y)\, dy \right).$$

E_2 and B_3 given by (28),(30) satisfy similar conditions as E_1: They are globally Lipschitz continuous uniformly with respect to t and μ. and there are constants K_2, K_3, M_2, M_3, such that

$$(31) \qquad |E_2[\mu.](t,x_1) - E_2[\nu.](t,x_1)| \leq K_2\rho(\mu_t,\nu_t) + M_2 \int_0^t \rho(\mu_\tau,\nu_\tau)\, d\tau$$

$$|B_3[\mu.](t,x_1) - E_3[\nu.](t,x_1)| \leq K_3\rho(\mu_t,\nu_t) + M_3 \int_0^t \rho(\mu_\tau,\nu_\tau)\, d\tau \quad .$$

However, these constants K_2, K_3, M_2, M_3 depend on the bound of the supports of μ_t, i.e. on R.
The vectorfield $V[\mu.]$, defined by

$$V[\mu.](t,x_1,v) := \begin{pmatrix} v_1 \\ -E_1[\mu.] - \dfrac{1}{c} v_2 B_3[\mu.] \\ -E_2[\mu.] + \dfrac{1}{c} v_1 B_3[\mu.] \end{pmatrix} \quad .$$

satisfies condition (I) of lecture 1; condition (II) has to be generalized to

$$(II') \qquad \int_{\mathbb{P}^k} \|V[\mu.](t,P) - V[\nu.](t,P)\|\, d\mu_t(P) \cdot$$

$$\cdot K\rho(\mu_t,\nu_t) + M \int_0^t \rho(\mu_\tau,\nu_\tau)\, d\tau .$$

There k = 3; the additional term $M \int_0^t \rho(\mu_\tau, \nu_\tau) \, d\tau$ causes no trouble in any proof. What is really unpleasant is the fact, that (I) and (II') are only satisfied, if μ_t, ν_t, $t \in [0,T]$ have uniformly bounded support. We need an a priori estimate: If $\mu.$ is in $[0,T]$ a solution of the initial value problem with $\overset{0}{\mu}$ having bounded support, then μ_t, $t \in [0,T]$ has uniformly bounded support.

In order to prove this we consider again the kinetic energy

$$(16') \qquad W(t) := \int_{\mathbb{R}^3} \|v\|^2 \, d\mu_t(x_1, v)$$

(We use W instead of E in order to avoid confusion with the electric field). Using the fact, that $\mu.$ is a solution of (2") we get

$$\dot{W}(t) = -2 \int_{\mathbb{R}^3} (v_1 E_1(t, x_1) + v_2 E_2(t, x_1)) \, d\mu_t(x_1, v) \; .$$

From (27), (28) and (30), we get

$$W(t) \leq W(0) + 2\alpha \left(t + \int_0^t W(\tau) \, d\tau \right) + 4\pi c_\delta t \cdot$$

$$\cdot \int_0^t \left(\int_{\mathbb{R}^3} |v_2|^2 \, d\mu_\tau(x_1, v) \right) \, d\tau$$

(with $\alpha = c+4\pi$) and finally

$0 \leq W(t) \leq C_1 + C_2 \int_0^t W(\tau) \, d\tau$, where C_1, C_2 are dependent on $W(0)$, T and δ.

An application of Gronwall's lemma yields $0 \leq W(t) \leq A$ for $t \in [0,T]$, where A depends only on $W(0), \delta$ and T.

As $\int_{\mathbb{R}^3} |v_i| \, d\mu_t(x_1, v) \leq 1 + W(t)$, $i = 1, 2$, we get a uniform bound for these integrals and therefore also for E_1, E_2 and B_3. Hence

$$\|V[\mu.](t,P)\| \leq L_1 + L_2 \|P\| \qquad \text{for } t \in [0,T], \; P \in \mathbb{R}^3$$

with constants depending only on $\overset{0}{\mu}, \delta$ and T. It follows that

$$\|T_{t,0}[\mu.]Q\| \leq (\|Q\| + L_1 T) e^{L_2 T}, \qquad t \in [0,T], \; Q \in \mathbb{R}^3$$

Choosing $R := (r + L_1 T) e^{L_2 T}$, we see that for $Q \in \text{supp } \overset{0}{\mu}$ the point

$T_{t,0}[\mu.]Q$ is contained in κ_R and therefore

$$\text{supp } \mu_t = \text{supp}(\overset{o}{\mu} \circ T_{0,t}[\mu.]) \subset \kappa_R.$$

This gives the desired a priori estimate.

The remaining part of an uniqueness and existence proof is simple:
One first shows a local (with respect to time) theorem by one of the
standard methods and continues the solution by using the a priori
estimate. The details can be found in [25]. This result is

Theorem 6: For $\overset{o}{f} \in L^1(\mathbb{R}^3)$ with compact support the corresponding
initial value problem given by (2"), (1a'), (1c'), (1d'),
(25") together with the initial-boundary condition (26)
has in any intervall [0,T] a unique (weak) solution. The
solution is classical if the initial data $\overset{o}{f}$, $\overset{o}{E}_2$ and $\overset{o}{B}_3$
are continuously differentiable.

Remark:

The following questions arise but are not answered till now.

(1) Could a similar theorem be proved without reducing the dimension
of the problem, i.e. for the sixdimensional "mollified" case?
Since no essential use of the dimension is made and the advan-
tage consists only in getting solutions of the Maxwell equations,
which are easy to handle, there should be a good chance for a
positive answer.

(2) Is it possible to take the limit $\delta \longrightarrow 0$ to get solutions for
the unmodified equations? It should be by far easier to get
existence but not uniqueness of weak solutions. (Inspite of not
having shown, that a δ-independent bound for the kinetic energy
W exists) than to prove something about classical solutions.

Theorem 6 might be considered only as a first step in solving a
problem, which is of high interest for applications in plasma
physics.

