

Lattice Boltzmann Simulation of Plane Strain Problems

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The Lattice Boltzmann Method (LBM), e.g. in [3] and [4], can be interpreted as an alternative method for the numerical solution of partial differential equations. The LBM is usually applied to solve fluid flows. However, the interpretation of the LBM as a general numerical tool, allows to extend the LBM to solid mechanics as well. In this spirit, the LBM has been studied in recent years. First publications [5], [6] present a LBM scheme for the numerical solution of the dynamic behavior of a linear elastic solid under simplified deformation assumptions. For so-called anti-plane shear deformation, the only non-zero displacement component is governed by the two-dimensional wave equation. In this work, the existing LBM for the two-dimensional wave equation is extended to more general plane strain problems. The algorithm reduces the plane strain problem to the solution of two separate wave equations for the volume dilatation and the non-zero component of the rotation vector, respectively.

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1 Representing the Plane Strain Problem by Wave Equations

For a linear elastic body with density ρ and Lamé parameters λ and μ under plane strain assumption, the volume dilatation $\nabla \cdot \mathbf{u} = \phi$ as well as the only non-zero component of the rotation vector $(\nabla \times \mathbf{u})_z = \psi$ are governed by the separate wave equations

$$c_d^2 \Delta \phi = \frac{\partial^2 \phi}{\partial t^2}, \quad \text{where } c_d = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad (1)$$

and

$$c_s^2 \Delta \psi = \frac{\partial^2 \psi}{\partial t^2}, \quad \text{where } c_s = \sqrt{\frac{\mu}{\rho}}. \quad (2)$$

Herein, \mathbf{u} denotes the displacement field. The dilatation and the rotation vector are coupled via the Navier equation

$$c_d^2 \nabla (\nabla \cdot \mathbf{u}) - c_s^2 \nabla \times (\nabla \times \mathbf{u}) = \ddot{\mathbf{u}}, \quad (3)$$

which results from the balance of linear momentum, Hooke's law for isotropic linear elastic material and the definition of the linearized strain tensor.

2 A Lattice Boltzmann Method for Plane Strain Problems

In the LBM, information is represented by distribution functions that are defined on a discretized lattice. The lattice consists of spatially discrete lattice points, which are connected via lattice links α . The lattice links are associated with a lattice speed c^α , that determines to which neighbor information may travel in one time step Δt . The distribution functions are updated via the explicit rule

$$f^\alpha(\mathbf{x} + c^\alpha \Delta t, t + \Delta t) = f^\alpha(\mathbf{x}, t) - \frac{1}{\tau} [f^\alpha(\mathbf{x}, t) - f_{\text{eq}}^\alpha(\mathbf{x}, t)], \quad (4)$$

where τ is a relaxation time. The distribution functions must be interpreted in relation to the macroscopic fields. We consider two sets of distribution functions for the dilatation ϕ and the non-zero displacement component ψ of the rotation vector. These sets of distribution functions are interpreted according to

$$\sum_{\alpha} f_{\psi}^{\alpha} = \psi, \quad \text{and} \quad \sum_{\alpha} f_{\phi}^{\alpha} = \phi. \quad (5)$$

The particular mesoscopic evolution law is then given by (4), where the equilibrium distribution functions $f_{\psi,\text{eq}}^{\alpha}$, $f_{\phi,\text{eq}}^{\alpha}$ and the relaxation times τ_{ψ} and τ_{ϕ} have to be chosen such that (4) together with the interpretation (5) yields the desired macroscopic

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behavior (1) and (2). In order to accomplish this, we choose the LBM for the wave equation, proposed by [2]. This model is two-dimensional and has five lattice velocities at each lattice point, i.e. it is referred to as a D2Q5-model. The same lattice and time step Δt is used for both sets of distribution functions. This is possible due to the fact, that the model [2] allows to adjust the macroscopic wave speed independently of the lattice spacing and the time step by adapting the f_{eq} .

The overall algorithm computes the accelerations $\ddot{\mathbf{u}}^n$ at a particular time step t_n from the Navier equation (3) and finite difference approximations for $\nabla\phi^n$ as well as $\nabla \times \psi^n$. The acceleration $\ddot{\mathbf{u}}^n$ at boundary lattice points needs to be determined from boundary conditions. For Neumann boundary conditions this involves the evaluation of a local balance of momentum. Once $\ddot{\mathbf{u}}^n$ is known at each lattice point, the displacement \mathbf{u}^{n+1} is computed by integration with the explicit Newmark scheme. The LBM method is required to update the dilatation and the rotation, i.e. to determine ψ^{n+1} and ϕ^{n+1} , at each lattice point.