5. Numerical Methods, in particular Simulation Procedures

There exist a lot of numerical methods in order to solve the
Vlasov equation approximately. Difference schemes, Fourier expan-
sions with respect to x, Hermite polynomials are used as well as
the so-called waterbag model, which uses the fact that if $\overset{o}{f}$ is a
stepfunction, then the solution $f(t,\cdot)$ is of the same kind. As far
as I know, for none of these methods convergence is proved. Besides
these methods plasmaphysicists very often use so-called simulation
methods, especially the "particle-in-cell"-method (PIC), developed
for the Vlasov equation by R. Morse. Since the convergence of this
method can be proved by the methods developed in lecture 1, we will
describe it in more details.

The main idea of a simulation in the kinetic theory is simply as
follows: A real gas consists of say 10^{23} particles. The Vlasov
equation describes a gas of infinitely many particles (in the sense
given in lecture 1). Let's try to create a gas of about 10^4 par-
ticles, whose motion is governed by the rules of classical mechanics
such that this gas behaves "as similar as possible" to the infinite
particle system.

In order to make this more precise, we reformulate the idea:

Let N be the number of particles in a real system, $\overset{o}{\omega}_N$ the initial
state and $\overset{o}{\mu} = \int \overset{o}{f} \, dP$ a good approximation for $\delta_{\overset{o}{\omega}_N}$.

Let n be the number of particles in the "simulation gas". Now let
us try to find an initial state $\overset{o}{\omega}_n$ for these particles, such that
the distance

$$\rho(\delta_{\omega_n}(t), \mu_t)$$

of the state of the simulated n-particle system at time t to the
solution of the Vlasov equation at the same time is as small as
possible.

It should be clear that the only possibility to be successful is
in constructing $\overset{o}{\omega}_n$ for given $\overset{o}{f}$. Then $\delta_{\omega_n}(t)$ is given by solving
the Newtonian system (3), which can be done at least numerically
in a satisfying manner, if n is not too large. But it is again
necessary to smooth the interaction forces or, equivalently, to
"smear out" the particles. This is done by PIC - the particles are

smeared out over cells.

We want to get some information about how the complexity of the method increases if we increase the dimension k; k=3 is the normal case of 3 space and 3 velocity variables. The interaction force is then given by

$$G(x,y) = - \gamma \frac{x-y}{\| x-y \|^k} \quad , \quad k=1,2,3.$$

The steps of PIC are:

<u>Step 1</u>: Choose $\overset{0}{\omega}_n$ such that $\rho(\delta_{\overset{0}{\omega}_n}, \overset{0}{\mu})$ is as small as possible

("as small as possible" means: as small as you can make it):

$$\overset{0}{\omega}_n = \{ (\overset{0}{x}_1, \overset{0}{v}_1), \dots, (\overset{0}{x}_n, \overset{0}{v}_n) \}.$$

<u>Step 2</u>: Instead of ω_δ given in lecture 1 we choose a less smooth but simpler mollifier

$$\omega_\delta(x) = \begin{cases} \dfrac{1}{\gamma_k \delta^k} & \text{for} \quad \| x \| \leq \delta \\ \\ 0 & \text{otherwise} \end{cases} \quad ,$$

where γ_k is the volume of the k-dimensional unit ball, $\gamma_1 = 2, \ \gamma_2 = \pi, \ \gamma_3 = \dfrac{4\pi}{3}$.

We use this ω_δ in order to smear out the particles over balls of radius δ in the x-space (i.e. over "cells"); therefore we substitute the spatial distribution

$$\frac{1}{n} \sum_{j=1}^{n} \delta_{\overset{0}{x}_j}$$

of the n-particle system by a distribution with the spatial density

$$(32) \quad \overset{0}{\rho}_n(x) := \frac{1}{n} \sum_{j=1}^{n} \omega_\delta(x - \overset{0}{x}_j) = \int_{\mathbb{R}^k} \omega_\delta(x-z) d\delta_{\overset{0}{\omega}_n}(z,v)$$

Then

$$\int_{\mathbb{R}^k} G(x,y) \overset{0}{\rho}_n(y) dy = \int_{\mathbb{R}^k} G(x,y) \left(\int_{\mathbb{R}^k} \omega_\delta(y-z) d\delta_{\overset{0}{\omega}_n}(z,v) \right) dy$$

$$= \int_{\mathbb{R}^k} (\int_{\mathbb{R}^k} G(x,y)\omega_\delta(y-z)dy)d\delta_{\overset{0}{\omega}_n}(z,v)$$

$$= \int_{\mathbb{R}^k} G_\delta(x,z)d\delta_{\overset{0}{\omega}_n}(z,v)$$

so that this procedure is equivalent to our former modification, where we substitute G by G_δ.

Remark: Our procedure consists in smearing out a particle over a δ-ball centered at the particle. The normal PIC uses fixed cells not depending on the positions of the particles, but adds afterwards a socalled area weighting, which gives at least for k=1 exactly the same values of $\overset{0}{\rho}_n$. For k > 1 the differences are not important and can be avoided by using another ω_δ.

Step 3: Calculate

$$\overset{0}{E}_n(x) = E_n(0,x) = \int G(x,y)\overset{0}{\rho}_n(y)dy$$

for $x = \overset{0}{x}_j$, j=1,...,n. Since $\overset{0}{\rho}_n$ is a step function, the integration can be done explicitly; one only has to evaluate n values of an explicit function.

Step 4: Propagate the particles for a time step $\Delta t > 0$ by

$$(33) \quad \begin{array}{l} \hat{v}_i(\Delta t) = \overset{0}{v}_i + \Delta t \, E_n(0,\overset{0}{x}_i) \\ \\ \hat{x}_i(\Delta t) = \overset{0}{x}_i + \Delta t \, v_i(\Delta t) \end{array}$$

This provides the first step of a numerical integration of the characteristic equations

$$\dot{x} = v$$
$$\dot{v} = E[\delta_{\omega_n}(\cdot)](t,x).$$

$(\hat{x}_j(\Delta t), \hat{x}_j(\Delta t))$ is an approximation for $T_{\Delta t,0}[\delta_{\omega_n}(\cdot)]\overset{0}{P}_i$.

Choosing the symmetric difference scheme (33) has some advantages. The mapping

$$(\overset{0}{x}_i, \overset{0}{v}_i) \longrightarrow (\hat{x}_i(\Delta t), \hat{v}(\Delta t)) = \hat{P}_i(\Delta t)$$

has some of the properties $T_{\Delta t,0}[\mu\cdot]$ has: It is measure preserving and bijective.

Step 5: Repeat the process with $(\hat{P}_1(\Delta t),\ldots,\hat{P}_n(\Delta t))$ instead of $(\overset{o}{P}_1,\ldots,\overset{o}{P}_N)$ in order to get $\rho_n(\Delta t, x)$, $E_n(\Delta t, x)$ and finally $\hat{P}_1(2\Delta t),\ldots,\hat{P}_n(2\Delta t))$ and so on.

Remark: As the numerical method is uniformly convergent for $\Delta t \to 0$, we have $\|P_i(m\Delta t) - \hat{P}_i(m\Delta t)\| \leq \varepsilon$ for $0 \leq m\Delta t \leq T$, if Δt is small enough. Hence

$$\rho(\delta_{\omega_n}(t), \delta_{\hat{\omega}_n}(t)) = \sup_{\varphi \in D} \left| \int \varphi\, d\delta_{\omega_n}(t) - \int \varphi\, \delta_{\hat{\omega}_n}(t) \right|$$

$$\leq \sup_{\varphi \in D} \frac{1}{n} \sum_{j=1}^{n} \left| \varphi(P_i(t)) - \varphi(\hat{P}_i(t)) \right|$$

$$\leq \max_{i=1,\ldots,n} \|P_i(t) - \hat{P}_i(t)\| < \varepsilon$$

for $t = m\Delta t$, Δt small enough.

According to theorem 1

$$\rho(\mu_t, \delta_{\omega_n}(t)) \leq e^{Ct}\rho(\overset{o}{\mu}, \delta_{\overset{o}{\omega}_n});$$

therefore

$$\rho(\mu_t, \delta_{\hat{\omega}_n}(t)) \leq \rho(\mu_t, \delta_{\omega_n}(t)) + \rho(\delta_{\omega_n}(t), \delta_{\hat{\omega}_n}(t)) \leq \varepsilon + e^{Ct}\rho(\overset{o}{\mu}, \delta_{\overset{o}{\omega}_n}).$$

Consequently, PIC converges in the sense that for given $\varepsilon > 0$, $T > 0$ one can choose Δt small enough and n large enough such that

$$\rho(\mu_t, \delta_{\hat{\omega}_n}(t)) < \varepsilon \qquad \text{for } t = m\Delta t, \ 0 \leq m\Delta t \leq T.$$

This entails that the numerically calculated electric field E_n approximates the real electric field E uniformly, i.e.

$$\|E_n(m\Delta t, \cdot) - E(m\Delta t, \cdot)\|_o < \varepsilon \qquad \text{for } 0 \leq m\Delta t \leq T, \ \Delta t \text{ sufficiently small.}$$

Looking a bit more carefully, how the constant $C = K + L$ in the preceding estimate depends on δ and the dimension k, one gets

$$\rho(\mu_t, \delta_{\omega_n}(t)) \leq e^{C(k,\delta)t}\rho(\overset{o}{\mu}, \delta_{\overset{o}{\omega}_n})$$

with

$$C(k,\delta) = \sqrt{2}\left(1 + \frac{\sqrt{k}}{\delta^k}\left(1 - \frac{\sqrt{2}}{\delta^{k-1}}\right)\right).$$

If δ is chosen near to 1 (i.e. in physical dimensions: approximately equal to the socalled Debye length), then $C(k,\delta)$ is of order $\sqrt{\bar{k}}$.

The only thing which remains to be done and which is of big practical importance is to find an appropriate $\overset{o}{\omega}_n$ for given $\overset{o}{\mu}$. The problem of finding the optimal $\overset{o}{\omega}_n$ with respect to all n-particle systems has not even been attacked.

So we will discuss a method to construct a "good" (not a best) $\overset{o}{\omega}_n$ for given $\overset{o}{\mu} = \int \overset{o}{f}\, dP$. For the special case $\overset{o}{f} = \chi_E$, where E is the unit cube in $\mathbb{R}^{2k}$, $E = \{P \in \mathbb{R}^{2k} \mid 0 \le p_i < 1,\ i=1,\ldots,2k\}$ and χ_E is the characteristic function of E, there are some results available. Therefore we reduce the general case to the special one.

We assume, that there exists a convex domain B in $\mathbb{R}^{2k}$ such that $\overset{o}{f}(P) > 0$ for $P \in B$, $\overset{o}{f}(P) = 0$ for $P \notin B$. Then one is able to construct a mapping $T : B \longrightarrow E$, which has the following properties:
T is differentiable and bijective, and its Jacobian is

$$J_T(P) = \overset{o}{f}(P), \quad P \in B.$$

Then $\overset{o}{\mu} = \mu_E \circ T$, where $\mu_E = \int \chi_E dP$ is the uniform distribution in E:

$$\overset{o}{\mu}(M) = \int_B \chi_M \overset{o}{f}\, dP = \int_B \chi_M J_T dP = \int_E \chi_M \circ T^{-1} dP = \int_E \chi_{T(M)} dP = \mu_E(T(M))$$

for arbitrary M.