As a test for the algorithm, we consider a quadratic domain

$$\Omega = \left\{ (x, y) \mid -\frac{L}{2} \leq x + \frac{L}{2} \wedge -\frac{L}{2} \leq y + \frac{L}{2} \right\} \quad (6)$$

that is subjected to a tensile traction load $\mathbf{t}^* = \sigma(t)\mathbf{e}_y$ at the top and the bottom edges, where $\sigma(t) = 0.01 \mu u_r t/LT$ for $t \leq T$ and $\sigma(t) = 0.01 \mu u_r/L$ otherwise. The time scale is given by $T = L/c_s$ and u_r is a reference displacement. The Lamé parameters have identical value, i.e. $\lambda = \mu$.

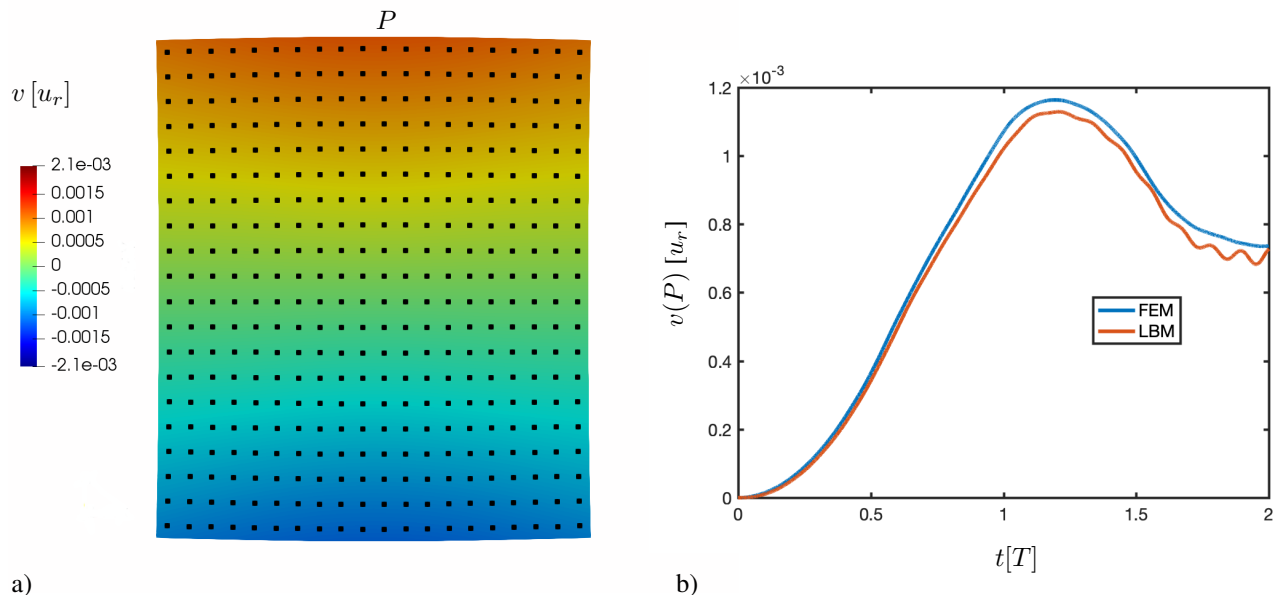


Fig. 1: Comparison of the LBM and the FEM for a tension loaded quadratic domain. a) displays the deformed configuration at $t = 1.0T$ for the LBM (dots represent the lattice points) and the FEM (contour plot of the vertical displacement component v in the background). The deformation is scaled. b) displays the displacement component in vertical direction v at point $P = (0, L/2 - \Delta x/2)$ for LBM and FEM, where Δx is the lattice spacing.

The deformed configurations for the LBM simulation as well as for a FEM simulation for the same setup are depicted in Fig. 1 a). Note that the LBM is able to reproduce transversal contraction of the originally square domain and agrees generally well with the benchmark FEM results. This can also be verified in Fig. 1 b), which plots the vertical displacement at the top center location P . However, the LBM also reveals oscillations in the later stages of the simulation.

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