The construction of T is given in a paper by Hlawka and Mück in 1972. It is simple when $\overset{o}{f}$ factorises, i.e.

$$\overset{o}{f}(P) = \overset{o}{f}_1(p_1) \ \ldots \ \overset{o}{f}_{2k}(p_{2k}) \qquad (p_i = x_i \text{ and } p_{k+i} = v_i \text{ for }$$
$$i=1,\ldots,k).$$

Then

$$T_i(P) := T_i(p_i) = \int_{-\infty}^{p_i} \overset{o}{f}_i(\xi)d\xi, \quad i=1,\ldots,2k, \quad P \in B.$$

If now $\tilde{\omega}_n = \{\tilde{P}_1,\ldots,\tilde{P}_n\}$ is a good approximation for μ_E, then $\overset{o}{\omega}_n := \{T^{-1}\tilde{P}_1,\ldots,T^{-1}\tilde{P}_n\}$ is a good approximation for $\overset{o}{\mu}$: If T^{-1} is Lipschitz continuous with a Lipschitz constant λ , then

$$\rho(\delta_{\overset{o}{\omega}_n}, \overset{o}{\mu}) \le \lambda \rho(\delta_{\tilde{\omega}_n}, \mu_E).$$

T^{-1} is relatively easy to calculate in the special case mentioned

above. The discrepancy, defined in (6) is also a metric which measures the distance of $\delta_{\substack{o \\ \omega_n}}$ to $\overset{o}{\mu}$. For the purposes discussed here it is easier to handle. For example, for a factorising $\overset{o}{f}$, a set $R \in \mathcal{R}$ is transformed by T into a set of the same kind whose corner Q is in E. It follows that for these kind of initial conditions even the quality

$$D(\delta_{\substack{o \\ \omega_n}}, \overset{o}{\mu}) = D(\delta_{\substack{\sim \\ \omega_n}}, \mu_E)$$

holds.

Therefore, it remains to construct $\delta_{\substack{\sim \\ \tilde{\omega}_n}}$. We have to point out, that this problem cannot be solved in a nice way by using a random generator: The only property $\tilde{\omega}_n$ has to have is, that the discrepancy $D(\delta_{\substack{\sim \\ \tilde{\omega}_n}}, \mu_E)$ is small, therefore things like correlations are irrelevant. $\tilde{\omega}_n$ might be distributed very regularly - this is what physicists call a "quiet start".

There is a method proposed by Niedereiter how to calculate $D(\delta_{\substack{\sim \\ \tilde{\omega}_n}}, \mu_E)$ without taking the l.u.b. over all $R \in \mathcal{R}$; a finite number of property selected R is enough.

Just to give you an example how real calculations are done, we consider the one dimensional case k=1. For n we choose only the Fibonacci numbers $n = \alpha_k$, $k \in \mathbb{N}$ with $\alpha_o = \alpha_1 = 1$, $\alpha_{k+1} = \alpha_k + \alpha_{k-1}$.

Then we choose $\{\tilde{P}_1, \ldots, \tilde{P}_n\}$ with $\tilde{P}_i = (x_i, v_i)$ in the following way:

$$x_i = \frac{2i-1}{2\alpha_k}, \quad v_i = [\frac{2(i-1)\alpha_{k-1}+1}{2\alpha_k}], \quad i = 1, \ldots, \alpha_k = n,$$

where $[\xi]$ denotes the fractional part of the positive real number ξ.

One obtains the following values for $D(\delta_{\substack{\sim \\ \tilde{\omega}_n}}, \mu_E)$

k	$\alpha_k = n$	$D(\delta_{\substack{\sim \\ \omega_n}}, \mu_E)$
12	144	$1,5 \cdot 10^{-2}$
14	377	$7,6 \cdot 10^{-3}$
16	987	$3,3 \cdot 10^{-3}$
18	2584	$1,4 \cdot 10^{-3}$
20	6765	$5,6 \cdot 10^{-4}$

For 6765 points, generated by the IBM-random-generator, the discrepancy is $1,4 \cdot 10^{-2}$; that means, that you need only 144 points with the construction given above to obtain a similar result. The computing time however differs by a factor 47.

In the table given above one realizes that the convergence of $D(\delta_{\tilde{\omega}_n}, \mu_E) = D_n$ to zero is rather slow. In fact it has been shown by numbertheoretists, that there exists a constant C_k, depending only on the dimension, such that

$$D_n > \frac{C_k}{n} (\ln n)^{\frac{k-1}{2}}$$

So, one cannot expect a very quick convergence of the method. Nevertheless, it is better than most of the other methods used in praxis - and its convergence is proved!

6. Stationary Solutions

Not very much is known about stationary (i.e. time independent) solutions. The only exeption is the one-dimensional case, where all solutions can be explicitly constructed. These solutions, discovered in 1957, are the socalled "Bernstein-Green-Kruskal"-modes (BGK-modes).

The problem is to construct all solutions of the system

(34a) $\quad v \frac{\partial f_-}{\partial x} + E(x) \frac{\partial f_-}{\partial v} = 0$

(34b) $\quad v \frac{\partial f_+}{\partial x} - E(x) \frac{\partial f_+}{\partial v} = 0$

(35) $\quad c_- \int\limits_{-\infty}^{+\infty} f_-(x,v)dx - c_+ \int\limits_{-\infty}^{+\infty} f_+(x,v)dv = E'(x),$

where c_- und c_+ are some positive constants. f_-, f_+ are the distribution functions of the electrons and the ions respectively. We consider here a two-component gas, because in this case the solutions become even simpler.

By solving (35) together with some boundary values for E and substituting the solution into (34) one realizes that the problem is again nonlinear.

However, the main idea is to make the problem linear by prescribing appropriate data of f. If we look for solutions of (34), (35) with

$$(36) \qquad \rho(x) := c_- \int_\infty^\infty f_-(x,v)\,dv - c_+ \int_{-\infty}^{+\infty} f_+(x,v)\,dx$$

given, then E'(x) is known from the data, E is - up to a boundary condition - a known function and we have to solve the linear equation (34) with the rather unusual condition (36).

For a given E, let us consider (34a). As f_ depends only on two variables x and v, we know that a function f_ is a solution of (34a) if and only if it depends only on the Hamiltonian

$$(37) \qquad H(x,v) = v^2 - 2U(x)$$

where U is an arbitrary integral of E

$$(38a) \qquad f_-(x,v) = \phi_-(v^2 - 2U(x)).$$

Similarly f+ has to be of the form

$$(38b) \qquad f_+(x,v) = \phi_+(v^2 + 2U(x)).$$

The figure on the following page gives you a typical picture for the level curves of f_+, f_- for a given U. To different branches of such a level curve one might assign different values of the functions $f_\pm$. These level curves for f_+ degenerate to a single point at x_1, where U has a local minimum, if we choose $c = 2U(x_1)$. For $c < 2U(x_1)$ the level curve is an empty set.
The level curves have an intersection with the x-axis, if $\frac{c}{2}$ is contained in the range of U; at these intersection points, the level curves are orthogonal to the x-axis.

The level curves for f_ behave somewhat antisymmetric. They degenerate at maximum points x_o of U, do not exist for $c < -2U(x_o)$ and intersect with the x-axis, if $\frac{c}{2}$ is contained in the range of -U.

We restrict our considerations for a moment to an interval of strict monotonicity of U, say $[x_o, x_1]$ and choose an arbitrary ξ in (x_o, x_1).

Every point (x,v) in the strip $[\xi, x_1] \times \mathbb{R}$ lies on a level curve of f_-, which passes $x = \xi$. This is not true for the level curves of f_+.
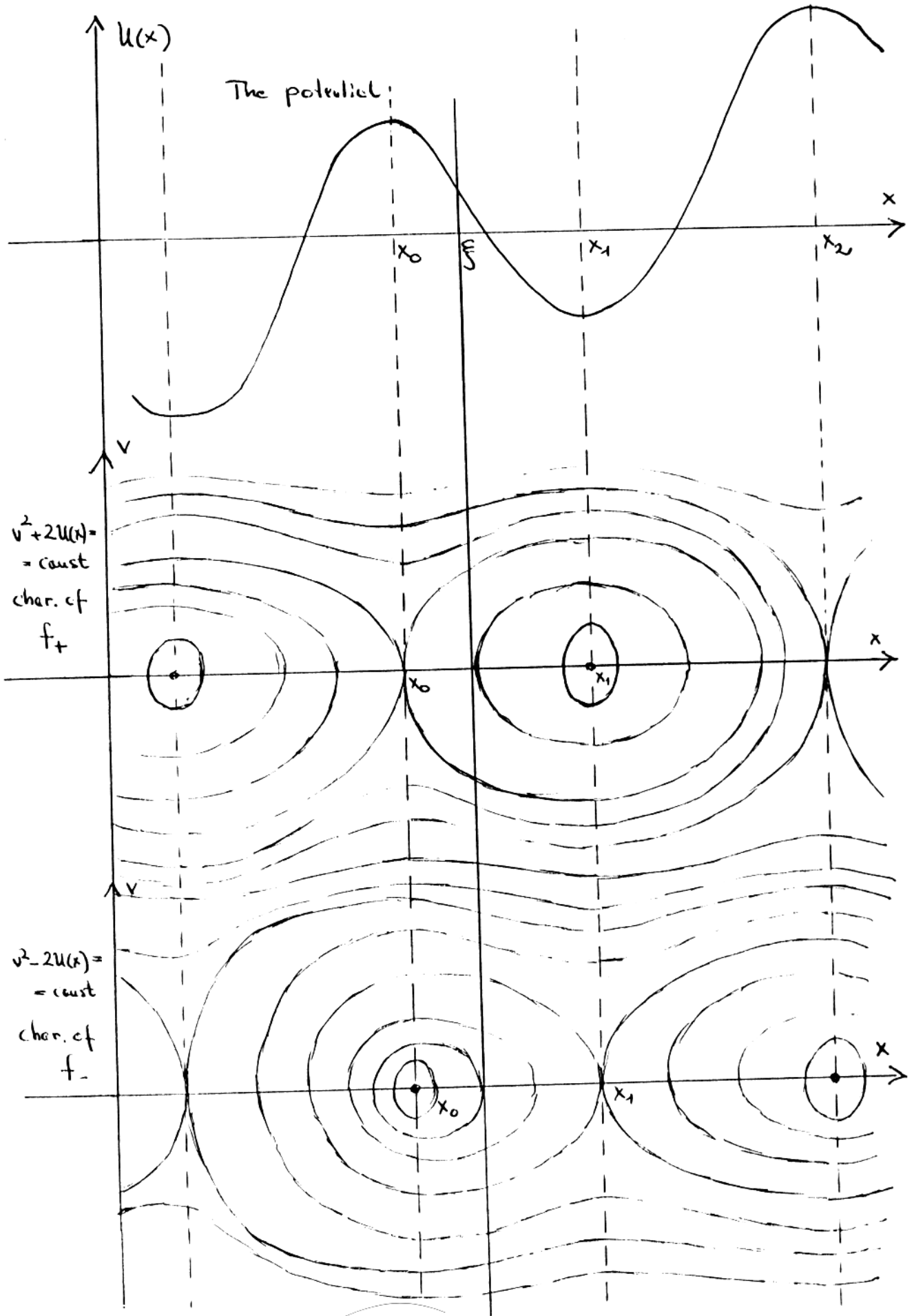
The level curve passing $(\xi, 0)$ is given by

$$v^2 + 2(U(x) - U(\xi)) = 0.$$

Therefore points $(x,v) \in [\xi, x_1] \times \mathbb{R}$ with

$$|v| < \sqrt{2(U(\xi) - U(x))}$$

are not reached by curves passing $(\xi, 0)$.

If, in addition to ρ according to (36), we prescribe the boundary values

$$(39) \qquad f_+(\xi, v) = \varphi_+(v), \quad f_-(\xi, v) = \varphi_-(v),$$

then f_- is completely determined in $[\xi, x_1] \times \mathbb{R}$; f_+ is determined only for (x,v) with $\xi \leq x \leq x_1$, $|v| \geq \sqrt{2(U(\xi) - U(x))}$.

Therefore $\int_{-\infty}^{+\infty} f_-(x,v) dv$ is determined by φ_-, $\int_{|v| \geq \sqrt{2(U(\xi)-U(x))}} f_+(x,v) dv$ by φ_+.

In order to satisfy (36), we have to solve the equation

$$\int_{-\sqrt{2(U(\xi)-U(x))}}^{+\sqrt{2(U(\xi)-U(x))}} f_+(x,v) dv = -\frac{\rho(x)}{c_+} + \frac{c_-}{c_+} \int_{-\infty}^{+\infty} f_-(x,v) dv - \int_{|v| \geq \sqrt{2(U(\xi)-U(x))}} f_+(x,v) dv =: h(x)$$

h is known in $[\xi, x_1]$; using (38b), we get

$$2 \int_{0}^{\sqrt{2(U(\xi)-U(x))}} \phi_+(v^2 + 2U(x)) dv = h(x) .$$

With $v^2 + 2U(x) = t$, we find

$$\int_{2U(x)}^{2U(\xi)} \phi_+(t) \frac{dt}{\sqrt{t - 2U(x)}} = h(x) .$$

U is strictly monotone in x; so $2U$ has an inverse function W and we obtain

$$\int_{y}^{2U(\xi)} \phi_+(t) \frac{dt}{\sqrt{t-y}} = h(W(y)) .$$

This is Abel's integral equation, which has the unique solution

$$(40) \quad \phi_+(z) = \frac{1}{\pi} \frac{d}{dz} \int_{z}^{2U(\xi)} \frac{h(W(t))}{\sqrt{z-t}}, \quad 2U(x) \leq z \leq 2U(\xi) .$$

We do not want to work out all the details. However, we note that $f_+(x,v) = \phi_+(v^2 + 2U(x))$ is now completely known:

For $|v| \geq \sqrt{2(U(\xi)-U(x))}$ it is given by φ_+, for $|v| < \sqrt{2(U(\xi)-U(x))}$ it is determined via (40).

Consequently f_+, f_- are determined in $[\xi, x_1] \times \mathbb{R}$ by (36), (39).

We can continue the procedure, e.g., to the right of x_1. $f_+(x_1, v)$, $f_-(x_1, v)$ are known. Therefore $f_+(x, v)$ is determined for $(x, v) \in [x_1, x_2] \times \mathbb{R}$, where x_2 is the point of the next maximum. In that strip f_- can be constructed by solving Abel's integral equation. In that way, we get the solution everywhere.

We have to mention, that we are looking for what is called a "mild" solution of (34), i.e. a solution, which is constant along the characteristics (the level curves) and continuous.
We get the following

Theorem 7: Suppose $E \in C^{(2)}(\mathbb{R})$ such that $E''(x) = O(E(x))$ for $x \to \eta$
for any η with $E(\eta) = 0$, and assume $\varphi_+ \in C_0 \cap L_1$ with the following properties:

(a) If $\Lambda_+(\xi) = \sup(U(x)-U(\xi))$, $\Lambda_-(\xi) := -\inf(U(x)-U(\xi))$,
then φ_+ is even and Hölder continuous for $\frac{v^2}{2} > \Lambda_+(\xi)$

(b) $E'(\xi) = c_- \int\limits_{-\infty}^{+\infty} {}_-dv - c_+ \int\limits_{-\infty}^{+\infty} \varphi_+ dv$

Then the system (34), (35) has a unique mild solution, where E is the given function and $f_\pm(\xi, v) = \varphi_\pm(v)$.

Remark: The assumptions on E and φ_+ are natural and necessary to get a mild solution. This solution can be explicitly constructed and is called a BGK-mode.
There is only one weak point: There are no simple conditions for E (or ρ respectively) and $\varphi_\pm$, such that $f_\pm$ is everywhere non-negative. So one really has to calculate the solutions to be sure that they are physically relevant.

I want to show you another derivation of the solution, which in my opinion is nice and shows something about the increase of complexity in higher dimensions.

In order to simplify the subject we consider again the interval $[x_0, x_1]$, where U is strictly monotonically decreasing, and let $\xi \in [x_0, x_1]$.

Again we assume, that E and f_- are already known in $[\xi, x_1] \times \mathbb{R}$, so we have to solve

$$v \frac{\partial f_+}{\partial x} - E \frac{\partial f_+}{\partial v} = 0$$

with $f_+(\xi, v) = \varphi_+(v)$ and $\int\limits_{-\infty}^{+\infty} f_+(x, v) dv = \rho_+(x)$ given.

Let us forget about characteristics but make a Fourier transform of the equation with respect to v:

With $u(x, y) := \int\limits_{-\infty}^{+\infty} e^{-ivy} f(x, v) dv$, we get formally

$$iu_{xy} - E i y u = 0$$

with $u(\xi, y) = \hat{\varphi}_+(y) = \int\limits_{-\infty}^{+\infty} e^{-ivy} \varphi_+(v) dv$ and $u(x, 0) = \rho_+(x)$.

We end up with a characteristic "initial value" problem for a hyperbolic equation:

$$(41) \quad u_{xy} - E(x) y u = 0, \quad u(\xi, y) = \hat{\varphi}_+(y), \quad u(x, 0) = \rho_+(x)$$

There is a simple trick to solve this equation, a trick, which seems not to be well-known: If you have to solve $u_{xy} + f(x) \cdot g(y) u = 0$, substitute

$$u(x, y) = U(F(x), G(y)),$$

where F, G are integrals of f, g respectively. Then you get

$$U_{\xi\eta} + U = 0$$

and the Riemann function for that equation is just

$$G(\xi, \eta; \xi', \eta') = I_0(2\sqrt{(\xi - \xi')(\eta - \eta')}).$$

Therefore, you get as a Riemann function for the original problem

$$(42) \quad G(x, y; x', y') = I_0(2\sqrt{(U(x) - U(x'))(\frac{y^2}{2} - \frac{y'^2}{2})})$$

where I_0 is the zeroth order Besselfunction of first kind. G is always real, even if the argument is complex.

The Riemann function G is defined to be a solution of $u_{xy} - E(x) y u = 0$ with respect to the variables x, y and satisfies the boundary

conditions

$$G(x',y,x',y') = G(x,y',x',y') = 1 \quad \text{for all } x,y,x',y'.$$

Using (42), one gets the solution of the "characteristic" boundary value problem (41) as

$$u(x,y) = d\, I_0\,(\sqrt{2\,(U(\xi)-U(x))}\,y) + \int_{\xi}^{x} I_0\,(\sqrt{2\,(U(t)-U(x))}\,y)\,\rho_{+}^{!}(t)\,dt$$

$$+ \int_{0}^{y} I_0\,(\sqrt{2\,(U(\xi)-U(x))\,(y^2-t^2)}\,)\,\hat{\varphi}_{+}^{!}(t)\,dt$$

with $d := u(\xi,0) = \hat{\varphi}_{+}(0) = \rho_{+}(\xi)$.

It is possible to calculate explicitly the inverse Fourier transform of u in order to get f_+. For example, the first term yields

$$\frac{d}{\pi}\ \frac{1}{\sqrt{2\,(U(\xi)-U(x))-v^2}}\quad \text{if } 2\,(U(\xi)-U(x))-v^2 > 0 \text{ and } 0 \text{ otherwise.}$$

Note the Hamiltonian. This approach yields exactly the same explicit solution of the problem as the method using Abel's integral equation.

What about higher dimensions?

If one is interested in a very special kind of solutions - those depending only on the Hamiltonian, i.e. for example

$$f_{-}(x,v) = \phi_{-}(\,\|v\|^2 - 2U(x))$$

everything works like in the one dimensional case; instead of Abel's integral equation, one gets an equation of the form

$$g(z) = \int_{Y}^{a} k(t)\,(t-y)^{\frac{k-2}{2}}\,dt, \quad \text{where } k=1,2,3$$

is the dimension.

This equation can be easily solved by using fractional derivates of g (for k=2 there is nothing to solve!) and one gets again explicit solutions, the socalled "more-dimensional BGK-modes".

But - in contrast to the one-dimensional case - these solutions are by far not all solutions of the problem - besides the energy, there are more integrals for the Newtonian system, which cannot be given explicitly. For example Horst remarked, that if k=2 and $\tilde{\phi}$ is a stationary solution for the stellardynamic case (for example a two-

dimensional BGK-mode), then

$$\phi(x,v) = \tilde{\phi}(x,v)(1+ \Psi(x_1 v_2 - x_2 v_1))$$

is also a stationary solution, if $\Psi: \mathbb{R} \longrightarrow [-1,1]$ is an odd function.
If one tries to play the Fourier transform trick, one gets for k=2

$$u_{x_1 y_1} + u_{x_2 y_2} \pm <E(x_1,x_2)y>u = 0.$$

This may be transformed to

$$\Lambda_\xi \tilde{u} - \Lambda_\eta \tilde{u} \pm <E(\xi+\eta),\xi-\eta>\tilde{u} = 0, \quad \xi := \frac{x+y}{2}, \quad \eta = \frac{x-y}{2}.$$

The difference of the two Laplacians show that this is a socalled
"ultrahyperbolic" equation; not very much is known about that kind
of partial differential equations. In particular, it is hard to
decide, what kind of boundary value problems are properly posed -
remember, that we need $u(x,0) = u(\xi,\xi)$ as data to linearize the
problem. The problem of determining all stationary solutions in more
than one dimension stays widely open.

Final remarks: Almost all of the material presented in these lectures
is mainly of mathematical interest. Now questions of real physical
interest arise. f.e.:

1) What is the qualitative character of a global solution? How does
   the system behave in a long run of time? When is it periodic?

2) Which of the stationary solutions are stable, which are not?

The second question can only be asked with respect to the BGK-modes.
A lot of results concerning the problem of linear stability are
available, most of them not really rigorous, but they can eventually
be made rigorous with some effort. Nothing rigorous is known on non-
linear stability as far as I know. But this should be a field of
great interest with respect to physical applications, especially
since BGK-modes have really been observed in a plasma a few years
ago.

Concerning the first question there is almost nothing known today.
The only exception is an example given by Kurth, which I want to put
on the end of these lectures. We consider the stellardynamic case
and are interested in a solution $f(t,\cdot)$, whose spatial density $\rho(t,\cdot)$

is constant in a certain ball whose diameter $r = r(t)$ depends on time

$$\rho(t,x) = \frac{1}{\frac{4\pi}{3}} \, r^{-3}(t) \qquad \text{if} \quad \|x\| < r(t)$$

and

$$\rho(t,x) = 0 \quad \text{otherwise.}$$

Choose $r(t)$ as a solution of

$$r^3 \ddot{r} + r = 1, \quad r(0) = 1, \quad \dot{r}(0) = H = \text{const.}$$

This solution can be calculated, depending on H. Then define

$$f(t,x,v) = \begin{cases} \frac{3}{2^3}[1 - \frac{\|x\|^2}{r^2(t)} - \|r(t)v - \dot{r}(t)x\|^2 + \|x \times v\|^2]^{-\frac{1}{2}} \\ \qquad\qquad\qquad \text{if} \quad \|x\| < r(t) \\ \\ 0 \qquad\qquad \text{otherwise} \end{cases}$$

Then f is a solution of the Vlasov-equation, whose spatial density is of the desired form. H might be interpreted as the (dimensionless) Hubble constant. As you may realize there is a lot of astrophysical experience needed to find such a solution. Its behaviour strongly depends on H: For $H < 1$ it behaves periodically, for $H > 1$ the density ρ goes to zero with respect to the $L^\infty(\mathbb{R}^3)$ - Norm.

One might realize by considering this example how complicated the answers for example on questions of the stability or instability of solutions might be.
Nevertheless, working in the field of the Vlasov equation one follows somehow the interest and the efforts of most of the great mathematicians of the 17th and 18th century: Most of the work they did were concerned with astronomy, the explanation of the behaviour of the universe (compare for example the highly interesting book "Mathematics, The Loss of Certainty" by Morris Kline, New York 1980). Therefore, the research on the Vlasov equation could be a challenge even for modern mathematicians.

LITERATURE

For the first lecture:

[1]   Braun, W.; Hepp K.: The Vlasov dynamics and its fluctuations
        in the 1/N limit of interacting classical particles. Comm.
        Math. Phys. 56 (1977) 101-113.

[2]   Dobrushin, R.L.: Vlasov's equation. Functional Anal. and
        its Appl. 13 (1979) 115-123.

[3]   Neunzert, H.: Neuere qualitative und numerische Methoden in
        der Plasmaphysik. Vorlesungsmanuskript, Paderborn 1975.

[4]   Neunzert, H.: Mathematical investigations on particle-in-
        cell methods. Fluid Dynamics Transactions 9, Warzawa 1978.

[5]   Spohn, H.: On the Vlasov Hierarchy. Math. Meth. in the Appl.
        Sci. 3 (1981).

For the second lecture:

Besides [3] and [4]

[6]   Batt, J.: Ein Existenzbeweis für die Vlasov-Gleichung der
        Stellardynamik bei gemittelter Dichte. Arch. Rational Mech.
        Anal. 13 (1963) 296-308.

[7]   Hannoschöck, G.; Rautmann, R.: Schwache Lösungen und Fehler-
        abschätzungen für die Vlasov-Gleichung bei gemittelter
        Dichte. Z. Angew. Math. Mech. 56 (1976) T254-T256.

[8]   Horst, E.: Zum statistischen Anfangswertproblem der Stellar-
        dynamik. Diplomarbeit München 1975.

[9]   Rautmann, R.: Zur globalen Lösbarkeit der Vlasovschen Glei-
        chung für Medien geringer Dichte mit relativistischer Korrek-
        tur. Z. Angew. Math. Mech. 48 (1968) T276-T279.

[10]   Strampp, W.: On Vlasov's equation with mollified density
        for an electron gas in an exterior magnetic field.
        Preprint 1978.

For the third lecture:

[11]   Arsen'ev, A.A.: Global existence of a weak solution of Vlasov's system of equations. U.S.S.R. Computational Math. and Math. Phys. 15 (1975) 131-143.

[12]   Arsen'ev, A.A.: Existence and uniqueness of the classical solutions of Vlasov's system of equations. U.S.S.R. Computational Math. and Math. Phys. 15 (1975) 252-258.

[13]   Batt, J.: Ein Existenzbeweis für die Boltzmann-Vlasov-Gleichung im eindimensionalen Fall. Berichte der Kernforschungsanlage Jülich Jül-126-Ma, 1963.

[14]   Batt, J.: Global symmetric solutions of the initial value problem of stellar dynamics. J. Differential Equations 25 (1977) 342-364.

[15]   Cooper, J.; Klimas A.: Boundary Value Problem for the Vlasov-Maxwell Equation in One Dimension. Journal of Math. An. and Appl. 75 (1980).

[16]   Horst, E.: Zur Existenz globaler klassischer Lösungen des Anfangswertproblems der Stellardynamik. Dissertation München 1979.

[17]   Horst, E.: On the existence of global classical solutions of the initial value problem of stellar dynamics. In "Math. Problems in the Kinetic Theory of Gases" (ed. by D.C. Pack and H. Neunzert), 1980.

[18]   Horst, E.: On the Classical Solutions of the Initial Value Problem for the Unmodified Non-Linear Vlasov Equation. Math. Meth. in the Appl. Sci. 3 (1981) 229-248.

[19]   Iordanskii, S.V.: The Cauchy problem for the kinetic equation of plasma. Amer. Math. Soc. Transl. Ser. 2 35 (1964) 351-363.

[20]   Illner, R.; Neunzert, H.: An Existence Theorem for the Unmodified Vlasov Equation. Math. Meth. in the Appl. Sci. 1 (1979) 530-554.

[21] Kurth, R.: Das Anfangswertproblem der Stellardynamik. Z. Astrophys. 30 (1951) 213-229.

[22] Strampp, W.: Special symmetric solutions of Vlasov's equation. Preprint 1978.

[23] Ukai, S.; Okabe, T.: On classical solutions in the large in time of two-dimensional Vlasov's equation. Osaka J. Math. 15 (1978) 245-261.

[24] Wollmann, S.: Global-in-time solutions on the two-dimensional Vlasov-Poisson system. Comm. Pure Appl. Math.

For the fourth lecture:

[25] Neunzert, H.; Petry, K.H.: Ein Existenzsatz für die Vlasov-Gleichung mit selbstkonsistentem Magnetfeld. Math. Meth. in the Appl. Sci. 2 (1980) 429-444.

[26] Petry, K.J.: Zur Existenz von Maßlösungen eines Anfangswertproblems für die gemittelte Vlasov-Gleichung. Dissertation Kaiserslautern 1980.

For the fith lecture:

Besides [3] and [4]

[27] Denavit, J., Walsh, J.M.: "Non-random initialization of particl codes." To appear in: Comm. Plasma Physics and Controlled Fusion

[28] Kürschner, P.: Numerische Lösung der Vlasov-Gleichung mit finiten Elementen. Dissertation Karlsruhe 1977.

[29] Neunzert, H.; Wick, J.: Die Darstellung von Funktionen mehrerer Veränderlicher durch Punktmengen. Berichte der Kernforschungsanlage Jülich Nr. 996 (1973).

[30] Neunzert, H.; Wick, J.: Die Theorie der asymptotischen Verteilung und die numerische Lösung von Integrodifferential-gleichungen. Num. Math. 21 (1973) 234-243.

[31]   Neunzert, H.; Wick, J.: The Convergence of Simulation
       Methods in Plasma Physics. In "Mathematical Methods of
       Plasmaphysics" (ed. by R. Kress and J. Wick), 1980.

[32]   Wick, J.: Zur Anwendung der Approximation durch endliche
       Punktmengen auf die Lösung von Integro-Differentialglei-
       chungen. Berichte der Kernforschungsanlage Jülich Nr. 1124
       (1974).

       For the last lecture:

[33]   Bernstein, I.B.; Green, J.M.; Kruskal, M.D.: Exact nonlinear
       plasma oscillations. Phys. Rev. 108 (1957).

[34]   Horst, E.: Private communication.

[35]   Kurth, R.: A global particular solution to the initial-
       value problem of stellar dynamics. Quarterly of Applied
       Math. Oct. 1978.

[36]   Neunzert, H.: Über ein Anfangswertproblem für die stationäre
       Boltzmann-Vlasov-Gleichung. Berichte der Kernforschungsanlage
       Jülich Nr. 297 (1965).